

## Unifying holographic inflation with holographic dark energy: A covariant approach

Shin'ichi Nojiri,<sup>1,2,\*</sup> S. D. Odintsov,<sup>3,4,†</sup> V. K. Oikonomou,<sup>5,6,7,‡</sup> and Tanmoy Paul<sup>8,9,§</sup>

<sup>1</sup>*Department of Physics, Nagoya University, Nagoya 464-8602, Japan*

<sup>2</sup>*Kobayashi-Maskawa Institute for the Origin of Particles and the Universe,  
Nagoya University, Nagoya 464-8602, Japan*

<sup>3</sup>*ICREA, Passeig Luis Companys, 23, 08010 Barcelona, Spain*

<sup>4</sup>*Institute of Space Sciences (IEEC-CSIC) C. Can Magrans s/n, 08193 Barcelona, Spain*

<sup>5</sup>*Department of Physics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece*

<sup>6</sup>*International Laboratory for Theoretical Cosmology, Tomsk State University of Control Systems and  
Radioelectronics (TUSUR), 634050 Tomsk, Russia*

<sup>7</sup>*Tomsk State Pedagogical University, 634061 Tomsk, Russia*

<sup>8</sup>*Department of Physics, Chandernagore College, Hooghly—712 136*

<sup>9</sup>*Department of Theoretical Physics, Indian Association for the Cultivation of Science,  
2A & 2B Raja S.C. Mullick Road Kolkata—700 032, India*



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In the present paper, we use the holographic approach to describe the early-time acceleration and the late-time acceleration eras of our Universe in a unified manner. Such “holographic unification” is found to have a correspondence with various higher curvature cosmological models with or without matter fields. The corresponding holographic cutoffs are determined in terms of the particle horizon and its derivatives, or the future horizon and its derivatives. As a result, the holographic energy density we propose is able to merge various cosmological epochs of the Universe from a holographic point of view. We find the holographic correspondence of several  $F(R)$  gravity models, including axion- $F(R)$  gravity models, of several Gauss-Bonnet  $F(G)$  models and finally of  $F(T)$  models, and in each case we demonstrate that it is possible to describe in a unified way inflation and late-time acceleration in the context of the same holographic model.

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### I. INTRODUCTION

The holographic principle originates from black hole thermodynamics and string theory and establishes a connection of the infrared cutoff of a quantum field theory, which is related to the vacuum energy, with the largest distance of this theory [1–4]. This consideration has been applied extensively in cosmological considerations, in particular, at the late-time era of the Universe, known currently as holographic dark energy models [5–23] which are also known to be in good agreement with observations [24–31]. At this stage we would like to mention that the most general holographic dark energy is given by the one with Nojiri-Odintsov cutoff [6] and it is interesting that it may be applied to covariant theories, too [32]. Apart from the dark energy model, the holographic energy density is also found to be useful to realize the early Universe

evolution like the inflationary evolution [33–38]. The first study of whether the Higgs inflation respects the holographic principle (within the effective field theoretical approach) was done in [33]. As a result it was found that the original model of Higgs inflation, where the Higgs field couples with the Ricci scalar, does not respect the holographic bound; however, a different Higgs inflationary model, where the coupling of the Higgs field is taken with the Einstein tensor rather than the Ricci scalar, can change the scenario; in particular, this new model passes the holographic test [33]. In the context of inflation, the holographic model has the advantage that since the largest distance (or the cutoff of the theory) of the early Universe is small, the holographic energy density is naturally large to successfully trigger the inflationary era. Moreover the application of the holographic principle at the early Universe studies, has been extended to the bouncing scenario by some of our authors in [39] where it was shown that the holographic energy density violates the null energy condition (a necessary condition for bounce [40–44]), which in turn generates the bouncing behavior of the Universe (see [45,46] for some more articles on holographic bounce).

\*nojiri@gravity.phys.nagoya-u.ac.jp

†odintsov@ieec.uab.es

‡v.k.oikonomou1979@gmail.com

§pul.tnmy9@gmail.com

Despite a considerable application of the holographic principle individually at the early and late time evolution of the Universe, to date it has not been attempted to provide a unified description of the inflationary era with the dark energy epoch. In the present work, we are interested in providing a unified framework of holographic inflation with holographic dark energy, providing a unified description of inflation with the late time accelerating Universe in a holographic context. There is too strong evidence that eventually modified gravity theories have a prominent role in describing the early and late-time acceleration eras of our Universe. Some of the higher curvature models which are well known to provide such a unified description of early and late-time acceleration can be found in Refs. [47–65] in the context of  $F(R)$  gravity, [66–78] in the  $f(R, \mathcal{G})$  gravity, etc. Recently, the axion- $F(R)$  gravity model was proposed in [55,61], where the axion field mimics the dark matter evolution and hence the model provides a description of dark matter along with the unification of early and late-time acceleration eras. Interestingly, in the present work, we propose several holographic models which, similar to these aforementioned higher curvature models, are able to describe the inflationary and the dark energy epoch of the Universe in an unified manner.

The plan of our paper is as follows: In Sec. II, we briefly discuss the essential features of the holographic model and the corresponding holographic cutoff. In the following sections, we propose various holographic models from a different perspective, which are able to unify the cosmological eras of the Universe. Finally the conclusions follow in the end of the paper.

## II. ESSENTIAL FEATURES OF HOLOGRAPHIC MODEL

According to the holographic principle, the holographic energy density is proportional to the inverse squared infrared cutoff  $L_{\text{IR}}$ , which could be related with the causality given by the cosmological horizon,

$$\rho_{\text{hol}} = \frac{3c^2}{\kappa^2 L_{\text{IR}}^2}. \quad (1)$$

Here  $\kappa^2$  is the gravitational constant and  $c$  is a free parameter. We now consider the Friedmann-Robertson-Walker (FRW) metric with the flat spatial part,

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} (dx^i)^2, \quad (2)$$

where  $a(t)$  is the scale factor. Then the Friedmann equation is given by

$$H^2 = \frac{\kappa^2}{3} \rho, \quad (3)$$

where  $\rho$  is the energy density of the generalized fluid driving the expansion of the Universe. We now assume that the energy density  $\rho$  is given by  $\rho_{\text{hol}}$  in (1). Then the Friedmann equation (3) can be rewritten as follows;

$$H = \frac{c}{L_{\text{IR}}}. \quad (4)$$

The infrared cutoff  $L_{\text{IR}}$  is usually assumed to be the particle horizon  $L_p$  or the future event horizon  $L_f$ , which are given as

$$L_p \equiv a \int_0^t \frac{dt}{a}, \quad L_f \equiv a \int_t^\infty \frac{dt}{a}. \quad (5)$$

Inserting these into (4) we obtain

$$\frac{d}{dt} \left( \frac{c}{aH} \right) = \frac{m}{a}. \quad (6)$$

The  $m = 1$  case corresponds to the particle horizon and  $m = -1$  case to the future event horizon. In the second case, if we choose  $c = 1$ , we obtain the solution describing the de Sitter spacetime,

$$a = a_0 e^{H_0 t}, \quad (7)$$

with  $a_0, H_0$  being two integration constants. Moreover for  $c \neq 1$ , Eq. (6) (with  $m = -1$ ) has the solution  $a(t) = a_0 [t(1-c) + b_0 c]^{-\frac{1}{1-c}}$ , thus we get the de Sitter solution only for the case  $c = 1$ . However in the following sections, when we determine the holographic cut-offs in terms of the particle or the future horizon, we will keep  $c$  as a free parameter.

In [8], a general form of the cutoff was proposed,

$$L_{\text{IR}} = L_{\text{IR}}(L_p, \dot{L}_p, \ddot{L}_p, \dots, L_f, \dot{L}_f, \dots, H, \dot{H}, \dots, R, R_{\mu\nu} R^{\mu\nu}, \dots). \quad (8)$$

The above cutoff could be chosen to be equivalent to a general covariant gravity model,

$$S = \int d^4x \sqrt{-g} F \times (R, R_{\mu\nu} R^{\mu\nu}, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \square R, \square^{-1} R, \nabla_\mu R \nabla^\mu R, \dots). \quad (9)$$

We will use the above expressions frequently in the following sections.

## III. HOLOGRAPHIC CORRESPONDENCE OF $F(R)$ GRAVITY WITHOUT/WITH MATTER FIELDS

In this section we establish the holographic correspondence of  $F(R)$  gravity with arbitrary form of  $F(R)$ . By the

term ‘‘holographic correspondence,’’ we mean there exists an equivalent holographic cutoff ( $L_{\text{IR}}$ ) which, along with the expression  $H = 1/L_{\text{IR}}$ , can reproduce the cosmological field equations for the corresponding  $F(R)$  gravity. We will show that such correspondence is not just confined to ‘‘vacuum  $F(R)$  models’’ but also holds true even for *nonvacuum*  $F(R)$  models, i.e., the  $F(R)$  gravity along with matter fields. We start with the action of a general  $F(R)$  gravity [see [47–49] for general reviews on  $F(R)$  gravity] in absence of matter fields,

$$S = \int d^4x \sqrt{-g} \left[ \frac{F(R)}{2\kappa^2} \right] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [R + f(R)], \quad (10)$$

where  $F(R)$  is decomposed as  $F(R) = R + f(R)$  in the second line and  $1/\kappa = M_{\text{P}}$  with  $M_{\text{P}}$  being the four-dimensional reduced Planck mass. The gravitational field equation for the above action in the FRW spacetime is given by

$$3H^2 = -\frac{f(R)}{2} + 3(H^2 + \dot{H})f'(R) - 3H \frac{df'(R)}{dt}, \quad (11)$$

where  $H = \frac{\dot{a}}{a}$  is the Hubble parameter of the Universe and  $f'(R) = \frac{df}{dR}$ . Equation (11) is the temporal component of Einstein’s field equation; however, the spatial component, containing  $\dot{H}$ , can be derived from Eq. (11) and thus we do not quote it here. Moreover, in the case of  $F(R)$  gravity, the off-diagonal component of Einstein’s field equations are trivial and do not give any new information for the dynamics of the cosmological equations. Comparing Eqs. (4) and (11), we can argue that the  $F(R)$  gravity has a holographic correspondence where the equivalent holographic cutoff is given by the following expression:

$$\frac{3c^2}{(L_{\text{IR}})^2} = -\frac{f(R)}{2} + 3(H^2 + \dot{H})f'(R) - 18H(\ddot{H} + 4H\dot{H})f''(R), \quad (12)$$

where we express  $\frac{df'(R)}{dt} = \dot{R}f''(R) = 6(\ddot{H} + 4H\dot{H})f''(R)$  (recall  $R = 12H^2 + 6\dot{H}$  in the FRW spacetime). As mentioned earlier, the holographic cutoff  $L_{\text{IR}}$ , in general, is a function of the particle horizon ( $L_p$ ), the future horizon ( $L_f$ ), the scale factor, and their derivatives [see Eq. (8)]. Keeping this in mind, here in the context of  $F(R)$  gravity, we determine the holographic cutoff in two different ways: (1)  $L_{\text{IR}}$  in terms of  $L_p$  and their derivatives, and (2)  $L_{\text{IR}}$  in terms of  $L_f$  and their derivatives. In order to determine the  $L_{\text{IR}}$  in terms of the particle horizon and their derivatives, we start from the expression  $L_p = a \int^t \frac{dt}{a(t)}$  as mentioned in Eq. (5). Upon differentiating both sides of this expression, one gets the Hubble parameter as  $H(L_p, \dot{L}_p) = \frac{\dot{L}_p}{L_p} - \frac{1}{L_p}$  which immediately leads to the Ricci scalar,

$$R^{(L_p)} = 6 \left[ \frac{\ddot{L}_p}{L_p} + \frac{\dot{L}_p^2}{L_p^2} - \frac{3\dot{L}_p}{L_p^2} + \frac{2}{L_p^2} \right]. \quad (13)$$

Plugging the above expressions of the Hubble parameter and the Ricci scalar into Eq. (12), one obtains the  $L_{\text{IR}} = L_{\text{IR}}(L_p, \dot{L}_p, \ddot{L}_p, \text{higher derivatives of } L_p)$  by the following relation:

$$\frac{3c^2}{(L_{\text{IR}})^2} = -\frac{f(R^{(L_p)})}{2} + 3 \left( \frac{\ddot{L}_p}{L_p} - \frac{\dot{L}_p}{L_p^2} + \frac{1}{L_p^2} \right) f'(R^{(L_p)}) - 3 \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right) \frac{df'(R^{(L_p)})}{dt}. \quad (14)$$

Similarly, to determine the holographic cutoff as a function of the future horizon ( $L_f$ ) and their derivatives, we use  $L_f = a \int_t^\infty \frac{dt}{a(t)}$ . The derivative on both sides of this expression yields the Hubble parameter and consequently the Ricci scalar as follows:

$$H(L_f, \dot{L}_f) = \frac{\dot{L}_f}{L_f} + \frac{1}{L_f}, \quad (15)$$

and

$$R^{(L_f)} = 6 \left[ \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f^2}{L_f^2} + \frac{3\dot{L}_f}{L_f^2} + \frac{2}{L_f^2} \right], \quad (16)$$

respectively. Equations (15) and (16) along with Eq. (12) immediately lead to  $L_{\text{IR}} = L_{\text{IR}}(L_f, \dot{L}_f, \ddot{L}_f, \text{higher derivatives of } L_f)$  as

$$\frac{3c^2}{(L_{\text{IR}})^2} = -\frac{f(R^{(L_f)})}{2} + 3 \left( \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f}{L_f^2} + \frac{1}{L_f^2} \right) f'(R^{(L_f)}) - 3 \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right) \frac{df'(R^{(L_f)})}{dt}. \quad (17)$$

Having established the holographic correspondence of the vacuum  $F(R)$  model, now we consider the  $F(R)$  gravity model in the presence of matter fields and the action is

$$S = \int d^4x \sqrt{-g} \frac{1}{2\kappa^2} [R + f(R) + L_{\text{mat}}], \quad (18)$$

where  $L_{\text{mat}}$  represents the matter field Lagrangian. The gravitational and the matter field equations for the above action in FRW spacetime are given by

$$3H^2 = -\frac{f(R)}{2} + 3(H^2 + \dot{H})f'(R) - 3H \frac{df'(R)}{dt} + \kappa^2 \rho_{\text{mat}}, \quad (19)$$

$$\dot{\rho}_{\text{mat}} + 3H(\rho_{\text{mat}} + p_{\text{mat}}) = 0,$$

with  $\rho_{\text{mat}}$  and  $p_{\text{mat}}$  being the energy density and pressure of the matter field, respectively. They are defined as the temporal and spatial component of matter energy-momentum tensor  $T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta L_{\text{mat}}}{\delta g^{\mu\nu}}$ , respectively. Comparing Eqs. (4) and (19), we can consider a holographic origin of nonvacuum  $F(R)$  models where the equivalent holographic cutoff is given by the same expression as shown in Eq. (12). Consequently, the above gravitational equation turns out to be

$$3H^2 = \frac{3c^2}{(L_{\text{IR}})^2} + \kappa^2 \rho_{\text{mat}}. \quad (20)$$

Thus Eq. (17) gives the equivalent holographic cutoff for any arbitrary vacuum/nonvacuum  $F(R)$  gravity in terms of the future horizon and their derivatives, while Eq. (14) does the same, however, in terms of the particle horizon and their derivatives. These clearly indicate that  $F(R)$  gravity has a holographic origin which mimics the cosmological field equations of the corresponding  $F(R)$  gravity. In the next sections, we will determine the holographic cutoffs [by using Eqs. (14) and (17)] for some explicit forms of  $F(R)$  gravity, which are known to give a unified picture of inflation and late time dark energy epoch.

At this stage it is worth mentioning that unlike in the two aforementioned approaches (where  $L_{\text{IR}}$  is determined in terms of particle horizon or the future horizon), the holographic cutoffs for various  $F(R)$  models are determined in another way and in particular, these are determined as an integral, as shown in Refs. [34,35]. With such integral forms, it is shown that the holographic correspondence for vacuum and nonvacuum  $F(R)$  models exists. In the spirit of these previous works, here we will present integral forms of holographic cutoffs for the considered forms of  $F(R)$  in the next sections.

### A. Holographic cutoff for $F(R)$ inflationary models

Before moving to the unified scenario of inflation and late-time accelerating epochs, we consider the  $F(R)$  inflationary models and  $F(R)$  dark energy models separately and will find the corresponding holographic cutoffs. The inflationary  $F(R)$  holographic cutoffs are shown in this subsection, while the same for dark energy models are treated in the next subsection. Some of the popular  $F(R)$  models which are known to trigger a viable inflationary era, are  $F(R) = R + \alpha R^2$ ,  $F(R) = e^{bR}$  [i.e., exponential  $F(R)$  gravity] etc. Thus these specific forms of  $F(R)$  are considered

here, to determine the holographic cutoffs. Moreover it has been shown earlier that  $F(R)$  models along with the second rank antisymmetric Kalb-Ramond (KR) fields also give a viable inflationary era [79,80]. In fact, the cubic curvature vacuum  $F(R)$  gravity, i.e.,  $F(R) = R + \beta R^3$  model does not produce a viable inflation; in particular, the theoretical expectations of spectral index ( $n_s$ ) and tensor to scalar ratio ( $r$ ) do not match with the Planck 2018 constraints; however, in the presence of the Kalb-Ramond field the cubic gravity model becomes compatible with the Planck constraints (i.e.,  $n_s = 0.9649 \pm 0.0042$  and  $r < 0.064$ ) [79].

Thus the impact of the KR field on the inflationary evolution is significant and will be clear from its cosmological evolution. The demonstration goes as follows: its equation of state (EoS) parameter is unity, and the conservation equation of the KR field is given by  $\dot{\rho}_{\text{KR}} + 6H\rho_{\text{KR}} = 0$ , solving which one obtains  $\rho_{\text{KR}} \propto 1/a^6$ ; i.e., the energy density of the KR field decreases with faster rate in comparison to pressureless matter and radiation. (The negligible footprint of the KR field in the present Universe can also be described from the higher dimensional point of view [81] where the KR field is generally considered to be a bulk field and our four-dimensional visible Universe is a brane embedded within the higher dimensional spacetime; further a nondynamical approach is also presented in [82] to explain the imperceptible signatures of the KR field in our Universe.) Thereby, it clearly depicts that the present Universe may be free from the direct signatures of the KR field; however, the KR field has considerable effects during the early Universe (when the scale factor is small compared to the present one). These arguments reveal the importance of the Kalb-Ramond field in inflationary models and thus, beside the vacuum  $F(R)$  models, here we also determine the holographic cutoffs for “ $F(R) + \text{KR}$ ” models.

#### 1. Quadratic curvature gravity without/with KR field

Consider the action  $S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [R + \alpha R^2]$  (i.e., quadratic gravity without the KR field and  $\alpha$  is a parameter having mass dimension  $[-2]$ ), for which the Friedmann equation takes the following form:

$$H^2 = 6\alpha[\dot{H}^2 - 2H\ddot{H} - 6\dot{H}\ddot{H}]. \quad (21)$$

The equivalent holographic cutoffs for  $f(R) = \alpha R^2$  in terms of  $L_p$  and their derivatives can be obtained from Eq. (14) and are given by

$$\frac{c^2}{(L_{\text{IR}})^2} = 6\alpha \left[ -\frac{2\ddot{L}_p \dot{L}_p}{L_p^2} + \frac{2\ddot{L}_p}{L_p^2} + \frac{\ddot{L}_p^2}{L_p^2} - \frac{2\ddot{L}_p \dot{L}_p^2}{L_p^3} + \frac{6\ddot{L}_p \dot{L}_p}{L_p^3} - \frac{4\ddot{L}_p}{L_p^3} + \frac{3\dot{L}_p^4}{L_p^4} - \frac{12\dot{L}_p^3}{L_p^4} + \frac{15\dot{L}_p^2}{L_p^4} - \frac{6\dot{L}_p}{L_p^4} \right]. \quad (22)$$

Similarly by plugging  $f(R) = \alpha R^2$  into Eq. (17), we get the holographic cu-off as a function of  $L_f$  and their derivatives as follows:

$$\frac{c^2}{(L_{\text{IR}})^2} = 6\alpha \left[ -\frac{2\ddot{L}_f \dot{L}_f}{L_f^2} - \frac{2\ddot{L}_f}{L_f^2} + \frac{\ddot{L}_f^2}{L_f^2} - \frac{2\dot{L}_f \dot{L}_f^2}{L_f^3} - \frac{6\ddot{L}_f \dot{L}_f}{L_f^3} - \frac{4\ddot{L}_f}{L_f^3} + \frac{3\dot{L}_f^4}{L_f^4} + \frac{12\dot{L}_f^3}{L_f^4} + \frac{15\dot{L}_f^2}{L_f^4} + \frac{6\dot{L}_f}{L_f^4} \right]. \quad (23)$$

Therefore, the  $R^2$  inflationary models which are known to be in good agreement with observations [83,84], have an equivalent holographic model, thanks to the holographic cutoffs obtained in Eqs. (22) and (23). However, in [34], a different kind of holographic model has been proposed, where the infrared cutoff takes the following form:

$$\frac{L_{\text{IR}}}{c} = -\frac{1}{6\alpha\dot{H}^2 a^6} \int dt a^6 \dot{H}, \quad (24)$$

with  $\alpha$  being the parameter of the model. Plugging this expression back into the holographic Friedmann equation  $H = \frac{c}{L_{\text{IR}}}$ , we obtain the cosmological equations for  $F(R) = R + \alpha R^2$  gravity [see Eq. (21)]. Thus, apart from the cutoffs determined earlier in terms of  $L_p$  or  $L_f$ , the holographic energy density with the cutoff given in Eq. (24) is also able to reproduce the Starobinsky  $R^2$  inflation.

The cutoffs in Eqs. (22) and (23) can provide an equivalent holographic model even for a nonvacuum quadratic gravity model where the Friedmann equation takes the form  $3H^2 = \frac{3c^2}{(L_{\text{IR}})^2} + \kappa^2 \rho_{\text{mat}}$  where  $\rho_{\text{mat}}$  is the matter energy density. We consider the Kalb-Ramond field as matter field (keeping the inflationary viability in mind) and the action for the “ $R + \alpha R^2 + \text{KR}$ ” model is

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} (R + \alpha R^2) - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right],$$

where  $H_{\mu\nu\lambda}$  is the field strength tensor of the KR field defined by  $H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]}$  where  $B_{\mu\nu}$  denotes the second rank antisymmetric KR field. The above model generates a viable inflationary scenario as explicitly shown in [79]. However the KR field indeed affects the Starobinsky inflationary model by the following ways: being the EoS parameter of the KR field is unity, the acceleration of the early Universe gets reduced due to the presence of the KR field in comparison to the case where the KR field is absent; moreover from the observational side, the KR field is found to enhance the tensor-to-scalar ratio with respect to the Starobinsky inflationary model. The integral form of the generalized infrared cutoff for the “ $R + \alpha R^2 + \text{KR}$ ” model [35] is obtained as

$$\frac{L_{\text{IR}}}{c} = -\frac{1}{6\alpha\dot{H}^2 a^6} \int dt a^6 \dot{H} \left( 1 - \frac{\kappa^2 \rho_0}{a^6 H^2} \right), \quad (25)$$

where  $\rho_0$  is the energy density of the KR field at the horizon crossing. Inserting the above expression into  $H = c/L_{\text{IR}}$  after some simple algebra, we obtain the following differential equation:

$$H^2 = 6\alpha[\dot{H}^2 - 2H\ddot{H} - 6\dot{H}H^2] + \frac{\kappa^2 \rho_0}{a^6}. \quad (26)$$

Equation (26) is actually a combination of two separate equations,

$$\frac{F(R)}{2} = 3(H^2 + \dot{H})F'(R) - 18(4H^2\dot{H} + H\ddot{H})F''(R) + \kappa^2 \rho_{\text{KR}}, \quad (27)$$

and

$$\frac{d\rho_{\text{KR}}}{dt} + 6H\rho_{\text{KR}} = 0, \quad (28)$$

respectively with  $F(R) = R + \alpha R^2$ . It may be observed that Eq. (28) is the conservation equation for the KR field having EoS parameter unity. Thereby the cutoff in Eq. (25) can reproduce the cosmological field equations for the “Starobinsky + KR” model. It may be mentioned that for  $\rho_0 = 0$ , the expression in Eq. (25) becomes the same as in Eq. (24), as expected. Therefore, the key equations which represent the equivalent holographic energy density for the  $R^2$  model without/with the KR field are given by Eqs. (22), (23), (24), and (25), respectively.

## 2. Cubic curvature gravity without/with KR field

The cubic curvature model without the KR field has the action  $S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [R + \beta R^3]$  ( $\beta$  is a constant parameter with mass dimension  $[-4]$ ) which, in the Friedmann spacetime, leads to the following gravitational equation:

$$H^2 = 36\alpha[2\dot{H}^3 - 6H\dot{H}\ddot{H} - 15H^2\dot{H}^2 + 4H^6 - 36H^4\dot{H} - 12H^3\ddot{H}]. \quad (29)$$

Consequently, by plugging the explicit form of  $f(R) = \beta R^3$  into Eqs. (14) and (17), we obtain  $L_{\text{IR}} = L_{\text{IR}}(L_p, \dot{L}_p, \ddot{L}_p, \text{higher derivatives of } L_p)$  and  $L_{\text{IR}} = L_{\text{IR}}(L_f, \dot{L}_f, \ddot{L}_f, \text{higher derivatives of } L_f)$  as follows:

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} &= \frac{36\alpha}{L_p^6} [2 - 3\dot{L}_p + \dot{L}_p^2 + L_p \ddot{L}_p] \\ &\times [2 - 27\dot{L}_p + 59\dot{L}_p^2 - 45\dot{L}_p^3 + 11\dot{L}_p^4 - 13L_p \ddot{L}_p \\ &- L_p \dot{L}_p^2 \ddot{L}_p + 2L_p^2 \ddot{L}_p^2 + 6L_p^2 \ddot{L}_p \\ &+ 18L_p \dot{L}_p \ddot{L}_p - 6L_p^2 \dot{L}_p \ddot{L}_p], \end{aligned} \quad (30)$$

and

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} = & \frac{36\alpha}{L_f^6} [2 + 3\dot{L}_f + \dot{L}_f^2 + L_f\ddot{L}_f][2 + 27\dot{L}_f + 59\dot{L}_f^2 + 45\dot{L}_f^3 + 11\dot{L}_f^4 - 13L_f\ddot{L}_f \\ & - L_f\dot{L}_f\ddot{L}_f + 2L_f^2\ddot{L}_f - 6L_f^2\ddot{\ddot{L}}_f - 18L_f\dot{L}_f\ddot{L}_f - 6L_f^2\dot{L}_f\ddot{\ddot{L}}_f], \end{aligned} \quad (31)$$

respectively. Moreover, the integral form of the holographic cutoff in the context of a vacuum cubic model is obtained as

$$\frac{L_{\text{IR}}}{c} = -\frac{\int dt a^{15/2} \dot{H}}{36[2\alpha\dot{H}^3 a^{15/2} - 4\alpha H \int dt a^{15/2} H\dot{H}(H^3 - 9H\dot{H} - 3\ddot{H})]}. \quad (32)$$

We can easily see that by inserting the above expression into the holographic Friedmann equation, we will finally yield Eq. (29). Therefore, the holographic cutoffs determined in Eqs. (30), (31), and (32) are equivalent to  $F(R) = R + \beta R^3$  model. Thereby, the cosmology of  $R^3$  gravity can be realized from the holographic origin with the specified infrared cutoffs just mentioned. It is well known that  $F(R) = R + \alpha R^3$  does not give a viable inflationary era, in particular, the theoretical values of  $n_s$  and  $r$  do not comply with the observable constraints

from Planck 2018. However, as mentioned earlier, in the presence of the second rank antisymmetric Kalb-Ramond field, the cubic gravity model becomes viable in respect to Planck 2018 constraints. In particular for  $0.03 \lesssim \kappa^2 \rho_0 \sqrt{\beta} \lesssim 0.3$  (where  $\rho_0$  is the energy density of the KR field at horizon crossing), the spectral index and the tensor to scalar ratio become simultaneously compatible with Planck 2018 constraints. In the  $R + \beta R^3 + \text{KR}$  model, the Friedmann and the KR field equations are given by

$$\begin{aligned} 0 = & -H^2 + 36\alpha[2\dot{H}^3 - 6H\dot{H}\ddot{H} - 15H^2\ddot{H}^2 + 4H^6 - 36H^4\dot{H} - 12H^3\ddot{H}] + \kappa^2\rho_{\text{KR}}, \\ 0 = & \frac{d\rho_{\text{KR}}}{dt} + 6H\rho_{\text{KR}}. \end{aligned} \quad (33)$$

As demonstrated in Sec. III, the holographic cutoffs for  $R + \beta R^3 + \text{KR}$  in terms of  $L_p$  or  $L_f$  (along with their derivatives) are the same as those obtained in Eqs. (30) and (31), respectively. On the other hand, the integral form of the holographic cutoff in the  $R + \beta R^3 + \text{KR}$  model is changed in comparison to Eq. (32) and is determined as

$$\frac{L_{\text{IR}}}{c} = -\frac{\int dt a^{15/2} \dot{H} (1 - \frac{\kappa^2 h_0}{\alpha^6 H^2})}{36[2\alpha\dot{H}^3 a^{15/2} - 4\alpha H \int dt a^{15/2} H\dot{H}(H^3 - 9H\dot{H} - 3\ddot{H})]}. \quad (34)$$

The above expression along with  $H = c/L_{\text{IR}}$  immediately leads to the cosmological field Eq. (33). Again one may note that for  $\rho_0 = 0$ , the expression in Eq. (34) is reduced to Eq. (32), as expected. Thus as a whole, the cosmological imprints of the cubic gravity model without/with the KR field can be reproduced by the holographic models having the cutoffs given in Eqs. (30), (31), (32), and (34), respectively.

### 3. Exponential $F(R)$ gravity without/with KR field

For exponential  $F(R) = ae^{bR}$  (with  $a$  and  $b$  being constant parameters, both having mass dimension  $[-2]$ ), the corresponding holographic cutoff in terms of  $L_p$  is directly obtained from Eq. (14) as

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} = & (abe^{bP(L_p, \dot{L}_p, \ddot{L}_p)} - 1) \left( \frac{1}{L_p^2} - \frac{\dot{L}_p}{L_p^2} + \frac{\ddot{L}_p}{L_p} \right) + \frac{1}{6} (P(L_p, \dot{L}_p, \ddot{L}_p) - ae^{bP(L_p, \dot{L}_p, \ddot{L}_p)}) \\ & + 6ab^2 e^{bP(L_p, \dot{L}_p, \ddot{L}_p)} \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right) \left( -4\frac{\dot{L}_p}{L_p^3} + 6\frac{\dot{L}_p^2}{L_p^3} - 2\frac{\dot{L}_p^3}{L_p^3} - 3\frac{\ddot{L}_p}{L_p^2} + \frac{\dot{L}_p\ddot{L}_p}{L_p^2} + \frac{\ddot{\ddot{L}}_p}{L_p} \right). \end{aligned} \quad (35)$$

Similarly the cutoff in terms of the future horizon can be determined from Eq. (17) as

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} = & (abe^{bQ(L_f, \dot{L}_f, \ddot{L}_f)} - 1) \left( \frac{1}{L_f^2} + \frac{\dot{L}_f}{L_f^2} + \frac{\ddot{L}_f}{L_f} \right) + \frac{1}{6} (Q(L_f, \dot{L}_f, \ddot{L}_f) - ae^{bQ(L_f, \dot{L}_f, \ddot{L}_f)}) \\ & + 6ab^2 e^{bQ(L_f, \dot{L}_f, \ddot{L}_f)} \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right) \left( -4\frac{\dot{L}_f}{L_f^3} - 6\frac{\dot{L}_f^2}{L_f^3} - 2\frac{\dot{L}_f^3}{L_f^3} + 3\frac{\ddot{L}_f}{L_f^2} + \frac{\dot{L}_f\ddot{L}_f}{L_f^2} + \frac{\ddot{\ddot{L}}_f}{L_f} \right), \end{aligned} \quad (36)$$

where the functions  $P$  and  $Q$  are

$$P(L_p, \dot{L}_p, \ddot{L}_p) = 6 \left[ \frac{2}{L_p^2} - \frac{3\dot{L}_p}{L_p^2} + \frac{\dot{L}_p^2}{L_p^2} + \frac{\ddot{L}_p}{L_p} \right],$$

$$Q(L_f, \dot{L}_f, \ddot{L}_f) = 6 \left[ \frac{2}{L_f^2} + \frac{3\dot{L}_f}{L_f^2} + \frac{\dot{L}_f^2}{L_f^2} + \frac{\ddot{L}_f}{L_f} \right],$$

respectively. Apart from these two kinds of holographic cutoffs, an integral form of  $L_{\text{IR}}$  in the context of exponential  $F(R)$  gravity is obtained as follows:

$$\frac{L_{\text{IR}}}{c} = \frac{\int dt a^4 (1 - 6\beta \dot{H}^2 / H^2)}{6\beta a^4 \dot{H} + H \int dt a^4 \left( \frac{1}{6\beta H^2} - \frac{\dot{H}}{H^2} \right)}. \quad (37)$$

The above expressions of  $L_{\text{IR}}$  along with the holographic equation  $H = c/L_{\text{IR}}$  lead to the following differential equation for  $H$ :

$$H^2 = 6\beta [H\dot{H} + 4H^2\dot{H}] + \frac{1}{6\beta} [1 - 6\beta\dot{H}], \quad (38)$$

which can be rewritten in the form

$$\frac{F(R)}{2} = 3(H^2 + \dot{H})F'(R) - 18(4H^2\dot{H} + H\ddot{H})F''(R),$$

with  $F(R) \propto e^{\beta R}$ . Therefore the holographic equation can mimic the cosmological equations of exponential  $F(R)$  gravity, thanks to the different kinds of holographic cutoffs in Eqs. (35), (36), and (37).

The generalized holographic cutoffs shown in Eqs. (35) and (36) are also valid for exponential  $F(R)$  gravity even in the presence of the Kalb-Ramond field where the Friedmann equation takes the following form:

$$3H^2 = \frac{3c^2}{(L_{\text{IR}})^2} + \kappa^2 \rho_{\text{KR}}.$$

However, in the presence of the KR field, the integral form of  $L_{\text{IR}}$  in the context of  $F(R) = \frac{1}{\kappa^2} e^{\beta R}$  comes with the following expression:

$$\frac{L_{\text{IR}}}{c} = \frac{\int dt a^4 \left( 1 - 6\beta \dot{H}^2 / H^2 - \frac{\kappa^2 \rho_0}{a^6 H^2} \right)}{6\beta a^4 \dot{H} + H \int dt a^4 \left( \frac{1}{6\beta H^2} - \frac{\dot{H}}{H^2} \right)}. \quad (39)$$

Equations (35), (36), (37), and (39) represent different types of holographic cutoffs which, along with  $H = c/L_{\text{IR}}$ , realize the cosmological scenario of the exponential  $F(R)$  gravity without/with the KR field. Regarding the observable viability, unlike to vacuum cubic curvature gravity, the vacuum exponential  $F(R)$  model is known to be in good agreement with Planck constraints. Moreover the exponential  $F(R)$  model in the presence of the KR field also leads to a viable inflationary scenario, in particular, for the parametric regime  $0.005 \lesssim \kappa^2 \rho_0 b \lesssim 0.1$  (where  $\rho_0$  is the KR field energy density at horizon crossing), the inflationary parameters like the spectral index and tensor-to-scalar ratio are found to comply with Planck 2018 constraints [35].

## B. Holographic cutoff for $F(R)$ dark energy models

As an  $F(R)$  dark energy model, we consider,

$$f(R) = f_0 R^m, \quad (40)$$

with constants  $f_0$  and  $m$  [51]. We consider the exponent  $m$  to be less than unity for which the term  $f_0 R^m$  dominates over the Einstein-Hilbert and the matter term(s) in the low curvature regime, as in the case of the late-time Universe. As a consequence, the Hubble rate  $H \equiv \dot{a}/a$  behaves as

$$H \sim \frac{-\frac{(m-1)(2m-1)}{m-2}}{t}, \quad (41)$$

with an effective EoS parameter,

$$w_{\text{eff}} = -\frac{(6m^2 - 7m - 1)}{3(m-1)(2m-1)}. \quad (42)$$

The above expression indicates that for  $m < -0.97$ , the theoretical expectations of the EoS parameter satisfies the observations coming from SNIa results ( $-1.57 < w_{\text{eff}} < -0.66$ ). On other hand, for  $m < -0.47$ , the EoS parameter in Eq. (42) satisfies the baryon acoustic oscillations (BAO) results ( $-2.19 < w_{\text{eff}} < -0.42$ ). Thereby for  $m < -0.97$ , the theoretical values of  $w_{\text{eff}}$  is consistent with both the SNIa and BAO results and thus we stick with  $m < -0.97$ , for which, our consideration that the term  $R^m$  dominates over the Einstein-Hilbert term in the low curvature regime is also valid. Thus  $f(R) = f_0 R^m$  with  $m < -0.97$  can act as a dark energy model in the context of  $F(R)$  gravity. Using Eqs. (14) and (17), we determine the equivalent holographic cutoff for this model as

$$\frac{3c^2}{(L_{\text{IR}}^{F(R)})^2} = f_0 \left[ -\frac{1}{2} \left\{ 6 \left( \frac{\ddot{L}_p}{L_p} + \frac{\dot{L}_p^2}{L_p^2} - \frac{3\dot{L}_p}{L_p^2} + \frac{2}{L_p^2} \right) \right\}^m + 3m \left( \frac{\dot{L}_p}{L_p} - \frac{\dot{L}_p}{L_p^2} + \frac{1}{L_p^2} \right) \left\{ 6 \left( \frac{\ddot{L}_p}{L_p} + \frac{\dot{L}_p^2}{L_p^2} - \frac{3\dot{L}_p}{L_p^2} + \frac{2}{L_p^2} \right) \right\}^{m-1} \right. \\ \left. - 18m \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right) \left( \frac{\ddot{L}_p}{L_p} + \frac{\dot{L}_p \ddot{L}_p}{L_p^2} - \frac{2\dot{L}_p^3}{L_p^3} - \frac{3\dot{L}_p}{L_p^2} + \frac{6\dot{L}_p^2}{L_p^3} - \frac{4\dot{L}_p}{L_p^3} \right) \left\{ 6 \left( \frac{\ddot{L}_p}{L_p} + \frac{\dot{L}_p^2}{L_p^2} - \frac{3\dot{L}_p}{L_p^2} + \frac{2}{L_p^2} \right) \right\}^{m-1} \right]$$

$$\begin{aligned}
&= f_0 \left[ -\frac{1}{2} \left\{ 6 \left( \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f^2}{L_f^2} + \frac{3\dot{L}_f}{L_f^2} + \frac{2}{L_f^2} \right) \right\}^m + 3m \left( \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f}{L_f} + \frac{1}{L_f^2} \right) \left\{ 6 \left( \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f^2}{L_f^2} + \frac{3\dot{L}_f}{L_f^2} + \frac{2}{L_f^2} \right) \right\}^{m-1} \right. \\
&\quad \left. - 18m \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right) \left( \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f \dot{L}_f}{L_f^2} - \frac{2\dot{L}_f^3}{L_f^3} + \frac{3\ddot{L}_f}{L_f^2} - \frac{6\dot{L}_f^2}{L_f^3} - \frac{4\dot{L}_f}{L_f^3} \right) \left\{ 6 \left( \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f^2}{L_f^2} + \frac{3\dot{L}_f}{L_f^2} + \frac{2}{L_f^2} \right) \right\}^{m-1} \right]. \quad (43)
\end{aligned}$$

The first expression in the right-hand side of Eq. (43) represents the  $L_{\text{IR}}$  in terms of  $L_p$  and its derivatives while the second expression gives  $L_{\text{IR}} = L_{\text{IR}}(L_f, \dot{L}_f, \ddot{L}_f, \text{higher derivatives of } L_f)$ . Therefore the model (40) can also be reproduced from the holographic energy density having the  $L_{\text{IR}}$  determined in Eq. (43). Clearly such holographic energy density is able to drive the dark energy epoch of our Universe. As a more realistic dark energy F(R) model, we may consider the exponential F(R) gravity which is known to provide a viable dark energy model as described in [85,86]. In particular, we consider

$$F_l(R) = R - 2\Lambda(1 - e^{-\frac{\beta R}{2\Lambda}}) \quad (44)$$

with  $\beta$  and  $\Lambda$  being two model parameters having the mass dimensions [0] and [+2] respectively. Due to the Supernovae Ia (Sne-Ia) [87], BAO [88], cosmic microwave background (CMB) [89], and H(z) [90] datasets, the parameters  $\beta$  and  $\Lambda$  are well constrained, in particular, the F(R) model (44) is best fitted with Sne-Ia + BAO + H(z) + CMB data for the parametric regimes given by  $\beta = 3.98_{-2.46}^{+\infty}$  and  $\Lambda = 1.2 \times 10^{-84} \text{ GeV}^2$  [85]. The equivalent holographic cutoffs (in terms of the particle and future horizon) for the above exponential F(R) model is determined as

$$\begin{aligned}
\frac{3c^2}{(L_{\text{IR}}^{F(R)})^2} &= \Lambda \left[ 1 - \exp \left[ -\frac{3\beta}{\Lambda} \left( \frac{\ddot{L}_p}{L_p} + \frac{\dot{L}_p^2}{L_p^2} - \frac{3\dot{L}_p}{L_p^2} + \frac{2}{L_p^2} \right) \right] \left\{ 1 - \frac{3\beta}{\Lambda} \left( \frac{\ddot{L}_p}{L_p} - \frac{\dot{L}_p}{L_p} + \frac{1}{L_p^2} \right) \right. \right. \\
&\quad \left. \left. - \frac{9\beta^2}{\Lambda^2} \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right) \left( \frac{\ddot{L}_p}{L_p} + \frac{\dot{L}_p \dot{L}_p}{L_p^2} - \frac{2\dot{L}_p^3}{L_p^3} - \frac{3\ddot{L}_p}{L_p^2} + \frac{6\dot{L}_p^2}{L_p^3} - \frac{4\dot{L}_p}{L_p^3} \right) \right\} \right] \\
&= \Lambda \left[ 1 - \exp \left[ -\frac{3\beta}{\Lambda} \left( \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f^2}{L_f^2} + \frac{3\dot{L}_f}{L_f^2} + \frac{2}{L_f^2} \right) \right] \left\{ 1 - \frac{3\beta}{\Lambda} \left( \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f}{L_f} + \frac{1}{L_f^2} \right) \right. \right. \\
&\quad \left. \left. - \frac{9\beta^2}{\Lambda^2} \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right) \left( \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f \dot{L}_f}{L_f^2} - \frac{2\dot{L}_f^3}{L_f^3} + \frac{3\ddot{L}_f}{L_f^2} - \frac{6\dot{L}_f^2}{L_f^3} - \frac{4\dot{L}_f}{L_f^3} \right) \right\} \right]. \quad (45)
\end{aligned}$$

Because the  $F(R)$  gravity model (44) generates the accelerating expansion of the late universe, the holographic model also plays the role of the dark energy.

### C. Unification of holographic inflation with holographic dark energy

In view of the previous sections, we are motivated to construct a model unifying the inflationary era in terms of the Starobinsky  $R^2$  inflation (which is known to lead to inflationary observables in a very good agreement with observations [91]), with the accelerating expansion of the late Universe by using the future event horizon  $L_f$  in (5). An example is given by

$$\begin{aligned}
\frac{L_{\text{IR}}}{c} &= \frac{a^{1+n}}{(a_0^1)^n + a^n} \int_t^\infty \frac{dt}{a} \\
&\quad - \frac{(a_0^e)^m}{6\alpha \dot{H}^2 a^6 ((a_0^e)^m + a^m)} \int^t dt a^6 \dot{H}. \quad (46)
\end{aligned}$$

Here  $n$ ,  $m$ ,  $a_0^1$ , and  $a_0^e$  are positive constants and we choose  $a_0^1$  to be smaller than the scale factor in the present Universe

and  $a_0^e$  to be larger than the scale factor in the Universe after inflation. We also assume  $a_0^1 \gg a_0^e$ . Then in the late Universe, where  $a \gg a_0^1$ , the first term dominates and behaves as the future horizon,

$$\frac{L_{\text{IR}}}{c} \sim a \int_t^\infty \frac{dt}{a}, \quad (47)$$

which along with the holographic Friedmann equation  $H = \frac{c}{L_{\text{IR}}}$  generates the accelerating expansion of the present Universe. On the other hand, in the early Universe, where  $a \ll a_0^e$ , the second term dominates and behaves as in (24),

$$\frac{L_{\text{IR}}}{c} \sim -\frac{1}{6\alpha \dot{H}^2 a^6} \int^t dt a^6 \dot{H}, \quad (48)$$

which generates the Starobinsky inflation. Thereby the cutoff proposed in Eq. (46) can provide a unified scenario of inflation and late time acceleration of the Universe from a holographic point of view.



### 1. Minimally coupled axion- $F(R)$ gravity model

A more realistic and a recent model which unifies various cosmological epochs of the Universe is the axion- $F(R)$  gravity model described in [61] and first proposed in [55]. The action of the model is

$$S = \int d^4x \sqrt{-g} \times \left[ \frac{1}{2\kappa^2} (R + f(R)) - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + L_{\text{mat}} \right], \quad (49)$$

where  $\phi$  is the axion scalar field endowed within the potential  $V(\phi)$ . The axion field acts as a dark matter component of the Universe, which, during the inflationary era, was frozen at its primordial vacuum expectation value.  $L_{\text{mat}}$  is the matter Lagrangian; however, in [61], the authors assumed the only perfect fluid to be present is the radiation fluid, which in fact comes as a viable consideration for the purpose of unification. Here it deserves mentioning that the model (49) does not describe the interaction between the ordinary/dark matter and the dark energy. However, in the next subsection in the model (62), the axion field i.e., the dark matter component, is considered to be nonminimally coupled with the curvature. The form of  $f(R)$  of action (49) is taken as

$$f(R) = \frac{R^2}{M^2} - \gamma R^\delta, \quad (50)$$

with  $\delta$  being a positive number in the interval  $0 < \delta < 1$ . Moreover the parameter  $M$  is chosen as  $M = 1.5 \times 10^{-5} (\frac{N}{50})^{-1}$  for early-time phenomenological reasons [61] where  $N$  is the e-foldings number. The first Friedmann equation of the action (49) is

$$3H^2 = -\frac{f(R)}{2} + 3(H^2 + \dot{H})f'(R) - 3H \frac{df'(R)}{dt} + \kappa^2 \left( \rho_r + \frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \quad (51)$$

with  $\rho_r$  being the energy density for the radiation (recall  $L_{\text{mat}}$  consists only of radiation, as mentioned earlier) and the scalar field is considered to be spatially homogeneous.

As described in [61], the model (49) successfully unifies various epochs of our Universe. The demonstration goes as follows: during the early epoch when the curvature is large, the term  $R^2$  dominates over  $R^\delta$  as  $0 < \delta < 1$ . Also in the early stage of the Universe, the axion field was frozen in its vacuum expectation value and for  $m_\phi \sim \mathcal{O}(10^{-14})$  eV, the axion field only contributes a very small cosmological

constant (compared to the other terms) in the equation of motion. The axion mass in the present model [i.e.,  $\sim \mathcal{O}(10^{-14})$  eV] respects the latest Planck data of the dark matter density given by  $\Omega_a h^2 = 0.12 \pm 0.001$ . This result is in agreement with [92] where it was shown that the dark matter density (i.e.,  $\Omega_a h^2 = 0.12 \pm 0.001$ ) requires the axion mass range as  $10^{-24}$  eV  $\leq m_\phi \leq 10^{-12}$  eV. For  $10^{14}$  GeV  $\lesssim \phi_i \lesssim 10^{17}$  GeV, where  $\phi_i$  is the vacuum expectation value obtained by the axion field. Further the ‘‘string anthropic boundary’’ is where  $\Omega_a h^2 = 0.12$  leads to  $m_\phi \simeq 10^{-19}$  eV for  $\phi_i = 10^{16}$  eV [92]. Here we would like to mention that in the present paper we discuss the axionlike particles (where the Peccei Quinn symmetry is not broken during inflation), not the QCD axion in which case the mass bound of the axion lies within  $10^{-5}$  eV  $\lesssim m_\phi \lesssim 10^{-2}$  eV [93]. Thus the mass range relevant for axion dark matter is wide, as demonstrated in various earlier literature. For the present higher curvature axion model, the axion mass comes as  $m_\phi \sim \mathcal{O}(10^{-14})$  eV with  $\phi_i \sim \mathcal{O}(10^{15})$  GeV. Coming back to Eq. (51), it is evident that the Ricci scalar related terms dominate the inflationary evolution, and specifically the  $R^2$ , hence the model is reduced to the  $R^2$  model, which yields a viable inflationary phenomenology compatible with the observational data coming from Planck 2018. With the expansion of the Universe, the Hubble parameter decreases and when  $H \lesssim m_\phi$ , the axion starts to oscillate. Assuming a slowly varying oscillation for the axion, it can be shown that the axion energy density scales as  $\rho_\phi \sim a^{-3}$ , thus the axion mimics the dark matter fluid with an average EoS parameter  $w_\phi \simeq 0$ . At late time of the Universe, the  $R^\delta$  term in the  $f(R)$  dominates and controls the dynamics. After demonstrating the contribution of each term, the full Friedmann equation is solved numerically for a wide range of redshift ( $z$ ), in particular for  $z = [0, 10]$ . Following the numerical solution, various parameters, namely the deceleration parameter  $q = -1 - \frac{\ddot{H}}{H^2}$ , the jerk parameter  $j = \frac{\dddot{H}}{H^3} - 3q - 2$ , the parameter  $s = \frac{(j-1)}{3(q-1/2)}$ , and the parameter  $Om(z) = \frac{H(z)-1}{H_0^2(1+z)^3-1}$  have been estimated. As a result the axion- $F(R)$  gravity model is found to produce results very similar to the  $\Lambda$ CDM model, in some cases almost identical for small redshifts, and in all cases compatible results with the latest Planck constraints on the cosmological parameters.

In order to map the axion- $F(R)$  gravity model with the holographic one, we put the form of  $f(R) = \frac{R^2}{M^2} - \gamma R^\delta$  into Eq. (14) and upon some simple algebra, we get the corresponding holographic cutoff in terms of  $L_p$  and its derivatives as follows:

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} &= 6 \left( \frac{1}{L_p} - \frac{\dot{L}_p}{L_p} \right) \left[ \frac{2}{M^2} + (6\Omega_{(L_p)})^{\delta-2} \gamma \delta (1-\delta) \right] \left[ -4 \frac{\dot{L}_p}{L_p^3} + 6 \frac{\dot{L}_p^2}{L_p^3} - 2 \frac{\dot{L}_p^3}{L_p^3} - 3 \frac{\ddot{L}_p}{L_p^2} + \frac{\dot{L}_p \ddot{L}_p}{L_p^2} + \frac{\ddot{\ddot{L}}_p}{L_p} \right] \\ &\quad + 6\Omega_{(L_p)} \left[ \frac{1}{L_p^2} - \frac{\dot{L}_p}{L_p^2} + \frac{\ddot{L}_p}{L_p} \right] \left[ \frac{2}{M^2} - (6\Omega_{(L_p)})^{\delta-2} \gamma \delta \right] - 6\Omega_{(L_p)} \left[ \frac{2}{M^2} - (6\Omega_{(L_p)})^{\delta-2} \gamma \right], \end{aligned} \quad (52)$$

with  $\Omega_{(L_p)} = \frac{2}{L_p^2} - 3 \frac{\dot{L}_p}{L_p^2} + \frac{\dot{L}_p^2}{L_p^2} + \frac{\ddot{L}_p}{L_p}$ . Similarly the cutoff in terms of the  $L_f$  and its derivatives is obtained from Eq. (17) as follows:

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} &= 6 \left( -\frac{1}{L_f} - \frac{\dot{L}_f}{L_f} \right) \left[ \frac{2}{M^2} + (6\Omega_{(L_f)})^{\delta-2} \gamma \delta (1-\delta) \right] \left[ -4 \frac{\dot{L}_f}{L_f^3} - 6 \frac{\dot{L}_f^2}{L_f^3} - 2 \frac{\dot{L}_f^3}{L_f^3} + 3 \frac{\ddot{L}_f}{L_f^2} + \frac{\dot{L}_f \ddot{L}_f}{L_f^2} + \frac{\ddot{\ddot{L}}_f}{L_f} \right] \\ &\quad + 6\Omega_{(L_f)} \left[ \frac{1}{L_f^2} + \frac{\dot{L}_f}{L_f^2} + \frac{\ddot{L}_f}{L_f} \right] \left[ \frac{2}{M^2} - (6\Omega_{(L_f)})^{\delta-2} \gamma \delta \right] - 6\Omega_{(L_f)} \left[ \frac{2}{M^2} - (6\Omega_{(L_f)})^{\delta-2} \gamma \right], \end{aligned} \quad (53)$$

where  $\Omega_{(L_f)} = \frac{2}{L_f^2} + 3 \frac{\dot{L}_f}{L_f^2} + \frac{\dot{L}_f^2}{L_f^2} + \frac{\ddot{L}_f}{L_f}$ . With the cutoffs determined in the above two expressions, the axion- $F(R)$  model (49) can be equivalently mapped to a holographic model where the Friedmann equation is of the form

$$3H^2 = \frac{3c^2}{(L_{\text{IR}})^2} + \kappa^2 \left( \rho_r + \frac{1}{2} \dot{\phi}^2 + V(\phi) \right). \quad (54)$$

The cutoffs determined in Eqs. (52) and (53) can be decomposed as

$$\frac{1}{(L_{\text{IR}})^2} = \frac{1}{(L_{\text{IR}}^{(1)})^2} + \frac{1}{(L_{\text{IR}}^{(2)})^2},$$

or the above expression can be rewritten as

$$\rho_{\text{hol}} = \rho_{\text{hol}}^{(1)} + \rho_{\text{hol}}^{(2)}, \quad (55)$$

where  $\rho_{\text{hol}}^{(i)} (= \frac{3c^2}{\kappa^2 (L_{\text{IR}}^{(i)})^2})$  is the holographic energy density with the cutoff  $L_{\text{IR}}^{(i)}$ . Furthermore  $\rho_{\text{hol}}^{(1)}$  and  $\rho_{\text{hol}}^{(2)}$  are given by

$$\begin{aligned} \rho_{\text{hol}}^{(1)} &= \frac{3c^2}{\kappa^2 (L_{\text{IR}}^{(1)})^2} \\ &= \frac{3}{\kappa^2} \left\{ \frac{12}{M^2} \left( \frac{1}{L_p} - \frac{\dot{L}_p}{L_p} \right) \left[ -4 \frac{\dot{L}_p}{L_p^3} + 6 \frac{\dot{L}_p^2}{L_p^3} - 2 \frac{\dot{L}_p^3}{L_p^3} - 3 \frac{\ddot{L}_p}{L_p^2} + \frac{\dot{L}_p \ddot{L}_p}{L_p^2} + \frac{\ddot{\ddot{L}}_p}{L_p} \right] + \frac{12}{M^2} \Omega_{(L_p)} \left[ \frac{1}{L_p^2} - \frac{\dot{L}_p}{L_p^2} + \frac{\ddot{L}_p}{L_p} \right] - \frac{12}{M^2} \Omega_{(L_p)} \right\} \\ &= \frac{3}{\kappa^2} \left\{ \frac{12}{M^2} \left( -\frac{1}{L_f} - \frac{\dot{L}_f}{L_f} \right) \left[ -4 \frac{\dot{L}_f}{L_f^3} - 6 \frac{\dot{L}_f^2}{L_f^3} - 2 \frac{\dot{L}_f^3}{L_f^3} + 3 \frac{\ddot{L}_f}{L_f^2} + \frac{\dot{L}_f \ddot{L}_f}{L_f^2} + \frac{\ddot{\ddot{L}}_f}{L_f} \right] + \frac{12}{M^2} \Omega_{(L_f)} \left[ \frac{1}{L_f^2} + \frac{\dot{L}_f}{L_f^2} + \frac{\ddot{L}_f}{L_f} \right] - \frac{12}{M^2} \Omega_{(L_f)} \right\}, \end{aligned} \quad (56)$$

and

$$\begin{aligned} \rho_{\text{hol}}^{(2)} &= \frac{3c^2}{\kappa^2 (L_{\text{IR}}^{(2)})^2} = \frac{3}{\kappa^2} \left\{ 6(6\Omega_{(L_p)})^{\delta-2} \gamma \delta (1-\delta) \left( \frac{1}{L_p} - \frac{\dot{L}_p}{L_p} \right) \left[ -4 \frac{\dot{L}_p}{L_p^3} + 6 \frac{\dot{L}_p^2}{L_p^3} - 2 \frac{\dot{L}_p^3}{L_p^3} - 3 \frac{\ddot{L}_p}{L_p^2} + \frac{\dot{L}_p \ddot{L}_p}{L_p^2} + \frac{\ddot{\ddot{L}}_p}{L_p} \right] \right. \\ &\quad \left. - (6\Omega_{(L_p)})^{\delta-1} \gamma \delta \left[ \frac{1}{L_p^2} - \frac{\dot{L}_p}{L_p^2} + \frac{\ddot{L}_p}{L_p} \right] + \gamma (6\Omega_{(L_p)})^{\delta-1} \right\} \\ &= \frac{3}{\kappa^2} \left\{ 16(6\Omega_{(L_f)})^{\delta-2} \gamma \delta (1-\delta) \left( -\frac{1}{L_f} - \frac{\dot{L}_f}{L_f} \right) \left[ -4 \frac{\dot{L}_f}{L_f^3} - 6 \frac{\dot{L}_f^2}{L_f^3} - 2 \frac{\dot{L}_f^3}{L_f^3} + 3 \frac{\ddot{L}_f}{L_f^2} + \frac{\dot{L}_f \ddot{L}_f}{L_f^2} + \frac{\ddot{\ddot{L}}_f}{L_f} \right] \right. \\ &\quad \left. - (6\Omega_{(L_f)})^{\delta-1} \gamma \delta \left[ \frac{1}{L_f^2} + \frac{\dot{L}_f}{L_f^2} + \frac{\ddot{L}_f}{L_f} \right] + \gamma (6\Omega_{(L_f)})^{\delta-1} \right\}, \end{aligned} \quad (57)$$

respectively. With such decomposition of  $L_{\text{IR}}$ , the holographic Friedmann Eq. (54) can be rewritten as

$$3H^2 = \kappa^2(\rho_{\text{hol}}^{(1)} + \rho_{\text{hol}}^{(2)}) + \kappa^2\left(\rho_r + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \quad (58)$$

Clearly  $\rho_{\text{hol}}^{(1)}$  corresponds to the  $R^2$  term and thus dominates over the  $\rho_{\text{hol}}^{(2)}$  and  $\rho_r$  term in the early epoch of the Universe. Moreover, the axion field was frozen during the inflationary era and thereby, we can safely neglect  $\rho_\phi$  from Eq. (58). Hence, in the early Universe when the curvature is large, Eq. (58) behaves as

$$3H^2 \simeq \kappa^2\rho_{\text{hol}}^{(1)}, \quad (59)$$

which successfully produces an inflationary scenario. As the Universe expands, the Hubble parameter decreases and from  $H \ll m_\phi$ , the axion field starts to oscillate and thus

contributes its effect to the dynamics, along with the term  $\rho_r$ . Moreover, in the low curvature regime, as in the case of the present Universe, the energy density  $\rho_{\text{hol}}^{(2)}$  dominates over the other terms of Eq. (58). In view of these arguments, after the inflationary scenario, Eq. (58) becomes

$$3H^2 \simeq \kappa^2\rho_{\text{hol}}^{(2)} + \kappa^2\left(\rho_r + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \quad (60)$$

where  $\rho_r$  and  $\rho_\phi$  denote the radiation and matter dominated epochs, respectively, while  $\rho_{\text{hol}}^{(2)}$  denotes the holographic dark energy density at late times. Therefore, Eq. (58) is able to unify various cosmological epochs of our Universe from a holographic point of view.

Before closing this section, we would like to mention that apart from the two aforementioned forms of  $L_{\text{IR}}$  [i.e., in Eqs. (52) and (53)], an integral form of  $L_{\text{IR}}$  for the axion- $F(R)$  model is also calculated and given by

$$\frac{L_{\text{IR}}}{c} = -\frac{1}{6\alpha\dot{H}^2 a^6} \int dt a^6 \dot{H} \left[ 1 - \gamma 6^{\delta-1} (2H^2 + \dot{H})^{\delta-2} \left\{ 2H^2(2-\delta) + \frac{\dot{H}^2}{H^2}(1-\delta) - \frac{\ddot{H}}{H}\delta(1-\delta) + \dot{H}(4-7\delta+4\delta^2) \right\} - \frac{\kappa^2}{H^2}(\rho_r + \rho_\phi) \right]. \quad (61)$$

Inserting the above expression of the cutoff into  $H = \frac{c}{L_{\text{IR}}}$ , one can reproduce the first Friedmann Eq. (51) for  $f(R) = \frac{R^2}{M^2} - \gamma R^\delta$ . Therefore, the cutoff in Eq. (61) also provides a corresponding holographic model for the axion- $F(R)$  action (49).

## 2. Nonminimally coupled axion- $F(R)$ gravity model

As an extension, we consider a second axion- $F(R)$  gravity model where the axion scalar field is nonminimally coupled with the curvature, unlike to the previous model (49) where the axion is minimally coupled with the gravity. The action of the second axion- $F(R)$  model is the following [55]:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} (R + f(R, \phi)) - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \quad (62)$$

where  $f(R, \phi)$  takes the form

$$f(R, \phi) = \frac{R^2}{M^2} + h(\phi)R^\delta, \quad (63)$$

where  $\delta$  is a dimensionless parameter with values in the interval  $0 < \delta < 1$ . It is evident that the axion field couples with the curvature with the coupling function  $h(\phi)$ . In the

spatially flat FRW spacetime, the first Friedmann equation of the action (62) becomes

$$3H^2 = -\frac{f(R, \phi)}{2} + 3(H^2 + \dot{H}) \frac{\partial f}{\partial R} - 3H \frac{d}{dt} \left( \frac{\partial f}{\partial R} \right) + \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (64)$$

with the consideration that the axion field is homogeneous in space. Similar to the previous model, the present model (62) is also able to unify various cosmological epochs of our Universe like inflation, dark matter epoch, and dark energy epoch as described in [55]. During the early epoch, the axion field in the action (62) was frozen at its vacuum expectation value (vev) and due to the consideration  $V(\phi) \gg \frac{1}{\kappa^2} h(\phi)R^\delta$ , the axion field merely contributes a cosmological constant (due to the presence of its potential) to the equation of motion. Again for  $m_\phi \sim \mathcal{O}(10^{-12})$  eV, the cosmological constant coming from the axion vev can be neglected compared to the  $R^2$  term which is a naturally dominant term in the large curvature regime, as in the early Universe. Moreover due to  $0 < \delta < 1$ , the  $R^\delta$  term is subdominant in comparison to the quadratic term and thus the dynamics of the early Universe is controlled by the  $R^2$  term which is known to produce a viable inflationary era compatible with Planck constraints. As the Universe expands, the Hubble parameter decreases and from

$H \sim m_\phi$ , the axion begins its dynamical evolution. Assuming slowly varying oscillating dynamics for the axion, it can be shown that the energy density of the same evolves as  $\rho_\phi \sim a^{-3}$ . Thus the axion field mimics the behavior of the dark matter of the Universe. Finally during the low curvature regime, i.e., in the present Universe, the term  $h(\phi)R^\delta$  starts to dominate and provides a dark energy model. These arguments clearly indicate that the qualitative nature of the models (49) and (62) are more or less the same. However it is worth mentioning that the model (62) predicts the existence of a stiff matter era for the axion field at a primordial preinflationary era, in which case the energy

density scales as  $\rho_\phi \sim a^{-6}$ , which is not predicted by the model (49). Actually the prediction of a stiff matter era from the model (62) arises due to the reason that the aforementioned condition  $V(\phi) \gg \frac{1}{\kappa^2} h(\phi)R^\delta$  should hold true during or after the inflationary era. Therefore, in the preinflationary epoch, one could have  $V(\phi) \sim \frac{1}{\kappa^2} h(\phi)R^\delta$  which in turn makes  $\rho_\phi \sim a^{-6}$  through the conservation equation of the axion field.

Plugging the form of  $f(R) = \frac{R^2}{M^2} + h(\phi)R^\delta$  into Eq. (14), we get the equivalent holographic cutoff in terms of the particle horizon ( $L_p$ ) and its derivatives as

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} = & 6 \left( \frac{1}{L_p} - \frac{\dot{L}_p}{L_p} \right) \left[ \frac{2}{M^2} - (6\Omega_{(L_p)})^{\delta-2} h(\phi(a)) \delta(1-\delta) \right] \left[ -4 \frac{\dot{L}_p}{L_p^3} + 6 \frac{\dot{L}_p^2}{L_p^3} - 2 \frac{\dot{L}_p^3}{L_p^3} - 3 \frac{\ddot{L}_p}{L_p^2} + \frac{\dot{L}_p \ddot{L}_p}{L_p^2} + \frac{\ddot{\dot{L}}_p}{L_p} \right] \\ & + 6\Omega_{(L_p)} \left[ \frac{1}{L_p^2} - \frac{\dot{L}_p}{L_p^2} + \frac{\ddot{L}_p}{L_p} \right] \left[ \frac{2}{M^2} + (6\Omega_{(L_p)})^{\delta-2} h(\phi(a)) \delta \right] - 6\Omega_{(L_p)} \left[ \frac{2}{M^2} + (6\Omega_{(L_p)})^{\delta-2} h(\phi(a)) \right], \end{aligned} \quad (65)$$

where  $\Omega_{(L_p)}$  is given after Eq. (52) and  $\phi = \phi(a)$  can be determined from the conservation equation of the axion field. The holographic cutoff for the model (62) in terms of the  $L_f$  and its derivatives is obtained from Eq. (17) as follows:

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} = & 6 \left( -\frac{1}{L_f} - \frac{\dot{L}_f}{L_f} \right) \left[ \frac{2}{M^2} - (6\Omega_{(L_f)})^{\delta-2} h(\phi(a)) \delta(1-\delta) \right] \left[ -4 \frac{\dot{L}_f}{L_f^3} - 6 \frac{\dot{L}_f^2}{L_f^3} - 2 \frac{\dot{L}_f^3}{L_f^3} + 3 \frac{\ddot{L}_f}{L_f^2} + \frac{\dot{L}_f \ddot{L}_f}{L_f^2} + \frac{\ddot{\dot{L}}_f}{L_f} \right] \\ & + 6\Omega_{(L_f)} \left[ \frac{1}{L_f^2} + \frac{\dot{L}_f}{L_f^2} + \frac{\ddot{L}_f}{L_f} \right] \left[ \frac{2}{M^2} + (6\Omega_{(L_f)})^{\delta-2} h(\phi(a)) \delta \right] - 6\Omega_{(L_f)} \left[ \frac{2}{M^2} + (6\Omega_{(L_f)})^{\delta-2} h(\phi(a)) \right], \end{aligned} \quad (66)$$

for  $\Omega_{(L_f)}$ , see the expression just after Eq. (53). Clearly the axion- $F(R)$  model (62) is equivalent to the holographic model with the cutoffs determined in the above two expressions. The corresponding holographic Friedmann equation takes the form:

$$3H^2 = \frac{3c^2}{(L_{\text{IR}})^2} + \kappa^2 \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right). \quad (67)$$

The cutoffs determined in Eqs. (65) and (66) are decomposed as

$$\begin{aligned} \rho_{\text{hol}}^{(1)} &= \frac{3c^2}{\kappa^2 (L_{\text{IR}}^{(1)})^2} \\ &= \frac{3}{\kappa^2} \left\{ \frac{12}{M^2} \left( \frac{1}{L_p} - \frac{\dot{L}_p}{L_p} \right) \left[ -4 \frac{\dot{L}_p}{L_p^3} + 6 \frac{\dot{L}_p^2}{L_p^3} - 2 \frac{\dot{L}_p^3}{L_p^3} - 3 \frac{\ddot{L}_p}{L_p^2} + \frac{\dot{L}_p \ddot{L}_p}{L_p^2} + \frac{\ddot{\dot{L}}_p}{L_p} \right] + \frac{12}{M^2} \Omega_{(L_p)} \left[ \frac{1}{L_p^2} - \frac{\dot{L}_p}{L_p^2} + \frac{\ddot{L}_p}{L_p} \right] - \frac{12}{M^2} \Omega_{(L_p)} \right\} \\ &= \frac{3}{\kappa^2} \left\{ \frac{12}{M^2} \left( -\frac{1}{L_f} - \frac{\dot{L}_f}{L_f} \right) \left[ -4 \frac{\dot{L}_f}{L_f^3} - 6 \frac{\dot{L}_f^2}{L_f^3} - 2 \frac{\dot{L}_f^3}{L_f^3} + 3 \frac{\ddot{L}_f}{L_f^2} + \frac{\dot{L}_f \ddot{L}_f}{L_f^2} + \frac{\ddot{\dot{L}}_f}{L_f} \right] + \frac{12}{M^2} \Omega_{(L_f)} \left[ \frac{1}{L_f^2} + \frac{\dot{L}_f}{L_f^2} + \frac{\ddot{L}_f}{L_f} \right] - \frac{12}{M^2} \Omega_{(L_f)} \right\}, \end{aligned} \quad (69)$$

and

$$\frac{1}{(L_{\text{IR}})^2} = \frac{1}{(L_{\text{IR}}^{(1)})^2} + \frac{1}{(L_{\text{IR}}^{(2)})^2},$$

and the above expression can be rewritten as

$$\rho_{\text{hol}} = \rho_{\text{hol}}^{(1)} + \rho_{\text{hol}}^{(2)}, \quad (68)$$

where, in the spirit of Eq. (1),  $\rho_{\text{hol}}^{(i)} = \frac{3c^2}{\kappa^2 (L_{\text{IR}}^{(i)})^2}$ . Furthermore  $\rho_{\text{hol}}^{(1)}$  and  $\rho_{\text{hol}}^{(2)}$  are given by

$$\begin{aligned}
\rho_{\text{hol}}^{(2)} &= \frac{3c^2}{\kappa^2(L_{\text{IR}}^{(2)})^2} \\
&= \frac{3}{\kappa^2} \left\{ -6(6\Omega_{(L_p)})^{\delta-2} h(\phi(a)) \delta(1-\delta) \left( \frac{1}{L_p} - \frac{\dot{L}_p}{L_p} \right) \left[ -4 \frac{\dot{L}_p}{L_p^3} + 6 \frac{\dot{L}_p^2}{L_p^3} - 2 \frac{\dot{L}_p^3}{L_p^3} - 3 \frac{\ddot{L}_p}{L_p^2} + \frac{\dot{L}_p \ddot{L}_p}{L_p^2} + \frac{\ddot{\dot{L}}_p}{L_p} \right] \right. \\
&\quad \left. + (6\Omega_{(L_p)})^{\delta-1} h(\phi(a)) \delta \left[ \frac{1}{L_p^2} - \frac{\dot{L}_p}{L_p^2} + \frac{\ddot{L}_p}{L_p} \right] - h(\phi(a)) (6\Omega_{(L_p)})^{\delta-1} \right\} \\
&= \frac{3}{\kappa^2} \left\{ -3(6\Omega_{(L_f)})^{\delta-2} h(\phi(a)) \delta(1-\delta) \left( -\frac{1}{L_f} - \frac{\dot{L}_f}{L_f} \right) \left[ -4 \frac{\dot{L}_f}{L_f^3} - 6 \frac{\dot{L}_f^2}{L_f^3} - 2 \frac{\dot{L}_f^3}{L_f^3} + 3 \frac{\ddot{L}_f}{L_f^2} + \frac{\dot{L}_f \ddot{L}_f}{L_f^2} + \frac{\ddot{\dot{L}}_f}{L_f} \right] \right. \\
&\quad \left. + (6\Omega_{(L_f)})^{\delta-1} h(\phi(a)) \delta \left[ \frac{1}{L_f^2} + \frac{\dot{L}_f}{L_f^2} + \frac{\ddot{L}_f}{L_f} \right] - h(\phi(a)) (6\Omega_{(L_f)})^{\delta-1} \right\}, \tag{70}
\end{aligned}$$

respectively. With such decomposition of  $L_{\text{IR}}$ , Eq. (67) can be rewritten as

$$3H^2 = \kappa^2(\rho_{\text{hol}}^{(1)} + \rho_{\text{hol}}^{(2)}) + \kappa^2 \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right). \tag{71}$$

As earlier,  $\rho_{\text{hol}}^{(1)}$  corresponds to the  $R^2$  term and due to the arguments demonstrated just after Eq. (64), the holographic Eq. (71) behaves during the early Universe as

$$3H^2 \simeq \kappa^2 \rho_{\text{hol}}^{(1)}, \tag{72}$$

which describes an inflationary scenario with good agreement in terms of the Planck observations. On other hand, after the inflationary scenario, the axion field starts to

contribute and also in the present Universe the  $h(\phi)R^\delta$  term dominates over the quadratic curvature. As a result, after inflation, Eq. (71) becomes

$$3H^2 \simeq \kappa^2 \rho_{\text{hol}}^{(2)} + \kappa^2 \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \tag{73}$$

where  $\rho_\phi$  denotes the matter dominated epoch, while  $\rho_{\text{hol}}^{(2)}$  stands for the holographic dark energy density during late time of the Universe. Therefore, Eq. (71) is able to unify various cosmological epochs of our Universe like inflation, dark matter, and dark energy epochs respectively, from a holographic point of view.

Similar to an earlier case, here we also determine the integral form of  $L_{\text{IR}}$  for the nonminimally coupled axion- $F(R)$  model, which is

$$\begin{aligned}
\frac{L_{\text{IR}}}{c} &= -\frac{1}{6\alpha \dot{H}^2 a^6} \int dt a^6 \dot{H} \left[ 1 + h(\phi(a)) 6^{\delta-1} (2H^2 + \dot{H})^{\delta-2} \right. \\
&\quad \left. \times \left\{ 2H^2(2-\delta) + \frac{\dot{H}^2}{H^2}(1-\delta) - \frac{\ddot{H}}{H} \delta(1-\delta) + \dot{H}(4-7\delta+4\delta^2) \right\} - \frac{\kappa^2}{H^2} (\rho_r + \rho_\phi) \right]. \tag{74}
\end{aligned}$$

Inserting the above expression of the cutoff into  $H = \frac{c}{L_{\text{IR}}}$ , one can reproduce Eq. (64) for  $f(R, \phi) = \frac{R^2}{M^2} + h(\phi)R^\delta$ . Therefore, the axion- $F(R)$  model (62) can also be mapped to a holographic model with the cutoff determined in Eq. (74).

#### IV. HOLOGRAPHIC CORRESPONDENCE OF $f(\mathcal{G})$ GRAVITY

We now establish the holographic correspondence and consequently determine the holographic cutoff for  $f(\mathcal{G})$  gravity whose action is given by [see [47,66] for different aspects of  $f(\mathcal{G})$  gravity]

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + f(\mathcal{G}) + \mathcal{L}_{\text{mat}} \right], \tag{75}$$

where  $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$  is the Gauss-Bonnet invariant which, in the FRW spacetime, takes the form as

$$\mathcal{G} = 24H^2[H^2 + \dot{H}]. \tag{76}$$

With this expression of  $\mathcal{G}$ , the first Friedmann equation can be written as

$$0 = -\frac{3}{\kappa^2} H^2 - f(\mathcal{G}) + \mathcal{G}f'(\mathcal{G}) - 24\dot{\mathcal{G}}f''(\mathcal{G})H^3 + \rho_{\text{mat}}. \tag{77}$$

It is interesting to note that for  $f(\mathcal{G}) = \mathcal{G}$ , Eq. (77) reduces to the standard Friedmann equation for Einstein gravity. This is expected because the Gauss-Bonnet term in 3 + 1 dimensional spacetime becomes a topological surface term and therefore vanishes identically. However the functional form other than  $f(\mathcal{G}) = \mathcal{G}$  indeed contributes in the equation of motion as reflected from the above equation. Comparing Eq. (77) with the holographic Friedmann equation  $H^2 = \frac{1}{(L_{\text{IR}})^2}$ , we can immediately conclude that  $f(\mathcal{G})$  gravity (without/with matter fields) has an equivalent holographic correspondence with the holographic cutoff given by

$$\frac{3c^2}{(L_{\text{IR}})^2} = \kappa^2[-f(\mathcal{G}) + \mathcal{G}f'(\mathcal{G}) - 24\dot{\mathcal{G}}f''(\mathcal{G})H^3], \quad (78)$$

and as a consequence, Eq. (77) is rewritten as  $3H^2 = \frac{3}{(L_{\text{IR}})^2} + \kappa^2\rho_{\text{mat}}$ . Thereby, any arbitrary  $f(\mathcal{G})$  gravity can be mapped to an equivalent holographic model with the cutoff given by Eq. (78). In a similar way as in  $F(R)$  gravity, here we also determine  $L_{\text{IR}}$  in two different ways—namely in terms of the particle horizon ( $L_p$ ) and its derivatives or in terms of the future horizon ( $L_f$ ) and its derivatives. Recall that the Hubble parameter in terms of  $L_p$  or  $L_f$  can be expressed as  $H(L_p, \dot{L}_p) = \frac{\dot{L}_p}{L_p} - \frac{1}{L_p}$  or  $H(L_f, \dot{L}_f) = \frac{\dot{L}_f}{L_f} + \frac{1}{L_f}$  respectively. These considerations lead to the Gauss-Bonnet invariant as

$$\mathcal{G}^{(L_p)} = 24 \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right)^2 \left( \frac{\ddot{L}_p}{L_p} - \frac{\dot{L}_p}{L_p^2} + \frac{1}{L_p^2} \right), \quad (79)$$

$$\mathcal{G}^{(L_f)} = 24 \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right)^2 \left( \frac{\ddot{L}_f}{L_f} + \frac{\dot{L}_f}{L_f^2} + \frac{1}{L_f^2} \right). \quad (80)$$

Using the above expressions, the holographic cutoff in terms of  $L_p$  and its derivatives can be determined as

$$\begin{aligned} \frac{3c^2}{(L_{\text{IR}})^2} &= f(\mathcal{G}^{(L_p)}) - \mathcal{G}^{(L_p)} f'(\mathcal{G}^{(L_p)}) \\ &+ 24 \frac{d\mathcal{G}^{(L_p)}}{dt} f''(\mathcal{G}^{(L_p)}) \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right)^3. \end{aligned} \quad (81)$$

Similarly,  $L_{\text{IR}} = L_{\text{IR}}(L_f, \dot{L}_f, \ddot{L}_f, \text{higher derivatives of } L_f)$  takes the following form:

$$\begin{aligned} \frac{3c^2}{(L_{\text{IR}})^2} &= f(\mathcal{G}^{(L_f)}) - \mathcal{G}^{(L_f)} f'(\mathcal{G}^{(L_f)}) \\ &+ 24 \frac{d\mathcal{G}^{(L_f)}}{dt} f''(\mathcal{G}^{(L_f)}) \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right)^3. \end{aligned} \quad (82)$$

Equations (81) and (82) are the key equations that will determine the equivalent holographic cutoff for any arbitrary  $f(\mathcal{G})$  gravity. As an example, we may consider a simple model like

$$f(\mathcal{G}) = f_0 \mathcal{G}^m, \quad (83)$$

where  $f_0$  and  $m$  are dimensionless parameters [78]. Plugging back this explicit form of  $f(\mathcal{G})$  into Eq. (81), we get

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} &= \frac{f_0(1-m)}{3(1-\dot{L}_p/L_p + L_p\ddot{L}_p)^2} \left[ \frac{24(1-\dot{L}_p)^2(1-\dot{L}_p/L_p + L_p\ddot{L}_p)}{L_p^4} \right]^m [1 - 4m\dot{L}_p^3 + (2-3m)L_p\ddot{L}_p \\ &+ \dot{L}_p^2(1+8m+3mL_p\ddot{L}_p) + L_p^2((1-2m)\ddot{L}_p^2 + m\ddot{L}_p) - \dot{L}_p(2+4m+2L_p\ddot{L}_p + mL_p^2\ddot{L}_p)]. \end{aligned} \quad (84)$$

Moreover Eq. (82) leads to the holographic cutoff for  $f(\mathcal{G}) = f_0 \mathcal{G}^m$  in terms of the future horizon as

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} &= \frac{f_0(1-m)}{3(1+\dot{L}_f/L_f + L_f\ddot{L}_f)^2} \left[ \frac{24(1+\dot{L}_f)^2(1+\dot{L}_f/L_f + L_f\ddot{L}_f)}{L_p^4} \right]^m [1 + 4m\dot{L}_f^3 + (2-3m)L_f\ddot{L}_f \\ &+ \dot{L}_f^2(1+8m+3mL_f\ddot{L}_f) + L_f^2((1-2m)\ddot{L}_f^2 - m\ddot{L}_f) + \dot{L}_f(2+4m+2L_f\ddot{L}_f - mL_f^2\ddot{L}_f)]. \end{aligned} \quad (85)$$

Clearly the above two holographic cutoffs can mimic the cosmological field equations and thus we established the holographic correspondence for the  $f(\mathcal{G}) = f_0 \mathcal{G}^m$  model.

It may be observed from Eq. (83) that for  $m > 1/2$ , the term  $f(\mathcal{G}) \sim \mathcal{G}^m$  dominates over the Einstein and the matter term(s) in the large curvature regime, while for  $m < 1/2$ , the Gauss-Bonnet function  $f(\mathcal{G})$  becomes the dominating one in the low curvature regime. Here we consider a case in

which the contributions from the Einstein and matter terms can be neglected compared to  $f(\mathcal{G}) \sim \mathcal{G}^m$ . In such situation, the scale factor evolves as  $a(t) = a_0 t^{h_0}$  where the exponent  $h_0$  is given by  $h_0 = 1-4m$ . Such evolution of the scale factor immediately leads to the effective EoS parameter as

$$w_{\text{eff}} = -1 + \frac{2}{3h_0} = -1 + \frac{2}{3(1-4m)}, \quad (86)$$

where in the last line, we used the expression for  $h_0$ . Therefore, if  $m > 0$ , the Universe is accelerating ( $w_{\text{eff}} < -1/3$ ), and if  $m > 1/4$ , the Universe is in a phantom phase ( $w_{\text{eff}} < -1$ ). Hence, depending on the parameter  $m$ , the model  $f(\mathcal{G}) = f_0 \mathcal{G}^m$  can act as inflationary or as late-time accelerating model. However, this model does not give a unified scenario of inflation and late time dark energy epoch of our Universe. Keeping this in mind, we will consider a different form of  $f(\mathcal{G})$ , in the next subsection, that is able to describe inflation and the late-time acceleration of the Universe in a unified manner.

### A. Unification of holographic inflation with holographic dark energy

In the spirit of the power law model and the above discussions, we consider the following model [47]:

$$f(\mathcal{G}) = f_1 \mathcal{G}^{\beta_1} + f_2 \mathcal{G}^{\beta_2}, \quad (87)$$

where the exponents are assumed to take values in the intervals,

$$\beta_1 > \frac{1}{2}, \quad \frac{1}{4} < \beta_2 < \frac{1}{2}. \quad (88)$$

Thus in the large curvature regime, as in the early Universe, the first term dominates compared to the second term and the Einstein term, which in turn leads to the effective EoS parameter as  $w_{\text{eff}}^{(1)} = -1 + \frac{2}{3(1-4\beta_1)}$  [using Eq. (86)]. Due to  $\beta_1 > 1/2$ ,

$$-\frac{5}{3} < w_{\text{eff}}^{(1)} < -1. \quad (89)$$

On the other hand, when the curvature is small, as is the case in the late Universe, the second term in (87) dominates compared with the first term and the Einstein term and yields

$$w_{\text{eff}}^{(1)} = -1 + \frac{2}{3(1-4\beta_2)} < -\frac{5}{3}. \quad (90)$$

Therefore, the theoretical framework (87) produces a model that describes the unified scenario of inflation and dark energy epochs of our Universe. By inserting the explicit form of  $f(\mathcal{G}) = f_1 \mathcal{G}^{\beta_1} + f_2 \mathcal{G}^{\beta_2}$  into Eq. (81), we get the holographic cutoff in terms of  $L_p$  and its derivatives as

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} = & \frac{f_1}{3} \left[ \frac{24(1 - \dot{L}_p)^2(1 - \dot{L}_p + L_p \ddot{L}_p)}{L_p^4} \right]^{\beta_1} \left[ 1 - \beta_1 - \beta_1(1 - \beta_1) \frac{X(L_p, \dot{L}_p, \ddot{L}_p, \ddot{\ddot{L}}_p)}{(1 - \dot{L}_p + L_p \ddot{L}_p)^2} \right] \\ & + \frac{f_2}{3} \left[ \frac{24(1 - \dot{L}_p)^2(1 - \dot{L}_p + L_p \ddot{L}_p)}{L_p^4} \right]^{\beta_2} \left[ 1 - \beta_2 - \beta_2(1 - \beta_2) \frac{X(L_p, \dot{L}_p, \ddot{L}_p, \ddot{\ddot{L}}_p)}{(1 - \dot{L}_p + L_p \ddot{L}_p)^2} \right]. \end{aligned} \quad (91)$$

Equation (82) leads to the  $L_{\text{IR}}$  as a function of the future horizon and its derivatives as

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} = & \frac{f_1}{3} \left[ \frac{24(1 + \dot{L}_f)^2(1 + \dot{L}_f + L_f \ddot{L}_f)}{L_f^4} \right]^{\beta_1} \left[ 1 - \beta_1 - \beta_1(1 - \beta_1) \frac{\Upsilon(L_f, \dot{L}_f, \ddot{L}_f, \ddot{\ddot{L}}_f)}{(1 + \dot{L}_f + L_f \ddot{L}_f)^2} \right] \\ & + \frac{f_2}{3} \left[ \frac{24(1 + \dot{L}_f)^2(1 + \dot{L}_f + L_f \ddot{L}_f)}{L_f^4} \right]^{\beta_2} \left[ 1 - \beta_2 - \beta_2(1 - \beta_2) \frac{\Upsilon(L_f, \dot{L}_f, \ddot{L}_f, \ddot{\ddot{L}}_f)}{(1 + \dot{L}_f + L_f \ddot{L}_f)^2} \right], \end{aligned} \quad (92)$$

where the functions  $X$  and  $\Upsilon$  are given by

$$X(L_p, \dot{L}_p, \ddot{L}_p, \ddot{\ddot{L}}_p) = 4\dot{L}_p - 8\dot{L}_p^2 + 4\dot{L}_p^3 - 3L_p \dot{L}_p^2 \ddot{L}_p + L_p^2 \dot{L}_p \ddot{\ddot{L}}_p + L_p(3\ddot{L}_p + 2L_p \ddot{L}_p^2 - L_p \ddot{\ddot{L}}_p),$$

and

$$\Upsilon(L_f, \dot{L}_f, \ddot{L}_f, \ddot{\ddot{L}}_f) = -4\dot{L}_f - 8\dot{L}_f^2 - 4\dot{L}_f^3 - 3L_f \dot{L}_f^2 \ddot{L}_f + L_f^2 \dot{L}_f \ddot{\ddot{L}}_f + L_f(3\ddot{L}_f + 2L_f \ddot{L}_f^2 + L_f \ddot{\ddot{L}}_f), \quad (93)$$

respectively. The cutoffs determined in Eq. (91) or Eq. (92) along with the expression  $H^2 = \frac{1}{(L_{\text{IR}})^2}$  can reconstruct the cosmological field equations and thus provide an equivalent holographic scenario for the considered  $f(\mathcal{G})$  model (87). The cutoffs determined in Eqs. (91) and (92) can be decomposed as

$$\frac{1}{(L_{\text{IR}})^2} = \frac{1}{(L_{\text{IR}}^{(1)})^2} + \frac{1}{(L_{\text{IR}}^{(2)})^2},$$

or the above expression can be rewritten as

$$\rho_{\text{hol}} = \rho_{\text{hol}}^{(1)} + \rho_{\text{hol}}^{(2)}, \quad (94)$$

where  $\rho_{\text{hol}}^{(i)} = \frac{3c^2}{\kappa^2(L_{\text{IR}}^{(i)})^2}$ . Furthermore  $\rho_{\text{hol}}^{(1)}$  and  $\rho_{\text{hol}}^{(2)}$  are given by

$$\begin{aligned} \rho_{\text{hol}}^{(1)} &= \frac{f_1}{\kappa^2} \left[ \frac{24(1 - \dot{L}_p)^2(1 - \dot{L}_p + L_p \ddot{L}_p)}{L_p^4} \right]^{\beta_1} \left[ 1 - \beta_1 - \beta_1(1 - \beta_1) \frac{X(L_p, \dot{L}_p, \ddot{L}_p, \ddot{\ddot{L}}_p)}{(1 - \dot{L}_p + L_p \ddot{L}_p)^2} \right] \\ &= \frac{f_1}{\kappa^2} \left[ \frac{24(1 + \dot{L}_f)^2(1 + \dot{L}_f + L_f \ddot{L}_f)}{L_f^4} \right]^{\beta_1} \left[ 1 - \beta_1 - \beta_1(1 - \beta_1) \frac{\Upsilon(L_f, \dot{L}_f, \ddot{L}_f, \ddot{\ddot{L}}_f)}{(1 + \dot{L}_f + L_f \ddot{L}_f)^2} \right], \end{aligned} \quad (95)$$

and

$$\begin{aligned} \rho_{\text{hol}}^{(2)} &= \frac{f_2}{\kappa^2} \left[ \frac{24(1 - \dot{L}_p)^2(1 - \dot{L}_p + L_p \ddot{L}_p)}{L_p^4} \right]^{\beta_2} \left[ 1 - \beta_2 - \beta_2(1 - \beta_2) \frac{X(L_p, \dot{L}_p, \ddot{L}_p, \ddot{\ddot{L}}_p)}{(1 - \dot{L}_p + L_p \ddot{L}_p)^2} \right] \\ &= \frac{f_2}{\kappa^2} \left[ \frac{24(1 + \dot{L}_f)^2(1 + \dot{L}_f + L_f \ddot{L}_f)}{L_f^4} \right]^{\beta_2} \left[ 1 - \beta_2 - \beta_2(1 - \beta_2) \frac{\Upsilon(L_f, \dot{L}_f, \ddot{L}_f, \ddot{\ddot{L}}_f)}{(1 + \dot{L}_f + L_f \ddot{L}_f)^2} \right], \end{aligned} \quad (96)$$

respectively. With such decomposition, the holographic Friedmann equation turns out to be

$$3H^2 = \kappa^2(\rho_{\text{hol}}^{(1)} + \rho_{\text{hol}}^{(2)}). \quad (97)$$

Clearly  $\rho_{\text{hol}}^{(1)}$  corresponds to  $f_1 \mathcal{G}^{\beta_1}$  and thus dominates over  $\rho_{\text{hol}}^{(2)}$  in the large curvature regime, while in the low curvature regime  $\rho_{\text{hol}}^{(2)}$  is the dominant compared to the other one. Therefore, during the early Universe when the curvature is large, Eq. (97) can be approximated as  $3H^2 \simeq \kappa^2 \rho_{\text{hol}}^{(1)}$  which produces an inflationary scenario. On other hand due to low curvature in the present Universe, Eq. (97) goes as  $3H^2 \simeq \kappa^2 \rho_{\text{hol}}^{(2)}$  which provides a holographic dark energy model during late time. Thereby the holographic Friedmann Eq. (97) [see Eqs. (95) and (96) for the expressions of  $\rho_{\text{hol}}^{(i)}$ ] is able to describe inflation and dark energy epochs of the Universe in a unified way.

## V. HOLOGRAPHIC CORRESPONDENCE OF $F(T)$ GRAVITY

We extend our discussion of holographic correspondence to the generalized teleparallel cosmology, i.e.,  $F(T)$  cosmology [94,95]. The teleparallel gravity (TEGR) is described by the Weitzenbock connection which is determined by two dynamical variables, namely the tetrads and the spin connection. Moreover unlike the connection in Einstein's general relativity, the Weitzenbock connection comes as a curvature-free quantity. Recall, in the current work, we consider the spatially flat FRW metric, i.e.,

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2], \quad (98)$$

with  $a(t)$  being the scale factor of the Universe. Generally the tetrad for this metric is considered as

$$e_\mu^a = \text{diag}(1, a(t), a(t), a(t)). \quad (99)$$

This form of the FRW metric is very advantageous because its spin connection vanishes [95], and so no extra contribution is needed in the  $F(T)$  field equations. For the aforementioned tetrad in Eq. (99), the torsion scalar turns out to be  $T = -6H^2$ , with  $H$  being the Hubble parameter.

Following the same reasoning as  $F(R)$  gravity, the action of TEGR can be generalized to  $F(T)$  gravity, in particular

$$S = \frac{1}{2\kappa^2} \int d^4x |e| F(T), \quad (100)$$

where  $e = \det e_\mu^a$  (we use  $e = \det e_\mu^a$  in italics and  $e$  for Napier's constant  $e = 2.718281828 \dots$ ) and  $F(T)$  is an analytic function of  $T$ . Variation of the action with respect to the tetrad in the FRW spacetime yields the following equation:

$$H^2 = -\frac{(F(T) - T)}{6} - 2H^2 \frac{dF}{dT}. \quad (101)$$

The above differential equation can be mapped to the holographic Friedmann equation  $H = \frac{1}{L_{\text{IR}}}$ , with the holographic cutoff being



$$\frac{c^2}{(L_{\text{IR}})^2} = -\frac{(F(T) - T)}{6} - 2H^2 \frac{dF}{dT}. \quad (102)$$

As similar to earlier gravity theories, here we also determine the cutoff in terms of the particle horizon and the future horizon. For this purpose, what we need is the following expression:

$$T^{(L_p)} = -6 \left[ \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right]^2, \quad T^{(L_f)} = -6 \left[ \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right]^2.$$

The above expressions along with Eq. (102) immediately lead to the  $L_{\text{IR}} = L_{\text{IR}}(L_p, \dot{L}_p, \ddot{L}_p, \text{higher derivatives of } L_p)$  and  $L_{\text{IR}} = L_{\text{IR}}(L_f, \dot{L}_f, \ddot{L}_f, \text{higher derivatives of } L_f)$  as follows:

$$\frac{c^2}{(L_{\text{IR}})^2} = -\frac{(F(T^{(L_p)}) - T^{(L_p)})}{6} - 2 \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right)^2 \frac{dF}{dT} \Big|_{T=T^{(L_p)}}, \quad (103)$$

and

$$\frac{c^2}{(L_{\text{IR}})^2} = -\frac{(F(T^{(L_f)}) - T^{(L_f)})}{6} - 2 \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right)^2 \frac{dF}{dT} \Big|_{T=T^{(L_f)}}, \quad (104)$$

respectively. The holographic cutoffs determined in Eqs. (103) and (104) constitute the cosmological field equations and thus can provide an equivalent holographic model for any arbitrary  $F(T)$  gravity model. As an example, we may consider [94]

$$F(T) = T - \alpha(-T)^p, \quad (105)$$

with  $\alpha$  being a model parameter having mass dimension  $[2 - 2p]$  and  $p$  is a dimensionless quantity. Earlier it was shown that the  $F(T)$  model in Eq. (105) (along with suitable initial conditions) allow the Universe to evolve from an initial phase of radiation domination to a cosmic acceleration at late times for  $p \neq 1$  [94]. With the help of Eqs. (103) and (104), we determine two different forms of holographic cutoff for the model (105) as

$$\begin{aligned} \frac{c^2}{(L_{\text{IR}})^2} &= 6^{p-1} \alpha (1 + 2p) \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right)^{2p} - 2 \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right)^2 \\ &= 6^{p-1} \alpha (1 + 2p) \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right)^{2p} - 2 \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right)^2. \end{aligned} \quad (106)$$

The first line in the above equation gives the cutoff in terms of the particle horizon and its derivatives, while the second line gives the same however in terms of  $L_f$  and its derivatives. Clearly the holographic energy density with

the  $L_{\text{IR}}$  of Eq. (106) reproduces the cosmological field equations for the model (105) and hence drives the late-time accelerating epoch of our Universe.

## VI. CONCLUSION

In this paper, we applied the holographic principle to describe the early and late-time acceleration epochs of our Universe in a unified manner. Although holographic energy density has been well studied at late times and recently it has also been applied in inflation studies, giving rise to holographic dark energy and inflationary realization respectively; however, to date it has not been incorporated to unify various cosmological epochs of the Universe. Such ‘‘holographic unification’’ is demonstrated in the present paper, in the context of  $F(R)$  and  $f(\mathcal{G})$  gravity theory without/with matter fields, where the corresponding holographic cutoffs ( $L_{\text{IR}}$ ) are determined in terms of the particle horizon and its derivatives or the future horizon and its derivatives. For this purpose, we first prove the holographic correspondence for general  $F(R)$  or  $f(\mathcal{G})$  theory and then consider several specific forms of  $F(R)$  or  $f(\mathcal{G})$  (which are known to be viable models as per the unification of inflation where the dark energy epoch is concerned) to show the ‘‘holographic unification’’ explicitly. One of the models considered here is the axion- $F(R)$  gravity in the presence of radiation fluid, where the corresponding holographic energy density that we propose is found to unify inflation with the radiation, dark matter, and dark energy epochs of the Universe in a holographic context.

Moreover in the context of  $F(R)$  gravity, apart from the two aforementioned ways (where  $L_{\text{IR}}$  is determined in terms of particle horizon or the future horizon), we also establish the holographic cutoff in a different way, in particular, by an integral form which along with  $H = 1/L_{\text{IR}}$  mimics the cosmological dynamics of the corresponding model. The integral form of  $L_{\text{IR}}$  has been discussed in earlier literature; however, these studies were focused on inflationary models. Here we extended the determination of the integral form of  $L_{\text{IR}}$  to the unified description of our Universe.

In summary, the holographic principle (where the cutoffs are in terms of the particle horizon, or in terms of the future horizon or in an integral form) proves to be very useful to unify the cosmological eras of the Universe. However, our understanding for the choice of fundamental viable cutoff still remains to be lacking. The comparison of such cutoffs for realistic description of the universe evolution in a unified manner may help in better understanding the holographic principle.

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