

Unifying Hidden-Variable Problems from Quantum Mechanics by Logics of Dependence and Independence

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Abstract

We study hidden-variable models from quantum mechanics and their abstractions in purely probabilistic and relational frameworks by means of logics of dependence and independence, which are based on team semantics. We show that common desirable properties of hidden-variable models can be defined in an elegant and concise way in dependence and independence logic. The relationship between different properties and their simultaneous realisability can thus be formulated and proven on a purely logical level, as problems of entailment and satisfiability of logical formulae. Connections between probabilistic and relational entailment in dependence and independence logic allow us to simplify proofs. In many cases, we can establish results on both probabilistic and relational hidden-variable models by a single proof, because one case implies the other, depending on purely syntactic criteria. We also discuss the ‘no-go’ theorems by Bell and Kochen-Specker and provide a purely logical variant of the latter, introducing *non-contextual choice* as a team-semantical property.

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1 Introduction

Hidden-variable models have been proposed since the 1920s as an alternative to the standard interpretation, sometimes called the Copenhagen interpretation, of quantum mechanics with the goal to explain and remove counterintuitive aspects of quantum mechanics. In particular, quantum systems behave, according to the standard interpretation, probabilistically rather than deterministically, non-local interactions between agents or particles that are widely separated in space are possible through entanglement, and there is an unavoidable dependence between an observer of a quantum mechanical system and the observed properties. Due to such features, Einstein, Podolsky, and Rosen [19] considered a quantum mechanical state an ‘incomplete’ description of physical reality. The basic idea of hidden-variable models is to ‘complete’ quantum mechanics by adding unobservable, ‘hidden’, variables to the description of a system, to obtain models that are consistent with the predictions of quantum mechanics, but which do not exhibit counterintuitive behavior.

The question to what extent hidden-variable models can indeed explain quantum mechanical effects in a satisfactory way, has been studied for a long time by many researchers. John von Neumann [35], who has coined the term ‘hidden-variable models’, claimed to have established the impossibility of such an endeavour:

It should be noted that we need not go any further into the mechanism of the ‘hidden parameters,’ since we now know that the established results of quantum mechanics can never be re-derived with their help.

However, this claim was not generally accepted. Some of the most outspoken criticism came from Grete Hermann, Davin Mermin, and John Bell [8], who even went as far as calling von Neumann’s proof “not merely false, but foolish” (see [14]). Further, David Bohm and Lois de Broglie developed non-standard, deterministic interpretations of quantum mechanics using hidden-variables referred to as Bohmian mechanics or de Broglie-Bohm theory [11, 12, 17].

Many different variants of determinism, locality, and independence properties of hidden-variable models have been studied. The attempts to build hidden-variable models that jointly realise these can be seen as attempts to avoid the counter-intuitive consequences of the standard formalism of quantum mechanics. However, the famous ‘no-go’ theorems of quantum mechanics, such as the ones by Bell [7] and Kochen-Specker [31], show that one cannot really escape such consequences, and that there are severe limitations for the hidden-variable programme in general. The dominant attitude today is to accept the quantum mechanical formalism at face value and to make use of its counter-intuitive features such as entanglement and contextuality in modern applications like quantum computing and quantum cryptology.

A survey of the most important properties of determinism, locality, and independence in hidden-variable models has been given in an influential paper by Brandenburger and Yanofsky [13], in an abstract and purely probabilistic framework that does not make explicit reference to quantum mechanics. It makes precise the relationship between the different properties and discusses the question which combinations of them can be realised simultaneously in a probabilistic hidden-variable model. Despite the general limitations of the hidden-variable programme, there are interesting combinations of properties that are jointly realisable, and the work of Brandenburger and Yanofsky [13] gives a detailed account of what is possible in a probabilistic setting. A further fundamental step for the understanding of hidden-variable phenomena has been Abramsky’s proposal of a purely relational (rather than probabilistic) framework [1] for hidden-variable models, with discrete analogues of the probabilistic dependence and independence properties studied in [13]. He showed that the main structure of the theory is preserved under this simplification. Abramsky’s work opens the possibility to study hidden-variable questions on a much more general level leaving aside quantum mechanical details. But on the other side, Abramsky also discusses the issue of quantum realisations of empirical models (which has also been studied on a more general level in [4]). An important point is that among the empirical models that cannot be extended to hidden-variable models with specific locality and independence properties there are models that are not just abstract mathematical constructions but do indeed arise from quantum mechanical experiments, such as Bell test.

We propose here a study of hidden-variable properties by means of modern logics of dependence and independence. These logics are based on team semantics, introduced by Hodges [29]. While a classical logical formula (from first-order logic, for instance) is evaluated for a single assignment, mapping its free variables to values in some mathematical structure, team semantics evaluates a formula for a *set* of such assignments, called a team. Further, while previous formalisms for studying dependence and independence (such as Henkin quantifiers or independence-friendly logic) had modeled dependencies by special quantifiers, the modern dependence and independence logics treat them, following a proposal by Väänänen [34], as *atomic properties of teams*. While these logics are syntactically very simple, extending the atomic team properties by the standard first-order operators \vee , \wedge , \exists and \forall , they are semantically rather powerful. Indeed, team semantics admits the manipulation of second-order objects by first-order syntax and, in fact, independence logic [24] has the full expressive power of existential second-order and can thus define all team properties in the complexity class NP [21].

There are several reasons why logics of dependence and independence are natural tools for reasoning about hidden-variable models. First of all, the desirable properties of hidden-variable models, in particular the ones studied in [1] and [13], are obviously properties of dependence and independence. We shall see that in the logics we are using, their definition is extremely simple and transparent, mostly just conjunctions of dependence or independence atoms. Second, the models studied in hidden-variable theories, both empirical models and hidden-variable models, and both in the relational and the probabilistic setting, can readily be understood and presented as teams. This means that formulae in dependence and independence logic can be directly evaluated on the models we are interested in. Moreover, it turns out that operators on the team side are naturally compatible with the structure of hidden-variable models – for instance, the connection between probabilistic and relational models corresponds directly to the relationship between relational and probabilistic team semantics.

Although our study does not establish new results on quantum mechanical hidden-variable models as such, we think that our spelling out of the connections between logics with team semantics and hidden-variable models in detail is useful and provides the following contributions:

- We show that the properties of empirical and hidden-variable models can be defined in a very concise and elegant way in logics of dependence and independence. This also helps to make implicit assumptions in the definition of such properties explicit.
- Our team-semantical framework enables a full unification of the probabilistic and relational theory. In fact, for all the properties that we study, the same formula can be used for both settings, evaluated over relational teams in the first case, and over probabilistic teams in the second case. This provides a very elegant perspective on the connection between relational and probabilistic hidden-variables, complementing earlier work on the connection between the two such as [4], which used a more algebraic approach based on distributions over semirings.
- Connections between different properties of hidden-variable models can be formulated in terms of *logical entailment* between the team-semantical formulae defining these properties and can thus be proved on a purely logical level. This may also be beneficial for formalising such proofs in an appropriate proof system or theorem prover.
- Similarly, the existence of an empirically equivalent hidden-variable model that satisfies some combination of desirable properties corresponds to the *satisfiability* of a suitable formula by an extension of the team which represents a given empirical model.
- On a purely logical level, we establish connections between probabilistic entailment and relational entailment of formulae from dependence and independence logic, which we believe to be of independent interest. Applying these connections to the entailment and satisfiability problems related to properties of hidden-variable models, we often need only one proof to establish both the probabilistic and the relational case because, depending on purely syntactic criteria, one case implies the other. This provides more general reasons why the relational variant of hidden-variable models is so closely related to their probabilistic counterpart, and further motivates Abramsky’s translation as a special case of a more general framework.
- We further show that the famous ‘no-go’ theorems by Bell and Kochen-Specker can be formulated in a natural way in the team semantical framework. In particular we provide logical variants of the Kochen-Specker Theorem, highlighting the aspect of contextuality in a different way compared to the classical formulation, and discuss the related notion of *non-contextual choice*.

By different methods, not directly related to team semantics, but based on sheaf theory, a full unification of the relational and probabilistic theory has already been obtained in [4]. This work further gives a unified treatment of non-locality and contextuality in a very general setting. It remains to be seen whether meaningful connections between this approach and logics with team semantics can be established.

Acknowledgments. Researchers studying logics of dependence and independence have been aware for some time of the possibility to use these logics for reasoning about hidden-variable models in quantum mechanics, and this idea was informally discussed in this research community at several occasions. However, when we started our work, no systematic study in this direction had been conducted yet. Only at a point where this paper was almost finished, it was brought to our attention that Joni Puljujärvi and Jouko Väänänen at the University of Helsinki have simultaneously pursued a similar line of research, and we had interesting discussions with them on this topic. Meanwhile they have written a joint paper with Samson Abramsky on this work [6]. We also wish to thank an anonymous referee for valuable comments and additional references that helped us to improve this paper.

2 Hidden-Variable Models and Teams

We introduce the setup required to understand both relational and probabilistic hidden-variable models. We also define teams and explain how hidden-variable models can be cast as teams. Finally, we show that the structure of hidden-variable models is compatible with operators on the team side and thereby demonstrate that teams are very well suited to express hidden-variable phenomena.

2.1 Relational Models and Teams

A purely relational setting to speak about hidden-variable models was introduced by Abramsky [1].

► **Definition 2.1.** Let M_1, \dots, M_n and O_1, \dots, O_n be finite sets. We set $M = \prod_{i=1}^n M_i$ and $O = \prod_{i=1}^n O_i$. An arbitrary relation $e \subseteq M \times O$ is called an empirical model over (M, O) . In this setting, n is called the arity of the system, M the measurement set, and O set of outcomes.

We usually interpret such a system as having n components which may but need not be separated in space. We interpret M_i as the set of measurements that can be performed in component i and O_i as potential outcomes in that component. The measurement-outcome pairs $(\bar{m}, \bar{o}) \in e$ are interpreted to be *possible* in the model. To make sure that the underlying sets M and O can be inferred from the empirical model, we tacitly assume that every value $m_i \in M_i$ and $o_j \in O_j$ appears in at least one tuple of the relation e ; in database terms this means that $\bigcup_{i \leq n} M_i \cup \bigcup_{i \leq n} O_i$ coincides with the active domain of e .

► **Example 2.2.** Let us consider a system with 2 components, called Alice and Bob. Let $M_1 = \{a_1, a_2\}$ and $M_2 = \{b_1\}$. Thus, Alice has a choice between two measurements while Bob can perform only one. In this example, all measurements reveal a single bit of information, i.e. we let $O_1 = O_2 = \{+, -\}$. An example of an empirical model over (M, O) is

e	(+, +)	(+, -)	(-, +)	(-, -)
(a_1, b_1)	1	0	0	1
(a_2, b_1)	0	1	1	0

In this example, the outcome of Alice's measurement is always identical to Bob's if Alice chooses a_1 and always opposite to Bob's if she chooses a_2 .

► **Definition 2.3.** *Let M and O be as in Definition 2.1, and let Λ be a finite set. A hidden-variable model over (M, O, Λ) is a relation $h \subseteq M \times O \times \Lambda$. The elements $\lambda \in \Lambda$ are called hidden-variables.*

The interpretation of hidden-variable models is similar to that of empirical models. The hidden-variables are assumed to be unobservable parameters of the system that may influence outcomes. A triplet $(\bar{m}, \bar{o}, \lambda) \in h$ means that the measurement-outcome pair (\bar{m}, \bar{o}) is possible in the system when the hidden-variable takes the value λ . Again, we tacitly assume that every $\lambda \in \Lambda$ appears in at least one tuple of h . The original motivation behind the introduction of hidden-variables was that seemingly unintuitive phenomena on the empirical side in quantum mechanics might be explained by ignorance about the hidden-variables and not by an intrinsically non-classical system. However, as we shall see later, certain non-classical phenomena remain even if hidden-variables are introduced.

Hidden-variable models induce empirical models by projecting back onto the empirically observable parameters $M \times O$, giving rise to a notion of empirical equivalence, i.e. a notion of which systems we can distinguish by experiment and which not.

► **Definition 2.4.** *Let h be a hidden-variable model over (M, O, Λ) . We call*

$$e := \{(\bar{m}, \bar{o}) \in M \times O : (\exists \lambda \in \Lambda)(\bar{m}, \bar{o}, \lambda) \in h\}$$

its induced empirical model, and say that h and e empirically equivalent.

Now that we have defined relational models, we present them as teams.

► **Definition 2.5.** *A team is a set X of assignments $s : D \rightarrow A$ with a common finite domain $D = \text{dom}(X)$ of variables and values in a set A . For a tuple of variables $\bar{x} = (x_1, \dots, x_m) \in X$, we write $X(\bar{x}) := \{(s(x_1), \dots, s(x_m)) : s \in X\} \subseteq A^m$ for the set of values of \bar{x} in X . Thinking of an arbitrary but fixed enumeration of the finite domain of a team X as $\text{dom}(X) = \{x_1, \dots, x_k\}$, we often identify X with its relational encoding $X(\bar{x}) = \{s(\bar{x}) : s \in X\} \subseteq A^k$ and assignments $s \in X$ with corresponding tuples.*

With that in mind, the interpretation of hidden-variable models as teams is very natural. To make sure that the underlying sets of measurements $M = \prod_{i=1}^n M_i$, outcomes $O = \prod_{i=1}^n O_i$, and hidden-variables Λ can be inferred from the team we again assume that every $m_i \in M_i, o_j \in O_j$ and $\lambda \in \Lambda$ appears in at least one assignment of the team. Then we have a one-to-one correspondence between hidden-variable models and teams (and similarly for empirical models).

► **Definition 2.6.** *A team X over variables $\text{Var}_n^h := \{m_1, \dots, m_n, o_1, \dots, o_n, \lambda\}$ induces a hidden-variable model, denoted h_X , with $M_i := X(m_i)$, $O_i := X(o_i)$, and $\Lambda := X(\lambda)$ such that*

$$(\bar{a}, \bar{b}, c) \in h_X \quad :\iff \quad (\exists s \in X) s(\bar{m}) = \bar{a}, s(\bar{o}) = \bar{b}, \text{ and } s(\lambda) = c.$$

We denote by e_X the empirical model that is induced by h_X .

Analogously, empirical models $e \subseteq M \times O$ are represented by teams over $\text{Var}_n^e := \{m_1, \dots, m_n, o_1, \dots, o_n\}$.

2.2 Probabilistic Models and Probabilistic Teams

Systems arising from quantum mechanics are probabilistic in nature. Thus, the literature primarily investigates probabilistic hidden-variable models. A comprehensive discussion of such models can be found in [13]. However, the relevant definitions of probabilistic models and their properties are presented here in a somewhat different way, on the basis of our team semantical framework.

► **Definition 2.7.** *Let (M, O) be as in Definition 2.1. A probability distribution $e_{\mathbb{P}}: M \times O \rightarrow [0, 1]$ is called a probabilistic empirical model over (M, O) . Analogously, a probabilistic hidden-variable model is a probability distributions $h_{\mathbb{P}}$ over $M \times O \times \Lambda$. We make use of standard notation for marginalization and conditionals when speaking about probabilistic models. For example, for a probabilistic hidden-variable model $h_{\mathbb{P}}: M \times O \times \Lambda \rightarrow [0, 1]$, we call the marginalization $e_{\mathbb{P}}: M \times O \rightarrow [0, 1]$ with $e_{\mathbb{P}}(\bar{m}, \bar{o}) := h_{\mathbb{P}}(\bar{m}, \bar{o}) = \sum_{\lambda \in \Lambda} h_{\mathbb{P}}(\bar{m}, \bar{o}, \lambda)$ its induced empirical model. In this case, $h_{\mathbb{P}}$ and $e_{\mathbb{P}}$ are called empirically equivalent.*

The intuition behind empirical equivalence is that empirically equivalent models cannot be distinguished by experiment as they agree on the observable parameters of the system. For this purpose, the conditional distributions $h_{\mathbb{P}}(\bar{o} \mid \bar{m})$ are actually slightly more relevant than the joint probabilities $h_{\mathbb{P}}(\bar{o}, \bar{m})$ as they define the outcome distributions for fixed measurements; $h_{\mathbb{P}}(\bar{m})$ has no meaningful interpretation without going into discussions about the experimenters' free will. Thus, the literature regards a slightly different notion of empirical equivalence. [13] defines an empirical model $e_{\mathbb{P}}$ and a hidden-variable model $h_{\mathbb{P}}$ to be empirically equivalent if they agree on all suitable conditional probabilities, i.e. if $h_{\mathbb{P}}(\bar{o} \mid \bar{m}) = e_{\mathbb{P}}(\bar{o} \mid \bar{m})$ always holds. We think that the definition we have chosen, i.e. requiring the slightly stronger $h_{\mathbb{P}}(\bar{o}, \bar{m}) = e_{\mathbb{P}}(\bar{o}, \bar{m})$, is more natural and elegant for our purposes since empirical equivalence reduces to marginal equivalence in the standard probability theoretic sense. For all relevant results it makes no difference which choice is made. The probability distribution $h_{\mathbb{P}}(\bar{m}, \bar{o}, \lambda)$ can be decomposed into $h_{\mathbb{P}}(\bar{m})$, $h_{\mathbb{P}}(\lambda \mid \bar{m})$ and $h_{\mathbb{P}}(\bar{o} \mid \bar{m}, \lambda)$. For all properties of interest only the latter two components matter. Thus, we could adjust $h_{\mathbb{P}}(\bar{m})$ to enforce agreement with the empirical model without harming properties of $h_{\mathbb{P}}$. In particular, all existence and non-existence results which we will show later in this work are independent of this technical detail.

The connection between probabilistic and relational models is one of possibilistic collapse, i.e. we consider the set of tuples with probability greater than zero. This corresponds to our understanding of relational models where we interpret $(\bar{m}, \bar{o}) \in e$ as statement “it is possible in e to obtain the measurement-outcome pair (\bar{m}, \bar{o}) ”.

► **Definition 2.8.** *A probabilistic empirical model $e_{\mathbb{P}}$ over (M, O) induces the relational model $e := \{(\bar{m}, \bar{o}) \in M \times O : e_{\mathbb{P}}(\bar{m}, \bar{o}) > 0\}$. Analogously, probabilistic hidden-variable models induce relational hidden variable models.*

Again, we cast probabilistic models as teams, using the notion of probabilistic teams, which have been considered for instance in [18, 25, 26].

► **Definition 2.9.** *A probabilistic team is a pair $\mathbb{X} = (X, \mathbb{P}_{\mathbb{X}})$, where X is a relational team and $\mathbb{P}_{\mathbb{X}}: X \rightarrow (0, 1]$ is a probability distribution over X . X is referred to as the underlying team of \mathbb{X} . We denote \mathbb{P} instead of $\mathbb{P}_{\mathbb{X}}$ if the context is clear.*

As in the relational case, there is a one-to-one correspondence between probabilistic hidden-variable models and probabilistic teams.

► **Definition 2.10.** A probabilistic team $\mathbb{X} = (X, \mathbb{P})$ over variables $\text{Var}_n^h = \{m_1, \dots, m_n, o_1, \dots, o_n, \lambda\}$ induces a hidden-variable model $h_{\mathbb{X}}$ with $M_i := X(m_i)$, $O_i := X(o_i)$, and $\Lambda := X(\lambda)$, such that, for every $\bar{a} \in M$, $\bar{b} \in O$, and $c \in \Lambda$

$$h_{\mathbb{X}}(\bar{a}, \bar{b}, c) := \begin{cases} \mathbb{P}(s) & \text{if } s(\bar{m}, \bar{o}, \lambda) = (\bar{a}, \bar{b}, c) \\ 0 & \text{if } (\bar{a}, \bar{b}, c) \notin X(\bar{m}, \bar{o}, \lambda) \end{cases}$$

We denote the underlying probabilistic empirical model by $e_{\mathbb{X}}$.

2.3 Compatibility of Models and Teams

Team semantics admits elegant formulations of hidden-variable phenomena because the structure of hidden-variable and empirical models nicely interplays with operators on the team side. First, we observe the following relationship between probabilistic and relational hidden-variable models.

► **Lemma 2.11.** Let $\mathbb{X} = (X, \mathbb{P})$ be a probabilistic team over Var_n^h and let $h_{\mathbb{X}}$ be the corresponding hidden-variable model. Then X is the team representation of the induced relational model h of $h_{\mathbb{X}}$.

Indeed, X is the underlying support team of \mathbb{X} , containing all assignments with non-zero probability, and this corresponds to the notion of possibilistic collapse. This holds of course also for empirical models. Further, we observe that the induced empirical model of a hidden-variable model corresponds to the restriction to Var_n^e on the team side.

► **Lemma 2.12.** Let X be a relational team over Var_n^h , let h_X be its corresponding hidden-variable model and e_X its induced empirical model. Then, $X \upharpoonright \text{Var}_n^e$ is the team representation of e_X . Similarly, let \mathbb{X} be a probabilistic team over Var_n^h representing the probabilistic hidden-variable model $h_{\mathbb{X}}$, and let $e_{\mathbb{X}}$ be its induced empirical model. Then, $\mathbb{X} \upharpoonright \text{Var}_n^e$ is the team representation of $e_{\mathbb{X}}$.

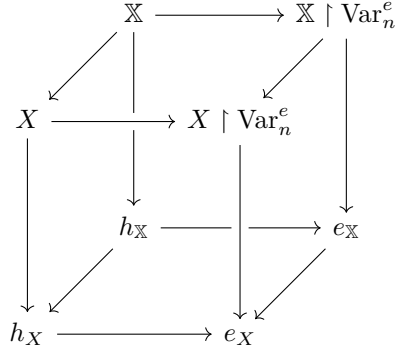
This allow us to cast properties of the underlying model as properties of the hidden-variable model. We can also formulate sort of a reverse.

► **Corollary 2.13.** Let X be a team over Var_n^e with corresponding model e_X . A team Y over Var_n^h represents an empirically equivalent hidden-variable model if, and only if, $Y \upharpoonright \text{Var}_n^e = X$. Analogously, let \mathbb{X} be a probabilistic team over Var_n^h . A team \mathbb{Y} over Var_n^h represents an empirically equivalent hidden-variable model if, and only if, $\mathbb{Y} \upharpoonright \text{Var}_n^e = \mathbb{X}$.

All these correspondences are neatly summarized by the commutative diagram in Figure 1.

3 Relational and Probabilistic Team Semantics

We now introduce the logical machinery that we need to reason about teams and the hidden-variable models that they represent. We first recall the main definitions for logics with relational and probabilistic team semantics. Then, we develop novel correspondences between the two frameworks that will then allow us to unify relational and probabilistic hidden-variables models and enable us to develop their theory in parallel.



■ **Figure 1** Compatibility of team and model structure

3.1 Relational Team Semantics

Modern logics of dependence and independence are based on atomic properties of teams and on an interpretation of the classical logical operators \wedge , \vee , \exists and \forall in the framework of teams. There are many atomic team properties that have been studied in the context of such logics. In this paper, we shall need only dependence, inclusion, and independence, which are defined as follows, for any team X , and for tuples $\bar{x}, \bar{y}, \bar{z}$ of variables in the domain of X .

Dependence: $X \models \text{dep}(\bar{x}, \bar{y})$ if for all $s, s' \in X$ such that $s(\bar{x}) = s'(\bar{x})$, also $s(\bar{y}) = s'(\bar{y})$;

Inclusion: $X \models \bar{x} \subseteq \bar{y}$ if $X(\bar{x}) \subseteq X(\bar{y})$;

Independence: $X \models \bar{x} \perp_{\bar{z}} \bar{y}$ if for all $s, s' \in X$ such that $s(\bar{z}) = s'(\bar{z})$, there exists some $s'' \in X$ such that $s''(\bar{z}) = s(\bar{z})$, $s''(\bar{x}) = s'(\bar{x})$, and $s''(\bar{y}) = s(\bar{y})$.

Thus $\text{dep}(\bar{x}, \bar{y})$ describes functional dependence of \bar{y} from \bar{x} , in the sense that there exist a function $f: X(\bar{x}) \rightarrow X(\bar{y})$ such that $s(\bar{y}) = f(s(\bar{x}))$ for every assignment $s \in X$. Independence however is more than the absence of dependence. The atom $\bar{x} \perp_{\bar{z}} \bar{y}$ expresses that for any fixed value for \bar{z} in the team, the values of \bar{x} and \bar{y} are completely independent in the sense that additional information about the value of \bar{x} does not constrain the possible values of \bar{y} in any way, and vice versa. This variant of independence is sometimes called *conditional independence*. A *simple independence atom* instead has the form $\bar{x} \perp \bar{y}$. We then have that $X \models \bar{x} \perp \bar{y}$ if, and only if, $X(\bar{x}\bar{y}) = X(\bar{x}) \times X(\bar{y})$, so any value for \bar{x} in the team co-exists with any value for \bar{y} in a common assignment in X .

Syntactically, dependence logic $\text{FO}(\text{dep})$ is built from dependence atoms $\text{dep}(\bar{x}, \bar{y})$ and first-order literals, i.e. atoms and negated atoms of a fixed vocabulary τ , by the operators \wedge , \vee , \exists and \forall . All formulae are written in negation normal form and negation is only applied to first-order atoms, not to dependence atoms. Independence logic $\text{FO}(\perp)$ is built analogously, using independence atoms $\bar{x} \perp_{\bar{z}} \bar{y}$ instead.

We now present the relational team semantics of $\text{FO}(\text{dep})$ and $\text{FO}(\perp)$. Recall that the traditional Tarski semantics for first-order formulae $\varphi(\bar{x})$ is based on single assignments s whose domain must comprise the variables in $\text{free}(\varphi)$; we write $\mathfrak{A} \models \varphi[s]$ for saying that \mathfrak{A} satisfies φ with the assignment s .

The *team semantics* for a formula $\varphi(\bar{x})$ in a logic of dependence and independence is instead defined by inductive clauses for the satisfaction relation $\mathfrak{A} \models_X \varphi$ saying that the

team X satisfies φ in \mathfrak{A} . Since the underlying structure \mathfrak{A} is not of interest for the purposes of this paper, we shall simplify notation and simply write $X \models \varphi$ instead. For a classical first-order literal, i.e. an atom or its negation, we simply say that $X \models \varphi$ if $\varphi[s]$ is true in the underlying structure for all $s \in X$. To present the inductive rules of team semantics, we shall need two basic operations that (possibly) extend the domain of a given team X (with values in A) to new variables.

► **Definition 3.1.** *Given an assignment s , a variable x , and a value $a \in A$, we write $s[x \mapsto a]$ for the assignment that extends, or updates, s by mapping x to a (and leaving the values of all other variables unchanged). The unrestricted generalisation of X over A is $X[x \mapsto A] := \{s[x \mapsto a] : s \in X, a \in A\}$, and the Skolem-extension for a function $F: X \rightarrow \mathcal{P}(A) \setminus \{\emptyset\}$ is $X[x \mapsto F] := \{s[x \mapsto a] : s \in X, a \in F(s)\}$.*

The following inductive rules then extend the team semantics of atomic team properties and first-order literals to arbitrary formulae in $\text{FO}(\text{dep})$ or $\text{FO}(\perp)$:

- $X \models \varphi_1 \wedge \varphi_2$ if $X \models \varphi_i$ for $i = 1, 2$;
- $X \models \varphi_1 \vee \varphi_2$ if $X = X_1 \cup X_2$ for two teams X_i such that $\mathfrak{A} \models_{X_i} \varphi_i$;
- $X \models \forall x \varphi$ if $X[x \mapsto A] \models \varphi$;
- $X \models \exists x \varphi$ if $X[x \mapsto F] \models \varphi$, for some suitable Skolem extension of X by a function $F: X \rightarrow \mathcal{P}(A) \setminus \{\emptyset\}$.

Since $\text{dep}(\bar{x}, \bar{y}) \equiv \bar{y} \perp_{\bar{x}} \bar{y}$, we can freely use dependence atoms in $\text{FO}(\perp)$, and it is known [21] that inclusion atoms $\bar{x} \subseteq \bar{y}$ are definable in $\text{FO}(\perp)$ as well. In fact, the logic $\text{FO}(\text{dep}, \subseteq)$ with both inclusion and dependence atoms is equivalent to $\text{FO}(\perp)$ and it suffices to use simple independence atoms to get the full power of $\text{FO}(\perp)$.

3.2 Probabilistic Team Semantics

To understand probabilistic hidden-variable phenomena, we make use of a probabilistic variant of team semantics. Our definitions are essentially the same as in [18, 25, 26] with deviations in some details. However, we stress the fact that we consciously select notations that invite jumping back and forth between probabilistic and relational semantics. In particular, we use precisely the same syntax for the probabilistic variants of $\text{FO}(\text{dep})$ and $\text{FO}(\perp)$, but we evaluate the formulae now over probabilistic teams $\mathbb{X} = (X, \mathbb{P}_{\mathbb{X}})$. To define the probabilistic semantics for these logics, we first have to describe the meaning of first-order literals and of dependence and independence atoms.

- For first-order literals φ , we let $\mathbb{X} \models \varphi$ if, and only if, $X \models \varphi$ in the relational sense.
- $\mathbb{X} \models \text{dep}(\bar{x}, \bar{y})$ if for all $\bar{a} \in X(\bar{x})$ there is a $\bar{b} \in X(\bar{y})$ such that $\mathbb{P}(\bar{y} = \bar{b} \mid \bar{x} = \bar{a}) = 1$. Note that this corresponds to a deterministic sense of dependence.
- $\mathbb{X} \models \bar{x} \perp_{\bar{z}} \bar{y}$ if we have conditional stochastic independence between \bar{x} and \bar{y} for any given \bar{z} . Formally, this means that for all $\bar{a} \in X(\bar{x}), \bar{b} \in X(\bar{y}), \bar{c} \in X(\bar{z})$,

$$\mathbb{P}(\bar{x} = \bar{a}, \bar{y} = \bar{b} \mid \bar{z} = \bar{c}) = \mathbb{P}(\bar{x} = \bar{a} \mid \bar{z} = \bar{c}) \cdot \mathbb{P}(\bar{y} = \bar{b} \mid \bar{z} = \bar{c}).$$

The probabilistic team semantics of the logical operators generalises the relational semantics. To make this precise, we have to give appropriate definitions for the split, the generalisation, and the Skolem extension of a probabilistic team.

► **Definition 3.2.** Let $\mathbb{X} = (X, \mathbb{P}_{\mathbb{X}})$ be a probabilistic team with A as its set of values, and let $\Delta(A)$ denote the set of all probability distributions over A . Consider a function $\mathcal{F}: X \rightarrow \Delta(A)$ that maps every $s \in X$ to a probability distribution $\mathcal{F}_s \in \Delta(A)$ on A . It induces a function $F: X \rightarrow \mathcal{P}(A) \setminus \{\emptyset\}$ via the support of the distributions \mathcal{F}_s , i.e. by setting $F(s) := \{a \in A : \mathcal{F}_s(a) > 0\}$. Then, the Skolem extension $\mathbb{X}[x \mapsto F]$ is defined as the probabilistic team over $X[x \mapsto F]$ with distribution

$$\mathbb{P}(s[x \mapsto a]) := \sum_{\substack{t \in X, \\ t[x \mapsto a] = s[x \mapsto a]}} \mathbb{P}_{\mathbb{X}}(t) \cdot F_t(a).$$

Note that the right-hand side simplifies to $\mathbb{P}_{\mathbb{X}}(s) \cdot F_s(a)$ if x is a new variable.

Intuitively, this means that the probability mass of s is split over multiple extensions $s[x \mapsto a]$ where the proportion assigned to each a is given by $F_s(a)$. Thus, we can interpret $F_s(a)$ as the probability of the event $x = a$ conditional on agreement with s on $\text{dom}(s)$. The function F tells us how we have to extend our underlying relational support team.

Note that the marginalization of Skolem extensions by \mathcal{F} gives back the original team. More formally, for $\mathbb{X} = (X, \mathbb{P})$ with $x \notin \text{dom}(X)$ and a Skolem extension $\mathbb{Y} = \mathbb{X}[x \mapsto \mathcal{F}]$, it holds that $\mathbb{X} = \mathbb{Y} \upharpoonright \text{dom}(X)$.

► **Definition 3.3.** The uniform extension $\mathbb{X}[x \mapsto A]$ of \mathbb{X} is the special case of a Skolem extension $\mathbb{X}[x \mapsto \mathcal{F}]$ where \mathcal{F} maps all $s \in X$ to the uniform distribution over A . Explicitly, we get the distribution

$$\mathbb{P}(s[x \mapsto a]) = \frac{1}{|A|} \sum_{\substack{t \in X, \\ t[x \mapsto a] = s[x \mapsto a]}} \mathbb{P}_{\mathbb{X}}(t).$$

Again this simplifies to $\mathbb{P}(s[x \mapsto a]) = \mathbb{P}_{\mathbb{X}}(s)/|A|$ if x is a new variable. This corresponds to a uniform split of the probability mass of s over all $|A|$ extensions.

► **Definition 3.4.** Let \mathbb{X} be a probabilistic team with X as its underlying team. We now define the probabilistic team semantics of the logical operators as follows:

- $\mathbb{X} \models \varphi_1 \wedge \varphi_2$ if $\mathbb{X} \models \varphi_i$ for $i = 1, 2$. Thus, conjunctions are defined in the straightforward way, as in relational team semantics.
- $\mathbb{X} \models \varphi_1 \vee \varphi_2$ if there are probabilistic teams $\mathbb{X}_1, \mathbb{X}_2$ and $\lambda \in [0, 1]$ with $X = X_1 \cup X_2$ and $\mathbb{P}_{\mathbb{X}} = (1 - \lambda)\mathbb{P}_{\mathbb{X}_1} + \lambda\mathbb{P}_{\mathbb{X}_2}$ such that $\mathbb{X}_i \models \varphi_i$ for $i = 1, 2$. In other words, this means that we can split \mathbb{X} into a convex combination of two teams that satisfy the formulae φ_i .
- $\mathbb{X} \models \forall x \varphi$ if $\mathbb{X}[x \mapsto A] \models \varphi$, i.e. if the uniform extension of \mathbb{X} satisfies φ .
- $\mathbb{X} \models \exists x \varphi$ if $\mathbb{X}[x \mapsto \mathcal{F}] \models \varphi$ for some $\mathcal{F}: X \rightarrow \Delta(A)$, i.e. if a suitable Skolem extension of \mathbb{X} satisfies φ .

The existential quantifier is most relevant for our purposes and we want to provide more intuition for it. For a variable $x \notin \text{dom}(X)$, $\mathbb{X} \models \exists x \varphi$ expresses the existence of another probabilistic team \mathbb{Y} over $\text{dom}(X) \cup \{x\}$ such that

1. \mathbb{Y} satisfies φ , and
2. \mathbb{Y} marginalizes to \mathbb{X} if restricted to $\text{dom}(X)$. This means that for all $s \in X$ we have that $\mathbb{P}_{\mathbb{X}}(s) = \sum_{a \in A} \mathbb{P}_{\mathbb{Y}}(s[x \mapsto a])$.

3.3 Back and Forth between Relational and Probabilistic Teams

We now compare probabilistic and relational team semantics. We start with the satisfaction relation. Note that it is essential for the formulation of this theorem that we use the same syntax for both variants.

► **Theorem 3.5.** *Let \mathbb{X} be a probabilistic team and X its underlying team. For every formula $\psi \in \text{FO}(\perp)$, we have that $\mathbb{X} \models \psi$ implies $X \models \psi$. For every $\varphi \in \text{FO}(\text{dep})$, we also have the converse, so that $\mathbb{X} \models \varphi$ if, and only if, $X \models \varphi$.*

Proof. We proceed by induction.

- For first-order literals φ , we obviously have that $\mathbb{X} \models \varphi$ if, and only if, $X \models \varphi$.
- Assume that $\mathbb{X} \models \bar{x} \perp_{\bar{z}} \bar{y}$. Now let $s, s' \in X$ with $s(\bar{z}) = s'(\bar{z}) = \bar{c}$. With $\bar{a} = s(\bar{x})$ and $\bar{b} = s'(\bar{y})$ we get that

$$\mathbb{P}(\bar{x} = \bar{a}, \bar{y} = \bar{b} \mid \bar{z} = \bar{c}) = \underbrace{\mathbb{P}(\bar{x} = \bar{a} \mid \bar{z} = \bar{c})}_{\geq \mathbb{P}(s) > 0} \cdot \underbrace{\mathbb{P}(\bar{y} = \bar{b} \mid \bar{z} = \bar{c})}_{\geq \mathbb{P}(s') > 0} > 0.$$

Since X is the support team of \mathbb{X} , there exists $s'' \in X$ with $s''(\bar{x}, \bar{y}, \bar{z}) = (\bar{a}, \bar{b}, \bar{c})$ which proves that $X \models \bar{x} \perp_{\bar{z}} \bar{y}$.

- For dependence atom, the implication from left to right follows from the fact that dependence atoms can be understood as special cases of independence atoms, the other direction is straightforward since X is defined to be the support of \mathbb{P} .

It remains to be shown that the logical operators preserve the implications in both directions. This is obvious for conjunctions.

- For $\mathbb{X} \models \psi \vee \vartheta$, we choose $\mathbb{Y} \models \psi$ and $\mathbb{Z} \models \vartheta$ with $\mathbb{X} = (1 - \lambda)\mathbb{Y} + \lambda\mathbb{Z}$ so that $X = Y \cup Z$. By induction hypothesis, $Y \models \psi$ and $Z \models \vartheta$, hence $X \models \psi \vee \vartheta$. Conversely, given a probabilistic team \mathbb{X} and a decomposition $X = Y \cup Z$ of the underlying team with $Y \models \psi$ and $Z \models \vartheta$, it is straightforward to construct \mathbb{Y} and \mathbb{Z} with underlying Y and Z such that $\mathbb{P}_{\mathbb{X}} = (1 - \lambda)\mathbb{P}_{\mathbb{Y}} + \lambda\mathbb{P}_{\mathbb{Z}}$ for some $\lambda \in [0, 1]$. By induction hypothesis $\mathbb{Y} \models \psi$ and $\mathbb{Z} \models \vartheta$, hence $\mathbb{X} \models \psi \vee \vartheta$.
- For formulae $\forall x\psi$, the induction step follows from the fact that the underlying team of $\mathbb{X}[x \mapsto A]$ is $X[x \mapsto A]$.
- If $\mathbb{X} \models \exists x\psi$ by means of $\mathcal{F}: X \rightarrow \mathcal{P}(A) \setminus \{\emptyset\}$, we put $F(s) = \{a \in A : \mathcal{F}_s(a) > 0\}$ to get the Skolem extension $X[x \mapsto F]$ as underlying team of $\mathbb{X}[x \mapsto \mathcal{F}]$, so $X[x \mapsto F] \models \psi$ and hence $X \models \exists x\psi$. Conversely, let $X \models \exists x\psi$, hence $X[x \mapsto F] \models \psi$ for some $F: X \rightarrow \mathcal{P}(A) \setminus \{\emptyset\}$. Let $\mathcal{F}: X \mapsto \Delta(A)$ be the function that maps every $s \in X$ to the uniform distribution over $F(s)$. Then, $\mathbb{X}[x \mapsto \mathcal{F}]$ has $X[x \mapsto F]$ as underlying team and the induction hypothesis is applicable. ◀

We next address the matter of logical entailment.

► **Definition 3.6.** *Let $\psi, \varphi \in \text{FO}(\perp)$. We write $\psi \models_{\text{rel}} \varphi$ if for all suitable relational teams X with $X \models \psi$ also $X \models \varphi$ holds. Analogously, we define $\psi \models_{\text{prob}} \varphi$ as entailment according to probabilistic team semantics. If both $\psi \models_{\text{rel}} \varphi$ and $\psi \models_{\text{prob}} \varphi$ hold, we write $\psi \models_{\text{all}} \varphi$.*

Results that are analogous to the theorem below have been established for databases but apparently not applied to teams so far.

► **Theorem 3.7.**

1. For formulae in $\text{FO}(\text{dep})$, \models_{rel} and \models_{prob} coincide.
2. If $\psi \in \text{FO}(\text{dep})$, $\varphi \in \text{FO}(\perp)$, then $\psi \models_{\text{prob}} \varphi \implies \psi \models_{\text{rel}} \varphi$.
3. If $\psi \in \text{FO}(\perp)$ and $\varphi \in \text{FO}(\text{dep})$, then $\psi \models_{\text{rel}} \varphi \implies \psi \models_{\text{prob}} \varphi$.
4. For $\varphi, \psi \in \text{FO}(\perp)$, neither implication holds in general. Conjunctions of conditional independence atoms suffice to generate counterexamples.

Proof. Notice that (2) and (3) follow immediately from Theorem 3.5 and that (1) follows by combining (2) and (3) since $\text{FO}(\text{dep}) \subseteq \text{FO}(\perp)$. (4) remains to be shown. We adapt formulae and counterexamples from [36].

We first show that, in general, $\psi \models_{\text{rel}} \varphi$ does not imply $\implies \psi \models_{\text{prob}} \varphi$. Let $\psi_1 = z \perp_x w \wedge z \perp_y w \wedge x \perp_{wz} y$ and $\varphi_1 = z \perp_{xy} w$. The intuition behind $\psi_1 \models_{\text{rel}} \varphi_1$ is that we can combine $z \perp_x w$ and $z \perp_y w$ to get $z \perp_{xy} w$ by using $x \perp_{wz} y$ to enforce agreement on xy .

Let $X \models \psi_1$ and $s_1, s_2 \in X$ with $(a, b) = s_1(x, y) = s_2(x, y)$. Further, let $c := s_1(z)$ and $d := s_2(w)$. Since $X \models z \perp_x w$, there exists $s_3 \in X$ with $s_3(x) = a$ and $s_3(z, w) = (c, d)$. Similarly, we obtain $s_4 \in X$ with $s_4(y) = b$ and $s_4(z, w) = (c, d)$ via $X \models z \perp_y w$. By applying $X \models x \perp_{wz} y$ on s_3 and s_4 , we finally obtain $s_5 \in X$ with $s_5(x, y, z, w) = (a, b, c, d)$ which shows that $X \models \varphi_1$. Thus, $\psi_1 \models_{\text{rel}} \varphi_1$. However, it is straightforward to verify that the probabilistic team given below satisfies ψ_1 but not φ_1 .

x	y	z	w		\mathbb{P}		
0	0	0	0		0.2		
0	0	0	1		0.2		
0	0	1	0		0.2		
0	0	1	1		0.1		
0	1	1	1		0.1		
1	0	1	1		0.1		
1	1	1	1		0.1		
$\psi_1 \not\models_{\text{prob}} \varphi_1$							

x	y	z	w	
0	0	0	0	
0	0	0	1	
0	1	0	0	
1	0	0	0	
1	1	0	0	
1	1	1	0	
$\psi_2 \not\models_{\text{rel}} \varphi_2$				

To prove that $\psi \models_{\text{prob}} \varphi$ does not necessarily imply that $\psi \models_{\text{rel}} \varphi$, we use the fact from [33, equation (A3), p. 15] that $\psi_2 \models_{\text{prob}} \varphi_2$, for $\psi_2 = x \perp_{yz} y \wedge z \perp_x w \wedge z \perp_y w \wedge x \perp_y$ and $\varphi_2 = z \perp w$ (proven in a different context with techniques employing measure theory and information theory). On the other side, a team proving that $\psi_2 \not\models_{\text{rel}} \varphi_2$ is given above (to the right). ◀

We remark that if $\varphi, \psi \in \text{FO}(\perp)$ are conjunctions of *simple independence atoms*, then $\psi \models_{\text{rel}} \varphi \iff \psi \models_{\text{prob}} \varphi$. This follows by observing that the proof calculi provided in the literature, see [22] for the relational and [23] for the probabilistic case, coincide.

4 Reasoning about Hidden-Variable Properties in Independence Logic

We now define and investigate properties of hidden-variable models by means of dependence and independence logic.

4.1 Logical Definitions of Properties of Hidden-Variable Models

In the literature, many properties of hidden-variable and empirical models are investigated. Given that these models can be seen as (relational or probabilistic) teams, we can define properties of such models by logical formulae that are evaluated over such teams. Actually, since natural properties are defined for models of arbitrary arity, their logical definition is not given by a single formula, but by a uniform family of such formulae, one for each arity. That is, for a property P of probabilistic hidden-variable models, we present a family of formulae $\psi_n \in \text{FO}(\perp)$, with free variables in Var_n^h such that for any probabilistic team \mathbb{X} that represents a hidden-variable model $h_{\mathbb{X}}$ of arity n , we have that $h_{\mathbb{X}}$ has property P if, and only if, $\mathbb{X} \models \psi_n$. Similarly, for empirical models, and for the relational case.

Actually, it is a fundamental observation that for all elementary properties, syntactically identical formulae work for the relational and the probabilistic case simultaneously. Also, our formulae are in fact very simple; essentially just conjunctions of atoms. This is quite elegant and another indication that team semantics is very well suited to express hidden-variable phenomena.

Properties of empirical models We start with presenting three fundamental properties that empirical models may or may not have. Note that in our formulae, the role of n is hidden in notations such as \bar{m} for $m_1 \dots m_n$ and \bar{o} for $o_1 \dots, o_n$. The following notation is helpful: For a tuple $\bar{z} = (z_1 \dots z_n)$, let $\bar{z}_{-i} = (z_1 \dots z_{i-1} z_{i+1} \dots z_n)$.

Weak Determinism: An empirical model is weakly deterministic if the combined measurements \bar{m} in all components deterministically determine the outcome \bar{o} of all components. However, this property does not forbid that the outcome of the i -th component depends on the measurement choice in components other than i . This property is defined by

$$\text{WeakDet}_n^e := \text{dep}(\bar{m}, \bar{o}).$$

Strong Determinism: If we strengthen Weak Determinism and require that the outcome of the i -th component is uniquely determined by the measurement choice in that component, we obtain the notion of Strong Determinism, defined by

$$\text{StrongDet}_n^e := \bigwedge_{i=1}^n \text{dep}(m_i, o_i).$$

No-Signalling: This property is a bit more complicated. Roughly speaking, the idea is to formalize that no information can be transmitted to component i by choice of measurement in components other than i . Thus, the outcome of the i -th component (which for example Alice receives) may, conditional on her choice of measurement, not depend on other measurements. An illustration of this point is given in Example 4.1 below. No-Signalling is closely related to the famous No-Communication Theorem in Quantum Mechanics. Formally,

$$\text{NoSig}_n^e := \bigwedge_{i=1}^n o_i \perp_{m_i} \bar{m}_{-i}.$$

It may not be completely obvious why the formulae NoSig_n^e have the intended meaning. Intuitively, an independence atom $\bar{x} \perp_{\bar{z}} \bar{y}$ expresses that conditional on knowing \bar{z} , getting information about \bar{x} does not give additional information about \bar{y} . Thus, NoSig_n^e states that if m_i is known, the other measurements \bar{m}_{-i} do not give additional information about o_i .

► **Example 4.1.** To illustrate how a violation of No-Signalling allows communication, consider the empirical model defined by the following team X :

m_1	m_2	o_1	o_2
a	b_1	+	-
a	b_2	-	-

In this case, $X \not\models o_1 \perp_{m_1} m_2$ which allows Bob to instantly transmit information to Alice by choice of his measurement. If Bob makes measurement b_1 , Alice receives a + in her experiment; if he chooses b_2 , Alice receives a -. In this configuration, Bob can send a full bit of information to Alice by means of his measurement choice; No-Signalling is rightfully violated. Also note that this example satisfies Weak Determinism but not Strong Determinism.

It is a relevant feature of team semantics, often called locality, that the meaning of a formula depends only on those variables that occur free in it. Thus, even if we evaluate a formula over a hidden-variable model, we can express properties of the underlying empirical model (if the variable λ is not used in the formula). Properties of the underlying empirical model can therefore be cast as properties of the hidden-variable model itself. This is made precise in the following proposition.

► **Proposition 4.2.** *Let \mathbb{X} be a probabilistic team over Var_n^h , and let $\psi^e \in \text{FO}(\perp)$ be a formula over Var_n^e that formalizes a property P of probabilistic empirical models. Then $\mathbb{X} \models \psi^e$ if, and only if, $e_{\mathbb{X}}$ has property P .*

Proof. By locality, $\mathbb{X} \models \psi^e$ is equivalent to $\mathbb{X} \upharpoonright \text{Var}_n^e \models \psi^e$. By Lemma 2.12, $\mathbb{X} \upharpoonright \text{Var}_n^e$ is the team representation of $e_{\mathbb{X}}$. ◀

By the same argument, the analogous statement for relational teams and models also holds.

Properties of hidden-variable models We now show, in a similar fashion, that common properties of hidden-variable models are definable in dependence and independence logic. The reader may want to compare our formula with the definitions in [1, 13]. Except in the case of locality (which we shall discuss separately), it is straightforward to see that our definitions capture the properties from the literature. We feel that our definitions by formulae of team semantics are more compact and precise, and they highlight the essential features better than an explicit unrolling of the semantic of dependence and independence atoms in every instance.

Weak Determinism: For hidden-variable models, Weak Determinism means that for every fixed value of a hidden-variable the chosen measurements in all components determine the outcomes. It is essential that the outcomes may also depend on the hidden-variable, which allows explaining non-deterministic empirical models by deterministic hidden-variable models with more fine-grained internal states. Formally

$$\text{WeakDet}_n^h := \text{dep}(\bar{m}\lambda, \bar{o}).$$

Strong Determinism: Strong Determinism strengthens Weak Determinism by requiring that for every fixed value of the hidden-variable the measurement in the i -th component determines the outcome in the i -th component. Formally,

$$\text{StrongDet}_n^h := \bigwedge_{i=1}^n \text{dep}(m_i\lambda, o_i).$$

Single-Valuedness: A hidden-variable model is single valued if the hidden-variable set Λ has only one element. Such a model is essentially just an empirical model, because a single-valued variable does not provide any additional freedom.

$$\text{SingVal}_n^h := \text{dep}(-, \lambda).$$

λ -Independence: This formalizes the idea that the measurement process and the hidden-variable should be independent. It is related to the intuition that the physical reality we measure shall be independent of the experimenters' choices. A particularly pathological counterexample to λ -Independence would be a system where the hidden-variable of the system uniquely determines which measurements will be done by the experimenters.

$$\lambda\text{-Indep}_n^h := \bar{m} \perp \lambda.$$

Outcome Independence: This property states that outcomes of measurements in different components shall be independent from each other when conditioning on $\bar{m}\lambda$. This means that when we perform the same measurements in the same hidden states, the outcomes of the various components shall not provide information about each other. Note that violation of this property is related to the phenomenon of quantum entanglement where we obtain correlations between outcomes even in case of spatially separated particles.

$$\text{Out-Indep}_n^h := \bigwedge_{i=1}^n o_i \perp_{\bar{m}\lambda} \bar{o}_{-i}.$$

Parameter Independence: This is essentially the hidden-variable analogue of No-Signalling. For fixed $m_i\lambda$, the outcome o_i of component i shall be independent of the measurements taken in the other components:

$$\text{Par-Indep}_n^h := \bigwedge_{i=1}^n o_i \perp_{m_i\lambda} \bar{m}_{-i}.$$

Locality: A hidden-variable model is local if all components are independent of each other. Roughly speaking, to understand a system satisfying Locality, one only needs to understand every component and piece them together independently. Jarrett has shown that this is adequately expressed by the conjunction of Parameter Independence and Outcome Independence [30]:

$$\text{Loc}_n^h := \text{Out-Indep}_n^h \wedge \text{Par-Indep}_n^h.$$

Heuristically speaking, Outcome Independence gives independence of the i -th component of the other outcomes while Parameter Independence gives independence of the other measurements. Put together, they give full independence the other components. This argument is made precise below.

► **Example 4.3.** We adapt Example 4.1 to a hidden-variable model satisfying Single-Valuedness:

m_1	m_2	o_1	o_2	λ
a	b_1	$+$	$-$	λ_1
a	b_2	$-$	$-$	λ_1

The resulting model does satisfy neither Parameter Independence nor Strong Determinism but satisfies Weak Determinism and Outcome Independence.

An intuitive and precise semantic characterization of Locality is shown in the following two lemmata for the relational and probabilistic case respectively.

► **Lemma 4.4.** *Let X be a relational team of arity n over Var_n^h . $X \models \text{Loc}_n^h$ if, and only if, for every tuple $(\bar{a}, c) \in X(\bar{m}, \lambda)$ and for all b_1, \dots, b_n with $b_i \in X(o_i)$ the following condition (*) holds: If there is an assignment $s_i \in X$ with $s_i(m_i, o_i, \lambda) = (a_i, b_i, c)$ for all i , then there is an $s \in X$ with $s(\bar{m}, \bar{o}, \lambda) = (\bar{a}, \bar{b}, \bar{c})$.*

Proof. Assume that $X \models \text{Loc}_n^h$, i.e., $X \models \text{Out-Indep}_n^h \wedge \text{Par-Indep}_n^h$. Let $(\bar{a}, c) \in X(\bar{m}, \lambda)$ and $b_i \in X(o_i)$ for every $i \leq n$. Let $s_i \in X$ with $s_i(m_i, o_i, \lambda) = (a_i, b_i, c)$. We need to construct some $t \in X$ with $t(\bar{m}, \bar{o}, \lambda) = (\bar{a}, \bar{b}, \bar{c})$. Since $(\bar{a}, c) \in X(\bar{m}, \lambda)$, there exists $\tilde{s} \in X$ with $\tilde{s}(\bar{m}, \lambda) = (\bar{a}, c)$. By applying Parameter Independence on s_i and \tilde{s} we get $\tilde{s}_i \in X$ which fulfills $\tilde{s}_i(\bar{m}, \lambda) = (\bar{a}, c)$ and $\tilde{s}_i(o_i) = b_i$. By inductively applying Outcome Independence, we can now construct a sequence t_1, \dots, t_n with $t_i(\bar{m}, \lambda) = (\bar{a}, c)$ and $t_i(o_1, \dots, o_i) = (b_1, \dots, b_i)$. Choosing $t := t_n$ completes the argument for the first implication.

For the converse, suppose that X satisfies condition (*). We prove that $X \models \text{Out-Indep}_n^h$. Let $s_1, s_2 \in X$ with $s_1(\bar{m}, \lambda) = s_2(\bar{m}, \lambda)$ and let $i \leq n$. Condition (*) immediately gives that s_3 given by $s_3(\bar{m}, \lambda) = s_1(\bar{m}, \lambda)$, $s_3(o_i) = s_1(o_i)$, and $s_3(\bar{o}_{-i}) = s_2(\bar{o}_{-i})$ is indeed in X ; this proves Outcome Independence. Parameter Independence is shown similarly. ◀

Condition (*) in Lemma 4.4 is a common alternative definition of Locality. In words it essentially states that the question whether a measurement-outcome pair (\bar{a}, \bar{b}) is possible for a fixed value of λ reduces to whether the measurement-outcome pair (a_i, b_i) is possible in every component i . However, a slight technical complication is that we require this to only hold for measurements \bar{a} that are possible for a fixed value of λ : We explicitly do not require that $(a_i, c) \in X(m_i, \lambda)$ for all i implies $(\bar{a}, c) \in X(\bar{m}, \lambda)$. The latter condition is called Measurement Locality in [1] and rather natural but not entailed in the usual definition of Locality.

► **Example 4.5.** Consider the hidden-variable model given by the following team:

m_1	m_2	o_1	o_2	λ
a	b	1	0	λ_1
a	b	1	1	λ_1
a	c	0	1	λ_1
a	c	0	0	λ_1
a	b	0	0	λ_2
a	c	0	1	λ_2

This model does not satisfy Locality: For the hidden-variable λ_1 , it is possible that a results in 0 and that b results in 1. However, $(ab, 01, \lambda_1)$ is not an element of the team even though $(ab, \lambda_1) \in X(\bar{m}, \lambda)$. Thus, Locality is falsified according to the previous lemma. Note that, in this instance, Parameter Independence is violated since the choice of measurement in the second component influences the possible results in the first component. However, Outcome Independence holds. Also, this team satisfies λ -Independence since all possible measurements, i.e. ab and ac , appear for both values of the hidden-variable.

In the probabilistic case, Locality means that the probability factors over the various components. This is in a sense a quantitative analogue of the qualitative statement from the previous lemma.

► **Lemma 4.6.** *Let $\mathbb{X} = (X, \mathbb{P})$ be a probabilistic team over Var_n^h . Then $\mathbb{X} \models \text{Loc}_n^h$ if, and only if the following condition (**) holds: For all $(\bar{a}, c) \in X(\bar{m}, \lambda)$ and all b_1, \dots, b_n with $b_i \in X(o_i)$*

$$\mathbb{P}(\bar{o} = \bar{b} \mid \bar{m} = \bar{a}, \lambda = c) = \prod_{i=1}^n \mathbb{P}(o_i = b_i \mid m_i = a_i, \lambda = c).$$

Proof. Assume that $\mathbb{X} \models \text{Loc}_n^h$. Let $(\bar{a}, c) \in X(\bar{m}, \lambda)$ and $b_i \in X(o_i)$ for $i \leq n$. To simplify notation, let $\bar{o}_k := o_1 \dots o_k$ and $\bar{b}_k := b_1 \dots b_k$. We apply induction over k to obtain

$$\begin{aligned} & \mathbb{P}(\bar{o}_{k+1} = \bar{b}_{k+1} \mid \bar{m} = \bar{a}, \lambda = c) \\ &= \mathbb{P}(o_{k+1} = b_{k+1} \mid \bar{o}_k = \bar{b}_k, \bar{m} = \bar{a}, \lambda = c) \cdot \mathbb{P}(\bar{o}_k = \bar{b}_k \mid \bar{m} = \bar{a}, \lambda = c) \\ &= \mathbb{P}(o_{k+1} = b_{k+1} \mid \bar{m} = \bar{a}, \lambda = c) \cdot \prod_{i=1}^k \mathbb{P}(o_i = b_i \mid \bar{m} = \bar{a}, \lambda = c) \\ &= \prod_{i=1}^{k+1} \mathbb{P}(o_i = b_i \mid \bar{m} = \bar{a}, \lambda = c) = \prod_{i=1}^{k+1} \mathbb{P}(o_i = b_i \mid m_i = a_i, \lambda = c), \end{aligned}$$

by applying Outcome Independence, Parameter Independence, and the induction hypothesis. Setting $k + 1 = n$ completes the argument.

For the converse, let \mathbb{X} satisfy condition (**). We first show that $\mathbb{X} \models \text{Par-Indep}_n^h$:

$$\begin{aligned} \mathbb{P}(o_i = b_i \mid \bar{m} = \bar{a}, \lambda = c) &= \sum_{\bar{b}_{-i}} \mathbb{P}(\bar{o} = \bar{b}, \bar{m} = \bar{a}, \lambda = c) \\ &= \sum_{\bar{b}_{-i}} \prod_{j=1}^n \mathbb{P}(o_j = b_j \mid m_j = a_j, \lambda = c) \\ &= \mathbb{P}(o_i = b_i \mid m_i = a_i, \lambda = c) * \prod_{j \neq i} \sum_{b_j} \mathbb{P}(o_j = b_j \mid m_j = a_j, \lambda = c) \\ &= \mathbb{P}(o_i = b_i \mid m_i = a_i, \lambda = c). \end{aligned}$$

Outcome Independence is then shown by the following equation for all i :

$$\begin{aligned} \mathbb{P}(o_i = b_i, \bar{o}_{-i} = \bar{b}_{-i} \mid \bar{m} = \bar{a}, \lambda = c) &= \prod_{j=1}^n \mathbb{P}(o_j = b_j \mid m_j = a_j, \lambda = c) \\ &= \mathbb{P}(o_i = b_i \mid m_i = a_i, \lambda = c) * \prod_{j \neq i} \mathbb{P}(o_j = b_j \mid m_j = a_j, \lambda = c) \\ &= \mathbb{P}(o_i = b_i \mid m_i = a_i, \lambda = c) * \mathbb{P}(\bar{o}_{-i} = \bar{b}_{-i} \mid \bar{m}_{-i} = \bar{a}_{-i}, \lambda = c) \\ &= \mathbb{P}(o_i = b_i \mid \bar{m} = \bar{a}, \lambda = c) * \mathbb{P}(\bar{o}_{-i} = \bar{b}_{-i} \mid \bar{m} = \bar{a}, \lambda = c), \end{aligned}$$

where the last equality follows by Parameter Independence. ◀

Unifying Relational and Probabilistic Properties. An advantage of our framework is that it gives an elegant logical argument why properties of probabilistic models carry over to the underlying relational models. This behaviour is, by Theorem 3.5, an immediate consequence of the fact that the properties in question are defined in both cases by syntactically the same formulae from independence logic. This novel and purely logical perspective complements earlier work on unifying relational and probabilistic properties employing algebraic and category-theoretical tools [4].

► **Proposition 4.7.** *Let $\psi \in \text{FO}(\perp)$ capture a property P of both probabilistic and relational models. If a probabilistic model satisfies P , its underlying relational model also satisfies P . For $\psi \in \text{FO}(\text{dep})$, the converse also holds.*

In particular, this holds for every single property defined in Section 4.1.

► **Corollary 4.8.** *If a probabilistic hidden-variable model satisfies any of the elementary properties defined above (e.g. λ -Independence, Locality, No-Signalling, ...), its underlying relational model also satisfies the corresponding property. For the properties which rely only on dependence atoms (Strong-Determinism, Weak-Determinism, Single-Valuedness), the converse also holds.*

The proposition applies of course much more generally than just to the handful of properties that we listed explicitly. Of course, every family of formulae in $\text{FO}(\perp)$ induces properties of relational and probabilistic models in a way that satisfies the assumptions of Proposition 4.7. Since independence logic is a rather powerful logic – recall that it can express all NP properties of teams by [21] – this induces a very large class of properties with the aforementioned relationship between probabilistic and relational models.

4.2 Connections between Properties via Logical Entailment

Implications of the form that “all models with property P also satisfy property Q ” have been proved for instance in [1] and [13]. In our framework, such statements are elegantly cast as entailments between formulae of independence logic $\text{FO}(\perp)$. Thus, they can be proven directly on the logical level. This allows us to make use of the relationship between \models_{rel} and \models_{prob} to develop the results for both cases in parallel.

► **Theorem 4.9.**

1. *Single-Valuedness implies λ -Independence:*

$$\text{SingVal}_n^h \models_{\text{all}} \lambda\text{-Indep}_n^h.$$

2. *Weak Determinism implies Outcome Independence:*

$$\text{WeakDet}_n^h \models_{\text{all}} \text{Out-Indep}_n^h.$$

3. *Strong Determinism corresponds to the conjunction of Weak Determinism and Parameter Independence:*

$$\text{StrongDet}_n^h \equiv_{\text{all}} \text{WeakDet}_n^h \wedge \text{Par-Indep}_n^h.$$

Proof. By Theorem 3.7, the probabilistic case suffices for (1) and (2). Claim (1) immediately follows from the fact that point distributions are stochastically independent of all distributions.

For (2) we need to show that $\text{WeakDet}_n^h \models_{\text{prob}} \text{Out-Indep}_n^h$. Let $\mathbb{X} = (X, \mathbb{P}) \models \text{WeakDet}_n^h$. For every $(\bar{a}, c) \in X(\bar{m}, \lambda)$ the distribution $\mathbb{P}(\bar{o} \mid \bar{m} = \bar{a}, \lambda = c)$ is a point distribution. This implies that $\mathbb{X} \models o_i \perp_{\bar{m}\lambda} \bar{o}_{-i}$ and thereby $\mathbb{X} \models \text{Out-Indep}_n^h$.

For (3) we first observe that the entailment $\text{StrongDet}_n^h \models \text{WeakDet}_n^h$ is obvious in both semantics. For $\text{StrongDet}_n^h \models_{\text{all}} \text{WeakDet}_n^h \wedge \text{Par-Indep}_n^h$, it remains to show that $\text{StrongDet}_n^h \models_{\text{prob}} \text{Par-Indep}_n^h$. But this is trivial as well since the distribution over o_i for a fixed pair (m_i, λ) assigns, by Strong Determinism, probability 1 to a single value. This suffices because point distributions are stochastically independent of every other distribution and thus in particular from the conditional distribution on \bar{m}_{-i} .

For the converse, we show that $\text{WeakDet}_n^h \wedge \text{Par-Indep}_n^h \models_{\text{rel}} \text{StrongDet}_n^h$. Let $X \models \text{WeakDet}_n^h \wedge \text{Par-Indep}_n^h$, and let $s_1, s_2 \in X$ agree on m_i and λ . Then, by Parameter Independence, there exists $s_3 \in X$ that takes the same value on m_i and λ and also satisfies $s_3(o_i) = s_1(o_i)$ and $s_3(\bar{m}_{-i}) = s_2(\bar{m}_{-i})$. This entails that $s_3(\bar{m}) = s_2(\bar{m})$ which in turn gives, by Weak Determinism, that $s_3(\bar{o}) = s_2(\bar{o})$ and in particular $s_1(o_i) = s_3(o_i) = s_2(o_i)$. ◀

It is instructive to think about the intuition for the classification of Strong Determinism as conjunction of Weak Determinism and Parameter Independence: Consider a model which satisfies Weak Determinism but not Strong Determinism. In this case, full measurements \bar{m} uniquely determine outcomes \bar{o} but single m_i do not determine o_i . However, it is then necessary that the other measurements \bar{m}_{-i} contain information about o_i which contradicts Parameter Independence. In a sense, the independence of other measurements is precisely the missing component to go from Weak Determinism to Strong Determinism.

Recall that we defined Locality as conjunction of Parameter Independence and Outcome Independence. Thus, we immediately obtain the following.

► **Corollary 4.10.** *Strong Determinism implies Locality:* $\text{StrongDet}_n^h \models_{\text{all}} \text{Loc}_n^h$.

Because of Proposition 4.2, we can formulate implications of the form “if h is a hidden-variable model satisfying P^h , then its underlying empirical model e satisfies P^e ” also by entailment of properties.

► **Theorem 4.11.** *The underlying empirical models of hidden-variable models that satisfy Parameter Independence and λ -Independence fulfill No-Signalling:*

$$\lambda\text{-Indep}_n^h \wedge \text{Par-Indep}_n^h \models_{\text{all}} \text{NoSig}_n^e.$$

Before we prove this we want to give some intuition. Recall that the only difference between Parameter Independence and No-Signalling is that the former requires independence of o_i and \bar{m}_{-i} conditional on (m_i, λ) instead of just m_i . Intuitively, we use the λ -Independence to obtain a “copy” of s_2 – namely \tilde{s}_2 – which agrees with s_1 on the hidden-variable to allow us to apply Parameter Independence.

Proof (relational case). Let $X \models \text{Par-Indep}_n^h \wedge \lambda\text{-Indep}_n^h$. Let $i \in \{1, \dots, n\}$ and $s_1, s_2 \in X$ with $s_1(m_i) = s_2(m_i)$. Choose by λ -Independence an assignment $\tilde{s}_2 \in X$ with $\tilde{s}_2(\lambda) = s_1(\lambda)$ and $\tilde{s}_2(\bar{m}) = s_2(\bar{m})$. Then $s_1(m_i, \lambda) = \tilde{s}_2(m_i, \lambda)$ and by applying Parameter Independence to s_1 and \tilde{s}_2 , we get $s_3 \in X$ with $s_3(o_i) = s_1(o_i)$ and $s_3(\bar{m}_{-i}) = \tilde{s}_2(\bar{m}_{-i}) = s_2(\bar{m}_{-i})$. Thus, $X \models \text{NoSig}_n^e$. ◀

Proof (probabilistic case). Let $\mathbb{X} = (X, \mathbb{P}) \models \text{Par-Indep}_n^h \wedge \lambda\text{-Indep}_n^h$ and $i \in \{1, \dots, n\}$. Let $o \in X(o_i)$, $a \in X(m_i)$, and $\bar{a}_{-i} \in X(\bar{m}_{-i})$. We put $\Lambda = X(\lambda)$. We have

$$\begin{aligned} \mathbb{P}(o_i = o \mid \bar{m} = \bar{a}) &= \sum_{c \in \Lambda} \mathbb{P}(o_i = o, \lambda = c \mid \bar{m} = \bar{a}) \\ &= \sum_{c \in \Lambda} \mathbb{P}(o_i = o \mid \lambda = c, \bar{m} = \bar{a}) * \mathbb{P}(\lambda = c \mid \bar{m} = \bar{a}) \\ &= \sum_{c \in \Lambda} \mathbb{P}(o_i = o \mid \lambda = c, m_i = a_i) * \mathbb{P}(\lambda = c \mid \bar{m} = \bar{a}) \quad (\text{by Par-Indep}_n^h) \\ &= \sum_{c \in \Lambda} \mathbb{P}(o_i = o \mid \lambda = c, m_i = a_i) * \mathbb{P}(\lambda = c \mid m_i = a_i) \quad (\text{by } \lambda\text{-Indep}_n^h) \\ &= \mathbb{P}(o_i = o \mid m_i = a_i) \end{aligned}$$

and thus $\mathbb{X} \models o_i \perp_{m_i} \bar{m}_{-i}$. ◀

5 Existence of Hidden-Variable Models via Satisfiability

We now investigate the question whether a given empirical models admits an empirically equivalent hidden-variable model with certain given properties. Again, our team semantical framework allows us to treat this in an elegant fashion. Essentially, the question reduces to the satisfiability of formulae of the form $\exists\lambda\psi$ on extensions of the given team by a finite set of values for the hidden variable λ , where ψ encodes the properties of interest.

Notice that a team X of assignments $s: \text{dom}(X) \rightarrow A$ can of course also be understood as a team of assignments $s: \text{dom}(X) \rightarrow A \cup \Lambda$ for any set Λ , and since elements of Λ do not occur in the team, this does not change any of the *atomic* dependence and independence properties of X . However, for an existential formula $\exists\lambda\varphi$ (or a universal one) the universe of values that are available for λ does of course matter.

► **Definition 5.1.** *For a team X with values in A and a set Λ , we write $X + \Lambda$ for the team with the additional supply Λ of values. We say that a formula $\varphi \in \text{FO}(\perp)$ is satisfiable by a (finite) extension of X if there exists a (finite) set Λ such that $X + \Lambda \models \varphi$. All of this applies as well to probabilistic teams.*

The following observation, which we formulate for probabilistic teams, connects existence of suitable hidden-variable models with team semantics.

► **Proposition 5.2.** *Let \mathbb{X} be a probabilistic team representing an empirical model $e_{\mathbb{X}}$. For an arbitrary property P of hidden-variable models captured by a formula $\psi \in \text{FO}(\perp)$, the following two statements are equivalent:*

1. *There is a hidden-variable model $h_{\mathbb{P}}$ which satisfies P and is empirically equivalent to $e_{\mathbb{X}}$.*
2. *$\exists\lambda\psi$ is satisfiable by a finite extension of \mathbb{X} .*

Proof. A hidden-variable model $h_{\mathbb{P}}$ (where λ takes values in Λ) which is empirically equivalent to $e_{\mathbb{X}}$ corresponds to a team that can be written as a Skolem extension $\mathbb{Y} = (\mathbb{X} + \Lambda)[\lambda \mapsto \mathcal{F}]$ for a suitable function $\mathcal{F}: X \rightarrow \Delta(\Lambda)$. Meanwhile, every Skolem extension of $\mathbb{X} + \Lambda$ represents a hidden-variable model that is equivalent to $e_{\mathbb{X}}$. The proposition follows since $\mathbb{X} + \Lambda \models \exists\lambda\psi$ expresses precisely that a suitable Skolem extension of $\mathbb{X} + \Lambda$ satisfies ψ . ◀

We immediately get the following connection between probabilistic and relational existence results by our general results on the level of team semantics.

► **Proposition 5.3.** *Let $\psi \in \text{FO}(\perp)$ formalize a property P of both relational and probabilistic hidden-variable models. If a probabilistic empirical model has an empirically equivalent hidden-variable model satisfying P , then this also holds for its induced relational model. If $\psi \in \text{FO}(\text{dep})$, the converse also holds.*

Proof. Assume that the probabilistic case holds for a probabilistic team $\mathbb{X} = (X, \mathbb{P})$ over Var_n^e . Thus, there is a finite set Λ such that $\mathbb{X} + \Lambda \models \exists\lambda\psi$. This implies by Theorem 3.5 that $X + \Lambda \models \exists\lambda\psi$, which was to be shown. For $\psi \in \text{FO}(\text{dep})$, the converse follows by similar arguments. ◀

Again, our elementary properties can be treated as a special case of this general result. If a probabilistic empirical model possesses an equivalent hidden-variable model satisfying any of the elementary properties defined above (such as Locality, λ -Independence, Strong Determinism, ...), then the same holds for its underlying relational model. For the properties that

rely only on dependence atoms (Strong-Determinism, Weak-Determinism, Single-Valuedness), the converse also holds.

We next give logical proofs of some existence theorems – which are already known from [1] and [13] – in our framework. We will start with a trivial one.

► **Proposition 5.4.** *Every (relational and probabilistic) empirical model is realized by an equivalent hidden-variable model satisfying Single-Valuedness.*

Proof. Recall that $\text{SingVal}_n^h := \text{dep}(-, \lambda)$, and for any team X , we can take an arbitrary set Λ to get that $X + \Lambda \models \exists \lambda \text{dep}(-, \lambda)$. This establishes the relational case, which also implies the probabilistic one. ◀

► **Proposition 5.5.** *Every (relational and probabilistic) empirical model is realized by an empirically equivalent hidden-variable model satisfying Strong Determinism.*

Proof. Let X be a team and put $\Lambda := X$. We need to show that

$$X + \Lambda \models \exists \lambda \bigwedge_{i=1}^n \text{dep}(m_i \lambda, o_i).$$

Consider the Skolem extension $Y = (X + \Lambda)[\lambda \mapsto F]$ with $F(s) = \{s\}$. For all $s, s' \in Y$, $s(\lambda) = s'(\lambda)$ implies that $s = s'$. This shows that Y satisfies Strong Determinism and thus proves the relational case. Since $\text{StrongDet}_n^h \in \text{FO}(\text{dep})$, the probabilistic case also follows. ◀

This construction shows that Strong Determinism is hardly a satisfying property in itself; letting every fixed hidden variable deterministically determine a measurement-outcome pair is far from desirable. A natural additional assumption is λ -Independence, which states that the measurement process is independent from the hidden-variables of the system to be measured. This is by no means satisfied in the construction above; indeed, the hidden-variables deterministically determine the measurement taken.

Proposition 5.5 implies a result for Locality since $\text{StrongDet}_n^h \models_{\text{all}} \text{Loc}_n^h$.

► **Corollary 5.6.** *Every (relational and probabilistic) empirical model is realized by an empirically equivalent hidden-variable model satisfying Locality.*

Terms such as “local realism” appearing in the literature refer in fact to the existence of hidden-variable models satisfying the conjunction of Locality and λ -Independence. This is not proven above and indeed in general false as we shall see in Section 6 when we discuss Bell’s Theorem. But if we weaken Strong Determinism to Weak Determinism, we can realize it together with λ -Independence.

► **Proposition 5.7.** *Every relational empirical model and every probabilistic empirical model with rational probabilities is realized by an empirically equivalent hidden-variable model satisfying Weak Determinism and λ -Independence.*

Proof. Note that we only need to prove the probabilistic case, as the restriction to rational probabilities does not impair the argument in the proof of Proposition 5.3. We adapt a construction from [13] to our team semantic framework.

Let $\mathbb{X} = (X, \mathbb{P}_{\mathbb{X}})$ be a probabilistic team over Var_n^e , with $|X(\bar{m})| = L$ and $|X(\bar{o})| = K$. For every pair $z = (\bar{a}, \bar{b}) \in X(\bar{m}) \times X(\bar{o})$, let

$$\mathbb{P}_{\mathbb{X}}(\bar{o} = \bar{b} \mid \bar{m} = \bar{a}) = p(z) = \frac{r(z)}{s(z)}$$

with $r(z), s(z) \in \mathbb{N}$ such that $s(z) \neq 0$ and $r(z)$ and $s(z)$ are co-prime for each z . Let N be the least common multiple of all numbers $s(z)$, and choose a set Λ with N points. The idea is to make all $\lambda \in \Lambda$ equally likely and independent from the measurements. However, we assign a different number of hidden-variables to each measurement-output pair as follows. For every z , let $N(z) := p(z)N \in \mathbb{N}$. Clearly, for every $\bar{a} \in X(\bar{m})$ we have that $\sum_{\bar{b} \in X(\bar{o})} N(\bar{a}, \bar{b}) = N$. For every $\bar{a} \in X(\bar{m})$ we thus get a partition of Λ into a collection $(\Lambda(\bar{a}, \bar{b}))_{\bar{b} \in X(\bar{o})}$ of disjoint sets where $\Lambda(z)$ has $N(z)$ elements. We now define the Skolem extension $\mathbb{Y} = \mathbb{X}[\lambda \mapsto \mathcal{F}]$ by the function $\mathcal{F} : X \rightarrow \Delta(\Lambda)$ that maps $s \in X$ with $s(\bar{m}, \bar{o}) = z$ to the uniform distribution over $\Lambda(z)$. We claim that \mathbb{Y} satisfies Weak Determinism and λ -Independence.

Weak Determinism: Weak Determinism follows immediately from the construction of \mathbb{Y} . Regard arbitrary $\bar{a} \in X(\bar{m}), c \in \Lambda$. There exists a unique $\bar{b} \in X(\bar{o})$ such that $c \in N((\bar{a}, \bar{b}))$. This is by construction the unique $\bar{b} \in X(\bar{o})$ with $\mathbb{P}(\bar{o} = \bar{b}, \lambda = c \mid \bar{m} = \bar{a}) > 0$.

λ -Independence: To prove λ -Independence we have to show that $\mathbb{Y} \models \bar{m} \perp \lambda$. Regard arbitrary $\bar{a} \in X(\bar{m})$ and $c \in \Lambda$ and choose the unique \bar{b} with non-zero probability implied by Weak Determinism. We observe that

$$\begin{aligned} \mathbb{P}(\lambda = c \mid \bar{m} = \bar{a}) &= \mathbb{P}(\bar{o} = \bar{b}, \lambda = c \mid \bar{m} = \bar{a}) \\ &= \mathbb{P}(\lambda = c \mid \bar{o} = \bar{b}, \bar{m} = \bar{a}) \cdot \mathbb{P}(\bar{o} = \bar{b} \mid \bar{m} = \bar{a}) = \frac{1}{N(z)} p(z) = \frac{1}{N}, \end{aligned}$$

independent of \bar{a} . This completes the proof. \blacktriangleleft

Next we provide a normal form for hidden-variable models satisfying Locality and λ -Independence, i.e. “local realism”. This normal form is well-known from prior literature, cf. e.g. [20, 4, 1] – nevertheless we provide an elementary and accessible proof for the benefit of the reader new to the study of hidden-variables.

► **Theorem 5.8.** *Every relational empirical model and every probabilistic empirical model with rational probabilities which admits an empirically equivalent hidden-variable satisfying Locality and λ -Independence also admits an equivalent model satisfying Strong Determinism and λ -Independence.*

Proof (relational case). Let X be a team over Var_n^e such that $X + \Lambda \models (\exists \lambda \in \Lambda) \lambda\text{-Indep}_n^h \wedge \text{Loc}_n^h$. Thus, there is a Skolem extension $Y = (X + \Lambda)[\lambda \mapsto F]$ such that $Y \models \lambda\text{-Indep}_n^h \wedge \text{Loc}_n^h$. For $\bar{a} \in X(\bar{m}), c \in \Lambda$ we let

$$O(\bar{a}, c) := \{\bar{b} \in X(\bar{o}) : (\bar{a}, \bar{b}, c) \in Y(\bar{m}, \bar{o}, \lambda)\}.$$

By λ -Independence, $O(\bar{a}, c) \neq \emptyset$. Also, for each $i \leq n$ and $a \in X(m_i)$, we set

$$O_i(a, c) := \{b \in X(o_i) : (a, b, c) \in Y(m_i, o_i, \lambda)\}.$$

Note that $Y \models \text{Loc}_n^h$ and Lemma 4.4 imply that $O(\bar{a}, c) = \prod_{i=1}^n O_i(a_i, c)$.

Let F_c^i be the set of all functions $f_i : X(m_i) \rightarrow X(o_i)$ where $f_i(a) \in O_i(a, c)$ for all $a \in X(m_i)$. The set Λ' is now defined to contain all pairs $(c, f) = (c, (f_1, \dots, f_n))$ where $c \in Y(\lambda)$ and $f_i \in F_c^i$ for $i \leq n$.

Let $F' : X \rightarrow \mathcal{P}^+(\Lambda')$ be the function with $F'(s) = \{(c, f) \in \Lambda' : f_i(s(m_i)) = s(o_i) \text{ for all } i\}$. It is straightforward to see that $F'(s)$ is indeed non-empty. We claim that $Z := X[\lambda \mapsto F']$ satisfies Strong Determinism and λ -Independence.

Strong Determinism: Let $s, s' \in Z$ with $s(\lambda) = s'(\lambda) = (c, f)$ and $s(m_i) = s'(m_i) = a$. By definition of F' , it holds that $s(o_i) = f_i(s(m_i)) = f_i(s'(m_i)) = s'(o_i)$.

λ -Independence: We show the slightly stronger claim that for every $\bar{a} \in X(\bar{m})$ and $(c, f) \in \Lambda'$ there exists some $s \in Z$ with $s(\bar{m}) = \bar{a}$ and $s(\lambda) = (c, f)$. By definition of Λ' , it holds that $b_i := f_i(a_i) \in O_i(a_i, c)$ for all $i \leq n$. Thus, $\bar{b} \in \prod_{i=1}^n O_i(a_i, c) = O(\bar{a}, c)$. Choose $s \in X$ with $s(\bar{m}) = \bar{a}$, $s(\bar{o}) = \bar{b}$. Since $(c, f) \in F'(s)$, our claim and thereby λ -Independence follows. \blacktriangleleft

Proof (probabilistic case). Let now $\mathbb{X} = (X, \mathbb{P}_{\mathbb{X}})$ be a probabilistic team over Var_n^e with $\mathbb{X} + \Lambda \models (\exists \lambda \in \Lambda) \lambda\text{-Indep}_n^h \wedge \text{Loc}_n^h$, and let $\mathbb{Y} = (\mathbb{X} + \Lambda)[\lambda \mapsto \mathcal{F}]$ be a Skolem extension with $\mathbb{Y} \models \lambda\text{-Indep}_n^h \wedge \text{Loc}_n^h$. As in the relational case we set for $\bar{a} \in X(\bar{m}), c \in \Lambda$

$$\begin{aligned} O(\bar{a}, c) &= \{\bar{b} \in X(\bar{o}) : (\bar{a}, \bar{b}, c) \in Y(\bar{m}, \bar{o}, \lambda)\} \\ &= \{\bar{b} \in X(\bar{o}) : \mathbb{P}_{\mathbb{Y}}(\bar{o} = \bar{b} \mid \bar{m} = \bar{a}, \lambda = c) > 0\}, \end{aligned}$$

which is non-empty by λ -Independence. Also, we let for all $i \leq n$ and $a \in X(m_i)$

$$O_i(a, c) := \{b \in X(o_i) : (a, b, c) \in Y(m_i, o_i, \lambda)\}$$

such that, by Locality, $O(\bar{a}, c) = \prod_{i=1}^n O_i(a_i, c)$.

Let $Z_i = Y(m_i) \times Y(o_i) \times \Lambda$. For every triple $z_i = (a_i, b_i, c) \in Z_i$, let

$$\mathbb{P}_{\mathbb{Y}}(o_i = b_i \mid m_i = a_i, \lambda = c) = p_i(z_i) = \frac{r_i(z_i)}{s_i(z_i)}.$$

where $r_i(z)$ and $s_i(z)$ are co-prime natural numbers with $s_i(z_i) \neq 0$. Let N_i be the least common multiple of the numbers $s_i(z_i)$, let Λ_i be a set with N_i elements and define

$$\tilde{\Lambda} = \Lambda \times (\Lambda_1 \times \cdots \times \Lambda_n).$$

We then put $N_i(z_i) = p_i(z_i)N_i$ and construct, for every pair $(a_i, c) \in Y(m_i) \times \Lambda$ a partition $(\Lambda_i(a_i, b_i, c))_{b_i \in Y(o_i)}$ of Λ_i with $|\Lambda_i(z_i)| = N_i(z_i)$ for all $z_i \in Z_i$. This is possible because

$$\begin{aligned} \sum_{b_i \in Y(o_i)} N_i(a_i, b_i, c) &= \sum_{b_i \in Y(o_i)} p_i(a_i, b_i, c)N_i \\ &= N_i \sum_{b_i \in Y(o_i)} \mathbb{P}_{\mathbb{Y}}(o_i = b_i \mid m_i = a_i, \lambda = c) = N_i. \end{aligned}$$

We now define a function $\mathcal{F}' : X \rightarrow \Delta(\tilde{\Lambda})$ as follows. An assignment $s \in X$ with $s(\bar{m}, \bar{o}, \lambda) = (\bar{a}, \bar{b}, c)$ defines the tuple (z_1, \dots, z_n) where $z_i = (a_i, b_i, c) \in Z_i$. The function \mathcal{F}' maps s to the probability distribution \mathcal{F}'_s with

$$\mathcal{F}'_s(c, (c_1, \dots, c_n)) = \frac{\mathbb{P}_{\mathbb{Y}}(\lambda = c \mid \bar{m} = \bar{a}, \bar{o} = \bar{b})}{N_1(z_1) \dots N_n(z_n)},$$

if $(c_1, \dots, c_n) \in \Lambda_1(z_1) \times \cdots \times \Lambda_n(z_n)$, and $\mathcal{F}'_s(c, (c_1, \dots, c_n)) = 0$, otherwise. It is straightforward to see that this is indeed always a probability distribution. We then define $\mathbb{Z} = \mathbb{X}[\lambda \mapsto \mathcal{F}']$ and claim that \mathbb{Z} satisfies Strong Determinism and λ -Independence:

Strong Determinism: Regard an arbitrary $a_i \in X(m_i)$ and $(c, (c_1, \dots, c_n)) \in \tilde{\Lambda}$. Suppose that $\mathbb{P}_{\mathbb{Z}}(o_i = b_i \mid m_i = a_i, \lambda = (c, (c_1, \dots, c_n))) > 0$. By definition of \mathcal{F}' it then follows that $c_i \in \Lambda_i(z_i)$ for $z_i = (a_i, b_i, c)$. Since a_i, c, c_i are fixed, this uniquely determines b_i and shows Strong Determinism.

λ -Independence: Here, we abuse notation a bit and write $\lambda = (\lambda_1, \lambda_2)$ to refer to the two components of elements of $\tilde{\Lambda}$. Let $\bar{a} \in X(\bar{m})$ and $(c, (c_1, \dots, c_n)) \in \tilde{\Lambda}$. There is a unique $\bar{b} \in X(\bar{o})$ with $c_i \in \Lambda(a_i, b_i, c)$ for all i . Thus,

$$\begin{aligned} \mathbb{P}_{\mathbb{Z}}(\lambda = (c, (c_1, \dots, c_n)) \mid \bar{m} = \bar{a}) &= \mathbb{P}_{\mathbb{Z}}(\lambda = (c, (c_1, \dots, c_n), \bar{o} = \bar{b} \mid \bar{m} = \bar{a})) \\ &= \mathbb{P}_{\mathbb{Z}}(\lambda_2 = (c_1, \dots, c_n) \mid \lambda_1 = c, \bar{o} = \bar{b}, \bar{m} = \bar{a}) \cdot \mathbb{P}_{\mathbb{Z}}(\lambda_1 = c, \bar{o} = \bar{b} \mid \bar{m} = \bar{a}). \end{aligned}$$

► **Lemma 5.9.** $\mathbb{P}_{\mathbb{Z}}(\bar{o} = \bar{b}, \lambda_1 = c \mid \bar{m} = \bar{a}) = \mathbb{P}_{\mathbb{Y}}(\lambda = c) \prod_{i=1}^n p_i(z_i)$.

Indeed, $\mathbb{P}_{\mathbb{Z}}(\lambda_1 = c, \bar{o} = \bar{b}, \bar{m} = \bar{a}) = \mathbb{P}_{\mathbb{Y}}(\lambda = c, \bar{o} = \bar{b}, \bar{m} = \bar{a})$ since marginalizing \mathbb{Z} to $(\bar{m}, \bar{o}, \lambda_1)$ gives rise to \mathbb{Y} . Hence

$$\begin{aligned} \mathbb{P}_{\mathbb{Z}}(\bar{o} = \bar{b}, \lambda_1 = c \mid \bar{m} = \bar{a}) &= \mathbb{P}_{\mathbb{Y}}(\bar{o} = \bar{b}, \lambda = c \mid \bar{m} = \bar{a}) \\ &= \mathbb{P}_{\mathbb{Y}}(\bar{o} = \bar{b} \mid \lambda = c, \bar{m} = \bar{a}) \cdot \mathbb{P}_{\mathbb{Y}}(\lambda = c \mid \bar{m} = \bar{a}), \end{aligned}$$

and by Locality and λ -Independence of \mathbb{Y} this coincides with

$$\prod_{i=1}^n \mathbb{P}_{\mathbb{Y}}(o_i = b_i \mid m_i = a_i, \lambda = c) * \mathbb{P}_{\mathbb{Y}}(\lambda = c) = \mathbb{P}_{\mathbb{Y}}(\lambda = c) \prod_{i=1}^n p_i(z_i).$$

This proves the lemma. Putting it together with the equation above we get

$$\begin{aligned} \mathbb{P}_{\mathbb{Z}}(\lambda = (c, (c_1, \dots, c_n)) \mid \bar{m} = \bar{a}) \\ = \prod_{i=1}^n \frac{1}{N_i \cdot p_i(z_i)} \cdot \mathbb{P}_{\mathbb{Y}}(\lambda = c) \prod_{i=1}^n p_i(z_i) = \frac{\mathbb{P}_{\mathbb{Y}}(\lambda = c)}{N_1 \cdots N_n}, \end{aligned}$$

which is independent from \bar{a} . This proves that $\mathbb{Z} \models \lambda\text{-Indep}_n^h$. ◀

6 Bell's Theorem and Non-Locality

The famous theorem by Bell, originally formulated in the groundbreaking work [7], showed that a certain flavor of local hidden-variable theories cannot reproduce some predictions by quantum mechanics. This property corresponds to the conjunction of Strong-Determinism and λ -Independence – which is by Theorem 5.8 closely related to the conjunction of Locality and λ -Independence – in our framework. Commonly, the class of theories refuted by Bell's Theorem is referred to by the name of “local realism” (cf. [16]).

Since Bell's work contains one of the most influential results on hidden-variables, we think it is worthwhile to discuss how its assumptions compare to our formulation. All important assumptions of Bell, which are led to a contradiction, are contained in equation (2) from the original paper [7], namely

$$P(\vec{a}, \vec{b}) = \int \mathbb{P}(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda) d\lambda.$$

There, \vec{a}, \vec{b} range over measurements of two components A and B , λ ranges over a space of hidden-variables, and A and B denote functions that provide the outcomes of the respective components deterministically depending on their arguments. Thus, $P(\vec{a}, \vec{b})$ denotes the expectation value of the product of the outcomes of A and B when measurements \vec{a}, \vec{b} are chosen.

The assumed properties are implicitly encoded in the dependencies of the equation above. Writing $A(\vec{a}, \lambda)$ and $B(\vec{b}, \lambda)$ entails the assumption that the outcome of a component is deterministically given by the measurement in the component together with the hidden-variable, i.e. Strong Determinism. Writing $\mathbb{P}(\lambda)$ instead of $\mathbb{P}(\lambda \mid \vec{a}, \vec{b})$ assumes that the distribution over λ is independent from the measurements chosen, i.e. λ -Independence. Thus, proving that Strong Determinism and λ -Independence cannot generally be realized together constitutes a proof of Bell's No-Go-Theorem.

The proof of Bell's Theorem is probabilistic in nature. By Proposition 5.3, a relational analogue implies the probabilistic formulation. Therefore, we provide here a relational no-go theorem based on the Hardy Paradox from [27] (instead of the original argument due to Bell), following the presentation in [1]. We add some further remarks. The significance of Bell's Theorem goes far beyond the theoretical no-go result as it allows a way to experimentally test Non-Localities (cf. for example [16]). In this context it should be noted, that the empirical models and teams appearing in this and related results [1] are not just arbitrary mathematical constructions, but arise from quantum mechanical states and measurements. We do not go into details here, but refer to [6] where, based on [1], the notion of *quantum realisable teams* has been spelled out explicitly.

► **Theorem 6.1.** *There exists a relational empirical model without an equivalent hidden-variable model satisfying the conjunction of Strong Determinism and λ -Independence.*

Proof. Let X be an arbitrary team over Var_2^c which satisfies the following assumptions:

1. $X(m_1, m_2) = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}$.
2. $X(o_1, o_2) \subseteq \{R, G\}^2$ where $R \neq G$
3. $(a_1 b_1, RR) \in X$
4. $(a_1 b_2, RR) \notin X$
5. $(a_2 b_1, RR) \notin X$
6. $(a_2 b_2, GG) \notin X$

Of course many such teams exist. Assume that X has an equivalent hidden-variable model represented by a team Y with $Y \models \text{StrongDet}_2^h \wedge \lambda\text{-Indep}_2^h$. Note that $Y \models \text{Par-Indep}_2^h$ since $\text{StrongDet}_n^h \models_{\text{rel}} \text{Par-Indep}_n^h$. Due to (3), there exists $s_1 \in Y$ with $s_1(m_1 m_2) = a_1 b_1$ and $s_1(o_1 o_2) = RR$. Define $c = s_1(\lambda)$. There also exists an $s_2 \in Y$ with $s_2(m_1 m_2) = a_1 b_2$ and $s_2(\lambda) = c$ due to λ -Independence and $(a_1, b_2) \in X(\vec{m}) = Y(\vec{m})$. Applying Parameter Independence on s_1 and s_2 gives $s_3 \in Y$ with $s_3(\vec{m}) = s_2$ and $s_3(o_1) = s_1(o_1) = R$. Assumption (4) gives $s_3(o_2) = G$.

Analogously to s_2 , we have $s_4 \in Y$ with $s_4(\vec{m}) = a_2 b_2$ and $s_4(\lambda) = c$. Applying Parameter Independence on s_2 and s_4 gives $s_5 \in Y$ with $s_5(\vec{m}) = a_2 b_1$ and $s_5(\vec{o}) = RG$.

Next, we construct $s_6 \in Y$ with $s_6(\vec{m}) = a_2 b_1$ and $s_6(\vec{o}) = RG$. However, $s_1(m_2) = b_1 = s_6(m_2)$ but $s_1(o_2) = R \neq G = s_6(o_2)$ which contradicts Strong Determinism. ◀

By Theorem 5.8, we immediately obtain a no-go result for local realism; formalizing the notion that quantum mechanics is non-local.

► **Corollary 6.2.** *There exist a relational empirical model and a probabilistic empirical model without empirically equivalent hidden-variable models satisfying the conjunction of Locality and λ -Independence.*

We want to conclude this section with a historical note: It shall be remarked that Bell himself did not regard his theorem as a refutation of hidden-variables. In fact, Bell was a proponent of Bohmian Mechanics introduced in [11, 12] – an explicitly non-local hidden-variable interpretation of quantum mechanics. Bell clearly emphasizes (see [10], p. 53) that the essential argument of Bell’s Theorem does not depend on determinism or any other property of hidden-variables but just on the kind of correlations quantum mechanics predicts. Thus, he said, “it is a merit of the de Broglie-Bohm version to bring this out so explicitly that it cannot be ignored” ([9], p. 159).

7 The Kochen-Specker Theorem and Non-Contextuality

The celebrated Kochen-Specker Theorem about the contextuality of quantum mechanics is one of the most important results about hidden-variable models. It is rather difficult to understand, which is also due to the fact that it is often formulated in an imprecise way and that the notion of (non-)contextuality is used inconsistently in the literature. We attempt to shed light on the Kochen-Specker Theorem by discussing several variants of it, including a formulation purely in terms of linear algebra and two different formulations in terms of logics with team semantics, highlighting (non-)contextuality as a team-semantic property.

7.1 The Linear-Algebraic Version of the Kochen-Specker Theorem

We first recall the Kochen-Specker Theorem in essentially its original form. A *measurement context* on a Hilbert space is a set X of observables (i.e. Hermitian operators) that are compatible in the sense that they share a common eigenbasis (and thus commute). In quantum mechanical terms, the Kochen-Specker Theorem states that there is a finite set M of observables such that it is impossible to assign a unique meaningful value to every observable in M in a *non-contextual* way, i.e. independent of its measurement context $X \subseteq M$.

► **Definition 7.1.** *Let M be a set of linear operators on a Hilbert space \mathcal{H} , A valuation $v: M \rightarrow \mathbb{R}$ respects the algebraic structure of \mathcal{H} if for any collection $\{A_i : i \in I\}$ of pairwise commuting operators in M we have that*

- if $\sum_{i \in I} A_i = A \in M$, then $v(A) = \sum_{i \in I} v(A_i)$, and
- if $A_i A_j = A_j A_i \in M$, then $v(A_i A_j) = v(A_i) v(A_j)$.

Note that the algebraic structure needs to be respected by v only inside of measurement contexts, i.e. for operators that commute. The product of non-commuting observables, on the other side, is in general not an observable. The Kochen-Specker says that, for certain small sets of observables, the algebraic structure cannot be respected even in this weak sense.

► **Theorem 7.2 (Kochen-Specker).** *For every Hilbert space \mathcal{H} with $\dim \mathcal{H} \geq 3$, there exists a finite set M of Hermitian operators on \mathcal{H} such that no $v: M \rightarrow \mathbb{R}$ with $v \neq 0$ respects the algebraic structure of \mathcal{H} .*

A modern proof [15] for $\dim \mathcal{H} \geq 4$ relies on a simple combinatorial lemma about orthonormal bases (ONB) of \mathbb{R}^4 .

► **Lemma 7.3.** *There exists a set $Z = \{a_1, \dots, a_{18}\} \subseteq \mathbb{R}^4$ and nine sets X_1, \dots, X_9 , each consisting of four elements of Z that form an ONB of \mathbb{R}^4 , such that no subset $S \subseteq Z$ fulfills $|S \cap X_j| = 1$ for all j .*

This lemma relies on an explicit construction of vectors $a_1, \dots, a_{18} \in \mathbb{R}^4$, and a collection of subsets X_1, \dots, X_9 that use every a_i exactly twice. That is, for every $1 \leq i \leq 18$ there exist precisely two indices j with $a_i \in X_j$. Once this construction is complete, the lemma follows by a simple parity argument. The explicit construction is not important for our purposes and can be found in [15].

The Kochen-Specker Theorem (for $\dim \mathcal{H} \geq 4$) can now be established as follows: Each vector $a \in Z$ induces an orthonormal projection operator A . Let M be the collection of all operators A_i , induced by $a_i \in Z$, together with the identity operator I . For an ONB $\{a, b, c, d\}$ the induced projection operators A, B, C, D , together with I , form a measurement context of compatible observables, since they share $\{a, b, c, d\}$ as a common eigenbasis. Let now $v: M \rightarrow \mathbb{R}$ be a valuation that respects the algebraic structure. We have to show that $v = 0$. Otherwise, assume that $v(A) \neq 0$ for some $A \in M$. Since $v(I) \cdot v(A) = v(I \cdot A) = v(A)$ it follows that $v(I) = 1$. Further, every projection operator $A \in M$ is idempotent, so $v(A) = v(A)^2$ and hence $v(A) \in \{0, 1\}$. Finally, if A, B, C, D are projection operators corresponding to an orthonormal basis it holds that $A + B + C + D = I$ and thereby that $v(A) + v(B) + v(C) + v(D) = 1$. Since $v(A), \dots, v(D) \in \{0, 1\}$, it follows that precisely one of the four projections is mapped to 1. But then the set $S := \{a_i \in Z : v(A_i) = 1\}$ would have the property that $|S \cap X_j| = 1$ for all j , contradicting Lemma 7.3.

For the proof of dimension three, we refer the interested reader to [28, 31].

7.2 Non-Contextuality as a Team Semantical Property

We now formulate the Kochen-Specker theorem in the language of teams, focusing on the notion of non-contextuality. We provide two alternative formulations each of which has its own advantages. To formulate this as elegantly as possible, we first define new team semantic atoms.

► **Definition 7.4.** For k -tuples \bar{x}_1, \bar{x}_2 and m -tuples \bar{y}_1, \bar{y}_2 of variables, let

$$X \models \text{dep}((\bar{x}_1, \bar{x}_2), (\bar{y}_1, \bar{y}_2)) \text{ if for all } s, t \in X \text{ with } s(\bar{x}_1) = t(\bar{x}_2), \\ \text{also } s(\bar{y}_1) = t(\bar{y}_2).$$

Further, we define that $X \models \text{nc}(x_1 \dots x_k, y)$ if for all $s, t \in X$ with $t(y) \in \{s(x_1), \dots, s(x_k)\}$ it follows that $s(y) = t(y)$. We use this to define the atom $\text{ncc}(x_1 \dots x_k)$ for non-contextual choice, with semantics given by

$$\text{ncc}(x_1 \dots x_k) \equiv \exists y \left(\bigvee_{i=1}^k y = x_i \wedge \text{nc}(\bar{x}, y) \right).$$

Obviously, this new dependence atom is an extension of the regular one, with $\text{dep}(\bar{x}, \bar{y}) \equiv \text{dep}((\bar{x}, \bar{x}), (\bar{y}, \bar{y}))$. The atom $\text{ncc}(x_1 \dots x_k)$ for non-contextual choice expresses that one can choose for every assignment $s \in X$ an element $s(y) \in \{s(x_1), \dots, s(x_k)\}$ with the additional constraint that an element that is selected in an assignment s is also selected in every other assignment t in which it appears. This explains the name “non-contextual choice”.

Since these all atoms are obviously in NP and downwards-closed, it follows by [32] that they are definable in dependence logic. Explicit formulae for them are given in A.

► **Definition 7.5.** A team X over variables $\text{Var}_n^e = \{m_1, \dots, m_n, o_1, \dots, o_n\}$ (and the empirical model it represents) is non-contextual if there exists in every component $i \leq n$ a

valuation $v_i: X(m_i) \rightarrow X(o_i)$ that is consistent with the empirical model in the sense that for measurements $\bar{m} = \bar{a}$ the outcome $\bar{o} = v_1(a_1) \dots v_n(a_n)$ is possible. Formally, for every $n \in \mathbb{N}$, non-contextuality can be defined by the formula

$$\text{NonContext}_n^e := \exists v_1 \dots v_n \left(\bigwedge_{1 \leq i < j \leq n} \text{dep}((m_i, m_j), (v_i, v_j)) \wedge \bar{m}\bar{v} \subseteq \bar{m}\bar{o} \right).$$

The generalized dependence atom ensures that if the same measurement appears in components i and j then the valuations v_i, v_j agree there. Notice that NonContext_n^e uses both (extended) dependence atoms and inclusion atoms, so it is a formula of independence logic.

Thus, $X \not\models \text{NonContext}_n^e$ means that there is no single, non-contextual assignment of values to all measurements that is consistent with the empirical model. The term *contextuality* refers to the fact that in this situation measurement outcomes are inherently dependent on their measurement context, i.e. other measurements performed simultaneously. There is no way to assign values to measurements independent of each other in a consistent way.

With this formula we can formulate an analogue of the Kochen-Specker Theorem. Let $e_i = (\delta_{i1}, \dots, \delta_{i4}) \in \{0, 1\}^4$ with $\delta_{ij} = 1$ if, and only if, $i = j$.

► **Theorem 7.6** (Logical formulation of the Kochen-Specker Theorem). *There exists a team X over variables $\{m_1, \dots, m_4\}$ such that every extension Y of X to variables $\{m_1, \dots, m_4, o_1, \dots, o_4\}$ which satisfies the constraint that $Y(\bar{o}) \subseteq \{e_1, \dots, e_4\}$ violates non-contextuality, i.e. $Y \not\models \text{NonContext}_n^e$.*

The Kochen-Specker Theorem gives us a class of empirical models which are necessarily contextual: no non-contextual outcome of measurements is possible, due to the violation of NonContext_n^e . Note that the constraint $Y(\bar{o}) \subseteq \{e_1, \dots, e_4\}$ corresponds to the requirement that valuations must respect the algebraic structure of the measurement operators. This implies that in each ONB the valuations actually define a choice of one of the basis vectors in the ONB. It is a very remarkable feature of the theorem that the measurement setup alone, encoded by X , suffices to ensure that all compatible extensions are contextual.

Using our non-contextual choice atom we can give a different team semantical formulation of the Kochen-Specker Theorem that makes this aspect of choice more explicit. While the version given above is closer to the rest of our framework and structurally more similar to the general formulation of the Kochen-Specker Theorem, the alternative formulation below directly corresponds to the linear algebraic argument formalized by Lemma 7.3 and is more compact and elegant.

► **Theorem 7.7.** *There is a team X over the variables $\{m_1, \dots, m_4\}$ such that $X \not\models \text{ncc}(\bar{m})$. Furthermore, it is possible to choose X with values in \mathbb{R}^4 so that for every $s \in X$ the set $\{s(m_1), \dots, s(m_4)\}$ forms an ONB of \mathbb{R}^4 .*

Proof. By Lemma 7.3 we can form the team X consisting of assignments s_1, \dots, s_9 with values in \mathbb{R}^4 such that for each $i \leq 9$, the set $B_i := \{s_i(m_1), \dots, s_i(m_4)\}$ is an ONB, consisting of four vectors from the collection $Z = \{a_1, \dots, a_{18}\}$ constructed in the proof of Lemma 7.3. We claim that $X \not\models \text{ncc}(\bar{m})$. Otherwise there exists a choice function f mapping each of the sets B_1, \dots, B_9 to one of its elements such that, whenever $f(B_i) = a$ and $a \in B_k$, then also $f(B_k) = a$. But this means that the image of f forms a set $S \subseteq Z$ such that $|S \cap X_i| = 1$ for all $i \leq 9$, contradicting Lemma 7.3. ◀

Thus, Theorem 7.7 is a team semantical formulation of Lemma 7.3 which readily implies the Kochen-Specker Theorem.

It is an interesting question, how the treatment of (non-)contextuality via team semantics can be taken further, to more general scenarios. There is a vast literature on various contextuality properties, and it is not clear that the setup of teams as it is chosen here is adequate for more general settings. The scenarios studied here, sometimes called Bell scenarios, assume different sets of measurements each of which can be performed independently by some agent or site. A context is a combined choice of measurements, one by each agent. We represent this by having for each agent a pair of variables, one of which evaluates to the choices of measurement, the other to the observed values.

There are more general scenarios, such as the Specker triangle or the Peres-Mermin magic square, where certain subsets of measurements can be performed simultaneously, but not all of them, and there are constraints on the (Boolean) values that are observed when a possible set of measurement is performed. Together, these constraints result in an impossibility of a global Boolean assignment to all measurements. Abramsky and Brandenburger [4] propose an approach where variables represent the measurements themselves (not the agents) and the values assigned to these variables correspond the outcomes. A context is a domain of simultaneously performable measurements (i.e. variables) and a behaviour in this context is a relational or probabilistic team on this domain. An empirical model is thus no longer described by a team, but by a set of teams with different domains, one for each context. Contextuality is then the question whether there exist a team on the domain of all variables, whose set of restrictions to the contexts coincides with the given empirical model. We further refer to [2, 5, 3] for more details and for connecting these issues to applications in other areas, including logic, constraint satisfaction problems, databases, etc. It is an interesting challenge for future research whether logics of dependence and independence can be successfully applied also this more general scenario.

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A Explicit Formulae for Extended Dependence and Non-Contextual Choice

► **Proposition A.1.** *The following formula $\psi \in \text{FO}(\text{dep})$ is equivalent to the generalized dependence atom $\text{dep}((\bar{x}_1, \bar{x}_2), (\bar{y}_1, \bar{y}_2))$:*

$$\psi = \forall \bar{z}_1 \bar{z}_2 \forall \bar{w}_1 \bar{w}_2 \exists u_1 \exists u_2 \exists u_3 \left(\bigwedge_{i=1}^3 \text{dep}(\bar{z}_i \bar{z}_2 \bar{w}_1 \bar{w}_2, u_i) \wedge \left[\begin{array}{l} (u_1 = u_2 = u_3 \wedge \bar{z}_1 \bar{w}_1 \mid \bar{x}_1 \bar{y}_1) \\ \vee (u_1 = u_2 \neq u_3 \wedge \bar{z}_2 \bar{w}_2 \mid \bar{x}_2 \bar{y}_2) \\ \vee (u_1 \neq u_2 = u_3 \wedge (\bar{z}_1 \neq \bar{z}_2 \vee \bar{w}_1 = \bar{w}_2)) \end{array} \right] \right)$$

Proof. In team semantics, we can compare equality only within an assignment. However, we need to compare values of $x_1 y_1$ and $x_2 y_2$ across different tuples s, t . We therefore create copies of $\bar{x}_1 \bar{y}_1$ and $\bar{x}_2 \bar{y}_2$ and regard all possible recombinations: $\bar{z}_1 \bar{w}_1$ is to be interpreted as copy of $\bar{x}_1 \bar{y}_1$ and similarly $\bar{z}_2 \bar{w}_2$ as copy of $\bar{x}_2 \bar{y}_2$. With ‘recombining’ we mean that if $\bar{a}\bar{c}$ appears as value of $s(\bar{x}_1 \bar{y}_1)$ and $\bar{b}\bar{d}$ as value of $t(\bar{x}_2 \bar{y}_2)$, we want $\bar{a}\bar{b}\bar{c}\bar{d}$ to appear as value of $\bar{z}_1 \bar{z}_2 \bar{w}_1 \bar{w}_2$ within one assignment v . We can then check whether $s(\bar{x}_1) = t(\bar{x}_2) \rightarrow s(\bar{y}_1) = t(\bar{y}_2)$ holds by checking whether $v(\bar{z}_1) \neq v(\bar{z}_2) \vee v(\bar{w}_1) = v(\bar{w}_2)$ – and this is what is done in the third disjunct.

The role of the u_i and the first two disjuncts is to ensure that we only regard values that are actually taken on by $\bar{x}_1 \bar{y}_1$ and $\bar{x}_2 \bar{y}_2$ in the third disjunct.

We first extend the team via all possible values of $\bar{z}_1, \bar{z}_2, \bar{w}_1, \bar{w}_2$ using the universal quantifier. We then add flags u_1, u_2, u_3 to every assignment that are functionally dependent on $\bar{z}_1 \bar{z}_2 \bar{w}_1 \bar{w}_2$. This ensures that all assignments who agree on these copies share the same flags. The disjuncts then ensure that all assignments with identical flags must be assigned to the same disjunct. Therefore, we partition not really our assignments in the disjunction but rather the values of our copies.

The first disjunct handles all copies where the value of $\bar{z}_1\bar{w}_1$ does not in fact appear as value of $\bar{x}_1\bar{y}_1$. Similarly, the second disjunct handles all copies where the value of $\bar{z}_2\bar{w}_2$ does not appear as value of $\bar{x}_2\bar{y}_2$. The elements that cannot be handled in either the first or the second disjunct are precisely those of interest to us and must satisfy the $\bar{z}_1 = \bar{z}_2 \rightarrow \bar{w}_1 = \bar{w}_2$ condition from the final disjunct. This gives us the semantic of the generalized dependence atom. ◀

A similar approach works for the atom $\text{nc}(x_1 \dots x_k, y)$. Recall that $X \models \text{nc}(x_1 \dots x_k, y)$ if for all $s, t \in X$ with $t(y) \in \{s(x_1), \dots, s(x_k)\}$ it follows that $s(y) = t(y)$.

► **Proposition A.2.** *The following $\psi \in \text{FO}(\text{dep})$ is equivalent to $\text{nc}(x_1 \dots x_k, y)$:*

$$\begin{aligned} \psi = & \forall z_1 \dots z_k \forall w_1 w_2 \exists u_1 \exists u_2 \exists u_3 \left(\bigwedge_{i=1}^3 \text{dep}(\bar{z}\bar{w}, u_i) \wedge [\right. \\ & (u_1 = u_2 = u_3 \wedge \bar{z}w_1 \mid \bar{x}y) \\ & \vee (u_1 = u_2 \neq u_3 \wedge w_2 \mid y) \\ & \left. \vee (u_1 \neq u_2 = u_3 \wedge (w_2 = w_1 \vee \bigvee_{i=1}^k w_2 \neq z_i)) \right] \end{aligned}$$

The proof is structurally identical to the one above; $\bar{z}w_1$ serves as a copy of $\bar{x}y$ (to simulate s) and w_2 as another copy of y (to simulate t).