Unimodality of the freely selfdecomposable probability laws

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Abstract

We show that any freely selfdecomposable probability law is unimodal. This is the free probabilistic analog of Yamazato's result in [Ann. Probab. **6** (1978), 523-531].

Key words: free probability; free convolution; free selfdecomposability; unimodality; Yamazato's theorem

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1 Introduction

A. Ya. Khintchine introduced the class $\mathcal{L}$ of limit distributions of certain independent triangular arrays. It plays an important role in statistics and mathematical finance, mainly as a consequence of the following characterization established by P. Lévy in 1937: A (Borel-) probability measure μ belongs to $\mathcal{L}$, if and only if there exists, for any constant c in $(0,1)$, a probability measure μ_c on $\mathbb{R}$, such that

$$\mu = D_c\mu * \mu_c. \tag{1.1}$$

Here $D_c\mu$ is the push-forward of μ by the map $x \mapsto cx$ and $*$ denotes (classical) convolution of probability measures. To distinguish from the corresponding class $\mathcal{L}(\boxplus)$ in free probability (described below) we shall henceforth write $\mathcal{L}(*)$ instead of just $\mathcal{L}$. As a result of Lévy's characterization the measures in $\mathcal{L}(*)$ are called *selfdecomposable*. The class $\mathcal{L}(*)$ contains in particular the class $\mathcal{S}(*)$ of stable probability measures on $\mathbb{R}$ as a proper subclass (see e.g. [15]).

A probability measure μ on $\mathbb{R}$ is called *unimodal*, if, for some a in $\mathbb{R}$, it has the form

$$\mu(\mathrm{d}t) = \mu(\{a\})\delta_a(\mathrm{d}t) + f(t)\,\mathrm{d}t, \tag{1.2}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is increasing (meaning that $x \le y$ implies $f(x) \le f(y)$) on $(-\infty, a)$ and decreasing (meaning that $x \le y$ implies $f(x) \ge f(y)$) on (a, ∞), and where δ_a denotes the Dirac measure at a. The problem of unimodality of the measures in $\mathcal{L}(*)$ emerged in the 1940's. Already in the original 1949 Russian edition of the fundamental book [9] by B.V. Gnedenko and A.N. Kolmogorov it was claimed that all selfdecomposable distributions are unimodal. However, as explained in the English translation [9] (by K. L. Chung) there was an error in the proof, and it took almost 30 years before a correct proof was obtained by M. Yamazato in 1978 (see [19]). In the appendix to the paper [4] from 1999 it was proved by P. Biane that all measures in the class $\mathcal{S}(\boxplus)$ of stable measures with respect to free additive convolution $\boxplus$ (see Section 2) are unimodal. In the present paper we extend this result to the class $\mathcal{L}(\boxplus)$ of all selfdecomposable distributions with respect to $\boxplus$; thus establishing a full free probability analog of Yamazato's result.

In the paper [10] it was proved by U. Haagerup and the second named author that the free analogs of the Gamma distributions (which are contained in $\mathcal{L}(\boxplus)\setminus\mathcal{S}(\boxplus)$) are unimodal, and the present paper is based in part on techniques from that paper. Let us also point out that several results from Section 3 in the present paper (most notably Lemma 3.4) may be extracted from the more general and somewhat differently oriented theory developed in the papers [11]-[12] by H.-W. Huang. We prefer in the present paper to give a completely self-contained and elementary exposition in the specialized setup considered here. In particular our approach does not depend upon the rather deep complex analysis considered in Huang's papers and originating in the work of S.T. Belinschi and H. Bercovici (see e.g. [3]).

The remainder of the paper is organized as follows: In Section 2 we provide background material on $\boxplus$-infinite divisibility, the Bercovici-Pata bijection, selfdecomposability and unimodality. In Section 3 we establish unimodality for probability measures in $\mathcal{L}(\boxplus)$ satisfying in particular that the corresponding Lévy measure has a strictly positive C^2-density on $\mathbb{R}\setminus\{0\}$. In Section 4 we extend the unimodality result from such measures to general measures in $\mathcal{L}(\boxplus)$, using that unimodality is preserved under weak limits.

2 Background

2.1 Free and classical infinite divisibility

A (Borel-) probability measure μ on $\mathbb{R}$ is called infinitely divisible, if there exists, for each positive integer n, a probability measure $\mu^{1/n}$ on $\mathbb{R}$, such that

$$\mu = \underbrace{\mu^{1/n} * \mu^{1/n} * \cdots * \mu^{1/n}}_{n \text{ terms}}, \tag{2.1}$$

where $*$ denotes the usual convolution of probability measures (based on classical independence). We denote by $\mathcal{ID}(*)$ the class of all such measures on $\mathbb{R}$. We recall that a probability measure μ on $\mathbb{R}$ is infinitely divisible, if and only if its characteristic function (or Fourier transform) $\hat{\mu}$ has the Lévy-Khintchine representation:

$$\hat{\mu}(u) = \exp\Big[\mathrm{i}\eta u - \tfrac{1}{2}au^2 + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\big)\ \rho(\mathrm{d}t)\Big], \qquad (u \in \mathbb{R}), \tag{2.2}$$

where η is a real constant, a is a non-negative constant and ρ is a Lévy measure on $\mathbb{R}$, meaning that

$$\rho(\{0\}) = 0, \quad \text{and} \quad \int_{\mathbb{R}} \min\{1, t^2\}\ \rho(\mathrm{d}t) < \infty.$$

The parameters a, ρ and η are uniquely determined by μ and the triplet (a, ρ, η) is called the *characteristic triplet* for μ. Alternatively the Lévy-Khintchine representation may be written in the form:

$$\hat{\mu}(u) = \exp\Big[\mathrm{i}\gamma u + \int_{\mathbb{R}} \Big(\mathrm{e}^{\mathrm{i}ut} - 1 - \frac{\mathrm{i}ut}{1+t^2}\Big)\frac{1+t^2}{t^2}\ \sigma(\mathrm{d}t)\Big], \quad (u \in \mathbb{R}), \tag{2.3}$$

where γ is a real constant, σ is a finite measure on $\mathbb{R}$ and (γ, σ) is called the *generating pair* for μ. The relationship between the representations (2.3) and (2.2) is as follows:

$$\begin{aligned} a &= \sigma(\{0\}), \\ \rho(\mathrm{d}t) &= \frac{1+t^2}{t^2}\cdot 1_{\mathbb{R}\setminus\{0\}}(t)\ \sigma(\mathrm{d}t), \\ \eta &= \gamma + \int_{\mathbb{R}} t\Big(1_{[-1,1]}(t) - \frac{1}{1+t^2}\Big)\ \rho(\mathrm{d}t). \end{aligned} \tag{2.4}$$

For two probability measures μ and ν on $\mathbb{R}$, the free convolution $\mu \boxplus \nu$ is defined as the spectral distribution of $x + y$, where x and y are *freely independent* (possibly unbounded) selfadjoint operators on a Hilbert space with spectral distributions μ and ν, respectively (see [5] for further details). The class $\mathcal{ID}(\boxplus)$ of infinitely divisible probability measures with respect to free convolution $\boxplus$ is defined by replacing classical convolution $*$ by free convolution $\boxplus$ in (2.1).

For a (Borel-) probability measure μ on $\mathbb{R}$ with support $\mathsf{supp}(\mu)$, the Cauchy (or Stieltjes) transform is the mapping $G_\mu \colon \mathbb{C} \setminus \mathsf{supp}(\mu) \to \mathbb{C}$ defined by:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\, \mu(\mathrm{d}t), \qquad (z \in \mathbb{C} \setminus \mathsf{supp}(\mu)). \tag{2.5}$$

The *free cumulant transform* $\mathcal{C}_\mu$ of μ is then given by

$$\mathcal{C}_\mu(z) = zG_\mu^{\langle -1\rangle}(z) - 1 \tag{2.6}$$

for all z in a certain region R of $\mathbb{C}^-$ (the lower half complex plane), where the (right) inverse $G_\mu^{\langle -1\rangle}$ of G_μ is well-defined. Specifically R may be chosen in the form:

$$R = \{z \in \mathbb{C}^- \mid \tfrac{1}{z} \in \Delta_{\eta,M}\}, \quad \text{where} \quad \Delta_{\eta,M} = \{z \in \mathbb{C}^+ \mid |\mathsf{Re}(z)| < \eta \mathsf{Im}(z),\ \mathsf{Im}(z) > M\}$$

for suitable positive numbers η and M, where $\mathbb{C}^+$ denotes the upper half complex plane. It was proved in [5] (see also [14] and [18]) that $\mathcal{C}_\mu$ constitutes the free analog of $\log \hat{\mu}$ in the sense that it linearizes free convolution:

$$\mathcal{C}_{\mu \boxplus \nu}(z) = \mathcal{C}_\mu(z) + \mathcal{C}_\nu(z)$$

for all probability measures μ and ν on $\mathbb{R}$ and all z in a region where all three transforms are defined. The results in [5] are presented in terms of a variant, φ_μ, of $\mathcal{C}_\mu$, which is often referred to as the Voiculescu transform, and which is again a variant of the R-transform $\mathcal{R}_\mu$ introduced in [18]. The relationship is the following:

$$\varphi_\mu(z) = \mathcal{R}_\mu(\tfrac{1}{z}) = z\mathcal{C}_\mu(\tfrac{1}{z}) \tag{2.7}$$

for all z in a region $\Delta_{\eta,M}$ as above. In [5] it was proved additionally that $\mu \in \mathcal{ID}(\boxplus)$, if and only if φ_μ extends analytically to a map from $\mathbb{C}^+$ into $\mathbb{C}^- \cup \mathbb{R}$, in which case there exists a real constant γ and a finite measure σ on $\mathbb{R}$, such that φ_μ has the *free Lévy-Khintchine representation*:

$$\varphi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\, \sigma(\mathrm{d}t), \qquad (z \in \mathbb{C}^+). \tag{2.8}$$

The pair (γ, σ) is uniquely determined and is called the *free generating pair* for μ. In terms of the free cumulant transform $\mathcal{C}_\mu$ the free Lévy-Khintchine representation may be written as

$$\mathcal{C}_\mu(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-tz} - 1 - tz 1_{[-1,1]}(t) \Big)\, \rho(\mathrm{d}t), \tag{2.9}$$

where the relationship between the *free characteristic triplet* (a, ρ, η) and the free generating pair (γ, σ) is again given by (2.4).

In [4] Bercovici and Pata introduced a bijection Λ between the two classes $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$, which may formally be defined as the mapping sending a measure μ from $\mathcal{ID}(*)$ with generating pair (γ, σ) onto the measure $\Lambda(\mu)$ in $\mathcal{ID}(\boxplus)$ with *free* generating pair (γ, σ). It is then obvious that Λ is a bijection, and it turns out that Λ further enjoys the following properties (see [4] and [1]):

(a) If $\mu_1, \mu_2 \in \mathcal{ID}(*)$, then $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$.

(b) If $\mu \in \mathcal{ID}(*)$ and $c \in \mathbb{R}$, then $\Lambda(D_c\mu) = D_c\Lambda(\mu)$.

(c) For any constant c in $\mathbb{R}$ we have that $\Lambda(\delta_c) = \delta_c$, where δ_c denotes the Dirac measure at c.

(d) Λ is a homeomorphism with respect to weak convergence.

The property (d) is equivalent to the free version of Gnedenko's Theorem: Suppose $\mu, \mu_1, \mu_2, \mu_3, \ldots$ is a sequence of measures from $\mathcal{ID}(\boxplus)$ with *free* generating pairs: $(\gamma, \sigma), (\gamma_1, \sigma_1), (\gamma_2, \sigma_2), (\gamma$ respectively. Then

$$\mu_n \xrightarrow{\text{w}} \mu \iff \gamma_n \longrightarrow \gamma \text{ and } \sigma_n \xrightarrow{\text{w}} \sigma. \tag{2.10}$$

(cf. Theorem 3.8 in [1])

2.2 Selfdecomposability and Unimodality

The selfdecomposablity defined in (1.1) has an equivalent characterization: a probability measure μ is in $\mathcal{L}(*)$ if and only if μ is in $\mathcal{ID}(*)$ and the Lévy measure (cf. (2.2)) has the form

$$\rho(\mathrm{d}t) = \frac{k(t)}{|t|}\,\mathrm{d}t, \tag{2.11}$$

where $k\colon \mathbb{R} \setminus \{0\} \to [0, \infty)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ (see [15]).

In analogy with the class $\mathcal{L}(*)$, a probability measure μ on $\mathbb{R}$ is called $\boxplus$-*selfdecomposable*, if there exists, for any c in $(0, 1)$, a probability measure μ_c on $\mathbb{R}$, such that

$$\mu = D_c\mu \boxplus \mu_c. \tag{2.12}$$

Denoting by $\mathcal{L}(\boxplus)$ the class of such measures, it follows from the properties of Λ that

$$\Lambda(\mathcal{L}(*)) = \mathcal{L}(\boxplus) \tag{2.13}$$

(see [1]). By the definition of Λ and (2.13), if we let the term "Lévy measure" refer to the free Lévy-Khintchine representation (2.9) rather than the classical one (2.2), then we have exactly the same characterization of the measures in $\mathcal{L}(\boxplus)$: a probability measure μ is in $\mathcal{L}(\boxplus)$ if and only if its Lévy measure in (2.9) is of the form (2.11).

The definition of a unimodal probability measure μ given in Section 1 is equivalent to the existence of a real number a, such that the distribution function $t \mapsto \mu((-\infty, t])$ is convex (i.e. $\mu((-\infty, ps + qt]) \leq p\mu((-\infty, s]) + q\mu((-\infty, t])$ for all s, t and all $p, q \geq 0, p + q = 1$) on $(-\infty, a)$ and concave on (a, ∞). From this characterization it follows that for any sequence $\mu, \mu_1, \mu_2, \mu_3, \ldots$ of probability measures on $\mathbb{R}$ we have the implication:

$$\mu_n \text{ is unimodal for all } n \text{ and } \mu_n \xrightarrow{\text{w}} \mu \implies \mu \text{ is unimodal} \tag{2.14}$$

(see e.g. [9, §32, Theorem 4]).

2.3 Lindelöf's Theorem

In this subsection we present a variant (Lemma 2.1 below) of Lindelöf's Theorem (see [13] or [7, Theorem 2.2]), which plays a crucial role in Section 3 in combination with Stieltjes inversion. Before stating the lemma we introduce some notation: For any number δ in $(0, \pi)$ we put

$$\triangledown_\delta = \left\{ r\mathrm{e}^{\mathrm{i}\theta} \;\middle|\; \delta < \theta < \pi - \delta,\ r > 0 \right\}.$$

2.1 Lemma. *Let $G\colon \mathbb{C}^+ \to \mathbb{C}^-$ be an analytic function, and assume that there exists a curve $(z_t)_{t\in[0,1)}$ in $\mathbb{C}^+$, such that $\lim_{t\to1} z_t = 0$, and such that $\alpha := \lim_{t\to 1} G(z_t)$ exists in $\mathbb{C}$. Then for any number δ in $(0,\pi)$ we also have that $\lim_{z\to 0, z\in\nabla_\delta} G(z) = \alpha$, i.e., G has non-tangential limit α at 0.*

Lemma 2.1 may e.g. be derived from Theorem 2.2 in [7], which provides a similar result for (in particular) *bounded* analytic functions $f\colon \{x+\mathrm{i}y \mid x>0,\ y\in\mathbb{R}\} \to \mathbb{C}$. Recalling that the mapping $\zeta \mapsto \frac{\zeta-1}{\zeta+1}$ is a conformal bijection of $\{x+\mathrm{i}y \mid x>0,\ y\in\mathbb{R}\}$ onto the open unit disc in $\mathbb{C}$, Lemma 2.1 then follows by applying [7, Theorem 2.2] to the bounded function

$$f(z) = \frac{\mathrm{i}G(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}z)-1}{\mathrm{i}G(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}z)+1}, \qquad (z\in\{x+\mathrm{i}y \mid x>0,\ y\in\mathbb{R}\}).$$

3 The case of Lévy measures with positive density on $\mathbb{R}$

In this section we prove unimodality for measures in $\mathcal{L}(\boxplus)$ with Lévy measures in the form $\frac{k(t)}{|t|}$, where k satisfies the conditions (a)-(c) listed below. In a previous version of the manuscript we considered the case where k is compactly supported, but in that setting some proofs become more delicate and complicated than the ones to follow.

Throughout the remaining part of this section we consider a function $k\colon \mathbb{R}\setminus\{0\} \to [0,\infty)$ such that

(a) k is C^2 and $(1+t^2)^m k^{(n)}(t)$ are bounded for $m,n\in\{0,1,2\}$,

(b) k is increasing on $(-\infty,0)$, decreasing on $(0,\infty)$,

(c) k is strictly positive on $\mathbb{R}\setminus\{0\}$.

Next we define

$$\tilde{k}(t) = \mathrm{sign}(t)k(t), \qquad (t\in\mathbb{R}),$$

$$\tilde{G}_k(z) = \int_{\mathbb{R}} \frac{\tilde{k}(t)}{z-t}\,\mathrm{d}t, \qquad (z\in\mathbb{C}^+),$$

$$H_k(z) = z + z\tilde{G}_k(z), \qquad (z\in\mathbb{C}^+).$$

We note for later use that

$$H_k(z) = z + z\int_{\mathbb{R}} \frac{\tilde{k}(t)}{z-t}\,\mathrm{d}t = z + \int_{\mathbb{R}} \Big(1+\frac{t}{z-t}\Big)\tilde{k}(t)\,\mathrm{d}t = z+\gamma_k + \int_{\mathbb{R}} \frac{|t|k(t)}{z-t}\,\mathrm{d}t, \qquad (3.1)$$

where we have introduced $\gamma_k = \int_{\mathbb{R}} \tilde{k}(t)\,\mathrm{d}t$.

In the following we shall consider additionally the auxiliary function $F_k\colon \mathbb{C}^+ \to (0,\infty)$ given by

$$F_k(x+\mathrm{i}y) = \int_{\mathbb{R}} \frac{|t|k(t)}{(x-t)^2+y^2}\,\mathrm{d}t, \qquad (x+\mathrm{i}y\in\mathbb{C}^+), \qquad (3.2)$$

which satisfies $F_k(z)\mathsf{Im}(z) = \mathsf{Im}(z-H_k(z))$.

3.1 Lemma. (i) *For all x in $\mathbb{R}$ there exists a unique number $y=v_k(x)$ in $(0,\infty)$ such that*

$$F_k(x+\mathrm{i}v_k(x)) = \int_{\mathbb{R}} \frac{|t|k(t)}{(x-t)^2+v_k(x)^2}\,\mathrm{d}t = 1. \qquad (3.3)$$

(ii) *We have that*

$$\mathcal{G} := \{z \in \mathbb{C}^+ \mid H_k(z) \in \mathbb{R}\} = \{x + \mathrm{i}v_k(x) \mid x \in \mathbb{R}\}.$$

(iii) *We have that*

$$\mathcal{G}^+ := \{z \in \mathbb{C}^+ \mid H_k(z) \in \mathbb{C}^+\} = \{x + \mathrm{i}y \mid x \in \mathbb{R},\ y > v_k(x)\}.$$

(iv) *The function* $v_k \colon \mathbb{R} \to (0, \infty)$ *is analytic on* $\mathbb{R}$.

(v) *We have that*

$$\lim_{|x|\to\infty} v_k(x) = 0.$$

Proof. (i) For any x in $\mathbb{R}$ the function

$$y \mapsto \int_{\mathbb{R}} \frac{|t|k(t)}{(x-t)^2 + y^2}\,\mathrm{d}t, \qquad (y \in (0, \infty))$$

takes values in $(0, \infty)$ and is continuous (by dominated convergence) and strictly decreasing in y. Since k is strictly positive and continuous we find additionally that

$$\lim_{y\searrow 0} \int_{\mathbb{R}} \frac{|t|k(t)}{(x-t)^2 + y^2}\,\mathrm{d}t = \infty, \quad \text{and} \quad \lim_{y\nearrow\infty} \int_{\mathbb{R}} \frac{|t|k(t)}{(x-t)^2 + y^2}\,\mathrm{d}t = 0$$

by monotone and dominated convergence. Hence there is a unique $y = v_k(x)$ in $(0, \infty)$ such that $\int_{\mathbb{R}} \frac{|t|k(t)}{(x-t)^2+y^2}\,\mathrm{d}t = 1$.

(ii) For any x, y in $\mathbb{R}$, such that $y > 0$, we note that

$$\begin{aligned}\mathsf{Im}\big(H_k(x + \mathrm{i}y)\big) &= y + \mathsf{Im}\Big(\int_{\mathbb{R}} \frac{x + \mathrm{i}y}{x + \mathrm{i}y - t}\tilde{k}(t)\,\mathrm{d}t\Big) = y + \int_{\mathbb{R}} \frac{y(x-t) - yx}{(x-t)^2 + y^2}\tilde{k}(t)\,\mathrm{d}t \\ &= y\Big(1 - \int_{\mathbb{R}} \frac{t\tilde{k}(t)}{(x-t)^2 + y^2}\,\mathrm{d}t\Big) = y\Big(1 - \int_{\mathbb{R}} \frac{|t|k(t)}{(x-t)^2 + y^2}\,\mathrm{d}t\Big).\end{aligned} \tag{3.4}$$

Hence it follows that

$$\mathsf{Im}\big(H_k(x + \mathrm{i}y)\big) = 0 \iff \int_{\mathbb{R}} \frac{|t|k(t)}{(x-t)^2 + y^2}\,\mathrm{d}t = 1. \tag{3.5}$$

The right hand side of (3.5) holds, if and only if $y = v_k(x)$.

(iii) It is apparent that $\int_{\mathbb{R}} \frac{|t|k(t)}{(x-t)^2+y^2}\,\mathrm{d}t < 1$ for any x in $\mathbb{R}$ and all y in $(v_k(x), \infty)$. In combination with (3.4) this shows that $\mathcal{G}^+ = \{x + \mathrm{i}y \mid x \in \mathbb{R},\ y > v_k(x)\}$ as desired.

(iv) Consider the function $\tilde{F}_k \colon \mathbb{R} \times (0, \infty) \to \mathbb{R}$ given by

$$\tilde{F}_k(x, y) = F_k(x + \mathrm{i}y) = \int_{\mathbb{R}} \frac{|t|k(t)}{(x-t)^2 + y^2} = 1 - y^{-1}\mathsf{Im}\big(H_k(x + \mathrm{i}y)\big), \quad ((x, y) \in \mathbb{R} \times (0, \infty)).$$

Since H_k is analytic on $\mathbb{C}^+$ it follows that $\tilde{F}_k$ is analytic on $\mathbb{R} \times (0, \infty)$. By differentiation under the integral sign we note in particular that

$$\frac{\partial}{\partial y}\tilde{F}_k(x, y) = -2y \int_{\mathbb{R}} \frac{|t|k(t)}{((x-t)^2 + y^2)^2}\,\mathrm{d}t < 0$$

for all (x, y) in $\mathbb{R} \times (0, \infty)$. Since $v_k(x) > 0$ and $\tilde{F}_k(x + \mathrm{i}v_k(x)) = 1$ for all x in $\mathbb{R}$ it follows then from the Implicit Function Theorem (for analytic functions; see [8, Theorem 7.6]) that v_k is analytic on $\mathbb{R}$.

(v) By dominated convergence $\lim_{|x|\to\infty} F_k(x + \mathrm{i}y) = 0$ for any fixed y in $(0, \infty)$. Hence (v) follows from (3.3) and the fact that $y \mapsto F_k(x + \mathrm{i}y)$ is decreasing (for fixed x). ■

3.2 Lemma. *Let ν_k be the measure in $\mathcal{ID}(\boxplus)$ with free characteristic triplet $(0, \frac{k(t)}{|t|}\,\mathrm{d}t, \int_{-1}^{1}\tilde{k}(t)\,\mathrm{d}t)$. Then the Cauchy transform G_{ν_k} of ν_k satisfies the identity:*

$$G_{\nu_k}(H_k(z)) = \frac{1}{z}$$

for all z in $\mathbb{C}^+$ such that $H_k(z) \in \mathbb{C}^+$.

Proof. Let $\mathcal{C}_{\nu_k}$ denote the free cumulant transform of ν_k (extended to all of $\mathbb{C}^-$). For any w in $\mathbb{C}^-$ we then find (cf. formula (2.9)) that

$$\begin{aligned}
\mathcal{C}_{\nu_k}(w) &= w\int_{-1}^{1}\tilde{k}(t)\,\mathrm{d}t + \int_{\mathbb{R}}\Big(\frac{1}{1-wt} - 1 - wt1_{[-1,1]}(t)\Big)\frac{k(t)}{|t|}\,\mathrm{d}t \\
&= \int_{\mathbb{R}}\Big(\frac{1}{1-wt} - 1\Big)\frac{k(t)}{|t|}\,\mathrm{d}t \\
&= w\int_{\mathbb{R}}\frac{t}{1-wt}\frac{k(t)}{|t|}\,\mathrm{d}t = w\int_{\mathbb{R}}\frac{\tilde{k}(t)}{1-wt}\,\mathrm{d}t.
\end{aligned}$$

Setting $w = \frac{1}{z}$ it follows for any z in $\mathbb{C}^+$ that

$$\mathcal{C}_{\nu_k}\big(\tfrac{1}{z}\big) = \frac{1}{z}\int_{\mathbb{R}}\frac{\tilde{k}(t)}{1-\frac{t}{z}}\,\mathrm{d}t = \int_{\mathbb{R}}\frac{\tilde{k}(t)}{z-t}\,\mathrm{d}t = \tilde{G}_k(z).$$

By definition of the free cumulant transform it therefore follows that

$$\frac{1}{z}G_{\nu_k}^{\langle -1\rangle}\big(\tfrac{1}{z}\big) - 1 = \mathcal{C}_{\nu_k}\big(\tfrac{1}{z}\big) = \tilde{G}_k(z),$$

and hence that

$$G_{\nu_k}^{\langle -1\rangle}\big(\tfrac{1}{z}\big) = z\tilde{G}_k(z) + z = H_k(z)$$

for all z in a suitable region $\Delta_{\eta,M}$, where $\eta, M > 0$. We may thus conclude that

$$\frac{1}{z} = G_{\nu_k}(H_k(z)) \tag{3.6}$$

for all z in $\Delta_{\eta,M}$, but since $\{z \in \mathbb{C}^+ \mid H_k(z) \in \mathbb{C}^+\}$ is a connected region of $\mathbb{C}^+$ (cf. Lemma 3.1(iii)), the identity (3.6) extends to all z in this region by analytic continuation. ■

In the following we consider the function $P_k\colon \mathbb{R} \to \mathbb{R}$ defined by

$$P_k(x) = H_k(x + \mathrm{i}v_k(x)), \qquad (x \in \mathbb{R}). \tag{3.7}$$

3.3 Proposition. *For any x in $\mathbb{R}$ we have that*

$$G_{\nu_k}(z) \longrightarrow \frac{1}{x + \mathrm{i}v_k(x)} \qquad \text{as } z \to P_k(x) \text{ non-tangentially from } \mathbb{C}^+.$$

Proof. For any s in $[0,1]$ we put $w_s = x + \mathrm{i}(v_k(x) + s)$, so that $w_s \in \mathcal{G}^+$ for all s in $(0,1]$ according to Lemma 3.1(iii). Moreover, since H_k is analytic on $\mathbb{C}^+$, and $w_s \in \mathbb{C}^+$ for all s in $[0,1]$, it follows that

$$H_k(w_s) \longrightarrow H_k(w_0) = H_k(x + \mathrm{i}v_k(x)) = P_k(x) \in \mathbb{R} \qquad \text{as } s \searrow 0.$$

In addition it follows from Lemma 3.2 that

$$G_{\nu_k}(H_k(w_s)) = \frac{1}{w_s} = \frac{1}{x + \mathrm{i}(v_k(x)+s)} \longrightarrow \frac{1}{x+\mathrm{i}v_k(x)} \qquad \text{as } s \searrow 0.$$

Thus $G_{\nu_k}(z)$ has the limit $\frac{1}{x+\mathrm{i}v_k(x)}$ as $z \to P_k(x)$ along the curve $s \mapsto H_k(w_s)$. It follows then from Lemma 2.1 that in fact $G_{\nu_k}(z) \to \frac{1}{x+\mathrm{i}v_k(x)}$ as $z \to P_k(x)$ non-tangentially from $\mathbb{C}^+$, as desired. ■

3.4 Lemma. *The function P_k is a strictly increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$.*

Proof. We show first that $P_k(x) \to \pm\infty$ as $x \to \pm\infty$. From Lemma 3.1(v), formula (3.7) and formula (3.1) this will follow, if we show that

$$\sup_{y\in(0,1/2)} \Big| \int_{\mathbb{R}} \frac{|t|k(t)}{x+\mathrm{i}y-t}\,\mathrm{d}t \Big| \longrightarrow 0 \text{ as } |x| \to \infty. \tag{3.8}$$

Consider in the following x in $\mathbb{R} \setminus [-2, 2]$ and y, δ in $(0, \frac{1}{2})$. We then divide the integral as follows:

$$\int_{\mathbb{R}} \frac{|t|k(t)}{x+\mathrm{i}y-t}\,\mathrm{d}t = \int_{x-\delta}^{x+\delta} \frac{|t|k(t)}{x+\mathrm{i}y-t}\,\mathrm{d}t + \int_{\mathbb{R}\setminus[x-\delta,x+\delta]} \frac{|t|k(t)}{x+\mathrm{i}y-t}\,\mathrm{d}t. \tag{3.9}$$

To estimate the first term on the right hand side of (3.9), we perform integration by parts:

$$\int_{x-\delta}^{x+\delta} \frac{|t|k(t)}{x+\mathrm{i}y-t}\,\mathrm{d}t = \Big[-\log(x-t+\mathrm{i}y)|t|k(t)\Big]_{x-\delta}^{x+\delta} + \int_{x-\delta}^{x+\delta} \log(x-t+\mathrm{i}y)\frac{\mathrm{d}}{\mathrm{d}t}\big(|t|k(t)\big)\,\mathrm{d}t,$$

where log is the principal branch, i.e.,

$$\log(x-t+\mathrm{i}y) = \tfrac{1}{2}\log((x-t)^2+y^2) + \mathrm{i}\mathrm{Arg}(x-t+\mathrm{i}y),$$

where Arg is the principal argument. Given any positive number ϵ, we choose next δ in $(0, 1/2)$ such that

$$\int_{-\delta}^{\delta} \sqrt{\pi^2 + (\log|t|)^2}\,\mathrm{d}t \le \epsilon\Big(\sup_{|t|\ge 1}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\big(|t|k(t)\big)\Big|\Big)^{-1}.$$

Since $t \mapsto |\log(t)|$ is decreasing on $(0, 1)$, it follows then that

$$\begin{aligned}\Big|\int_{x-\delta}^{x+\delta} \log(x-t+\mathrm{i}y)\frac{\mathrm{d}}{\mathrm{d}t}\big(|t|k(t)\big)\,\mathrm{d}t\Big| &\le \int_{x-\delta}^{x+\delta} \sqrt{\pi^2+(\log|x-t|)^2}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\big(|t|k(t)\big)\Big|\,\mathrm{d}t \\ &\le \sup_{|t|\ge 1}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\big(|t|k(t)\big)\Big| \int_{-\delta}^{\delta} \sqrt{\pi^2+(\log|t|)^2}\,\mathrm{d}t \le \epsilon.\end{aligned}$$

Since $|t|k(t) \to 0$ as $|t| \to \infty$ (cf. condition (a) above), we note further that

$$\Big|\Big[-\log(x-t+\mathrm{i}y)|t|k(t)\Big]_{x-\delta}^{x+\delta}\Big| \le 2\cdot\sqrt{\pi^2+(\log\delta)^2}\cdot\max\big\{|x-\delta|k(x-\delta), |x+\delta|k(x+\delta)\big\} \le \epsilon,$$

for any y in $(0, 1/2)$ and all x with $|x|$ sufficiently large. Thus the first term of (3.9) is bounded by 2ϵ whenever $|x|$ is large enough, uniformly in $y \in (0, 1/2)$.

Regarding the second term on the right hand side of (3.9) we note first that $\lim_{|x|\to\infty}\int_{\mathbb{R}\setminus[x-\delta,x+\delta]}\frac{|t|k(t)}{|x-t|}\,\mathrm{d}t$
0 by dominated convergence. Therefore

$$\sup_{y\in\mathbb{R}}\Big|\int_{\mathbb{R}\setminus[x-\delta,x+\delta]}\frac{|t|k(t)}{x+\mathrm{i}y-t}\,\mathrm{d}t\Big|\le\int_{\mathbb{R}\setminus[x-\delta,x+\delta]}\frac{|t|k(t)}{|x-t|}\,\mathrm{d}t\le\epsilon,$$

whenever $|x|$ is sufficiently large. Thus we have established (3.8).

It remains now to show that P_k is injective and continuous on $\mathbb{R}$, since these properties are then automatically transferred to the inverse $P_k^{\langle -1\rangle}$. The continuity is obvious from the continuity of v_k (cf. formula 3.7). To see that P_k is injective on $\mathbb{R}$, assume that $x,x'\in\mathbb{R}$ such that $P_k(x)=P_k(x')$. Then Proposition 3.3 shows that

$$\frac{1}{x+\mathrm{i}v_k(x)}=\lim_{z\overset{\sphericalangle}{\to}P_k(x)}G_{\nu_k}(z)=\lim_{z\overset{\sphericalangle}{\to}P_k(x')}G_{\nu_k}(z)=\frac{1}{x'+\mathrm{i}v_k(x')},$$

where "$\overset{\sphericalangle}{\to}$" denotes non-tangential limits. Clearly the above identities imply that $x=x'$. ∎

3.5 Corollary. *The measure ν_k is absolutely continuous with respect to Lebesgue measure with a continuous density f_{ν_k} given by*

$$f_{\nu_k}(P_k(x))=\frac{v_k(x)}{\pi(x^2+v_k(x)^2)},\qquad(x\in\mathbb{R}).$$

In particular, the support of ν_k is $\mathbb{R}$.

Proof. This follows by Stieltjes-Inversion and Proposition 3.3. Indeed, for any x in $\mathbb{R}$ we have that

$$\lim_{y\searrow 0}G_{\nu_k}(P_k(x)+\mathrm{i}y)=\frac{1}{x+\mathrm{i}v_k(x)}.$$

Recalling (see e.g. Chapter XIII in [16]) that the singular part of ν_k is concentrated on the set

$$\big\{\xi\in\mathbb{R}\;\big|\;\lim_{y\searrow 0}|G_{\nu_k}(\xi+\mathrm{i}y)|=\infty\big\},$$

it follows in particular that ν_k has no singular part. For any x in $\mathbb{R}$ we find furthermore by the Stieltjes Inversion Formula that

$$f_{\nu_k}(P_k(x))=\frac{-1}{\pi}\lim_{y\searrow 0}\mathsf{Im}(G_{\nu_k}(P_k(x)+\mathrm{i}y))=\frac{-1}{\pi}\mathsf{Im}\Big(\frac{1}{x+\mathrm{i}v_k(x)}\Big)=\frac{v_k(x)}{\pi(x^2+v_k(x)^2)}.$$

In particular we see that $f_{\nu_k}(\xi)>0$ for any ξ in $\mathbb{R}$. Denoting by $P_k^{\langle -1\rangle}$ the inverse of P_k, we note finally that

$$f_{\nu_k}(\xi)=\frac{v_k(P_k^{\langle -1\rangle}(\xi))}{\pi(P_k^{\langle -1\rangle}(\xi)^2+v_k(P_k^{\langle -1\rangle}(\xi))^2)}\qquad(\xi\in\mathbb{R}),$$

which via the continuity of $P_k^{\langle -1\rangle}$ and v_k shows that f_{ν_k} is continuous too. ∎

3.6 Remark. Corollary 3.5 is a special case of Huang's density formula for freely infinitely divisible distributions [12, Theorem 3.10], which does not impose any assumptions on the Lévy measure. Our approach is similar to that of Biane in [6]. For example his function ψ_t resembles our function P_k.

The next lemma is key to the main result on unimodality.

3.7 Lemma. *Consider the function F_k defined by* (3.2). *Then for any r in $(0,\infty)$ there exists a number θ_r in $(0,\pi)$ such that the function*

$$\theta \mapsto F_k(r\sin(\theta)\mathrm{e}^{\mathrm{i}\theta})$$

is strictly decreasing on $(0,\theta_r]$ and strictly increasing on $[\theta_r,\pi)$.

Proof. We introduce a new variable u by setting $t=(r\sin\theta)u$. Then

$$F_k(r\sin(\theta)\mathrm{e}^{\mathrm{i}\theta}) = \int_{\mathbb{R}} \frac{|u|k(ru\sin\theta)}{1-2u\cos\theta+u^2}\,\mathrm{d}u, \qquad (\theta\in(0,\pi)).$$

Now consider any decreasing function $h\colon(0,\infty)\to(0,\infty)$ from $C^2((0,\infty))$ satisfying that the functions $(1+t^2)^m h^{(n)}(t)$ are bounded for any m,n in $\{0,1,2\}$. These assumptions ensure in particular that we may define $\psi_h\colon(-1,1)\to\mathbb{R}$ by

$$\psi_h(x) := \int_0^\infty \frac{u}{1-2xu+u^2} h(u\sqrt{1-x^2})\,\mathrm{d}u, \qquad (x\in(-1,1)).$$

Note then that if we define $k_r^{\pm}(u):=k(\pm ru)$ for u in $(0,\infty)$, and

$$\Psi_r(x) = \psi_{k_r^+}(x) + \psi_{k_r^-}(-x), \qquad (x\in(-1,1)), \tag{3.10}$$

then it holds that

$$F_k(r\sin(\theta)\mathrm{e}^{\mathrm{i}\theta}) = \Psi_r(\cos\theta), \qquad (\theta\in(0,\pi)). \tag{3.11}$$

We show in the following that

(1) $\psi_h'(x)>0$ for x in $(0,1)$,

(2) $\psi_h'(x)<0$ for x in $(-1,-\frac{\sqrt{2}}{2}]$,

(3) $\psi_h''(x)>0$ for x in $[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]$.

Before establishing these conditions we remark that the assumptions on h ensure, that we may perform differentiation under the integral sign and integration by parts as needed in the following, and we shall do so without further notice.

For any x in $(-1,1)$ we note first by differentiation under the integral sign that

$$\begin{aligned}\psi_h'(x) &= \int_0^\infty \frac{2u^2}{(1-2ux+u^2)^2} h(u\sqrt{1-x^2})\,\mathrm{d}u \\ &\quad - \int_0^\infty \frac{u^2}{1-2ux+u^2}\cdot\frac{x}{\sqrt{1-x^2}} h'(u\sqrt{1-x^2})\,\mathrm{d}u,\end{aligned} \tag{3.12}$$

which shows that (1) holds. Moreover, integration by parts yields that

$$\begin{aligned}\psi_h'(x) &= \int_0^\infty \frac{2u^2}{(1-2ux+u^2)^2} h(u\sqrt{1-x^2})\,\mathrm{d}u \\ &\quad + \int_0^\infty \frac{\partial}{\partial u}\Big(\frac{u^2}{1-2ux+u^2}\Big)\cdot\frac{x}{1-x^2} h(u\sqrt{1-x^2})\,\mathrm{d}u \\ &= \int_0^\infty \frac{2u((1-2x^2)u+x)}{(1-2xu+u^2)^2(1-x^2)} h(u\sqrt{1-x^2})\,\mathrm{d}u,\end{aligned} \tag{3.13}$$

which verifies (2).

Finally, we proceed to compute $\psi_h''(x)$. Using Leibniz' formula we find that

$$\begin{aligned}
\psi_h''(x) = &\int_0^\infty \frac{8u^3}{(1-2ux+u^2)^3}\, h(u\sqrt{1-x^2})\,\mathrm{d}u \\
&- \int_0^\infty \frac{4u^3}{(1-2ux+u^2)^2}\cdot\frac{x}{\sqrt{1-x^2}}\, h'(u\sqrt{1-x^2})\,\mathrm{d}u \\
&- \int_0^\infty \frac{u^2}{1-2ux+u^2}\left(\frac{1}{\sqrt{1-x^2}}+\frac{x^2}{(1-x^2)^{3/2}}\right) h'(u\sqrt{1-x^2})\,\mathrm{d}u \\
&+ \int_0^\infty \frac{u^3}{1-2ux+u^2}\cdot\frac{x^2}{1-x^2}\, h''(u\sqrt{1-x^2})\,\mathrm{d}u \\
= &\int_0^\infty \frac{8u^3}{(1-2ux+u^2)^3}\, h(u\sqrt{1-x^2})\,\mathrm{d}u \\
&- \int_0^\infty \frac{u^2(1+2ux+u^2)}{(1-2ux+u^2)^2\sqrt{1-x^2}}\, h'(u\sqrt{1-x^2})\,\mathrm{d}u \\
&- \int_0^\infty \frac{u^2}{1-2ux+u^2}\cdot\frac{x^2}{(1-x^2)^{3/2}}\, h'(u\sqrt{1-x^2})\,\mathrm{d}u \\
&+ \int_0^\infty \frac{u^3}{1-2ux+u^2}\cdot\frac{x^2}{1-x^2}\, h''(u\sqrt{1-x^2})\,\mathrm{d}u.
\end{aligned} \tag{3.14}$$

In the resulting expression above the first three integrals are positive for any x in $(-1,1)$, since $-h', h \geq 0$ and $u^2+2ux+1 = (u+x)^2+1-x^2 \geq 0$. By integration by parts, the last integral can be re-written as follows:

$$\begin{aligned}
&\int_0^\infty \frac{u^3}{1-2ux+u^2}\cdot\frac{x^2}{1-x^2}\, h''(u\sqrt{1-x^2})\,\mathrm{d}u \\
&\quad= -\frac{x^2}{(1-x^2)^{3/2}}\int_0^\infty \frac{\partial}{\partial u}\left(\frac{u^3}{1-2xu+u^2}\right)\cdot h'(u\sqrt{1-x^2})\,\mathrm{d}u \\
&\quad= -\frac{x^2}{(1-x^2)^{3/2}}\int_0^\infty \frac{u^2}{(1-2xu+u^2)^2}\big((u-2x)^2+3-4x^2\big)\, h'(u\sqrt{1-x^2})\,\mathrm{d}u.
\end{aligned} \tag{3.15}$$

Hence this integral is positive as well for any x in $[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$, and altogether the property (3) is established.

Recalling now formula (3.10), note that it follows from conditions (1)-(3) that $\Psi_r'(x) = \psi_{k_r^+}'(x) - \psi_{k_r^-}'(-x) > 0$, if $x \geq \frac{\sqrt{2}}{2}$, $\Psi_r'(x) < 0$, if $x \leq -\frac{\sqrt{2}}{2}$, and $\Psi_r''(x) = \psi_{k_r^+}''(x) + \psi_{k_r^-}''(-x) > 0$, if $|x| \leq \frac{\sqrt{3}}{2}$. Hence, Ψ_r' is strictly increasing on $(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$ and there exists a unique zero of Ψ_r' at some x_r in $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Therefore Ψ_r is strictly decreasing on $(-1, x_r]$ and strictly increasing on $[x_r, 1)$, and the lemma now follows readily from formula (3.11). ■

3.8 Proposition. *Consider a function $k\colon \mathbb{R}\setminus\{0\} \to [0,\infty)$ which satisfies conditions (a)-(c) listed in the beginning of this section. Then the associated measure ν_k (described in Lemma 3.2) is unimodal. In fact there exists a number ω in $\mathbb{R}$, such that the density f_{ν_k} (cf. Corollary 3.5) is strictly increasing on $(-\infty,\omega]$ and strictly decreasing on $[\omega,\infty)$.*

Proof. We show first for any number ρ in $(0,\infty)$ that the equality $f_{\nu_k}(\xi) = \rho$ has at most two solutions in ξ. Since P_k is a bijection of $\mathbb{R}$ onto itself, this is equivalent to showing that the equality

$$\rho = f_{\nu_k}(P_k(x)) = \frac{v_k(x)}{\pi(x^2+v_k(x)^2)}$$

has at most two solutions in x. For this we note first that

$$\big\{x+\mathrm{i}y \in \mathbb{C}^+ \;\big|\; \tfrac{y}{\pi(x^2+y^2)} = \rho\big\} = C_\rho \setminus \{0\},$$

where C_ρ is the circle in $\mathbb{C}$ with center $\frac{\mathrm{i}}{2\pi\rho}$ and radius $\frac{1}{2\pi\rho}$. Writing $x+\mathrm{i}y$ as $r\mathrm{e}^{\mathrm{i}\theta}$ ($r>0$, $\theta \in (-\pi,\pi]$) we find that C_ρ is given by

$$C_\rho = \big\{\tfrac{1}{\pi\rho}\sin(\theta)\mathrm{e}^{\mathrm{i}\theta} \;\big|\; \theta \in (0,\pi]\big\}$$

in polar coordinates. We need to show that C_ρ intersects the graph $\mathcal{G}$ of v_k in at most two points. By the defining property (3.3) of v_k, this is equivalent to showing that the equality

$$F_k\big(\tfrac{1}{\pi\rho}\sin(\theta)\mathrm{e}^{\mathrm{i}\theta}\big) = 1$$

has at most two solutions for θ in $(0,\pi)$. But this follows immediately from Lemma 3.7.

It is now elementary to check that ν_k is unimodal. Since f_{ν_k} is continuous and strictly positive on $\mathbb{R}$, and since $f_{\nu_k}(x) \to 0$ as $x \to \pm\infty$ (cf. Corollary 3.5), f_{ν_k} attains a strictly positive global maximum at some point ω in $\mathbb{R}$. If f_{ν_k} was not increasing on $(-\infty,\omega]$, then we could choose ξ_1,ξ_2 in $(-\infty,\omega)$ such that $\xi_1<\xi_2$, and $f_{\nu_k}(\xi_1) > f_{\nu_k}(\xi_2) > 0$. Choosing any number ρ in $(f(\xi_2), f(\xi_1))$, it follows then from the continuity of f_{ν_k}, that each of the intervals $(-\infty,\xi_1)$, (ξ_1,ξ_2) and (ξ_2,ω) must contain a solution to the equation $f_{\nu_k}(\xi)=\rho$, which contradicts what we established above. Subsequently the argumentation given above also implies that f_{ν_k} is in fact *strictly* increasing on $(-\infty,\omega]$. Similarly it follows that f_{ν_k} must be strictly decreasing on $[\omega,\infty)$, and this completes the proof. ■

4 The general case

In this section we extend Proposition 3.8 to general measures ν from $\mathcal{L}(\boxplus)$. The key step is the following approximation result.

4.1 Lemma. *Let $k\colon \mathbb{R}\setminus\{0\} \to [0,\infty)$ be a function as in (2.11) such that $\frac{k(t)}{|t|}1_{\mathbb{R}\setminus\{0\}}(t)\,\mathrm{d}t$ is a Lévy measure. Let further a be a non-negative number.*

Then there exists a sequence (k_n) of functions $k_n\colon \mathbb{R}\setminus\{0\} \to [0,\infty)$, satisfying the conditions (a)-(c) in Section 3, such that

$$\frac{|t|k_n(t)}{1+t^2}\,\mathrm{d}t \xrightarrow{\mathrm{w}} a\delta_0 + \frac{|t|k(t)}{1+t^2}\,\mathrm{d}t$$

as $n \to \infty$.

Proof. For each n in $\mathbb{N}$ we introduce first the function $k_n^0\colon \mathbb{R} \to [0,\infty)$ defined by

$$k_n^0(t) = \begin{cases} 0, & \text{if } t \in (-\infty,0],\\ k(\frac{1}{n}), & \text{if } t \in (0,\frac{1}{n}),\\ k(t), & \text{if } t \in [\frac{1}{n},n],\\ 0, & \text{if } t \in (n,\infty),\end{cases}$$

and we note that $k_n^0 \le k_{n+1}^0$ for all n. Next we choose a non-negative function φ from $C_c^\infty(\mathbb{R})$, such that $\mathsf{supp}(\varphi) \subseteq [-1,0]$, and $\int_{-1}^0 \varphi(t)\,\mathrm{d}t = 1$. We then define the function $\tilde{R}_n\colon \mathbb{R} \to [0,\infty)$ as the convolution

$$\tilde{R}_n(t) = n\int_{-1/n}^0 k_n^0(t-s)\varphi(ns)\,\mathrm{d}s = \int_0^1 k_n^0(t+\tfrac{u}{n})\varphi(-u)\,\mathrm{d}u, \qquad (t \in \mathbb{R}), \tag{4.1}$$

and we let R_n be the restriction of $\tilde{R}_n$ to $(0,\infty)$. Note also that

$$\tilde{R}_n(t) = n\int_{\mathbb{R}} \varphi(n(t-s))k_n^0(s)\,\mathrm{d}s, \qquad (t\in\mathbb{R}).$$

Since k_n^0 as well as the derivatives of φ and φ itself are all bounded functions, it follows then by differentiation under the integral sign that $\tilde{R}_n$ is a *bounded* C^∞-function on $\mathbb{R}$ with *bounded* derivatives, and so its restriction R_n to $(0,\infty)$ has bounded derivatives too. Since k_n^0 is decreasing on $(0,\infty)$, it follows immediately from (4.1) that so is R_n. Moreover, $\mathsf{supp}(R_n)\subseteq(0,n]$ by the definition of k_n^0.

For any t in $(0,\infty)$ and n in $\mathbb{N}$ note next that

$$R_n(t) \le \int_0^1 k_{n+1}^0(t+\tfrac{u}{n})\varphi(-u)\,\mathrm{d}u \le \int_0^1 k_{n+1}^0(t+\tfrac{u}{n+1})\varphi(-u)\,\mathrm{d}u = R_{n+1}(t).$$

Moreover, the monotonicity assumptions imply that k is continuous at almost all t in $(0,\infty)$ (with respect to Lebesgue measure). For such a t we may further consider n so large that $t+\frac{u}{n}\in[\frac{1}{n},n]$ for all u in $[0,1]$. For such n it follows then that

$$R_n(t) = \int_0^1 k(t+\tfrac{u}{n})\varphi(-u)\,\mathrm{d}u \underset{n\to\infty}{\longrightarrow} \int_0^1 k(t)\varphi(-u)\,\mathrm{d}u = k(t)$$

by monotone convergence. We conclude that $R_n(t)\nearrow k(t)$ as $n\to\infty$ for almost all t in $(0,\infty)$.

Applying the considerations above to the function $\kappa\colon(0,\infty)\to[0,\infty)$ given by $\kappa(t)=k(-t)$, it follows that we may construct a sequence $(L_n)_{n\in\mathbb{N}}$ of non-negative functions defined on $(-\infty,0)$ and with the following properties:

- For all n in $\mathbb{N}$ the function L_n has bounded support.
- For all n in $\mathbb{N}$ we have that $L_n\in C^\infty((-\infty,0))$, and $L_n^{(p)}$ is bounded for all p in $\mathbb{N}\cup\{0\}$.
- For all n in $\mathbb{N}$ the function L_n is increasing on $(-\infty,0)$.
- $L_n(t)\nearrow k(t)$ as $n\to\infty$ for almost all t in $(-\infty,0)$ (with respect to Lebesgue measure).

Next let $\psi(t)=\mathrm{e}^{-t^2}$, and note that $\int_{\mathbb{R}}|t|\psi(t)\,\mathrm{d}t=1$. We are then ready to define $k_n\colon\mathbb{R}\setminus\{0\}\to[0,\infty)$ by

$$k_n(t) = \begin{cases} an^2\psi(nt)+R_n(t), & \text{if } t>0,\\ an^2\psi(nt)+L_n(t), & \text{if } t<0.\end{cases}$$

It is apparent from the argumentation above that k_n satisfies the conditions (a)-(c) in Section 3, and it remains to show that $\frac{|t|k_n(t)}{1+t^2}\,\mathrm{d}t \overset{\mathrm{w}}{\to} a\delta_0+\frac{|t|k(t)}{1+t^2}\,\mathrm{d}t$ as $n\to\infty$. For any bounded continuous function $g\colon\mathbb{R}\to\mathbb{R}$ we find that

$$\begin{aligned}
\int_{\mathbb{R}} g(t)\frac{|t|k_n(t)}{1+t^2}\,\mathrm{d}t &= an^2\int_{\mathbb{R}} g(t)\frac{|t|\psi(nt)}{1+t^2}\,\mathrm{d}t + \int_{-\infty}^0 g(t)\frac{|t|L_n(t)}{1+t^2}\,\mathrm{d}t + \int_0^\infty g(t)\frac{tR_n(t)}{1+t^2}\,\mathrm{d}t\\
&= a\int_{\mathbb{R}} g(\tfrac{u}{n})\frac{|u|\psi(u)}{1+(\frac{u}{n})^2}\,\mathrm{d}u + \int_{-\infty}^0 g(t)\frac{|t|L_n(t)}{1+t^2}\,\mathrm{d}t + \int_0^\infty g(t)\frac{tR_n(t)}{1+t^2}\,\mathrm{d}t\\
&\underset{n\to\infty}{\longrightarrow} a\int_{\mathbb{R}} g(0)|u|\psi(u)\,\mathrm{d}u + \int_{-\infty}^0 g(t)\frac{|t|k(t)}{1+t^2}\,\mathrm{d}t + \int_0^\infty g(t)\frac{tk(t)}{1+t^2}\,\mathrm{d}t\\
&= ag(0)+\int_{\mathbb{R}} g(t)\frac{|t|k(t)}{1+t^2}\,\mathrm{d}t,
\end{aligned}$$

where, when letting $n \to \infty$, we used dominated convergence on each of the three integrals; note in particular that $\frac{|t|L_n(t)}{1+t^2}$ and $\frac{tR_n(t)}{1+t^2}$ are dominated almost everywhere by $\frac{|t|k(t)}{1+t^2}$ on the relevant intervals, and here $\int_{\mathbb{R}} \frac{|t|k(t)}{1+t^2}\,\mathrm{d}t < \infty$, since $\frac{k(t)}{|t|}\,\mathrm{d}t$ is a Lévy measure. This completes the proof. ■

4.2 Theorem. *Any measure ν in $\mathcal{L}(\boxplus)$ is unimodal.*

Proof. We note first that for any probability measure μ on $\mathbb{R}$ and any constant a in $\mathbb{R}$, the free convolution $\mu \boxplus \delta_a$ is the translation of μ by the constant a, and hence μ is unimodal, if and only if $\mu \boxplus \delta_a$ is unimodal for some (and hence all) a in $\mathbb{R}$. For $\boxplus$-infinitely divisible measures this means that the measure with free generating pair (γ, σ) (cf. (2.8)) is unimodal, if and only if the measure with free generating pair $(\gamma + a, \sigma)$ is unimodal for some (and hence all) a in $\mathbb{R}$. In other words, unimodality depends only on the measure σ appearing in the free generating pair.

Now let ν be a measure from $\mathcal{L}(\boxplus)$ with free characteristic triplet $(a, \frac{k(t)}{|t|}\,\mathrm{d}t, \eta)$, where $a \geq 0$, $\eta \in \mathbb{R}$ and $k\colon \mathbb{R} \setminus \{0\} \to [0, \infty)$ is a function as in (2.11). According to the discussion above, it suffices then to show that the measure ν^0 with free generating pair $(0, a\delta_0 + \frac{|t|k(t)}{1+t^2}\,\mathrm{d}t)$ is unimodal (cf. (2.4)). By application of Lemma 4.1 we may choose a sequence (k_n) of positive functions, satisfying (a)-(c) in Section 3, such that

$$\frac{|t|k_n(t)}{1+t^2}\,\mathrm{d}t \xrightarrow{\mathrm{w}} a\delta_0 + \frac{|t|k(t)}{1+t^2}\,\mathrm{d}t \quad \text{as } n \to \infty. \tag{4.2}$$

For such n it follows then from Proposition 3.8 and (2.4) that the measure ν_n^0 with free generating pair $(0, \frac{|t|k_n(t)}{1+t^2}\,\mathrm{d}t)$ is unimodal. From (4.2) and the free version of Gnedenko's Theorem (cf. (2.10)) it follows that $\nu_n^0 \xrightarrow{\mathrm{w}} \nu^0$ as $n \to \infty$, and hence (2.14) implies that ν^0 is unimodal, as desired. ■

4.3 Remark. A non-degenerate classically selfdecomposable probability measure is absolutely continuous with respect to Lebesgue measure (see [15, Theorem 27.13]). In the free case it was proved by N. Sakuma (see [17]) that non-degenerate freely selfdecomposable measures have no atoms. By definition (see formula 1.2), a unimodal measure does not have a continuous singular part, and via Theorem 4.2 we may thus conclude that also freely selfdecomposable measures are absolutely continuous with respect to the Lebesgue measure, unless they are degenerate. Moreover, from Huang's density formula [12, Theorem 3.10 (6)], which is a strengthened version of our Corollary 3.5, one can show that the density function of a freely selfdecomposable measure is continuous on $\mathbb{R}$. By contrast, the density of a classical selfdecomposable measure may have a single point of discontinuity (see [15, Theorem 28.4]).

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