Research Report
Statistical Research Unit
Department of Economics
Göteborg University
Sweden

Unimodal regression in the two-parameter exponential family with constant or known dispersion parameter

Kjell Pettersson

Research Report 2007:14
ISSN 0349-8034

Mailing address:	Fax	Phone	Home Page:
Statistical Research Unit	Nat: 031-786 12 74	Nat: 031-786 00 00	http://www.statistics.gu.se/
P.O. Box 640	Int: +46 31 786 12 74	Int: +46 31 786 00 00	
SE 405 30 Göteborg			
Sweden			

Unimodal regression in the two-parameter exponential family with constant or known dispersion parameter

KJELL PETTERSSON

Statistical Research Unit, Department of Economics, Göteborg University,
SE 405 30 Göteborg, Sweden
e-mail: kjell.pettersson@statistics.gu.se

SUMMARY.

In this paper we discuss statistical methods for curve-estimation under the assumption of unimodality for variables with distributions belonging to the two-parameter exponential family with known or constant dispersion parameter. We suggest a non-parametric method based on monotonicity properties. The method is applied to Swedish data on laboratory verified diagnoses of influenza and data on inflation from an episode of hyperinflation in Bulgaria.

KEY WORDS: Non-parametric; Order restrictions; Two-parameter exponential family; Known dispersion parameter; Poisson distribution

1 INTRODUCTION

One of the central subjects of statistics is the estimation of curves. There exists a vast literature on the subject. Examples of methods are regression analysis and time series analysis. Often one has some knowledge in advance of the studied phenomenon that may be used in the analysis of the data. Such knowledge may be that the shape of the curve is known. In, for example, the study of the evolution in time of influenza during a season it is known that the number of reported cases per week of influenza-like illness first tends to increase and after reaching a peak tends to decrease. In such applications, it is reasonable to assume that the curve has a unimodal shape. Existing theory may motivate the use of some parametric formulation of the model. In the absence of such knowledge of the functional form of the curve one may use methods with fewer assumptions.

Some smoothing method may be considered when no information of the shape of the curve is available. One may for example calculate a simple moving average with all non-zero weights equal. Since the weights may be regarded as a discrete log-concave function unimodality will be preserved as pointed out by Frisén [1]. Anderson and Bock [2] found, however, that the location of the maximum is generally not preserved.

Example of another kind of method, which may be considered when no information about the shape of the curve is available, is to use smoothing splines. One procedure is described by Silverman [3]. In order to produce a good fit to data and to get a smooth curve he minimizes the following quantity with respect to g:

$$\sum\left(x(t)-g(t)\right)^2+\alpha\int g''(u)^2du,$$

where $x(t)$ is the observed value at time t $(t=1,2,...n)$, $\int g''(u)^2du$ is a roughness penalty and α is a smoothing parameter. This method does not preserve unimodality since the weights are in general not log-concave [1].

Information about the shape of the curve can be of different kinds. Sometimes it is known that the curve is concave. Hildreth's [4] method of concave regression may then be used. This method gives consistent estimates of the curve [5]. Holm and Frisén [6] propose a method for estimating concave or convex and increasing or decreasing functions. Dahlbom [7] modified their algorithm and extended the method to estimate sigmoid and unimodal concave functions. In her paper, there is an extensive analysis of the properties of the estimators for different curve forms. The assumption of concavity is unrealistic in some applications. In, for example, a study of influenza it is shown by [8] that in the up-phase an exponential function seems to describe the number of laboratory diagnosed cases rather good. To the down-phase, an exponentially decreasing function can be fitted. Such a mixture of exponential functions is not concave. We do not consider methods for estimating unimodal functions under restrictions of concavity in the present paper.

Gill and Baron [9] consider a method for estimating a continuous change of the canonical parameter of an exponential family from a constant level to a linear function. By using a non-linear transformation of the time-scale the results can be generalized to a non-linear continuous change of the parameter. They give conditions for consistent estimators of the change-point. They consider parameters with known behaviour after the change-point.

As was seen above, there exist different methods to estimate regressions with order restrictions. However, here we concentrate on regression where the only restriction on shape is that of unimodality.

Davies and Kovac [10] describe methods for nonparametric regression controlling the number of local extremes. The methods considered are the run-method and the taut-string multiresolution method. In the run method there is a restriction on the maximum run-length of the sign of the residuals. The taut-string method was first proposed by [11] and extended to nonparametric regression by [12]. In the integrated process, one constructs upper and lower limits. A taut string is a function within those limits with the shortest length. The derivative of the taut string is the estimator of the curve. The estimates at the local extremes may be adjusted to get better results. The two methods have been used to estimate a unimodal curve.

In the estimation of a monotone function regression splines may be used as described by Ramsay [13]. The idea is to define a knot sequence which partitions the interval into subintervals. In each subinterval a non-negative linear combination of a small number of monotone splines are fitted, This type of method has been used by Meyer [14] to estimate a unimodal density with known mode. To each side of the mode she fits monotone regression splines under the restriction of continuity at the mode.

None of the methods, for unimodal regression mentioned above, however, give maximum likelihood estimators. Such estimators were constructed for the normal distribution with known variances in [1] but here we aim at maximum likelihood estimation for a wider class of members of the exponential family. These maximum likelihood-estimates are needed in surveillance, see e.g. [15].

In a study of influenza, [8] the Poisson distribution and the normal distribution were used to describe the distribution at given time points. The Poisson distribution belongs to the exponential family and has one parameter. The one-parameter exponential family may be regarded as a special case of the two-parameter exponential family with known or constant dispersion parameter. The normal distribution with constant variance is in the class of the exponential family with constant dispersion parameter.

These are the motivations of this paper, in which we study unimodal regression for variables with distributions belonging to the two-parameter exponential family with constant or known dispersion parameter. This kind of estimator is of interest for example in some economical problems and for outbreaks of infectious diseases as will be further discussed in Section 5. Andersson, Bock, Frisén and Pettersson have analyzed outbreaks of influenza in order to construct of methods for online detection of onsets and peaks of influenza [15-20]

The outline of the paper is the following: In Section 2 the model is described. In Section 3 we give the estimator. Some properties of the estimator are given in Section 4. Some applications of the method are described in Section 5. Concluding remarks are given in Section 6.

2 THE MODEL

2.1 The family of distributions

We observe a random process, $X(t)$, at n discrete values of the ordering variable t which will here be called time. The process may be defined as well in discrete time as in continuous time. In both cases, inferences are only for the behaviour of the process at the observed time points. We further restrict the attention to processes for which $X(t)$ has a distribution belonging to the two-parameter exponential family with constant or known dispersion parameter. The one-parameter exponential family may be regarded as a special case with a known dispersion parameter. We also assume that $X(t)$ and $X(u)$ are independent for $t \neq u$ and that $t, u \in (1, 2, ..., n)$. In applications, the assumption of independency may often be a realistic since we observe the process at discrete time points.

We write a probability function belonging to the exponential family in the canonical form as in [21]

$$f\left(x(t);\theta(t),\phi(t)\right)=\exp\left\{\frac{\left(x(t)\theta(t)-b\left(\theta(t)\right)\right)}{a\left(\phi(t)\right)}-c\left(x(t);\phi(t)\right)\right\} \tag{1}$$

where $\theta(t)\in\Theta(t)$ is constant for each t and $a(\cdot)$, $b(\cdot)$ and $c(.)$ are some functions. $\phi(t)$ is the dispersion parameter, which is regarded as a nuisance parameter. $a\left(\phi(t)\right)>0$ is of the form $a\left(\phi(t)\right)=\frac{\phi(t)}{\omega(t)}$ where $\omega(t)>0$ are known weights for all t. We also assume that the dispersion parameter $\phi(t)$ either is known or if unknown $\phi(t)=\phi$ for all t. f can be either the p.d.f. for a continuous random variable, e.g. the exponential distribution or the probability function for a discrete random variable such as the Poisson distribution.

It is also assumed that the family is regular as defined by Brown [22], i.e. that if $\Xi(t)=\left\{\theta(t):\int_{-\infty}^{\infty}\exp\left[\frac{\left(x(t)\theta(t)\right)}{a\left(\phi(t)\right)}-c\left(x(t);\phi(t)\right)\right]dx(t)<\infty\right\}$ then the parameter space $\Theta(t)$ is defined as $\Theta(t)=\operatorname{int}\left(\Xi(t)\right)$. The parameter space shall thus be an open set. If f is the probability function for a discrete random variable, then the integral should be replaced by a sum.

It is also assumed the first and second derivatives of $b(\theta)$ with respect to θ exist and that $\frac{\partial^2 b\left(\theta(t)\right)}{\partial\theta^2}>0$. It is a well-known fact that

$$\mu(t)=E\left(X(t)\right)=\frac{\partial b\left(\theta(t)\right)}{\partial\theta}$$

$$Var\left(X(t)\right)=a\left(\phi(t)\right)\frac{\partial^2 b\left(\theta(t)\right)}{\partial\theta^2}$$

2.2 The regression function

In the present paper, we restrict attention to the case when the expected value of the process first is increasing and after having reached a peak decreases. The methods are easily modified for the case when the expectations first decrease and after a trough increase.

We define unimodality as follows: There exists a t' such that

$$\left.\begin{array}{l}\mu_{\max}=\mu(1)\geq\ldots\geq\mu(n)\ \text{for } t'=1\\ \mu(1)\leq\ldots\leq\mu(t'\text{-}1)\leq\mu_{\max}\ \text{and}\ \mu_{\max}\geq\mu(t')\geq\ldots\geq\mu(n)\ \text{for } t'\in(2,3,\ldots,n)\\ \mu(1)\leq\ldots\leq\mu(n)=\mu_{\max}\ \text{for } t'=n+1\end{array}\right\} \tag{2}$$

where $\mu_{\max}=\max_{1\leq t\leq n}\mu(t)$ and there is at least one strict inequality in (2).

3 THE MAXIMUM LIKELIHOOD ESTIMATOR

For $X(1),X(2),\ldots,X(n)$, independently distributed random variables, $X(t)$, $t\in(1,2,\ldots,n)$, having a distribution belonging to the two-parameter exponential family (1) we assume that there are $m(t)$ observations on $X(t)$ for each *t*. In Lemma 1, we study the case of a monotone regression. Denote the maximum likelihood estimator of $\boldsymbol{\mu} = (\mu(1), \mu(2),...,\mu(n))'$ subject to $\mu(1) \le \mu(2) \le ... \le \mu(n)$ by $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}(1), \tilde{\mu}(2),...,\tilde{\mu}(n))'$.Then the following may be shown.

Lemma 1:
(a) $\tilde{\boldsymbol{\mu}}$ is given by minimizing

$$\sum_{t=1}^{n} (\bar{x}(t) - \mu(t))^2 \frac{m(t)}{\phi(t)} \tag{3}$$

with respect to $\mu(t)$ $(t = 1,2,...,n))$ under the restriction of isotonicity for $\mu(t)$ if $\phi(t)$ is known for all *t*.
(b) $\tilde{\boldsymbol{\mu}}$ is given by minimizing

$$\sum_{t=1}^{n} (\bar{x}(t) - \mu(t))^2 m(t) \tag{4}$$

with respect to $\mu(t)$, $(t = 1,2,...,n))$, under the restriction of isotonicity for $\mu(t)$ if $\phi(1) = \phi(2) = ... = \phi(n)$

The maximum likelihood estimator of $\boldsymbol{\mu}$, subject to $\mu(1) \ge \mu(2) \ge ... \ge \mu(n)$ and the family of distributions of Section 2.1, is obtained by minimizing (3) and (4) respectively under the restriction of antitonicity.

Proof: Silvapulle and Sen [23] consider the following case: Let $x_1(t),...,x_{m(t)}(t)$ be $m(t)$ independent observations on the random variable $X(t)$ from group t, $(t = 1,...,n)$.

We want to find the maximum likelihood estimator of $(\mu(1),...\mu(n))$ where $\mu(t) = E(X(t))$ under the restriction $A\boldsymbol{\mu} \ge 0$. A is a matrix in which each row is a permutation of $(-1,1,0,...,0)$ and $\mu = (\mu(1),...,\mu(n))'$. Assume that the distribution of $X(t)$ belongs to the exponential family with parameters $\theta(t)$ and $\phi(t)$. Part (a) of proposition 2.4.3 in [23] states that the maximum likelihood estimator $\tilde{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ under the restriction $A\boldsymbol{\mu} \ge 0$ is the value of $\boldsymbol{\mu}$ at which

$$\sum_{t=1}^{n} (\bar{x}(t) - \mu(t))^2 \frac{m(t)}{\phi(t)} \tag{5}$$

reaches its minimum subject to $A\boldsymbol{\mu} \ge 0$ if $\phi(1),\phi(2),...,\phi(n)$ are known constants. Part (b) of proposition 2.4.3 in [23] states that the maximum likelihood estimator $\tilde{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ under the restriction $A\boldsymbol{\mu} \ge 0$ is the value of $\boldsymbol{\mu}$ at which

$$\sum_{t=1}^{n} (\bar{x}(t) - \mu(t))^2 m(t) \tag{6}$$

reaches its minimum subject to $A\boldsymbol{\mu} \ge 0$ if $\phi(1) = \phi(2) = ... = \phi(n) = \phi$, say. In isotonic regression the i:th row of A has -1 in position i, +1 in position $i+1$ and 0 in all other positions. For antitonic regression the i:th row has +1 in position i, -1 in position $i+1$ and 0 in all other positions. Thus the maximum likelihood estimator $\tilde{\boldsymbol{\mu}}$ under the restriction of isotonicity and antitonicity respectively is thus given by minimizing $\sum_{t=1}^{n}\left(\overline{x}(t)-\mu(t)\right)^2\frac{m(t)}{\phi(t)}$ with respect to $\mu(t)$ under the restrictions of isotonicity and antitoncity respectively if $\phi(t)$ is known for all t. For $\phi(1)=\phi(2)=...=\phi(n)=\phi$ $\tilde{\boldsymbol{\mu}}$ is given by minimizing $\sum_{t=1}^{n}\left(\overline{x}(t)-\mu(t)\right)^2 m(t)$ with respect to $\mu(t)$ under the restrictions of isotonicity and antitoncity respectively □

An estimator $\tilde{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ under the restriction of unimodality may be constructed in the following way [1]. Regard the following partitions of the observations $x(1), x(2),..., x(n)$:

$$\left(\{\varnothing\},\{x(1),...,x(n)\}\right),\left(\{x(1)\},\{x(2),...,x(n)\}\right),...,\left(\{x(1),...,x(n)\},\{\varnothing\}\right). \tag{7}$$

For each of these partitions we fit monotone regressions to each part. To the first part is fitted an isotonic regression and to the second an antitonic regression. Denote the likelihood for the fitted unimodal regression for the i:th partition by L_i $\left(i=1,2,...n+1\right)$. $\tilde{\boldsymbol{\mu}}$ is given by the partition with gives $\max_{1\le i\le n+1}(L_i)$.

Theorem 1: $\tilde{\boldsymbol{\mu}}$ defined above is the maximum likelihood estimator of $\boldsymbol{\mu}$ under the restriction of unimodality for the regular exponential distribution with known or constant dispersion parameter as described in Section 2.1.

Proof: First, we regard the maximum likelihood estimator of $\boldsymbol{\mu}$ for a given t' as defined by (2) Assume that $t'=1$. Then $\mu(t)$ is an antitonic function of t. The maximum likelihood estimator of $\boldsymbol{\mu}$ is given by lemma 1. Denote the maximum of the likelihood function in this case by L_1. If $t'=n+1$ then $\mu(t)$ is an isotonic function of t. The maximum likelihood estimator of $\boldsymbol{\mu}$ follows from Lemma 1. Denote the maximum of the likelihood function by L_{n+1}. Now assume that $t'\in\left(2,3,...,n-1\right)$. According to the definition of t' in (2) then the curve is first increasing and then decreasing. If $\mu(1)\le...\le\mu(\text{t'-1})\le\mu_{\max}$ and $\mu_{\max}\ge\mu(t')...\ge\mu(n)$ for a given t', we fit an isotonic function according to lemma 1 to $x(1),...,x(t'-1)$. Denote the maximum of the likelihood by $L_{t'}^{I}$. Fit an antitonic function according to lemma 1 to $x(t'),...,x(n)$. Denote the maximum of the likelihood by $L_{t'}^{A}$.Then the maximal likelihood for the curve is $L_{t'}=L_{t'}^{I}\cdot L_{t'}^{A}$. Our maximum likelihood estimator $\tilde{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ under the restriction of unimodality is given by the partition, which maximizes $L_{t'}$ for $t'\in\left(1,2,...n+1\right)$ □

The method is illustrated by a numerical example in section 5.1.

The parameter $\phi(t)$ in (1) may be interpreted as a scale parameter and $\omega(t)$ as the number of observations on $X(t)$ for $t\in(1,2,...,n)$. For normally distributed variables, the scale parameter is the variance. When a unimodal regression is fitted to observations on normally distributed variables with varying variances, we correct for the differences in scale by weighting the observations inversely to their variances. If the dispersion parameter is constant for all observations, we have the same scale for all observations and we therefore do not correct for differences in scale and give all observations a weight, which is proportional to the number of observations for all time points.

In section 2 it was stated that $\mu = \frac{\partial b(\theta)}{\partial \theta}$ and since it was assumed that $\frac{\partial^2 b(\theta)}{\partial \theta^2} > 0$ then μ is a strictly increasing function of θ. This motivates the following corollary.

Corollary: The maximum-likelihood estimator of $\theta(t)$, $\tilde{\theta}(t)$, is given by $b_\theta'^{-1}\left(\tilde{\mu}(t)\right)$, where $b_\theta'^{-1}$ denotes the inverse of $\frac{\partial b(\theta)}{\partial \theta}$.

Proof: Since $\frac{\partial b\left(\theta(t)\right)}{\partial \theta}$ is strictly increasing in $\theta(t)$ the inverse exists. If the likelihood is maximized for $\mu(t) = \tilde{\mu}(t)$ it is also maximized for the value $\tilde{\theta}(t)$ of $\theta(t)$ for which

$$\tilde{\mu}(t) = \left(\frac{\partial b\left(\theta(t)\right)}{\partial \theta}\right)_{\theta = \tilde{\theta}} \quad \text{i.e. } \tilde{\theta}(t) = b_\theta'^{-1}\left(\tilde{\mu}(t)\right). \quad \square$$

4 PROPERTIES OF THE ESTIMATOR

The estimated curve preserves the unimodality since the transformation of the data is log-concave. See Frisén [1] for a proof. In this section we study consistency and bias.

4.1 Consistency.

When the number, *m(t),* of independent observations at each time point, *t*, tends to infinity, we have the following consistency property.

Theorem 2: In the class of distributions given in Section 2.1 $\tilde{\mu}(t)$ is a strongly consistent estimator of $\mu(t)$ for $t = 1, 2, ..., n$ when min *m(t)*→∞.

Proof: The theorem follows from the Kolmogorov law of large numbers since it is assumed in Section 2.1 that $\mu(t) = \frac{\partial b\left(\theta(t)\right)}{\partial \theta}$ exists for $t = 1, 2, ..., n$ $\square$

From this it follows that both the height and the time of the peak will be consistently estimated. Observe that the consistency do not prevail for the case where the number of time points tends to infinity.

If there is only one observation for each time point but the number of time points tends to infinity then $\hat{\mu}_{\max} = \max\left[\tilde{\mu}(t)\right]$ is in general an inconsistent estimator of $\mu_{\max} = \max\left[\mu(t)\right]$ as is pointed out by Frisén [1] and Dahlbom [24] for a regression with normal distribution and by Woodroofe and Sun [25] for density estimation in the exponential family.

4.2 Bias.

In the case when there is one observation per time point the estimators of the end-points and the maximum points are positively biased, as was pointed out by Dahlbom [24]. She also found that the bias of the estimators of other points at the curve often is negligible. Some results from her simulation experiments of certain curves and the normal distribution will now be reviewed.

We focus our interest to the problem of estimating $\mu_{\max} = \max_{t=1,2,..n}\left[\mu(t)\right]$ when the time point for the maximum is unknown and when there is one observation at each of a fixed number of time points. As an estimator of $\mu_{\max}$ one may use $\hat{\mu}_{\max} = \max_{t=1,2,..n}\left[\tilde{\mu}(t)\right]$. Dahlbom [24] studied, the case when $t_{\max}$ is unique, where $t_{\max} = \arg\max_{t=1,2,\ldots,n}\mu(t)$. One of the models for $\mu(t)$ was a symmetrical and concave second-degree polynomial. The bias of $\hat{\mu}_{\max}$ as an estimator of $\mu_{\max}$ was a decreasing function of the curvature normalised for the standard deviation. When the number of time points in an interval of fixed length increases then the bias tends to increase. Since deviations from symmetry may affect the bias, she also used a third-degree polynomial as a model for $\mu(t)$ with values of the coefficients giving unimodal and concave curves. It was found that moderate deviations from symmetry had small influence on the bias.

The errors in the simulation experiments by Dahlbom were normally distributed. There is no obvious reason to expect much different results for other members of the exponential family. The curves studied by Dahlbom in the simulation experiments are concave. The bias in the estimator of $\mu_{\max}$ for other curve forms and other distributions is not necessarily the same. As mentioned in Section 1 a mixture of exponential functions, one increasing and one decreasing, seems to give a good fit to laboratory diagnosed influenza in Sweden. Such mixtures of exponential functions are not concave.

5 EXAMPLES

5.1 A numerical example.

We have one observation on $X(t)$ at each of the time points $t = 1,2,3,4,5$ where $X(t)$ follows the Poisson distribution $P(\lambda(t))$. The Poisson distribution is in the exponential family of equation (1), with $\theta = \ln(\lambda)$, $b(\theta) = e^{\theta} = \lambda$, $a(\phi) = 1$, $c(x,\phi) = \ln(x!)$ and $\mu(t) = \dfrac{\partial b(\theta(t))}{\partial\theta} = e^{\theta} = \lambda$. We assume that $\mu(t)$ is unimodal as in (2).

We give the calculations for the case there the observed values of X are 1, 3, 1, 5, and 1. In the table below, we give the observed values and the estimates of $\mu(t)$ at different time points and the likelihood for different partitions of the observations. As an example, we get the following likelihood for the partition $\{1,3\},\{1,5,1\}$

$$L = \left(e^{-1}\cdot\frac{1^1}{1!}\right)\cdot\left(e^{-3}\cdot\frac{3^3}{3!}\right)\cdot\left(e^{-3}\cdot\frac{3^1}{1!}\right)\left(e^{-3}\cdot\frac{3^5}{5!}\right)\cdot\left(e^{-1}\cdot\frac{1^1}{1!}\right) = 0.457\cdot 10^{-3}$$

Table 1. The likelihood and the maximum likelihood estimators for each time conditional on each of the possible partitions of the dataset {1, 3, 1, 5, 1}

Partitions	t=1	t=2	t=3	t=4	t=5	Likelihood
$\{\varnothing\},\{1,3,1,5,1\}$	2.5	2.5	2.5	2.5	1	$0.221\cdot 10^{-3}$
$\{1\},\{3,1,5,1\}$	1	3	3	3	1	$0.457\cdot 10^{-3}$
$\{1,3\},\{1,5,1\}$	1	3	3	3	1	$0.457\cdot 10^{-3}$
$\{1,3,1\},\{5,1\}$	1	2	2	5	1	$1.160\cdot 10^{-3}$
$\{1,3,1,5\},\{1\}$	1	2	2	5	1	$1.160\cdot 10^{-3}$
$\{1,3,1,5,1\},\{\varnothing\}$	1	2	2	3	3	$0.271\cdot 10^{-3}$

The conclusion from Table 1 is that the maximum likelihood estimate of the curve is given in the rows for the partitions $(\{1,3,1\},\{5,1\})$ and $(\{1,3,1,5\},\{1\})$. Thus, the maximum likelihood estimator is $\tilde{\boldsymbol{\mu}} = (1,2,2,5,1)$.

5.2 An economical example.

An economical example where it is of interest to use an inverted U-shaped curve is in the study of inflation before, during and after periods of hyperinflation [26]. See for example monthly data on the inflation for Bulgaria during the time period from 1995 to 2000. After the end of central planning, there was a budget deficit and high monetary growth. The confidence in government was decreasing and the inflation was steadily increasing. During March, the twelve-month inflation was about 2000%. After reforms, Bulgarians became more confident in their currency and inflation decreased. In figure 1, we show the Bulgarian twelve-month inflation in percent during the time period from June 1995 to September 1999. We also show the unimodal regression function fitted under the assumption of normally distributed values with constant variances.

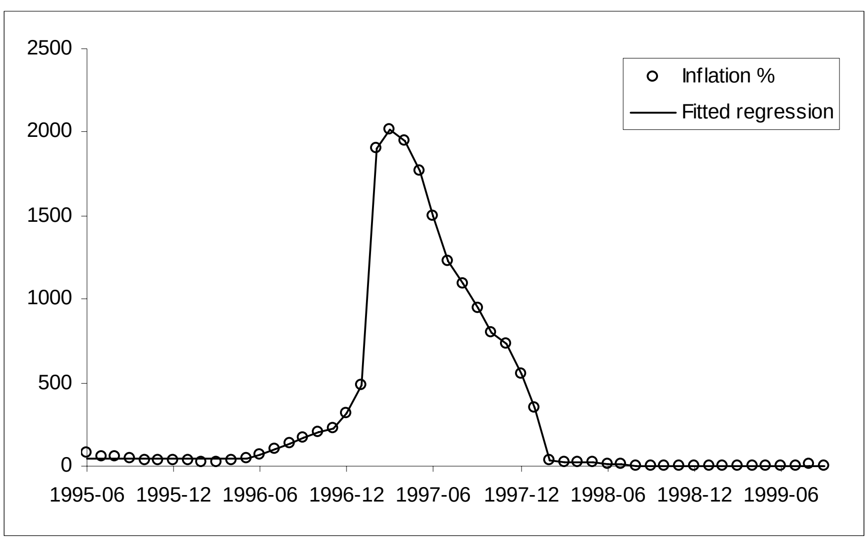

Figure 1. Bulgarian inflation and fitted regression for the years 1996 and 1997

5.3 Application to influenza incidences

In the study of the number of influenza cases during an ordinary season it may be reasonable to assume that the number of cases are initially increasing and that they after having reached a peak are decreasing. It is of interest to study the incidence, since influenza epidemics impose huge costs on society. The Swedish Institute for Infectious Disease control (SMI) publishes information on Swedish influenza incidence. Two types of weekly data are published, namely the number of influenza like illness (ILI) and laboratory diagnosed influenza cases (LDI). Details on the reporting can be found in [27].

The number of reported cases per week of influenza-like illness and the number of laboratory-diagnosed cases per week are counting variables. In some studies, it is assumed that the distribution of the number of cases can be approximated by the normal distribution. However, at the onset of an epidemic there are few cases. Assumption of normality then may assign a non-zero probability that cannot be ignored to a negative number of cases. The assumption of normality of the number of influenza cases has been criticized by Le Strat and Carrat [28]and Rath et al. [29]. Held et al [30] suggest the Poisson distribution for infectious surveillance data and also discuss the negative binomial distribution in cases of over-dispersion relative to the Poisson distribution. Sebastiani et al [31] use a log-normal distribution for ILI data. Andersson et al [8] studied the distributional properties for ILI and LDI in Swedish influenza data from five influenza seasons. They fitted piecewise exponential functions to each season and examined the residuals. The auto-correlations in the residuals were low all seasons for the two variables. Near the peak, there was no evident relation between the squared residuals and the estimated curve. Since there were numerous cases near the peak an assumption of normally distributed values with constant variance may be adequate for that part. In [19], they also studied the onset of an epidemic using ILI-data. They found that the squared residuals depend on the estimated means. Since there was no direct evidence against the Poisson distribution they suggested that this distribution may be used a first approximation.

Here we restrict our attention to the LDI data. The LDI-cases are reported weekly from five virus laboratories and between 15 and 20 microbiological laboratories. See [32] and [33] for details. The LDI cases are mostly patients in need of hospital care.

In figure 2, we show the number of cases and the unimodal regression for the number of laboratory diagnosed cases under the assumption of Poisson distributed values for the season 2005/2006

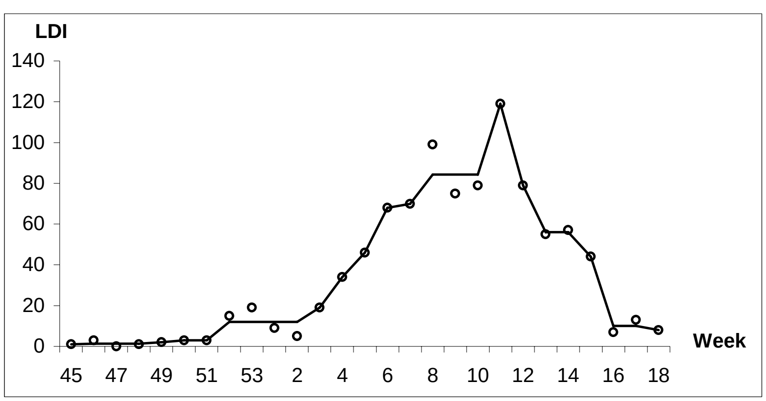

Figure 2. The number of LDI cases and the fitted regression during the season 2005/2006.

6 CONCLUDING REMARKS

We have given examples of situations where it is reasonable to assume that the evolution in time of a process may be described by a unimodal curve and when it can be assumed that the distribution of the observed process may be described by a distribution belonging to the one-parameter exponential distribution.

In unimodal regression with distributions belonging to the exponential family with varying dispersion parameter, the observations are weighted inversely proportional to that parameter if there is one observation per time point. For distributions belonging to a one-parameter exponential family or two-parameter exponential family with constant dispersion parameter, all observations shall have the same weight for one observation per time point. An example of a two-parameter distribution in the exponential family with constant dispersion parameter is the normal distribution with constant variance. An example of a distribution belonging to the one-parameter exponential family is the Poisson distribution. In this paper, we restrict attention to estimation of the unimodal regression curve under the assumptions given above. The results are also important in the construction of surveillance procedures in order to monitor different processes in time. An example is in timely detection of influenza peaks where the Poisson distribution is useful. In some financial time series, the assumption of a normal distribution with constant variance may be used as in the example of Bulgarian hyperinflation. The monitoring of such time series may be of interest to among others actors on the foreign exchange markets.

ACKNOWLEDGEMENTS

The author is grateful to Professor Marianne Frisén and Associate professor Eva Andersson for supervision of this work. Associate professor Dick Durevall has given valuable advice concerning the economical example. Research assistant Linus Schiöler has expertly given technical assistance. The research was supported by the Swedish Emergency Management Agency (grant 0622/204) and the Bank of Sweden Tercentenary Foundation (grant J2003-0558).

REFERENCES

[1] Frisén, M., 1986, Unimodal regression. *The Statistician,* **35,** 479-485.

[2] Andersson, E. & Bock, D. (2001) On seasonal filters and monotonicity. Department of Statistics, Göteborg University.

[3] Silverman, B.W., 1985, Some aspects of the spline smoothing approach to non-parametric regression curve fitting. *Journal of the Royal Statistical Society B,* **47,** 1-52.

[4] Hildreth, C., 1954, Point estimation of ordinates of concave functions. *Journal of the American Statistical Association,* **49,** 598-619.

[5] Hanson, D.L. & Pledger, G., 1976, Consistency in Concave Regression. *The Annals of Statistics,* **4,** 1038-1050.

[6] Holm, S. & Frisén, M. (1985) Nonparametric regression with simple curve characteristics. Department of Statistics, Göteborg University.

[7] Dahlbom, U. (1994) Estimation of regression functions with certain monotonicity and concavity/convexity restrictions. Göteborg, Ph.D. Thesis. Department of Statistics, Göteborg University.

[8] Andersson, E., Bock, D. & Frisén, M., 2007, Modeling influenza incidence for the purpose of on-line monitoring. *Statistical Methods in Medical Research***,** (in press).

[9] Gill, R. & Baron, M., 2004, Consistent estimation in generalized broken-line regression. *Journal of Statistical Planning and Inference,* **126,** 460.

[10] Davies, P.L. & Kovac, A., 2003, Local Extremes, Runs, Strings and Multiresolution. *The Annals of Statistics,* **29,** 1-65.

[11] Barlow, R.E., Bartholomew, D.J., Bremer, J.M. & Brunk, H.D., 1972, *Statistical inference under order restrictions,* (London: Wiley).

[12] Mammen, E. & Van Der Geer, S., 1997, Locally Adaptive Regression Splines. *The Annals of Statistics,* **26,** 387-413.

[13] Ramsay, J.O., 1988, Monotone regression Splines in Action. *Statistical Science,* **3,** 425-461.

[14] Meyer, M.C. (2007) Algorithms for the Estimation of a Decreasing or Unimodal Density using Shape-Restricted Regression Splines. *International Statistical Institute Conference 2007.* Lisboa.

[15] Bock, D., Andersson, E. & Frisén, M., 2007, Statistical surveillance of epidemics: Peak detection of influenza in Sweden. *Biometrical Journal***,** (in press).

[16] Andersson, E., 2004, Monitoring system for detecting starts and declines of influenza epidemics. *Morbidity and Mortality Weekly Report,* **53,** 229-229.

[17] Frisén, M. & Andersson, E. (2007) On-line detection of outbreaks. Manuscript.

[18] Frisén, M., Andersson, E. & Pettersson, K. (2007) Estimation of outbreak regression. Submitted manuscript.

[19] Frisén, M., Andersson, E. & Schiöler, L. (2007) A non-parametric system for on-line outbreak detection of epidemics. Manuscript.

[20] Bock, D. & Pettersson, K. (2006) Exploratory analysis of spatial aspects on the Swedish influenza data. *Smittskyddsinstitutets rapportserie.* Stockholm, Report from the Swedish Institute for Infectious Disease Control.

[21] Mccullagh, P. & Nelder, J.A., 1989, *Generalized Linear Models,* (London: Chapman and Hall).

[22] Brown, L.D., 1986, *Fundamentals of Statistical Exponential Families,* (Hayward, Calif . IMS).

[23] Silvapulle, M. & Sen, P.K., 2005, *Constrained statistical inference. Inequality, order and shape restriction,* (Wiley).

[24] Dahlbom, U. (1986) Some Properties of Estimates of Unimodal Regression. Licentiate thesis. Department of Statistics, Göteborg University.

[25] Woodroofe, M. & Sun, J.Y., 1993, A Penalized Maximum-Likelihood Estimate of f(0+) When f Is Non-Increasing. *Statistica Sinica,* **3,** 501-515.

[26] Burda, M. & Wyplosz, C., 2005, *Macroeconomics. A European Text,* (New York: Oxford University Press).

[27] Ganestam, F., Lundborg, C.S., Grabowska, K., Cars, O. & Linde, A., 2003, Weekly antibiotic prescribing and influenza activity in Sweden: a study throughout five influenza seasons. *Scandinavian Journal of Infectious Diseases,* **35,** 836-842.

[28] Le Strat, Y. & Carrat, F., 1999, Monitoring epidemiologic surveillance data using hidden Markov models. *Statistics in Medicine,* **18,** 3463-3478.

[29] Rath, T.M., Carreras, M. & Sebastiani, P., 2003, (Ed.) *Automated Detection of Influenza Epidemics with Hidden Markov Models,* (Berlin, Germany Springer-Verlag).

[30] Held, L., Höhle, M. & Hofmann, M., 2005, A statistical framework for the analysis of multivariate infectious disease surveillance data. *Statistical Modelling,* **5,** 187-199.
[31] Sebastiani, P., Mandl, K.D., Szolovits, P., Kohane, I.S. & Ramoni, M.F., 2006, A Bayesian dynamic model for influenza surveillance. *Statistics in Medicine,* **25,** 1803-1816.
[32] Linde, A., Brytting, M., Petersson, P., Mittelholzer, C., Penttinen, P. & Ekdahl, K. (2002) Annual Report September 2001 - August 2002: The National Influenza Reference Center. Swedish Institute for Infectious Disease Control.
[33] Linde, A., Brytting, M., Johansson, M., Wiman, Å., Högberg, L. & Ekdahl, K. (2004) Annual Report September 2003 - August 2004: The National Influenza Reference Center. Stockholm, Swedish Institute for Infectious Disease Control.

Research Report

2007:1	Andersson, E.:	Effect of dependency in systems for multivariate surveillance.
2007:2	Frisén, M.:	Optimal Sequential Surveillance for Finance, Public Health and other areas.
2007:3	Bock, D.:	Consequences of using the probability of a false alarm as the false alarm measure.
2007:4	Frisén, M.:	Principles for Multivariate Surveillance.
2007:5	Andersson, E., Bock, D. & Frisén, M.:	Modeling influenza incidence for the purpose of on-line monitoring.
2007:6	Bock, D., Andersson, E. & Frisén, M.:	Statistical Surveillance of Epidemics: Peak Detection of Influenza in Sweden.
2007:7	Andersson, E., Kühlmann-Berenzon, S., Linde, A., Schiöler, L., Rubinova, S. & Frisén, M.:	Predictions by early indicators of the time and height of yearly influenza outbreaks in Sweden.
2007:8	Bock, D., Andersson, E. & Frisén, M.:	Similarities and differences between statistical surveillance and certain decision rules in finance.
2007:9	Bock, D.:	Evaluations of likelihood based surveillance of volatility.
2007:10	Bock, D. & Pettersson, K.	Explorative analysis of spatial aspects on the Swedish influenza data.
2007:11	Frisén, M. & Andersson, E.	Semiparametric surveillance of outbreaks.
2007:12	Frisén, M., Andersson, E. & Schiöler, L.	Robust outbreak surveillance of epidemics in Sweden.
2007:13	Frisén, M., Andersson, E. & Pettersson, K.	Semiparametric estimation of outbreak regression.