Unifying the Dynkin and Lebesgue-Stieltjes formulae

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Abstract

We establish a local martingale M associate with (X, Y) where X is a sufficiently nice Markov process and Y is a process of bounded variation (on compact intervals). This martingale generalizes both Dynkin's formula for Markov processes and the Lebesgue-Stieltjes integration (change of variable) formula for (right continuous) functions of bounded variation. When further, relatively easily verifiable conditions are assumed then this local Martingale becomes an L^2 martingale. Convergence of the product of this Martingale with some deterministic function (of time) to zero both in L^2 and a.s. is also considered and sufficient conditions for functions for which this happens are identified.

Keywords: Lévy system, Markov process, jump diffusion, local martingales, Dynkin's formula.

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1 Introduction

Considering a sufficiently nice Markov process X with generator $\mathcal{A}$ (both to be defined), a multivariate process Y of bounded variation on finite intervals (FV) and a nice enough function f, the goal of this paper is to give a setup where

$$M_t = f(X_t, Y_t) - f(X_0, Y_0) - \int_0^t \mathcal{A}f(X_s, Y_s)ds - \int_0^t \nabla_y f(X_s, Y_s)^T dY_s^c - \sum_{0<s\le t} (f(X_s, Y_s) - f(X_s, Y_{s-})) \tag{1}$$

(with a minor abuse of the notation $\mathcal{A}$ to denote a natural operator to be defined), is a local martingale, to provide sufficient conditions for this process to be an L^2 martingale as well as satisfy $h(t)M_t \to 0$ in L^2 and a.s. (for some h) under appropriate conditions (see Lemma 1). When f does not depend on Y or Y is constant, then the second line of (1) is zero and (1) reduces to Dynkin's

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formula. When f does not depend on X or X is constant, then $\int_0^t \mathcal{A}f(X_s, Y_s)ds = M_t = 0$ and then (1) reduces to the Lebesgue-Stieltjes integration formula. When X is Brownian motion and Y is continuous then (1) becomes Itô's formula. Also we note that if X is a finite dimensional semimartingale, the sum part in the generalized (to discontinuous semimartingales) Itô's formula is different from the simple form

$$\sum_{0<s\le t} (f(X_s, Y_s) - f(X_s, Y_{s-})) \tag{2}$$

where we emphasize that X_s (not X_{s-}) appears in both $f(X_s, Y_s)$ and $f(X_s, Y_{s-})$.

When X is a real valued Lévy process with triplet (c, σ^2, ν) and Y is real valued, the operator $\mathcal{A}$ (for sufficiently nice functions) is given by

$$\begin{aligned}\mathcal{A}f(x, y) &= cf_x(x, y) + \frac{\sigma^2}{2} f_{xx}(x, y) \\ &\quad + \int_{\mathbb{R}} (f(x + z, y) - f(x, y) - f_x(x, y)z1_{\{|z|\le 1\}})\nu(dz)\end{aligned} \tag{3}$$

and in particular if we take $f_1(x, y) = \cos(\alpha(x+y))$ and $f_2(x, y) = \sin(\alpha(x+y))$, then for $f = f_1+if_2$ we have that

$$\mathcal{A}f(x, y) = \psi(\alpha)e^{i\alpha(x+y)} \tag{4}$$

where ψ is the Lévy exponent given by

$$\psi(\alpha) = ic\alpha - \frac{\sigma^2}{2}\alpha^2 + \int_{\mathbb{R}} (e^{i\alpha z} - 1 - i\alpha z1_{\{|z|\le 1\}})\nu(dz)\ . \tag{5}$$

For this case M is the (local) martingale from [7]. Originally, without further conditions, this process was shown to be only a local martingale, unless some further conditions were assumed, but it was discovered in [5] that it, as well as a generalized version of it, is in fact always an L^2 martingale and moreover $M_t/t \to 0$ a.s. and in L^2. The desire to place such results in a more general setting is what motivated the current study.

The paper is organized as follows. In Section 2 we show in some detail how the main ideas work for the case where X is a real valued Lévy process and the process Y is also one dimensional. In particular we show how the corresponding results from [7, 5] are obtained as immediate special cases. In Section 3 we expand the ideas to the case where X is a, so called, jump diffusion (both X and Y are still real valued) and show an application to reflected jump diffusions. In Section 4 the ideas are generalized to multivariate X and Y. In Section 5 we apply the results of Section 4 to certain Markov additive processes and generalize in more than one way corresponding results reported in [2]. Finally, in Section 6 we point out two possible future directions, motivated by the results of this paper and the observation that when $f(x, y) = \xi(x)\eta(y)$, then (1) is a local Martingale under far more general assumptions.

We are aware that Sections 2-4 could have been written in a reversed order, first starting with the most general results and then specializing. However, we believe that with the order that we chose, in every section pointing out only the new observations that are needed and leaving out similarities from earlier sections as exercises, makes the article easier to follow. An additional benefit is that it is more easily accessible to those who are only be interested in the Lévy or one dimensional jump-diffusion cases.

2 The case where X is a real valued Lévy process and Y one dimensional

Throughout we will assume that all processes are càdlàg semimartingales. $U = \{U_t | t \geq 0\}$ will denote a process with $U_{t-} = \lim_{s \uparrow t} U_s$, $\Delta U_t = U_t - U_{t-}$ and $U_{0-} \equiv 0$. When U is of bounded variation on finite intervals (FV) then we will also denote $U_t^c = U_t - \sum_{0 \leq s \leq t} \Delta U_s$ to be the continuous part of U. $[U, U]$ is the quadratic variation process associated with U and when V is also a semimartingale, $[U, V]$ denotes the covariation process. Also, $\mathbb{R}$ denotes the set of reals, $\mathbb{R}_+$ the set of nonnegative reals, $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$ and a.s. abbreviates *almost surely*.

As the ideas repeat themselves, we find it instructive to show in detail the arguments for the most basic case where Y is FV and X is a real valued Lévy process with associated:

- Lévy triplet $(c, \sigma^2, \nu(\cdot))$, where $c \in \mathbb{R}$, $\sigma \geq 0$, $\int_{\mathbb{R}} (z^2 \wedge 1) \nu(dz) < \infty$ and $\nu(\{0\}) = 0$.
- Wiener process W.
- Poisson random measure $N(dz, dt)$ with mean measure $\nu(dz)dt$.
- $\tilde{N}(dz, dt) = N(dz, dt) - \nu(dz)dt$.

That is,

$$X_t = ct + \sigma W_t + \int_{\mathbb{R} \setminus [-1,1] \times (0,t]} z N(dz, ds) + \int_{[-1,1] \times (0,t]} z \tilde{N}(dz, ds) \tag{6}$$

where it is well known that every real valued Lévy process has such a decomposition (e.g., Thm. 42, p. 31 of [10]).

If $f \in \mathcal{C}^{2,1}$ (e.g., [9]), as every Lévy process and every FV process are semimartingales (recall that all processes are assumed adapted and càdlàg), according to the standard generalization of Itô's lemma,

$$\begin{aligned} f(X_t, Y_t) = f(X_0, Y_0) &+ \int_{(0,t]} f_x(X_{s-}, Y_{s-}) dX_s + \int_{(0,t]} f_y(X_{s-}, Y_{s-}) dY_s \\ &+ \frac{1}{2} \int_0^t f_{xx}(X_s, Y_s) d[X, X]_s^c \\ &+ \sum_{0 < s \leq t} \big(f(X_s, Y_s) - f(X_{s-}, Y_{s-}) - f_x(X_{s-}, Y_{s-}) \Delta X_s \\ &\qquad\qquad - f_y(X_{s-}, Y_{s-}) \Delta Y_s \big) \, . \end{aligned} \tag{7}$$

Since Y is FV we may write

$$\int_{(0,t]} f_y(X_{s-}, Y_{s-}) dY_s = \int_0^t f_y(X_s, Y_s) dY_s^c + \sum_{0 < s \leq t} f_y(X_{s-}, Y_{s-}) \Delta Y_s \tag{8}$$

to obtain, after cancellation of the sum part, that

$$\begin{aligned} f(X_t,Y_t) &= f(X_0,Y_0) + \int_{(0,t]} f_x(X_{s-},Y_{s-})dX_s + \int_{(0,t]} f_y(X_s,Y_s)dY^c_s \\ &\quad + \frac{1}{2}\int_0^t f_{xx}(X_s,Y_s)d[X,X]^c_s \\ &\quad + \sum_{0<s\le t} \left(f(X_s,Y_s) - f(X_{s-},Y_{s-}) - f_x(X_{s-},Y_{s-})\Delta X_s\right) . \end{aligned} \tag{9}$$

Recalling (6)) and noting that

$$\int_{\mathbb{R}\setminus[-1,1]\times(0,t]} zN(dz,ds) = \sum_{0<s\le t} \Delta X_s 1_{\{|\Delta X_s|>1\}} \tag{10}$$

and

$$d[X,X]^c_s = d[\sigma W,\sigma W]_s = \sigma^2\, ds \tag{11}$$

we have that

$$\begin{aligned} f(X_t,Y_t) &= f(X_0,Y_0) + \int_{(0,t]} \left(cf_x(X_s,Y_s) + \frac{\sigma^2}{2} f_{xx}(X_s,Y_s)\right) ds \\ &\quad + \int_{(0,t]} f_y(X_s,Y_s)dY^c_s \\ &\quad + \sigma\int_0^t f_x(X_s,Y_s)dW_s + \int_{[-1,1]\times(0,t]} f_x(X_{s-},Y_{s-})z\tilde{N}(dz,ds) \\ &\quad + \sum_{0<s\le t} \left(f(X_s,Y_s) - f(X_{s-},Y_{s-}) - f_x(X_{s-},Y_{s-})\Delta X_s 1_{\{|\Delta X_s|\le 1\}}\right) . \end{aligned} \tag{12}$$

Next, note that

$$\begin{aligned} &\sum_{0<s\le t} \left(f(X_s,Y_s) - f(X_{s-},Y_{s-}) - f_x(X_{s-},Y_{s-})\Delta X_s 1_{\{|\Delta X_s|\le 1\}}\right) \\ &= \sum_{0<s\le t} \left(f(X_s,Y_s) - f(X_s,Y_{s-})\right) \\ &\quad + \sum_{0<s\le t} \left(f(X_{s-}+\Delta X_s,Y_{s-}) - f(X_{s-},Y_{s-}) - f_x(X_{s-},Y_{s-})\Delta X_s 1_{\{|\Delta X_s|\le 1\}}\right) \end{aligned} \tag{13}$$

and that

$$\begin{aligned} &\sum_{0<s\le t} \left(f(X_{s-}+\Delta X_s,Y_{s-}) - f(X_{s-},Y_{s-}) - f_x(X_{s-},Y_{s-})\Delta X_s 1_{\{|\Delta X_s|\le 1\}}\right) \\ &= \int_{\mathbb{R}\times(0,t]} \left(f(X_{s-}+z,Y_{s-}) - f(X_{s-},Y_{s-}) - f_x(X_{s-},Y_{s-})z1_{\{|z|\le 1\}}\right) N(dz,ds) \\ &= \int_{\mathbb{R}\times(0,t]} \left(f(X_{s-}+z,Y_{s-}) - f(X_{s-},Y_{s-}) - f_x(X_{s-},Y_{s-})z1_{\{|z|\le 1\}}\right) \tilde{N}(dz,ds) \\ &\quad + \int_0^t \left(\int_{\mathbb{R}} \left(f(X_{s-}+z,Y_{s-}) - f(X_{s-},Y_{s-}) - f_x(X_{s-},Y_{s-})z1_{\{|z|\le 1\}}\right)\nu(dz)\right) ds . \end{aligned} \tag{14}$$

If we denote

$$\mathcal{A}f(x,y) = cf_x(x,y) + \frac{\sigma^2}{2} f_{xx}(x,y) + \int_{\mathbb{R}} \left(f(x+z,y) - f(x,y) - f_x(x,y)z1_{\{|z|\le 1\}}\right) \nu(dz) \ , \tag{15}$$

then putting everything together, noting that

$$\int_{[-1,1]\times(0,t]} f_x(X_{s-},Y_{s-})z\tilde{N}(dz,ds) \tag{16}$$

in (14) cancels with the corresponding term in (12), gives

$$\begin{aligned} f(X_t,Y_t) = f(X_0,Y_0) &+ \int_0^t \mathcal{A}f(X_s,Y_s)ds + \int_0^t f_y(X_s,Y_s)dY_s^c + \sum_{0<s\le t} (f(X_s,Y_s) - f(X_s,Y_{s-})) \\ &+ \sigma \int_0^t f_x(X_s,Y_s)dW_s + \int_{\mathbb{R}\times(0,t]} (f(X_{s-}+z,Y_{s-}) - f(X_{s-},Y_{s-}))\tilde{N}(dz,ds) \ . \end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned} M_t &= f(X_t,Y_t) - f(X_0,Y_0) - \int_0^t \mathcal{A}f(X_s,Y_s)ds \\ &\quad - \int_0^t f_y(X_s,Y_s)dY_s^c - \sum_{0<s\le t} (f(X_s,Y_s) - f(X_s,Y_{s-})) \\ &= \sigma \int_0^t f_x(X_s,Y_s)dW_s + \int_{\mathbb{R}\times(0,t]} (f(X_{s-}+z,Y_{s-}) - f(X_{s-},Y_{s-}))\tilde{N}(dz,ds) \end{aligned} \tag{18}$$

is a local martingale (e.g., Subsection 4.3.2, p. 230-233 in [1], Thm. 29, p. 171 of [10] and Prop. 4.10 in [11]).

Next, if we denote

$$\begin{aligned} U_t &= \sigma \int_0^t f_x(X_s,Y_s)dW_s \\ V_t &= \int_{\mathbb{R}\times(0,t]} (f(X_{s-}+z,Y_{s-}) - f(X_{s-},Y_{s-}))\tilde{N}(dz,ds) \ , \end{aligned} \tag{19}$$

then V, being a compensated sum of jumps, is quadratic pure jump (e.g., [8]) with

$$[V,V]_t = \int_{\mathbb{R}\times(0,t]} (f(X_{s-}+z,Y_{s-}) - f(X_{s-},Y_{s-}))^2 N(dz,ds) \tag{20}$$

(note: N, not $\tilde{N}$), U has quadratic variation

$$[U,U]_t = \sigma^2 \int_0^t f_x^2(X_s,Y_s)ds \tag{21}$$

(e.g., Thm. 29, p. 75 of [10]) and $[V,U]_t = 0$.

Now, consider the following.

Assumption 1 *Let $B \subset \mathbb{R}^2$ be some closed set satisfying $P((X_t, Y_t) \in B) = 1$ for all $t \geq 0$, then $f_x(x, y)$ and $\int_{\mathbb{R}}(f(x+z, y) - f(x, y))^2 \nu(dz)$ are bounded on B.*

Under Assumption 1, it follows that

$$[M, M]_t = \int_0^t \left(\sigma^2 f_x^2(X_s, Y_s) + \int_{\mathbb{R}} (f(X_s + z, Y_s) - f(X_s, Y_s))^2 \nu(dz) \right) ds + \tilde{M}_t \tag{22}$$

where

$$\tilde{M}_t = \int_{\mathbb{R} \times (0,t]} (f(X_{s-} + z, Y_{s-}) - f(X_{s-}, Y_{s-}))^2 \tilde{N}(dz, ds) \tag{23}$$

is a zero mean martingale (e.g., prop. 4.10 in [11], or a generalized version of Lemma 1 of [9]).

Therefore, under assumption 1, we now have (as in [5]), that

$$E[M, M]_t \leq \int_0^t C(s) ds \tag{24}$$

where

$$C(s) = E \left(\sigma^2 f_x^2(X_s, Y_s) + \int_{\mathbb{R}} (f(X_s + z, Y_s) - f(X_s, Y_s))^2 \nu(dz) \right) \tag{25}$$

is bounded (in s).

Lemma 1 *Assume that M is a local martingale for which $E[M, M]_t$ is absolutely continuous (with respect to Lebesgue measure) and has a bounded (necessarily nonnegative) density C(s). Then*

(1) M is an L^2 martingale,

(2) for every (deterministic) h with $h(t)\sqrt{t} \to 0$ as $t \to \infty$; $h(t)M_t \to 0$ in L^2, as $t \to \infty$, and

(3) for every continuous, nonnegative, nonincreasing h, satisfying $\int_{t_0}^\infty h^2(s) ds < \infty$ for some $t_0 \geq 0$; $h(t)M_t \to 0$ a.s., as $t \to \infty$.

In particular, for every $\gamma > 1/2$, $M_t/t^\gamma \to 0$ in L^2 and a.s., as $t \to \infty$.

Proof: Let $C = \sup_{s \geq 0} C(s)$. From $E[M, M]_t \leq Ct < \infty$ it follows that M is an L^2 martingale with $EM_t^2 = E[M, M]_t$ (e.g., Cor. 3, p. 73 of [10]). This implies that

$$E(h(t)M_t)^2 = h^2(t)E[M, M]_t \leq Ct\, h^2(t) \ . \tag{26}$$

and thus (2) follows.

Next, we prove (3): if $h(t) = 0$ for some $t > 0$ then $h(s)M_s = 0$ for $s \geq t$. Also, if $h(t) > 1$ for some t then we may replace h by $h_1(t) = h(t) \wedge 1$ and clearly $h(t)M_t \to 0$ a.s. if and only if $h_1(t)M_t \to 0$ a.s. and $\int_0^\infty h_1^2(s) ds < \infty$. Thus, we may restrict ourselves to h with $0 < h(t) \leq 1$ for every $t \geq 0$, such that $\int_0^\infty h^2(s) ds < \infty$. With this assumption, $h \cdot M_t \equiv \int_{(0,t]} h(s) dM_s$ is a martingale with

$$\begin{aligned} E(h \cdot M_t)^2 = E[h \cdot M, h \cdot M]_t &= \int_0^t h^2(s) dE[M, M]_s \\ &= \int_0^t h^2(s) C(s) ds \leq C \int_0^\infty h^2(s) ds \end{aligned} \tag{27}$$

and thus converges a.s. Consider now $A(t) = \frac{1}{h(t)} - 1$. Then $A(\cdot)$ is continuous, nonnegative, nondecreasing and, as $t \to \infty$, $A(t) \to \infty$ with $\int_0^t \frac{dM_s}{1+A(s)}$ converging a.s. Hence, as in Ex. 14, p. 95 of [10], we also have that $h(t)M_t = M_t/(1 + A(t)) \to 0$ a.s. □

Thus, we can now conclude the following.

Theorem 1 *With càdlàg and adapted X, Y and with a function $f : \mathbb{R}^2 \to \mathbb{R}$, where*

- *X is a real valued Lévy process (with respect to the underlying filtration) with Lévy triplet $(c, \sigma^2, \nu(\cdot))$,*
- *Y a FV process,*
- *$f \in \mathcal{C}^{2,1}$.*

and with $\mathcal{A}$ defined in (15) then

$$\begin{aligned} M_t = f(X_t, Y_t) - f(X_0, Y_0) - \int_0^t \mathcal{A}f(X_s, Y_s)ds \\ - \int_0^t f_y(X_s, Y_s)dY_s^c - \sum_{0<s\le t} (f(X_s, Y_s) - f(X_s, Y_{s-})) \end{aligned} \tag{28}$$

is a local martingale.

If in addition Assumption 1 holds, then the assumptions and hence the conclusions of Lemma 1 hold.

Remark 1 We note the following regarding Assumption 1:

- A sufficient condition is that f and f_x are bounded on $\mathbb{R}^2$. For example, it holds for $f(x, y) = \sin(\alpha(x + y))$ or $f(x, y) = \cos(\alpha(x + y))$ and, thus, also for $f(x, y) = e^{i\alpha(x+y)}$.
- Another sufficient condition is that $f(x + z, y)$ is bounded on $(x, y) \in B$ and $z \in \mathbb{R} \setminus [-1, 1]/(0, 1]/[-1, 0))$ and $f_x(x + z, y)$ is bounded on $(x, y) \in B$ and $z \in [-1, 1]/(0, 1]/[-1, 0)$ for the general/spectrally positive/spectrally negative cases, respectively. For example, it holds for the spectrally positive case where $X_t + Y_t \ge 0$, a.s., and $f(x, y) = e^{-\alpha(x+y)}$ for $\alpha > 0$.

From Remark 1, recalling (4), ψ from (5), denoting $\varphi(\alpha) = \psi(i\alpha)$ (real valued) for the spectrally positive case and noting that

$$f(X_s, Y_s) - f(X_s, Y_{s-}) = e^{i\alpha(X_s+Y_s)} - e^{i\alpha(X_s+Y_{s-})} = e^{i\alpha(X_s+Y_s)}(1 - e^{-i\alpha\Delta Y_s}) \tag{29}$$

(with $-\alpha$ replacing $i\alpha$ for the spectrally positive case) we immediately reproduce the following (see [5, 7]).

Corollary 1 *With $Z_t = X_t + Y_t$, where X, Y are as in Theorem 1,*

$$\begin{aligned} M_t = \psi(\alpha) \int_0^t e^{i\alpha Z_s} ds + e^{i\alpha Z_0} - e^{i\alpha Z_t} \\ + i\alpha \int_0^t e^{i\alpha Z_s} dY_s^c + \sum_{0<s\le t} e^{i\alpha Z_s}(1 - e^{-i\alpha\Delta Y_s}) \end{aligned} \tag{30}$$

is an (complex valued) L^2 martingale that satisfies the assumptions and hence the conclusions of Lemma 1.

If in addition $Z_t \geq 0$ a.s. for all $t \geq 0$ and $\nu(-\infty, 0) = 0$ then the same holds for

$$M_t = \varphi(\alpha) \int_0^t e^{-\alpha Z_s} ds + e^{-\alpha Z_0} - e^{-\alpha Z_t} \\ - \alpha \int_0^t e^{-\alpha Z_s} dY_s^c + \sum_{0<s\leq t} e^{-\alpha Z_s}(1 - e^{\alpha \Delta Y_s}) \tag{31}$$

3 The case where X is a real valued jump diffusion and Y is real valued

In this section we replace the Lévy process X by a process satisfying the following stochastic differential equation:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_{\mathbb{R}\setminus[-1,1]\times(0,t]} K(X_{s-}, z)N(dz, ds) \\ + \int_{[-1,1]\times(0,t]} K(X_{s-}, z)\tilde{N}(dz, ds) \ . \tag{32}$$

Sufficient conditions for there to be a unique strong solution which is also strong Markov (see Chapter 6 of [1]) are that K is Borel,

(i) Lipschitz conditions: for all x_1, x_2,

$$|b(x_1) - b(x_2)| \vee |\sigma(x_1) - \sigma(x_2)| \vee \left(\int_{[-1,1]} (K(x_1, z) - K(x_2, z))^2 \nu(dz) \right)^{1/2} \leq \kappa |x_1 - x_2| \ ,$$

(ii) Finiteness condition: for some (hence all) x ,$\int_{[-1,1]} K^2(x, z)\nu(dz) < \infty$.

(iii) For each $z \notin [-1, 1]$, $K(\cdot, z)$ is continuous.

Note that in [1] the finiteness condition is expressed as a *growth* condition, but as the author remarks, under (i) both finiteness and growth conditions are equivalent. Therefore, we henceforth assume that these conditions are met.

3.1 A general study

We first observe that for each x

$$\int_{[-1,1]} |K(x, z)| 1_{\{|K(x,z)|>1\}} \nu(dz) \leq \int_{[-1,1]} K^2(x, z)\nu(dz) < \infty \tag{33}$$

and

$$\int_{\mathbb{R}\setminus[-1,1]} |K(x, z)| 1_{\{|K(x,z)|\leq 1\}} \nu(dz) \leq \nu(\mathbb{R} \setminus [-1, 1]) < \infty \ . \tag{34}$$

Next, we note that

$$\begin{aligned}\int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|z|>1\}}N(dz,ds) &= \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|\le 1\}}1_{\{|z|>1\}}N(dz,ds) \\ &\quad + \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|>1\}}1_{\{|z|>1\}}N(dz,ds) \\ &= \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|\le 1\}}1_{\{|z|>1\}}\tilde{N}(dz,ds) \\ &\quad + \int_0^t \int_{\mathbb{R}} K(X_{s-},z)1_{\{|K(X_{s-},z)|\le 1\}}1_{\{|z|>1\}}\nu(dz)ds \\ &\quad + \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|>1\}}1_{\{|z|>1\}}N(dz,ds)\end{aligned} \tag{35}$$

and, similarly, that

$$\begin{aligned}\int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|z|\le 1\}}\tilde{N}(dz,ds) &= \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|\le 1\}}1_{\{|z|\le 1\}}\tilde{N}(dz,ds) \\ &\quad + \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|>1\}}1_{\{|z|\le 1\}}\tilde{N}(dz,ds) \\ &= \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|\le 1\}}1_{\{|z|\le 1\}}\tilde{N}(dz,ds) \\ &\quad + \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|>1\}}1_{\{|z|\le 1\}}N(dz,ds) \\ &\quad - \int_0^t \int_{\mathbb{R}} K(X_{s-},z)1_{\{|K(X_{s-},z)|>1\}}1_{\{|z|\le 1\}}\nu(dz)ds\end{aligned} \tag{36}$$

Adding the right hand sides of (35) and (36) gives

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}} K(X_{s-},z)1_{\{|K(X_{s-},z)|\le 1\}}1_{\{|z|>1\}}\nu(dz)ds - \int_0^t \int_{\mathbb{R}} K(X_{s-},z)1_{\{|K(X_{s-},z)|>1\}}1_{\{|z|\le 1\}}\nu(dz)ds \\ &+ \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|>1\}}N(dz,ds) + \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|\le 1\}}\tilde{N}(dz,ds)\end{aligned} \tag{37}$$

and thus with

$$c(x) = b(x) + \int_{|z|\le 1} K(x,z)1_{\{|K(x,z)|>1\}}\nu(dz) - \int_{|z|>1} K(x,z)1_{\{|K(x,z)|\le 1\}}\nu(dz)\ , \tag{38}$$

we may rewrite (32) as

$$\begin{aligned}X_t = X_0 + \int_0^t c(X_s)ds + \int_0^t \sigma(X_s)dW_s &+ \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|>1\}}N(dz,ds) \\ &+ \int_{\mathbb{R}\times(0,t]} K(X_{s-},z)1_{\{|K(X_{s-},z)|\le 1\}}\tilde{N}(dz,ds)\ .\end{aligned} \tag{39}$$

Performing precisely the same steps that led to (18) (which is left to the reader) results in the following analogue, with Y a FV process:

$$\begin{aligned}M_t &= f(X_t, Y_t) - f(X_0, Y_0) - \int_0^t \mathcal{A}f(X_s, Y_s)ds \\ &\quad - \int_0^t f_y(X_s, Y_s)dY_s^c - \sum_{0<s\le t}(f(X_s, Y_s) - f(X_s, Y_{s-})) \\ &= \int_0^t \sigma(X_s)f_x(X_s, Y_s)dW_s + \int_{\mathbb{R}\times(0,t]}(f(X_{s-} + K(X_{s-}, z), Y_{s-}) - f(X_{s-}, Y_{s-}))\tilde{N}(dz, ds)\end{aligned} \tag{40}$$

where $\mathcal{A}$ is now given by

$$\begin{aligned}\mathcal{A}f(x, y) &= c(x)f_x(x, y) + \frac{\sigma^2(x)}{2}f_{xx}(x, y) \\ &\quad + \int_{\mathbb{R}}(f(x + K(x, z), y) - f(x, y) - f_x(x, y)K(x, z)1_{\{|K(x,z)|\le 1\}})\nu(dz)\end{aligned} \tag{41}$$

and we note that with the kernel $\mu(x, A) = \nu(\{z|\, K(x, z) \in A\})$ the last summand in (41) may be rewritten as follows:

$$\int_{\mathbb{R}}(f(x + z, y) - f(x, y) - f_x(x, y)z1_{\{|z|\le 1\}})\mu(x, dz)\ . \tag{42}$$

It is also straightforward to check that the modified versions of (22) and (23) become

$$[M, M]_t = \int_0^t \left(\sigma^2(X_s)f_x^2(X_s, Y_s) + \int_{\mathbb{R}}(f(X_s + K(X_s, z), Y_s) - f(X_s, Y_s))^2\nu(dz)\right) ds + \tilde{M}_t \tag{43}$$

and

$$\tilde{M}_t = \int_{\mathbb{R}\times(0,t]}(f(X_{s-} + K(X_{s-}, z), Y_{s-}) - f(X_{s-}, Y_{s-}))^2\tilde{N}(dz, ds)\ , \tag{44}$$

respectively, which can then be used to verify the validity of (24) and, thus, of Lemma 1, under

Assumption 2 *With B as in Assumption 1, $\sigma(x)f_x(x, y)$ and*

$$\int_{\mathbb{R}}(f(x + K(x, z), y) - f(x, y))^2\nu(dz) = \int_{\mathbb{R}}(f(x + z, y) - f(x, y))^2\mu(x, dz)$$

are bounded on B.

We therefore have the following.

Theorem 2 *With càdlàg and adapted X, Y and with a function $f : \mathbb{R}^2 \to \mathbb{R}$, where*

- *X is an real valued jump diffusion process (with respect to the underlying filtration) with $(b(\cdot), \sigma(\cdot), K(\cdot, \cdot), \nu(\cdot))$ satisfying (i)-(iii),*
- *Y a FV process,*

- $f \in \mathcal{C}^{2,1}$.

and with $\mathcal{A}$ defined via (41), then

$$
\begin{aligned}
M_t = f(X_t,Y_t) - f(X_0,Y_0) - \int_0^t \mathcal{A}f(X_s,Y_s)ds \\
- \int_0^t f_y(X_s,Y_s)dY_s^c - \sum_{0<s\le t}\left(f(X_s,Y_s) - f(X_s,Y_{s-})\right)
\end{aligned} \tag{45}
$$

is a local martingale.

If in addition Assumption 2 holds, then the assumptions and hence the conclusions of Lemma 1 hold.

3.2 An example with reflection

A small application of Theorem 2 is as follows. Assume that X is as described and that

$$Y_t = -\inf_{0\le s\le t} X_s^- \tag{46}$$

so that $Z_t = X_t + Y_t$ is the reflection of X. Then Z is also a Markov process. Now, according to Theorem 2, for every $g : \mathbb{R} \to \mathbb{R}$ such that $g \in \mathcal{C}^2$ we have from the fact that $Z_s = 0$ for every s for which $Y_{s-} < Y_u$ for $u > s$ (see [4]) and that $X_s + Y_{s-} = Z_s - \Delta Y_s$, that the following is a local martingale:

$$
\begin{aligned}
M_t &= g(Z_t) - g(Z_0) - \int_0^t \mathcal{A}g(Z_s)ds - \int_0^t g'(Z_s)dY_s^c - \sum_{0<s\le t}\left(g(Z_s) - g(Z_s - \Delta Y_s)\right) \\
&= g(Z_t) - g(Z_0) - \int_0^t \mathcal{A}g(Z_s)ds - g'(0)Y_t^c - \sum_{0<s\le t}\left(g(0) - g(-\Delta Y_s)\right)
\end{aligned} \tag{47}
$$

Therefore, if we assume that $g'(z) = 0$ for all $z \le 0$ (and thus also $g(z) = g(0)$ for $z \le 0$), then we have that for such g,

$$g(Z_t) - g(Z_0) - \int_0^t \mathcal{A}g(Z_s)ds \tag{48}$$

is local martingale. This implies that every such g is in the domain of the (extended) generator of Z and for such g, the generator of the process Z is the same as the one for X. This is a well known fact for Brownian motion, in which case it suffices to assume that $g'(0) = 0$.

Another observation for this example is that when X has no negative jumps (if $K \ge 0$), then Y is continuous. It is also known that $X_t/t \to \xi$ if and only if $(Y_t/t, Z_t/t) \to (-\xi^-, \xi^+)$ (e.g., see the proof of Theorem 1 of [6]). Therefore, for every g, bounded on $[0,\infty)$ ($g'(0)$ not necessarily zero) for which Assumption 2 holds with $f(x,y) = g(x+y)$, we have that a.s.,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{A}g(Z_s)ds = g'(0)\xi^- \ . \tag{49}$$

For example, if $g(z) = e^{-\alpha z}$ and X is a Lévy process with no negative jumps and Laplace-Stieltjes exponent $\varphi(\alpha) = \psi(i\alpha)$, then, provided that $E|X_1| < \infty$ (if and only if $\int_{(1,\infty)} x\nu(dx) < \infty$), then

$\xi = EX_1 = -\varphi'(0)$, $\mathcal{A}g(z) = \varphi(\alpha)e^{-\alpha z}$, $g'(0) = -\alpha$, g is bounded and satisfies Assumption 2. Hence, if $\varphi'(0) > 0$, we have that, a.s.,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-\alpha Z_s} ds = \frac{\alpha\varphi'(0)}{\varphi(\alpha)}, \tag{50}$$

where the right hand side is the well known generalized Pollaczeck-Khinchine formula. Of course, if $\varphi'(0) \le 0$ (and $\varphi(\alpha) \neq 0$), the right hand side is zero.

4 The case where X is a multivariate jump diffusion and Y is multivariate

The components of an n-dimensional jump-diffusion process are as follows (e.g., Chapter 6 of [1], noting the footnote on p. 363).

1. $W = (W_1, \dots, W_k)^T$, where W_i are independent Wiener processes.
2. $N(dz, dt)$ is a Poisson random measure on $\mathbb{R}^m \times \mathbb{R}_+$ with mean measure $\nu(dz)dt$, satisfying $\int_{\mathbb{R}^m}(\|x\|^2 \wedge 1)\nu(dx) < \infty$ ($\|x\|$ is the Euclidean norm) and $\nu(\{0\}) = 0$.
3. $\tilde{N}(dz, dt) = N(dz, dt) - \nu(dz)dt$.
4. $b_i : \mathbb{R}^n \to \mathbb{R}$, $\sigma_{ij} : \mathbb{R}^n \to \mathbb{R}$, $K_i : \mathbb{R}^{n+m} \to \mathbb{R}$, $i = 1, \dots, n$, $j = 1, \dots, k$. All are Borel.
5. Assumptions (i)-(iii) are replaced with:

   (i') Lipschitz conditions: for all $x_1, x_2 \in \mathbb{R}^n$, $1 \le i \le n$ and $1 \le j \le k$,

$$|b_i(x_1) - b_i(x_2)| \vee |\sigma_{ij}(x_1) - \sigma_{ij}(x_2)| \vee \left(\int_{\|z\|\le 1} (K_i(x_1, z) - K_i(x_2, z))^2 \nu(dz)\right)^{1/2} \le \kappa\|x_1 - x_2\| \ .$$

   (ii') Finiteness conditions: for each $1 \le i \le n$, for some (hence all) $x \in \mathbb{R}^n$,

$$\int_{\|z\|\le 1} K_i^2(x, z)\nu(dz) < \infty \ . \tag{51}$$

   (iii') For each z such that $\|z\| > 1$ and each $1 \le i \le n$, $K_i(\cdot, z)$ is continuous.

Then, X is the unique (strong Markov) strong solution of:

$$\begin{aligned}
X_{i,t} = X_{i,0} &+ \int_0^t b_i(X_s)ds + \sum_{j=1}^k \int_0^t \sigma_{ij}(X_s)dW_{j,s} + \int_{\mathbb{R}^m\times(0,t]} K_i(X_{s-}, z)1_{\{\|z\|>1\}}N(dz, ds) \\
&+ \int_{\mathbb{R}^m\times(0,t]} K_i(X_{s-}, z)1_{\{\|z\|\le 1\}}\tilde{N}(dz, ds) \qquad (52) \\
= X_{i,0} &+ \int_0^t c_i(X_s)ds + \sum_{j=1}^k \int_0^t \sigma_{ij}(X_s)dW_{j,s} + \int_{\mathbb{R}^m\times(0,t]} K_i(X_{s-}, z)1_{\{\|K_i(X_{s-},z)\|>1\}}N(dz, ds) \\
&+ \int_{\mathbb{R}^m\times(0,t]} K_i(X_{s-}, z)1_{\{\|K_i(X_{s-},z)\|\le 1\}}\tilde{N}(dz, ds)
\end{aligned}$$

where, as in (38),

$$c_i(x) = b_i(x) + \int_{\|z\|\le 1} K_i(x,z) 1_{\{|K_i(x,z)|>1\}} \nu(dz) - \int_{\|z\|>1} K_i(x,z) 1_{\{|K_i(x,z)|\le 1\}} \nu(dz) \tag{53}$$

In addition to X we also introduce an r dimensional FV process Y.

For some $f : \mathbb{R}^{n+r} \to \mathbb{R}$, by $f \in \mathcal{C}^{2,1}$ here we mean that f is twice continuously differentiable in the first n coordinates and continuously differentiable in the last r coordinates.

For $f : \mathbb{R}^{n+r} \to \mathbb{R}$ with $f \in \mathcal{C}^{2,1}$ we define the operator $\mathcal{A}$ in this case as follows, with $c = (c_i), \sigma = (\sigma_{ij}), K = (K_i)$ and $\mathbf{1}$ is an n-dimensional vector of ones.

$$\begin{aligned}\mathcal{A}f(x) &= c(x)^T \nabla_x f(x,y) + \frac{1}{2}\mathbf{1}^T \sigma(x) \nabla_{xx} f(x,y) \sigma(x)^T \mathbf{1} \\ &\quad + \int_{\mathbb{R}^m} (f(x + K(x,z), y) - f(x,y) - \nabla_x f(x,y)^T K(x,z) 1_{\{\|K(x,z)\|\le 1\}}) \nu(dz)\end{aligned} \tag{54}$$

where we may replace the last summand in (54) by

$$\int_{\mathbb{R}^m} (f(x+z,y) - f(x,y) - \nabla_x f(x,y)^T z 1_{\{\|z\|\le 1\}}) \mu(x,dz) \ , \tag{55}$$

where $\mu(x,A) = \nu(\{z |\ K(x,z) \in A\}$ as in (42) for the one dimensional case.

Finally consider the following.

Assumption 3 *Let $B \subset \mathbb{R}^{n+r}$ be a closed set with $P((X_t, Y_t) \in B) = 1$ for all $t \ge 0$. Then $\|\nabla_x f(x,y)^T \sigma(x)\|$ and*

$$\int_{\mathbb{R}^m} (f(x + K(x,z), y) - f(x,y))^2 \nu(dz) = \int_{\mathbb{R}^m} (f(x+z,y) - f(x,y))^2 \mu(x,dz)$$

are bounded on B.

A repetition of the arguments from the two previous sections gives the following result.

Theorem 3 *With càdlàg and adapted X, Y and with a function $f : \mathbb{R}^{n+r} \to \mathbb{R}$, where*

- *X is an $\mathbb{R}^n$ valued jump diffusion process (with respect to the underlying filtration) with $(b(\cdot), \sigma(\cdot), K(\cdot,\cdot), \nu(\cdot))$ satisfying (i')-(iii'),*
- *Y a $\mathbb{R}^r$ valued FV process,*
- *$f \in \mathcal{C}^{2,1}$*

and with $\mathcal{A}$ defined via (54), then

$$\begin{aligned}M_t &= f(X_t, Y_t) - f(X_0, Y_0) - \int_0^t \mathcal{A}f(X_s, Y_s) ds \\ &\quad - \int_0^t \nabla_y f(X_s, Y_s)^T dY_s^c - \sum_{0<s\le t} (f(X_s, Y_s) - f(X_s, Y_{s-}))\end{aligned} \tag{56}$$

is a local martingale.

If in addition Assumption 3 holds, then the assumptions and hence the conclusions of Lemma 1 hold.

Remark 2 An important special case of this setup is obtained when the measure ν is concentrated on the axes. That is, on $\bigcup_{i=1}^m \mathbf{1}_i\mathbb{R}$, where $\mathbf{1}_i$ is a unit vector with one in the i coordinate and zero elsewhere, so that $\mathbf{1}_i\mathbb{R}$ denotes the ith axis. For this case, we let ν_j be the jth marginal of ν and let $K^j(x,z) = K(x, \mathbf{1}_j z)$. For this case, we have in (54) that

$$c_i(x) = b_i(x) + \sum_{j=1}^m \left(\int_{|z|\le 1} K_i^j(x,z) 1_{\{|K_i^j(x,z)|>1\}} \nu_j(dz) - \int_{|z|>1} K_i^j(x,z) 1_{\{|K_i^j(x,z)|\le 1\}} \nu_j(dz) \right) , \tag{57}$$

the last summand becomes

$$\int_{\mathbb{R}^m} (f(x+K(x,z),y) - f(x,y) - \nabla_x f(x,y)^T K(x,z) 1_{\{\|K(x,z)\|\le 1\}}) \nu(dz) \tag{58}$$
$$= \sum_{j=1}^m \int_{\mathbb{R}} (f(x+K^j(x,z),y) - f(x,y) - \nabla_x f(x,y)^T K^j(x,z) 1_{\{\|K^j(x,z)\|\le 1\}}) \nu_j(dz)$$

and in Assumption 3 we have that

$$\int_{\mathbb{R}^m} (f(x+K(x,z),y) - f(x,y))^2 \nu(dz) = \sum_{j=1}^n \int_{\mathbb{R}} (f(x+K^j(x,z),y) - f(x,y))^2 \nu_j(dz) \tag{59}$$

which is bounded if and only if every term on the right is bounded. We also note that in this case we can replace the first equality in (52) by

$$X_{i,t} = X_{i,0} + \int_0^t b_i(X_s)ds + \sum_{j=1}^k \int_0^t \sigma_{ij}(X_s)dW_{j,s} + \sum_{j=1}^m \int_{\mathbb{R}\setminus[-1,1]\times(0,t]} K_i^j(X_{s-},z) N_j(dz,ds)$$
$$+ \sum_{j=1}^m \int_{[-1,1]\times(0,t]} K_i^j(X_{s-},z) \tilde{N}_j(dz,ds) \tag{60}$$

where $N_1,\dots,N_m$ are independent Poisson random measures (on $\mathbb{R}\times\mathbb{R}_+$) with intensities $\nu_j(dz)dt$, $j=1,\dots,m$.

This setup will prove useful for the next section.

5 An example of a Markov additive process with finite state space modulation

By a *Markov additive process with finite state space modulation* we mean a process (J,X) where J is a finite state space Markov chain with some rate transition matrix $Q = (q_{ij})$ and during epochs where $J(t) = i$, X behaves like a Lévy process with some triplet (c_i, σ_i^2, ν_i). In addition, at state change epochs of J from i to j the process X may incur independent jumps that have a distribution G_{ij} when the transition is from i to j. For a precise description see, e.g., [2]. Let us construct (X,J) as two dimensional jump diffusion with $n=2$, $k=1$ and $m = K(K+1)$.

Let us begin with the construction of J. In view of Remark 2, for every $1 \le i, j \le j$ with $i \neq j$, we let N_{ij} denote independent Poisson random measures on $\mathbb{R} \times \mathbb{R}_+$ with intensities $\nu_{ij}(\cdot) = q_{ij} G_{ij}(\cdot)$. Then J satisfies the following equation.

$$J_t = J_0 + \sum_{i \neq j} \int_{(0,t]} \int_{\mathbb{R}} (j-i) 1_{\{J_{s-}=i\}} N_{ij}(dz, ds) \tag{61}$$

Clearly one can find functions $K_1^{ij}((\cdot,\cdot),\cdot)$ that satisfy (i')-(iii') such that $K_1^{ij}((u,x),z) = (j-i)1_{\{u=i\}}$ for all i, j, u, x, z.

Next, with N_i being independent Poisson random measures (also on $\mathbb{R} \times \mathbb{R}_+$) which are also independent of $\{N_{ij} | 1 \le i, j \le K\}$ with intensities $\nu_i(dz)dt$, where ν_i is a Lévy measure and W is an independent Wiener process, then X satisfies

$$\begin{aligned} X_t = X_0 + \int_0^t c_{J_s} ds + \int_0^t \sigma_{J_s} dW_s &+ \sum_{i=1}^K \int_{(0,t]} \int_{\mathbb{R}\setminus[-1,1]} 1_{\{J_{s-}=i\}} z N_i(dz, ds) \\ &+ \sum_{i=1}^K \int_{(0,t]} \int_{[-1,1]} 1_{\{J_{s-}=i\}} z \tilde{N}_i(dz, ds) \\ &+ \sum_{i \neq j} \int_{(0,t]} \int_{\mathbb{R}} 1_{\{J_{s-}=i\}} z N_{ij}(dz, ds) \end{aligned} \tag{62}$$

where here also one can find $K_2^{ij}((\cdot,\cdot),\cdot)$, $K_2^{i}((\cdot,\cdot),\cdot)$,$c_2(\cdot,\cdot)$ and $\sigma_2(\cdot,\cdot)$ which satisfy (i')-(iii') and that agree with the corresponding values in the equation.

For appropriate f, the operator $\mathcal{A}$ in this case becomes

$$\begin{aligned} \mathcal{A}f(i,x,y) = c_i f_x(i,x,y) &+ \frac{1}{2}\sigma_i^2 f_{xx}(i,x,y) \\ &+ \int_{\mathbb{R}} (f(i,x+z,y) - f(i,x,y) - f_x(i,x,y) z 1_{\{|z|\le 1\}}) \nu_i(dz) \\ &+ \sum_{j \neq i} q_{ij} \int_{\mathbb{R}} (f(j,x+z,y) - f(i,x,y)) G_{ij}(dz) \ , \end{aligned} \tag{63}$$

noting that for this case there is no need to start with $b_i(x)$ and then modify to $c_i(x)$ as was done in (38) and (53). If we recall that $q_{ii} = -\sum_{j \neq i} q_{ij}$ and denote $G_{ii}(\{0\}) = 1$, then we may rewrite $\mathcal{A}$ as follows

$$\begin{aligned} \mathcal{A}f(i,x,y) = c_i f_x(i,x,y) &+ \frac{1}{2}\sigma_i^2 f_{xx}(i,x,y) \\ &+ \int_{\mathbb{R}} (f(i,x+z,y) - f(i,x,y) - f_x(i,x,y) z 1_{\{|z|\le 1\}}) \nu_i(dz) \\ &+ \sum_{j=1}^K q_{ij} \int_{\mathbb{R}} f(j,x+z,y) G_{ij}(dz) \end{aligned} \tag{64}$$

Now consider

Assumption 4 *Let $B \subset \{1, \dots, K\} \times \mathbb{R}^2$ be a closed set with $P((J_t, X_t, Y_t) \in B) = 1$ for all $t \ge 0$. Then, for each $1 \le i \le K$, $f_x(i,x,y)$ and $\int_{\mathbb{R}} (f(i,x+z,y) - f(i,x,y))^2 \nu_i(dz)$ are bounded on B.*

Together with Theorem 3 and observing that no continuity or differentiability with respect to the first variable is needed we now have the following.

Theorem 4 *With càdlàg and adapted J, X, Y and with a function $f : \mathbb{R}^3 \to \mathbb{R}$, where*

- *(J, X) is a Markov additive process as described above,*
- *Y an $\mathbb{R}^r$-valued FV process,*
- *$f(i, \cdot, \cdot) \in \mathcal{C}^{2,1}$ for each $1 \le i \le K$*

and with $\mathcal{A}$ defined via (64), then

$$\begin{aligned} M_t = f(J_t, X_t, Y_t) - f(J_0, X_0, Y_0) - \int_0^t \mathcal{A} f(J_s, X_s, Y_s) ds \\ - \int_0^t \nabla_y f(J_s, X_s, Y_s)^T dY_s^c - \sum_{0<s\le t} (f(J_s, X_s, Y_s) - f(J_s, X_s, Y_{s-})) \end{aligned} \tag{65}$$

is a local martingale.

If in addition Assumption 4 holds, then the assumptions and hence the conclusions of Lemma 1 hold.

We note that if we take $f(i, x, y) = g(x, y)\delta_{ii_0}$ for some i_0 ($\delta_{ii_0} = 1$ for $i = i_0$ and zero otherwise), and

$$\mathcal{A}_j g(x, y) = c_j g_x(x, y) + \frac{\sigma_j^2}{2} g_{xx}(x, y) + \int_{\mathcal{R}} \left(g(x + z, y) - g(x, y) - g_x(x, y) z 1_{\{|z|\le 1\}}\right) \nu_j(dz) \tag{66}$$

then $\mathcal{A}$ becomes

$$\mathcal{A} f(i, x, y) = \mathcal{A}_{i_0} g(x, y)\delta_{ii_0} + q_{ii_0} \int_{\mathbb{R}} g(x + z, y) G_{ii_0}(dz) \tag{67}$$

or if we prefer to write this in matrix notation where for each ij we compute the operator of $g(x, y)1_{\{i=j\}}$ then we have the following matrix valued operator $\mathcal{F}$ given by

$$\mathcal{F} g(x, y) = \operatorname{diag}\left(\mathcal{A}_1 g(x, y), \ldots, \mathcal{A}_K g(x, y)\right) + Q \circ \int_{\mathbb{R}} g(x + z, y) G(dz) \tag{68}$$

where $A \circ B = (a_{ij} b_{ij})$ and $\int_{\mathbb{R}} g(x + z, y) G(dz) = \left(\int_{\mathbb{R}} g(x + z, y) G_{ij}(dz)\right)$. Finally, recalling the notation $\mathbf{1}_i$, we now have the following.

Corollary 2 *With the assumptions of Theorem 3, if $g(x, y) \in \mathcal{C}^{2,1}$ then the following is a K-dimensional local martingale (a vector of local martingales)*

$$\begin{aligned} \mathbf{M}_t = g(X_t, Y_t)\mathbf{1}_{J_t}^T - g(X_0, Y_0)\mathbf{1}_{J_0}^T - \int_0^t \mathbf{1}_{J_s}^T \mathcal{F} g(X_s, Y_s) ds \\ - \sum_{i=1}^r \int_0^t g_{y_i}(X_s, Y_s)\mathbf{1}_{J_s}^T dY_{i,s}^c + \sum_{0<s\le t} (g(X_s, Y_s) - g(X_s, Y_{s-}))\mathbf{1}_{J_s}^T \ . \end{aligned} \tag{69}$$

If in addition $g_x(x, y)$ and $\int_{\mathbb{R}} (g(x+z, y) - g(x, y))^2 \nu_i(dz)$ are bounded on B from Assumption 4 for each $1 \le i \le K$, then the assumptions and hence the conclusions of Lemma 1 hold for each coordinate of $\mathbf{M}$.

We note that a special case of this last result was introduced in [2] for the case where Y is one dimensional and continuous, $g(x,y) = e^{\alpha(x+y)}$ under various restrictions on α (depending on whether the Lévy processes involved are general, spectrally positive or spectrally negative). In this case it is easy to check that

$$\mathcal{F}g(x,y) = e^{\alpha(x+y)}F(\alpha) \tag{70}$$

where

$$F(\alpha) = \mathrm{diag}\left(\psi_1(\alpha), \ldots, \psi_K(\alpha)\right) + Q \circ \int_{\mathbb{R}} e^{\alpha z} G(dz) \tag{71}$$

and

$$\psi_i(\alpha) = c_i\alpha + \frac{\sigma_i^2}{2}\alpha^2 + \int_{\mathbb{R}} (e^{\alpha z} - 1 - \alpha z 1_{\{|z|\le 1\}}\nu_i(dz) \tag{72}$$

are the Lévy exponents.

The above substantially generalizes the results in [2] and in particular we have the following.

Corollary 3 *Under the assumptions of Theorem 3, with $r = 1$, the notations above and $Z_t = X_t + Y_t$, if either*

1. $\Re(\alpha) = 0$,
2. $Z \ge 0$ *a.s.*, $\nu_i(-\infty, 0) = 0$ *for all* $1 \le i \le K$ *and* $\Re(\alpha) \le 0$,
3. $Z \le 0$ *a.s.*, $\nu_i(0, \infty) = 0$ *for all* $1 \le i \le K$ *and* $\Re(\alpha) \ge 0$,

Then,

$$\begin{aligned}\mathbf{M}_t = e^{\alpha Z_t}\mathbf{1}_{J_t}^T - e^{\alpha Z_0}\mathbf{1}_{J_0}^T - \int_0^t e^{\alpha Z_s}\mathbf{1}_{J_s}^T ds F(\alpha) \\ - \int_0^t e^{\alpha Z_s}\mathbf{1}_{J_s}^T dY_s^c - \sum_{0<s\le t} e^{\alpha Z_s}(1 - e^{-\alpha \Delta Y_s})\mathbf{1}_{J_s}^T\end{aligned} \tag{73}$$

is a zero mean vector valued L^2 martingale, of which each coordinate satisfies the assumptions and hence the conclusions of Lemma 1.

6 A final remark

Throughout the paper we have seen that under some conditions which in applications are typically relatively easy to verify, we have that M_t is an L^2 martingale satisfying the assumptions and hence the conclusions of Lemma 1. The question is what happens for $h(t) = 1/\sqrt{t}$. In general not much can be said, but we remark that under some considerably more restrictive conditions it is plausible that $M_t/\sqrt{t}$ may converge in distribution to a normal random variable (e.g., [12]).

Also, it also seems conceivable that our general martingale formula is valid for more general Markov processes. In particular, it would be nice to show that in a more general setup it holds for functions f such that with perhaps some further regularity conditions, $f(\cdot, y)$ is in the domain of the (extended) generator of an associated Markov process and $f(x, \cdot) \in \mathcal{C}^1$.

For example, for the case where $f(x,y)=\xi(x)\eta(y)$, where ξ is in the domain of a generator $\mathcal{A}$ and $\eta\in\mathcal{C}^1$, if we denote by M_t^ξ the local martingale for which

$$\xi(X_t)=\xi(X_0)+\int_0^t \mathcal{A}\xi(X_s)ds+M_t^\xi \tag{74}$$

and recall that

$$\eta(Y_t)=\eta(Y_0)+\int_0^t \nabla\eta(Y_s)^T dY_s^c+\sum_{0<s\le t}\Delta\eta(Y_s) \tag{75}$$

we can apply the integration by parts formula (with $\mathcal{A}$ applied only to ξ), to obtain that

$$\begin{aligned}
\xi(X_t)\eta(Y_t) &= \xi(X_0)\eta(Y_0)+\underbrace{\int_{(0,t]}\eta(Y_{s-})d\xi(X_s)}_{1}+\underbrace{\int_{(0,t]}\xi(X_{s-})d\eta(Y_s)}_{2}+\underbrace{[\xi(X),\eta(Y)]_t}_{3}\\
&= \xi(X_0)\eta(Y_0)+\underbrace{\int_0^t \mathcal{A}\xi(X_s)\eta(Y_s)ds+\int_{(0,t]}\eta(Y_{s-})dM_s^\xi}_{1} \qquad (76)\\
&\quad+\underbrace{\int_0^t \xi(X_s)\nabla\eta(Y_s)^T dY_s^c+\sum_{0<s\le t}\xi(X_{s-})\Delta f(Y_s)}_{2}+\underbrace{\sum_{0<s\le t}\Delta\xi(X_s)\Delta\eta(Y_s)}_{3}\\
&= \xi(X_0)\eta(Y_0)+\int_0^t \mathcal{A}\xi(X_s)\eta(Y_s)ds+\int_0^t \xi(X_s)\nabla\eta(Y_s)^T dY_s^c\\
&\quad+\sum_{0<s\le t}(\xi(X_s)\eta(Y_s)-\xi(X_s)\eta(Y_{s-}))+\int_{(0,t]}\eta(Y_{s-})dM_s^\xi \qquad (77)
\end{aligned}$$

and thus we have that for this case (1) is indeed a local martingale, as it is equal to $\int_{(0,t]}\eta(Y_{s-})dM_s^\xi$. Its quadratic variation is

$$\int_{(0,t]}\eta^2(Y_{s-})d[M^\xi,M^\xi]_s=\int_{(0,t]}\eta^2(Y_{s-})d[\xi(X),\xi(X)]_s \tag{78}$$

so that if in addition X is a real valued semimartingale and $\xi\in\mathcal{C}^1$, then (e.g., Lemma 2 of [5]) this quadratic variation has the form

$$\begin{aligned}
&\int_0^t \eta^2(Y_s)(\xi'(X_s))^2 d[X,X]_s^c+\sum_{0<s\le t}\eta^2(Y_{s-})(\Delta\xi(X_s))^2\\
&=\int_0^t f_x^2(X_s,Y_s)d[X,X]_s^c+\sum_{0<s\le t}(f(X_{s-}+\Delta X_s,Y_{s-})-f(X_{s-},Y_{s-}))^2 \qquad (79)
\end{aligned}$$

with a similar result when X is an $\mathbb{R}^n$ valued semimartingale for some $n\ge 2$, in which the continuous part is replaced by the quadratic form $\sum_{ij}\int_{(0,t]}f_{x_i}(X_s,Y_s)f_{x_j}(X_s,Y_s)d[X_i,X_j]_s^c$. Thus, for sufficiently nice cases Lemma 1 could be applied. Consequently, for any finite linear combination of products of the form $\xi(x)\eta(y)$ we will also have that (1) is a local Martingale, but we do not know yet if it holds in the generality that is described in the second paragraph.

We leave both this question and the one described in the first paragraph for future research.

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