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UNIMODULAR FUNCTIONS AND UNIFORM BOUNDEDNESS

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Abstract

In this paper we study the role that unimodular functions play in deciding the uniform boundedness of sets of continuous linear functionals on various function spaces. For instance, inner functions are a UBD-set in H^∞ with the weak-star topology.

1. Introduction

Let M be a topological vector space and M^* its dual space. We say that a set $S \subset M$ is a uniform boundedness deciding (UBD) set if whenever $\Phi \subset M^*$ satisfies $\sup \{| \varphi(u) |: \varphi \in \Phi\} < \infty$ for each $u \in S$, then Φ is uniformly bounded on an open neighborhood of 0. If M is the dual space of a normed space and we endow M with the weak-star topology, then Φ uniformly bounded on a weak-star open neighborhood of 0 implies that it is uniformly bounded on a norm open neighborhood of 0. In this case, we have

$$\sup_{\varphi \in \Phi} \| \varphi \|_{(M, \|\cdot\|)^*} < \infty.$$

By the uniform boundedness principle, any set of second category is a UBD-set.

Let H^∞ denote the Hardy space on the unit circle. We do not know if the set of inner functions is a UBD-set in H^∞ with the norm topology. Since the set of linear combinations of inner functions is of first category in $(H^\infty, \|\cdot\|)$[**9**], an affirmative answer cannot follow from the classical uniform boundedness principle. An affirmative answer would have as a simple consequence Marshall's theorem [**4**, p. 196]:

Theorem (Marshall). *H^∞ is the closed linear span of the Blaschke products.*

The question of whether the set of inner functions is a UBD-set in $(H^\infty, \|\cdot\|)$ was raised in [**3**]. In the same paper it is proved that the set of inner functions is a UBD-set in H^∞ with the weak-star topology, which we denote by (H^∞, w^*).

See [9] for a different proof. In [5], it is proved that the set of Blaschke products is a UBD–set in (H^∞, w^*).

Suppose (X, μ) is a positive σ–finite measure space. Let $L^p(X, \mu)$ denote the usual Lebesgue spaces, and unless indicated otherwise, $L^p(X, \mu)$ can be the space of real–valued functions or the space of complex–valued functions. In this paper we prove

Theorem 1. *The set of unimodular functions is a G_δ set in the closed unit ball of $L^\infty(X, \mu)$ with the weak–star topology.*

Since every closed set in a metric space is a G_δ, the conclusion of Theorem 1 applies also to the unimodular functions in any weak–star closed subset of $L^\infty(X, \mu)$. Combining this remark with Carathéodory's Theorem we have

Corollary 2. *The set of inner functions is a dense G_δ in the closed unit ball of (H^∞, w^*).*

Using a technique similar to that used in the proof of Theorem 1, we prove the following generalization of Carathéodory's Theorem.

Theorem 3. *The set of Blaschke products is a dense G_δ in the closed unit ball of (H^∞, w^*).*

By the Banach–Alaoglu Theorem, the closed unit ball of the dual space of a Banach space is compact and Hausdorff with the weak–star topology, and thus a Baire space [**10**, p. 200]. The results of [**3**], [**5**] then follows from Corollary 2, Theorem 3, and the following [**10**, p. 200].

Theorem. *Suppose that X is a Baire space and that Y is a dense G_δ subset of X. Then a family of continuous functions that is pointwise bounded on Y is uniformly bounded on an open subset of X.*

The proofs of Theorems 1 and 3 are in section 2. Section 3 contains an application of Theorem 1 and Section 4 contains examples related to the hypotheses of the theorems.

2. Proof of Theorem 1 and Theorem 3

Proof of Theorem 1: First suppose $\mu(X) < \infty$. For a finite partition P of X, let

$$\delta_P(f) = \max_{A \in P}\{1 - \frac{1}{\mu(A)} \mid \int_A f \, d\mu \mid\}.$$

It is clear that δ_P is weak–star continuous on the closed unit ball of $L^\infty(X, \mu)$. Let $\mathcal{P}$ be the collection of finite partitions of X and let

$$\delta(f) = \inf_{P \in \mathcal{P}} \delta_P(f).$$

Then δ is a weak-star upper semicontinuous function on the closed unit ball of $L^\infty(X,\mu)$ and clearly $0 \le \delta \le 1$. Since

$$\{\delta = 0\} = \bigcap_{n=1}^{\infty} \{\delta < \frac{1}{n}\},$$

we conclude that $\{\delta = 0\}$ is a G_δ.

We next show that $\{\delta = 0\}$ is the set of unimodular functions. Suppose P is a finite partition of X. Then for each $A \in P$, we have

$$\int_A | f | \, d\mu \ge (1 - \delta_P(f))\mu(A).$$

Summing over $A \in P$, we obtain

$$\delta_P(f) \ge 1 - \frac{1}{\mu(X)} \int_X | f | \, d\mu.$$

Therefore

$$0 \le 1 - \frac{1}{\mu(X)} \int_X | f | \, d\mu \le \delta(f).$$

Since $\|f\|_\infty \le 1$, we easily see that $\{\delta = 0\}$ is a subset of the unimodular functions.

Suppose f is unimodular. Divide the unit circle into n equal parts $S_1, \ldots, S_n$. Partition X with the sets of $\{f^{-1}(S_1), \ldots, f^{-1}(S_n)\}$ which have nonzero μ – measure. Call this partition P. Then for $A \in P$ we have

$$\frac{1}{\mu(A)} \left| \int_A f d\mu \right| \ge \cos\frac{\pi}{n}.$$

Therefore

$$0 \le \delta(f) \le \delta_P(f) \le 1 - \cos\frac{\pi}{n}.$$

Hence $\delta(f) = 0$ for f unimodular.

Thus if $\mu(X) < \infty$, the set of unimodular functions is a G_δ in the closed unit ball of $L^\infty(X,\mu)$ with the weak-star topology.

For the general case, let $X = \bigcup_{n=1}^{\infty} S_n$ with $S_0 = \emptyset, S_n \subset S_{n+1}$, and $0 < \mu(S_n \backslash S_{n-1}) < \infty$. Let

$$J(x) = \sum_{n=1}^{\infty} \frac{\chi_{S_n \backslash S_{n-1}}(x)}{2^n \mu(S_n \backslash S_{n-1})}.$$

Observe that $0 < J < \infty$. Define the measure v by $dv = Jd\mu$. Clearly v is a positive finite measure. It is also clear that $L^1(\mu) \subset L^1(v)$, that $L^\infty(\mu) = L^\infty(v)$, and that $F \in L^1(v)$ if and only if $FJ \in L^1(\mu)$. Therefore the weak-star topologies of $L^\infty(\mu)$ and $L^\infty(v)$ are identical. By the finite measure case, we conclude that the unimodular functions is a G_δ in the closed unit ball with the weak-star topology.

Proof of Theorem 3: We use the well known fact [4, p. 56] that if $f \in H^\infty$ with $\|f\| \leq 1$, then f is Blaschke product if and only if

$$\lim_{r \to 1} \int_0^{2\pi} \log |\, f(re^{i\theta}) \,|\, d\theta = 0.$$

Recall that weak-star convergence implies uniform convergence on compact subsets of the open unit disk. By Jensen's formula, it is easy to see that for $0 < r < 1$ the functions

$$\varphi_r(f) = \exp(\int_0^{2\pi} \log |\, f(re^{i\theta}) \,|\, d\theta)$$

are weak-star continuous on the closed unit ball of H^∞. Therefore the function

$$\begin{aligned}\varphi(f) &= \sup_{0<r<1} \varphi_r(f) \\ &= \exp(\lim_{r \to 1} \int_0^{2\pi} \log |\, f(re^{i\theta}) \,|\, d\theta)\end{aligned}$$

is weak-star lower semicontinuous. Hence

$$\{\varphi = 1\} = \bigcap_{n=1}^{\infty} \{\varphi > 1 - \frac{1}{n}\}$$

is a G_δ. Using the fact stated in the beginning of the proof we see that $\{\varphi = 1\}$ is the set of Blaschke products. ■

3. Application

As an application of Theorem 1, we have the following generalization of the main result in [3]. Let (X, μ) be a σ-finite positive measure space.

Theorem 4. *Suppose M is a weak-star closed subspace of $L^\infty(X, \mu)$. If the unimodular functions are weak-star dense in the closed unit ball of M, then the set of unimodular functions is a UBD-set in M with the weak-star topology.*

Theorem 4 is trivial when $M = L^\infty_{\mathbf{C}}(X, \mu)$ since every f in the unit ball of $L^\infty_{\mathbf{C}}(X, \mu)$ can be written as $f = (u_1 + u_2)/2$, where u_1, u_2 are unimodular. When $M = L^\infty_{\mathbf{R}}(X, \mu)$, Theorem 4 is a consequence of the following theorem of Nikodým [2, p. 309].

Theorem (Nikodým). *Suppose Φ is a subset of the space of countably additive measures defined on a σ-field Σ of subsets of X. If for each $E \in \Sigma$ we have*

$$\sup_{\varphi \in \Phi} | \varphi(E) | < \infty,$$

then

$$\sup_{E \in \Sigma} \sup_{\varphi \in \Phi} | \varphi(E) | < \infty.$$

An interesting question is whether Theorem 4 has an analogue when M is norm closed and $\Phi \subset L^\infty(X,\mu)^*$. Since the unimodular functions of M are never norm dense in the unit ball, a natural hypothesis seems to be the density of the convex combinations of the unimodular functions. However, there is an example in [**3**] which shows that this is not sufficient even if $\Phi \subset L^1(X,\mu)$. When $M = L^\infty(X,\mu)$, then the analogue of Theorem 4 is true with no density assumptions by the Nikodým–Grothendieck Theorem [1, p. 80].

A necessary condition for the conclusion of Theorem 4 to hold is the weak-star density of the linear span of the unimodular functions in M. Example 1 below shows that it is not sufficient. We need the following.

Theorem 5. *Let $n_1 < n_2 < n_3 < \ldots$ be a sequence of positive integers with the property that there is a sequence of positive integers $m_1 < m_2 < \ldots$ so that each m_j divides all but a finite number of the n_k's. Let M be the weak-star closure of the linear span of $\{1, z^{n_1}, z^{n_2}, \ldots\}$ in H^∞. Then a function in M that is unimodular on an arc of the unit circle has the form cz^{n_k}.*

Proof: Suppose $f = \sum_{j=0}^\infty a_j z^{n_j} \in M$ has unit modulus on the open arc γ. If $a_j = 0$ for $j \geq N$, a positive integer, then $| f |^2$ is a real analytic function and it follows that $| f |^2 = 1$ on the whole unit circle. Thus f is a finite Blaschke product. Since the only polynomial Blaschke products have the form cz^l, we are done if only a finite number of the a_j's are nonzero.

Suppose an infinite number of the a_j's are nonzero. Let $\omega_j = e^{2\pi i/m_j}$ and let

$$P_j(z) = f(z) - f(\omega_j z).$$

Since m_j divides all but a finite number of the n_k's, P_j is a polynomial. It is easy to see that

$$\limsup_{j\to\infty} \deg(P_j) = \infty.$$

Choose ω_j so small that $\gamma \cap \omega_j \gamma \neq \emptyset$ and that $\deg(P_j)$ is greater than the order of the zero of f at the origin. Suppose

$$P_j(z) = b_0 + b_1 z^{n_1} + \cdots + b_l z^{n_l},\ b_l \neq 0.$$

Since $f(z)$ and $f(\omega_j z)$ are unimodular on $\gamma \cap \omega_j\gamma$, we have on that interval

$$\frac{1}{f(z)} - \frac{1}{f(\omega_j z)} = \bar{b_0} + \frac{\bar{b_1}}{z^{n_1}} + \cdots + \frac{\bar{b_l}}{z^{n_l}}.$$

Equality then must persist throughout the open unit disk. We obtain a contradiction by multiplying both sides of the above by z^{n_l-1} and letting z tend to zero. This completes the proof. ■

Example 1. Consider the sequence

$$\{1, 2, 4, 6, 12, 18, 24, 48, 72, 96, 120, 240.\ldots\}.$$

Its structure is determined by taking arithmetic progressions of length 2,3,4, ... Hence it is not a Sidon set [6, p. 51]. Construct M as in Theorem 5. Then there is $f = \sum_{j=0}^{\infty} a_j z^{n_j} \in M$ with $\sum_{j=0}^{\infty} | a_j | = \infty$. It is clear that the above sequence satisfies the hypothesis of Theorem 5, and therefore the only unimodular functions are of the form cz^{n_k}. Let

$$\varphi_N = \sum_{n_j \le N} \lambda_j e^{-in_j\theta},$$

where $| \lambda_j | = 1$ and $\lambda_j a_j = | a_j |$. Clearly for each unimodular $u \in M$, we have

$$\sup_N \left| \int_0^{2\pi} \varphi_N u \frac{d\theta}{2\pi} \right| \le 1.$$

However

$$\sup_N \left| \int_0^{2\pi} \varphi_N f \frac{d\theta}{2\pi} \right| = \infty.$$

Therefore $\|\varphi_N\|_{M^*}$ is not bounded.

Further examples. The following example shows that the set of all unimodular functions in Theorem 4 cannot be replaced by the set of polynomials with unit norm.

Example 2. Let P be the set of polynomials with unit norm on the unit circle. It is well known that P is weak-star dense in the unit ball of H^∞ [4, p. 6]. Let $\varphi_n = \sum_{j=0}^{n} e^{-ij\theta}$. It is clear that if $p(z) = \sum_{j=0}^{N} a_j z^j$, then

$$\int_0^{2\pi} \varphi_n p \frac{d\theta}{2\pi} = \sum_{j=0}^{N} a_j$$

for $n > N$. Hence

$$\sup_n \left| \int_0^{2\pi} \varphi_n p \frac{d\theta}{2\pi} \right| < \infty.$$

However, by a theorem of Landau [4, p. 176], we have for large n,

$$\|\varphi_n\|_{(H^\infty)^*} \sim \frac{\log n}{\pi}.$$

Therefore P is not a UBD-set.

The following are examples of weak-star closed subspaces whose unimodular functions are weak-star dense in their unit balls.

Example 3. Let N be a positive integer and let S_N be the unit sphere in $\mathbb{C}^N$. Denote by σ_N the normalized Lebesgue measure on S_N. Then the inner functions are weak-star dense in $H^\infty(S_N)$. When $N = 1$, the weak-star density is a consequence of Carathéodory's Theorem [4, p. 6]. When $N \geq 2$, the weak-star density of the inner functions is a consequence of the existence of inner functions in $H^\infty(S_N)$[8, p. 36].

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