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# UNIMODULAR LATTICES IN DIMENSIONS 14 AND 15 OVER THE EISENSTEIN INTEGERS

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ABSTRACT. All indecomposable unimodular hermitian lattices in dimensions 14 and 15 over the ring of integers in $\mathbb{Q}(\sqrt{-3})$ are determined. Precisely one lattice in dimension 14 and two lattices in dimension 15 have minimal norm 3.

In 1978 W. Feit [10] classified the unimodular hermitian lattices of dimensions up to 12 over the ring $\mathbb{Z}[\omega]$ of Eisenstein integers, where $\omega$ is a primitive third root of unity. These lattices all have roots, that is, vectors of norm 2. In dimension 13, for the first time a unimodular lattice without roots appears [1, 3]. In [2] the unimodular lattices in dimension 13 are completely classified. The root-free lattice turns out to be unique. It has minimal norm 3, and its automorphism group is isomorphic to the group $\mathbb{Z}_6 \times \mathrm{PSp}_6(3)$ of order $2^{10}.3^{10}.5.7.13$. The remaining lattices all have roots; the rank of the root system is 12 in all cases.

In this paper, we classify the unimodular lattices in dimensions 14 and 15. There are exactly 58, respectively 259 classes of indecomposable lattices in these dimensions. Below, we list their root systems and the orders of their automorphism groups. Gram matrices for all lattices are available electronically via

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There is only one root-free unimodular lattice of rank 14, and there are two root-free unimodular lattices of rank 15.

The lattices without roots have minimal norm 3; they are *extremal* as introduced for unimodular Eisenstein lattices in [8], Chapter 10.7. They give rise to 3-modular extremal $\mathbb{Z}$-lattices in twice the dimension, as defined by Quebbemann in [17]. See [8, 19, 20] for more information on extremal and modular lattices and their relation to modular forms. In this context, the lattices classified in this paper can be considered as complex structures on (extremal) 3-modular lattices. The question for existence, uniqueness, and possibly a full classification of extremal modular lattices has been an ongoing challenge, both computationally and theoretically, after the appearance of the influential paper [17].

Let $V$ be a vector space over $\mathbb{Q}(\sqrt{-3})$ with a positive definite hermitian product $(,)$. A *lattice* $L$ in $V$ is a finitely generated $\mathbb{Z}[\omega]$-module contained in $V$ such that $L$ contains a basis of $V$ and $(x, y) \in \mathbb{Z}[\omega]$ for all $x$, $y \in L$. More precisely, one

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calls this an *integral lattice*. The ring $\mathbb{Z}[\omega]$ is a principal ideal domain. Thus every finitely generated torsion-free $\mathbb{Z}[\omega]$-module is free. The *discriminant* $d(L)$ of $L$ is the determinant of the Gram matrix $(a_i, a_j)$ with respect to some basis $a_1, \dots, a_n$ of $L$. The lattice is *unimodular* if $d(L) = 1$. The norm of a vector $x \in L$ is $N(x) = (x, x)$. The minimal norm of a lattice $L$ is $\min\{N(x,x) \mid x \in L,\ x \neq 0\}$. Let $L^{(2)}$ denote the sublattice of $L$ generated by all vectors of norm 2 in $L$, the so-called *roots* of $L$, and let $\mu_2(L)$ denote the number of roots in $L$. Finally, $G(L)$ denotes the group of all automorphisms of $L$ which preserve the form. The group $G(L)$ is finite. Every lattice can be uniquely decomposed into an orthogonal sum of orthogonally indecomposable lattices (it can be proved by standard reasoning similar to one in [12]). In view of the known classification in smaller dimensions it is therefore sufficient to list the indecomposable lattices.

We now come to the actual construction of unimodular lattices. We use various methods, such as representations of finite groups, lifting of linear codes, neighbour steps at suitable primes, and computer-assisted as well as hand computations of various kinds. We use extensively Schiemann's computer program [21] for calculation of invariants of hermitian lattices and their neighbours, and the Magma Computational Algebra System [4]. The crucial point is the computation of automorphism groups, which (in both systems) is based on the work [16]. After we have constructed (and distinguished) sufficiently many lattices, we use the mass formula as worked out in [10] to check the completeness of our list. An easy to calculate but important invariant of our lattices is their root systems. Due to the well-known classification of complex reflection groups [22, 5], there exist a few additional root systems over $\mathbb{Z}[\omega]$ which are not defined over $\mathbb{Z}$. Of course, they are familiar from the work on lattices of smaller dimension cited above. We use the following standard notation for root lattices. $I_n$ denotes a lattice of rank $n$ with an orthonormal basis. Equivalently,

$$I_n = \langle (a_1, \dots, a_n) \mid a_i \in \mathbb{Z}[\omega] \rangle$$

and if $x = (a_1, \dots, a_n)$, $y = (b_1, \dots, b_n)$, then $(x, y) = \sum_{i=1}^n a_i \bar{b}_i$. The lattices $A_n \subseteq I_{n+1}$, $E_8$, $E_7$ and $E_6$ are as usual. For $\alpha \in \mathbb{Z}[\omega]$ define $D_n(\alpha) \subseteq I_n$ by

$$D_n(\alpha) = \langle (a_1, \dots, a_n) \mid a_i \in \mathbb{Z}[\omega],\ \sum_{i=1}^n a_i \equiv 0 \pmod{\alpha} \rangle.$$

Finally,

$$U_5 = \langle A_5,\ (1/\sqrt{-3})(1, \omega, \omega^2, 1, \omega, \omega^2) \rangle,$$
$$U_6 = \langle D_6(\sqrt{-3}),\ (1/\sqrt{-3})(1, \dots, 1) \rangle.$$

The lattice $U_6$ is unimodular, and $U_5$ has discriminant 2.

The root systems of the lattices classified in this paper are given in Tables 1–3 below. For most of the lattices in dimension 14, we give "glue vectors" (additional generators) [8] together with the root systems.

In Table 1 we have collected separately all lattices that are constructed from self-dual codes of length 14 over $\mathbb{F}_4$. These codes have been classified in [6, 14]. Let $\varphi$ denote the composite mapping $\mathbb{Z}[\omega] \to \mathbb{Z}[\omega]/2\mathbb{Z}[\omega] \cong \mathbb{F}_4$. Then for any self-dual code $C$ of length 14 over $\mathbb{F}_4$ the lattice

$$L = \frac{1}{\sqrt{2}} \langle (a_1, \dots, a_{14}) \in I_{14} \mid (\varphi(a_1), \dots, \varphi(a_{14})) \in C \rangle$$

is a unimodular lattice of rank 14. In particular, lattice No. 10 comes from a quadratic residue code [13, 23] of length 14.

Now we come to a particular construction of a unimodular lattice of rank 15.

**Lemma 1.** *The exterior square $U_6 \wedge U_6$ is a unimodular lattice of rank 15 and minimal norm 3.*

*Proof.* We only need to show that the lattice $U_6 \wedge U_6$ has minimal norm 3. Our proof is similar to the proof of Theorem 2.1 in [9]. Any element $a \in U_6 \wedge U_6$ can be written in the form $a = \sum_{i=1}^r x_i \wedge y_i$, where $r \leq 3$ and $x_1, \ldots, x_r,\ y_1, \ldots, y_r$ are $2r$ linearly independent vectors in $U_6$. The lattice $L_{2r}$ generated by $x_1, \ldots, x_r$, $y_1, \ldots, y_r$ depends only on $a$. We have

$$\begin{aligned} (a,a) &= \left( \sum_{i=1}^r x_i \wedge y_i, \sum_{j=1}^r x_j \wedge y_j \right) \\ &= \sum_{i,j} [(x_i, x_j)(y_i, y_j) - (x_i, y_j)(y_i, x_j)] \\ &= \mathrm{Tr}(A\overline{B} - C\overline{C}), \end{aligned}$$

where $A = ((x_i, x_j))_{1\leq i,j \leq r}$ , $B = ((y_i, y_j))_{1\leq i,j\leq r}$ and $C = ((x_i, y_j))_{1\leq i,j\leq r}$. Define

$$H = \begin{pmatrix} A & C \\ {}^t\overline{C} & B \end{pmatrix},\ J = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}.$$

Then $H$ is the hermitian matrix of the lattice $L_{2r}$ and

$$(a,a) = \frac{1}{2}\mathrm{Tr}(H\,{}^t\overline{JH}J) \geq \frac{2r}{2}|H|^{1/2r}.|{}^t\overline{JH}J|^{1/2r} = r|H|^{1/r},$$

where we used the inequality $\mathrm{Tr}(MN) \geq m|M|^{1/m}|N|^{1/m}$ valid for any hermitian positive definite matrices $M$, $N$ of degree $m$ (it can be proved similarly to the proof of Lemma 7.1.3 in [11]). Now we show that $(a,a) \geq 3$. Indeed, for $r = 3$ we have $(a,a) \geq 3.|H|^{1/3} \geq 3$; for $r = 2$ we have $(a,a) \geq 2.|H|^{1/2} > 2$; for $r = 1$ we have $(a,a) \geq 1.|H| = d(L_{2r}) \geq 3$. Finally, it is easy to construct an element $a$ of norm 3: take $a = x \wedge y$ with $(x,x) = (y,y) = 2$ and $(x,y) = 1$. □

**Theorem 1.** *There are 58 hermitian indecomposable unimodular lattices over $\mathbb{Z}[\omega]$ of dimension 14. They are listed in Tables 1 and 2. One of them has no roots; the remaining lattices have root systems of ranks 6, 8, 10, 11, 12, 13 and 14. The root-free lattice $L_0$ has minimal norm 3 and automorphism group $\mathbb{Z}_6 \times \mathrm{G}_2(3).2$ of order $2^8.3^7.7.13$.*

**Theorem 2.** *There are 259 hermitian indecomposable unimodular lattices over $\mathbb{Z}[\omega]$ of dimension 15. They are listed in Table 3. For any integer $r$ from 0 to 15 there is an indecomposable unimodular lattice of dimension 15 with a root system of rank $r$. There are only two root-free lattices. They have minimal norm 3. One of the root-free lattices is isometric to the exterior square $U_6 \wedge U_6$ and has automorphism group $\mathbb{Z}_2 \times 3.\mathrm{U}_4(3).2$ of order $2^9.3^7.5.7$. The second root-free lattice has automorphism group $\mathbb{Z}_2 \times (3_+^{1+2} \times 3_+^{1+2}).\mathrm{SL}_2(3).2$ of order $2^5.3^7$.*

*Proof.* Let $\mathfrak{G}_n$ be a complete set of representatives of the isomorphism classes of lattices of rank $n$ and discriminant 1. Let $\mathfrak{G}_n^0$ be the set of all those lattices in $\mathfrak{G}_n$

that contain no elements of norm 1. Define

$$M_n = \sum_{L\in\mathfrak{G}_n} \frac{1}{|G(L)|}, \quad M_n(2) = \sum_{L\in\mathfrak{G}_n} \frac{\mu_2(L)}{|G(L)|},$$
$$Y_n = \sum_{L\in\mathfrak{G}_n^0} \frac{1}{|G(L)|}, \quad Y_n(2) = \sum_{L\in\mathfrak{G}_n^0} \frac{\mu_2(L)}{|G(L)|}.$$

These quantities can be calculated in principle well-known formulas: the mass formula for positive definite hermitian lattices, and a similar formula for average representation numbers like $\mu_2$. In our situation these formulas have been made explicit in [10]:

$$\frac{M_{2m}}{M_{2m-1}} = (-1)^{m-1}B_{2m}\frac{3^m+(-1)^m}{4m}, \quad \frac{M_{2m+1}}{M_{2m}} = \frac{a_{2m+1}}{3^m+(-1)^m},$$
$$\frac{M_{2m}(2)}{M_{2m}} = \frac{4m(2^{2m-1}+1)}{(3^m+(-1)^m)B_{2m}}, \quad \frac{M_{2m+1}(2)}{M_{2m+1}} = \frac{(2^{2m}-1)(3^m-(-1)^m)}{a_{2m+1}},$$
$$Y_n = M_n - \frac{1}{n!6^n} - \sum_{j=1}^{n-1} \frac{Y_j}{(n-j)!6^{n-j}},$$
$$Y_n(2) = M_n(2) - \sum_{j=1}^{n-1}\frac{Y_j(2)}{(n-j)!6^{n-j}} - \sum_{j=1}^{n-2}\frac{Y_j}{2(n-2-j)!6^{n-2-j}} - \frac{1}{2(n-2)!6^{n-2}},$$

where $a_{2m+1} = 3^{2m}(-1)^{m-1}B_{2m+1}(\frac{1}{3})/(2m+1)$, $B_n(t)$ is the Bernoulli polynomial, and $B_{2m}(0) = B_{2m}$ is the Bernoulli number $B_{2m}$. By definition $M_1 = 1/6$ and $M_1(2) = Y_1 = Y_1(2) = 0$.

Then one can calculate

$$M_{14} = \frac{689532652191539}{2^{25}.3^{19}.5^3.11}, \quad M_{14}(2) = \frac{1722885336811913}{2^{23}.3^{17}.5^3.11},$$
$$Y_{14} = \frac{902121810728981}{2^{24}.3^{14}.5^3.7^2.11.13}, \quad Y_{14}(2) = \frac{321547203435163}{2^{22}.3^{12}.5^3.11.13}.$$
$$M_{15} = \frac{4366489808207046403}{2^{26}.3^{21}.5^3.11}, \quad M_{15}(2) = \frac{1884491476714586441}{2^{24}.3^{18}.5^3.11},$$
$$Y_{15} = \frac{3619970721202760389}{2^{18}.3^{20}.5^3.7^2.11.13}, \quad Y_{15}(2) = \frac{312324214206248801}{2^{16}.3^{17}.5^2.7^2.11.13}.$$

Now we use an explicit list of Gram matrices of the constructed lattices. We use Schiemann's computer program and the Magma Computational Algebra System to calculate $\mu_2(L)$ and $|G(L)|$ in each case. Then a straightforward check of the mass formulas for $Y_{14}$ and $Y_{15}$ confirms the completeness of our list. The formulas for $Y_{14}(2)$ and $Y_{15}(2)$ give an additional check (ignoring the lattices without roots).

We now determine the structure of the automorphism groups of the three root-free lattices. The group $\mathrm{G}_2(3).2$ has a representation of dimension 14 over $\mathbb{Q}(\sqrt{-3})$ (see [7]) and it acts on a lattice $L$ of rank 14 over $\mathbb{Z}[\omega]$. Now results of [15] on rational maximal finite matrix groups in dimension 28 imply that this lattice is unimodular, and therefore is isometric to our root-free lattice $L_0$. The factor $\mathbb{Z}_6$ in $G(L_0)$ comes from roots of unity in $\mathbb{Z}[\omega]$.

Now we come to the automorphism group of the root-free lattice $U_6 \wedge U_6$ introduced in Lemma 1. Recall that $\mathrm{Aut}(U_6) \cong 6.\mathrm{U}_4(3).2$. There is a natural homomorphism $\varphi : \mathrm{Aut}(U_6) \mapsto \mathrm{Aut}(U_6 \wedge U_6)$, given by $\varphi(\sigma)(x \wedge y) = \sigma(x) \wedge \sigma(y)$

with Ker $\varphi = \{\pm\mathrm{id}\}$. Therefore, $\mathrm{Aut}(U_6 \wedge U_6) \supseteq 3.\mathrm{U}_4(3).2$. But $\{\pm\mathrm{id}\}.\varphi(\mathrm{Aut}(U_6))$ is a group of order $2^9.3^7.5.7$. The known orders of the two automorphism groups of root-free lattices in dimension 15 show that this group must be the full group. Therefore, $\mathrm{Aut}(U_6 \wedge U_6) \cong \mathbb{Z}_2 \times 3.\mathrm{U}_4(3).2$.

The structure of the automorphism group of the second root-free lattice, of order $2^5.3^7$ was determined with the help of Magma. □

The extremal 3-modular $\mathbb{Z}$-lattice in dimension 28 coming from our root-free lattice in dimension 14 appeared for the first time in [15] since its automorphism group is a rational irreducible maximal finite subgroup of $\mathrm{GL}_{28}(\mathbb{Q})$. The two root-free lattices of rank 15 have been found by A. Schiemann ([21], also mentioned in [19]), using a sophisticated computer search for lattices with a large minimum.

We want to add some comments on algebraic techniques for lattices which are related in one way or another to the automorphism group. One of the most important algebraic techniques for lattices is "gluing", which in the most general case means: (re)constructing a lattice $L$ from a known, "simpler" sublattice $M$. Then $L$ is generated by $M$ and certain additional vectors, called glue vectors. If $M$ has the same rank as $L$, that is, generates the vector space $V$, the glue vectors must come from the dual lattice $M^{\#}$, which can be analyzed once for all. Typically, $M$ is decomposable while $L$ is not, so the additional vectors "glue the components of $M$ together", this is where the terminology comes from. An important choice for $M$ is the lattice $L^{(2)}$ generated by the roots. Then $M$ is characteristic, that is, $\mathrm{Aut}(L)$ acts on $M$, and is a subgroup of $\mathrm{Aut}(M)$ if $M$ is of full rank. If $M$ is not of full rank, it must be glued with a lattice on its orthogonal subspace. Here are two cases where the rank of the root system is very small.

Let us consider the unimodular 15-dimensional lattice $\Lambda$ (No. 257) with root system $A_2$. This lattice is obtained by gluing of a 2-dimensional lattice with root system $A_2$ and a 13-dimensional lattice $\Lambda'$ with discriminant 3. Computations on Magma and [7] show that $\mathrm{Aut}(\Lambda) \cong 6.(\mathrm{S}_3 \times \mathrm{PSL}_2(27).3)$. Therefore, $\mathrm{Aut}(\Lambda') \cong 6.(\mathrm{PSL}_2(27).3)$ and $\Lambda'$ is associated with the irreducible complex character of the group $\mathrm{PSL}_2(27)$ of degree 13. It is the Weil character and it can indeed be realized over $\mathbb{Z}[\omega]$ (see [18]).

Similar reasoning shows that the 15-dimensional unimodular lattice No. 259 with root system $A_1$ has automorphism group $6.(\mathrm{S}_2 \times \mathrm{PSL}_2(13))$ and it is obtained from a 14-dimensional lattice of discriminant 2 with automorphism group $6.\mathrm{PSL}_2(13)$. This lattice generates an irreducible complex character of the group $\mathrm{PSL}_2(13)$ of degree 14 (the case of complex character of degree 13 is excluded since the description [1] of hermitian lattices of dimension 13 associated with the irreducible complex character of the group $\mathrm{PSL}_2(13)$ of degree 13 shows that this case is impossible).

Of course, root sublattices and their complements are not the only choice for characteristic sublattices. If the automorphism group is already known, one could proceed as in the following example. The complex representation associated with the action of the automorphism group $\mathbb{Z}_2 \times (3_+^{1+2} \times 3_+^{1+2}).\mathrm{SL}_2(3).2$ on the 15-dimensional root-free lattice $L_2$ is reducible, and is a sum of two irreducible representations of degrees 6 and 9 defined over $\mathbb{Q}(\sqrt{-3})$. And the complex representation, associated with the action of the subgroup $\mathbb{Z}_2 \times (3_+^{1+2} \times 3_+^{1+2}).\mathrm{SL}_2(3)$, is a sum of three irreducible representations of degrees 3, 3 and 9. This decomposition gives rise to invariant sublattices of $L_2$, and using the general method of gluing it should be possible to obtain a computer-free description of $L_2$.

Obviously, root lattices cannot be used directly to obtain lattices with minimal norm 3 or larger. But one can use a scaled copy of a root lattice as in the following example. We start with a root system over $\mathbb{Z}$, and in addition to scaling we also use the larger ground ring $\mathbb{Z}[\omega]$. Let $n$ be of the shape $n = 2r\bar{r}$ for some $r \in \mathbb{Z}[\omega]$. Let $C$ be a self-dual code over $\mathbb{F}_4$ of length $n$ and

$$\Lambda = \langle (a_1, \ldots, a_n) \in A_{n-1} \mid (\varphi(a_1), \ldots, \varphi(a_n)) \in C \rangle + \frac{1}{r}\langle (1-n, 1, \ldots, 1)\rangle.$$

Then $\frac{1}{\sqrt{2}}\Lambda$ is a unimodular hermitian lattice of rank $n-1$. This construction had been used in [2] to obtain the root-free lattice in dimension 13. It can also be applied in e.g. dimension 17.

We close with some remarks on the theta series of our lattices. There is one interesting observation in dimension 14 which might deserve further attention.

**Proposition 1.** *Any indecomposable unimodular lattice of rank* 14*, more generally any* 3*-modular* $\mathbb{Z}$*-lattice of rank* 28 *with minimum* $\geq 2$ *has exactly* $17472 = 2^6.3.7.13$ *vectors of norm* 3*.*

*Proof.* (See [19] for the method): The relevant algebra $\mathcal{M}^*(3,\chi)$ of modular forms (where $\chi$ is an appropriate character) is generated by the cusp form $\Delta_3(q) = \eta(q)^6\,\eta(3q)^6$ of weight 6 and the theta-series $\theta_3 := \theta_{A_2}$ of the $A_2$-root lattice, of weight 1. A linear basis for the space $\mathcal{M}_{14}(3,\chi)$ of forms of weight 14, containing the theta-series of the lattices in question is the following:

$$\begin{array}{lcccccccl}
\theta_3^{14} & = & 1 & + & 84q & + & 3276q^2 & + & 78708q^3 & +O(q^4) \\
{\theta_3}^8\,\Delta_3 & = & & & q & + & 42q^2 & + & 729q^3 & +O(q^4) \\
{\theta_3}^2\,{\Delta_3}^2 & = & & & & & q^2 & & & +O(q^4)
\end{array}$$

The theta series of a lattice without vectors of norm 1 is of the form

$$\theta_3^{14} - 84{\theta_3}^8\,\Delta_3 + c{\theta_3}^2\,{\Delta_3}^2$$

with $c$ given by the number of vectors of norm 2. The absence of $q^3$ in the third basis vector proves the claim. □

Note that $M_{13} \approx 0.00000014$ (there are 14 lattices of rank 13), $M_{14} \approx 0.000012$ (58 lattices), $M_{15} \approx 0.0045$ (259 lattices), $M_{16} \approx 6.57$, $M_{17} \approx 42188.20$. So beyond dimension 15, a full classification of unimodular Eisenstein lattices may no longer be possible.

*Notations in tables.* For each lattice $L$ we present the sublattice $L^{(2)}$ generated by its root system, in standard notation, the number of roots $\mu_2(L)$ and the order $|G(L)|$ of the automorphism group. In Tables 1 and 2 we give also glue vectors from $(L^{(2)})^\#$, as explained above. For instance, for lattice No. 24 the root system is $A_6D_5(2)A_2$, the glue vectors are given in terms of standard generators $y_1 \in (A_6)^\#$, $y_2 \in (D_5(2))^\#$, $y_3 \in (A_2)^\#$ and a vector $x$ of norm 84 orthogonal to $L^{(2)}$.

Table 1. Unimodular lattices of rank 14 (obtained from self-dual codes over $\mathbb{F}_4$)

| No. | $L$ | $L^{(2)}$ | $\lvert G(L)\rvert$ | $\mu_2(L)$ |
|---|---|---|---|---|
| 1 | $\langle L^{(2)}, \frac{1}{2}(\sqrt{-3}, 1, \ldots, 1)\rangle$ | $D_{14}(2)$ | $2^{24}.3^6.5^2.7^2.11.13$ | 1092 |
| 2 | $\langle L^{(2)}, y_1 + y_2\rangle$,<br>$y_i \in E_7^* - E_7$ | $E_7{}^2$ | $2^{21}.3^9.5^2.7^2$ | 756 |
| 3 | $\langle L^{(2)}, \frac{1}{2}(\sqrt{-3}, 1, \ldots, 1) + y_3$,<br>$\frac{1}{2}(-\sqrt{-3}, 1, \ldots, 1) + y_2\rangle$,<br>$y_2 = \frac{1}{2\sqrt{-3}}(-5, 1, 1, 1, 1, 1)$, $y_3 = \frac{1}{2}(-1, 1)$ | $D_8(2)U_5A_1$ | $2^{22}.3^7.5^2.7$ | 612 |
| 4 | $\langle L^{(2)}, y_1 + z, y_2 + z\rangle$<br>$z = \frac{1}{2}(\sqrt{-3}, 1, 1, 1)$,<br>$y_i = \frac{1}{2\sqrt{-3}}(-5, 1, 1, 1, 1, 1)$ | $U_5{}^2D_4(2)$ | $2^{21}.3^{11}.5^2$ | 612 |
| 5 | $\langle L^{(2)}$,<br>$\frac{1}{2}(1, \ldots, 1) + (1, 0, \ldots, 0)$,<br>$(1, 0, \ldots, 0) + \frac{1}{2}(\sqrt{-3}, 1, \ldots, 1)\rangle$ | $D_8(2)D_6(2)$ | $2^{23}.3^5.5^2.7$ | 516 |
| 6 | $\langle L^{(2)}$,<br>$(1, 0, \ldots, 0) + (1, 0, \ldots, 0) + \omega y_1 + \omega^2 y_2$,<br>$\frac{1}{2}(\sqrt{-3}, 1, \ldots, 1)_1 + y_1 + y_2$,<br>$\frac{1}{2}(\sqrt{-3}, 1, \ldots, 1)_1 - \frac{1}{2}(\sqrt{-3}, 1, \ldots, 1)_2\rangle$,<br>$y_i = \frac{1}{2}(-1, 1)$ | $D_6(2)^2A_1{}^2$ | $2^{21}.3^5.5^2$ | 372 |
| 7 | $\langle L^{(2)}$,<br>$(1, 0, \ldots, 0) + (1, 0, 0, 0) + (1, 0, 0, 0)$,<br>$(0, \ldots, 0, \omega) + \frac{1}{2}(1, 1, 1, 1) + \frac{1}{2}(1, 1, 1, 1)$,<br>$\frac{1}{2}(1, \ldots, 1) + \frac{1}{2}(\sqrt{-3}, 1, 1, 1)\rangle$ | $D_6(2)D_4(2)^2$ | $2^{22}.3^6.5$ | 324 |
| 8 | $\langle L^{(2)}$,<br>$(1, 0, 0, 0)_1 + (1, 0, 0, 0)_2 + (1, 0, 0, 0)_3$,<br>$y_1 + y_2 + y_4 + y_5$,<br>$y_2 + y_3 + \omega^2 y_4 + \omega^2 y_5$,<br>$(1, 0, 0, 0)_2 + (\omega^2, 0, 0, 0)_3 + \omega y_4 + \omega^2 y_5\rangle$,<br>$y_{1,2,3} = \frac{1}{2}(1, 1, 1, 1)$, $y_{4,5} = \frac{1}{2}(-1, 1)$ | $D_4(2)^3A_1{}^2$ | $2^{21}.3^5$ | 228 |
| 9 | $\langle L^{(2)}$,<br>$(1, 0, 0, 0)_1 + y_3 + y_4 + \omega y_5 + \omega y_6$,<br>$\frac{1}{2}(1, 1, 1, 1)_1 + \omega^2 y_2 + \omega^2 y_3 + y_4 + y_5$,<br>$(1, 0, 0, 0)_2 + y_2 + \omega y_3 + y_5 + \omega y_6$,<br>$\frac{1}{2}(1, 1, 1, 1)_2 + y_2 + y_3 + \omega^2 y_4 + \omega^2 y_5$,<br>$y_1 + y_2 + y_3 + y_4 + y_5 + y_6\rangle$,<br>$y_i = \frac{1}{2}(-1, 1)$ | $D_4(2)^2A_1{}^6$ | $2^{20}.3^5$ | 180 |
| 10 | $\langle L^{(2)}, \alpha_1 y_1 + \cdots + \alpha_{14} y_{14}\rangle$<br>$y_i = \frac{1}{2}(-1, 1)$,<br>$(\alpha_1, \ldots, \alpha_{14})_{\text{ mod } 2} \in QR(14)$ | $A_1{}^{14}$ | $2^{16}.3^2.7.13$ | 84 |

Table 2. Unimodular lattices of rank 14

| No. | $L$ | $L^{(2)}$ | $\|G(L)\|$ | $\mu_2(L)$ |
|---|---|---|---|---|
| 11 | $L_0$ | $\varnothing$ | $2^8.3^7.7.13$ | 0 |
| 12 | $\langle L^{(2)}, \frac{1}{\sqrt{-3}}(1,\dots,1)+y_2\rangle$,<br>$y_2 \in E_6^* - E_6$ | $D_8(\sqrt{-3})E_6$ | $2^{15}.3^{14}.5^2.7$ | 720 |
| 13 | $\langle L^{(2)}, (2+\sqrt{-3})y_1+y_2\rangle$,<br>$y_1=\frac{1}{14}(-13,1,\dots,1)$, $y_2=\frac{1}{2}(-1,1)$ | $A_{13}A_1$ | $2^{13}.3^6.5^2.7^2.11.13$ | 552 |
| 14 | Complex conjugate of 13 | | | |
| 15 | $\langle L^{(2)}, (1,0,0,0,0,0)+y_2+y_3$,<br>$\frac{1}{\sqrt{-3}}(1,1,1,1,1,1)-\sqrt{-3}y_3\rangle$,<br>$y_2 \in E_6^*-E_6$, $y_3=\frac{1}{3}(-2,1,1)$ | $D_6(\sqrt{-3})E_6A_2$ | $2^{13}.3^{13}.5^2$ | 504 |
| 16 | $\langle L^{(2)}$,<br>$\frac{\sqrt{-3}}{12}(-11,1,\dots,1)+\frac{\sqrt{-3}}{4}(-3,1,1,1)\rangle$, | $A_{11}A_3$ | $2^{14}.3^7.5^2.7.11$ | 432 |
| 17 | $\langle L^{(2)}, \sqrt{-3}y_1+y_2\rangle$,<br>$y_1=\frac{1}{9}(-8,1,\dots,1)$, $y_2\in E_6^*-E_6$ | $A_8E_6$ | $2^{15}.3^9.5^2.7$ | 432 |
| 18 | $\langle L^{(2)}, y_1+y_2+y_3,\ y_1-y_2+y_4\rangle$<br>$y_{1,2}=\frac{1}{\sqrt{-3}}(1,\dots,1)$, $y_{3,4}=\frac{1}{3}(-2,1,1)$ | $D_5(\sqrt{-3})^2{A_2}^2$ | $2^{11}.3^{13}.5^2$ | 396 |
| 19 | $\langle L^{(2)}\oplus\langle x\rangle$,<br>$y_1+y_2+\frac{2}{9}x$, $(1,0,0,0,0)+\frac{\sqrt{-3}}{3}x\rangle$,<br>$y_1=\frac{1}{9}(-8,1,\dots,1)$,<br>$y_2=\frac{1}{\sqrt{-3}}(1,1,1,1,1)$, $(x,x)=9$ | $A_8D_5(\sqrt{-3})$ | $2^{11}.3^{10}.5^2.7$ | 396 |
| 20 | $\langle L^{(2)}, (2+\sqrt{-3})y_1+y_2\rangle$,<br>$y_i=\frac{1}{8}(-7,1,\dots,1)$ | ${A_7}^2$ | $2^{15}.3^5.5^2.7^2$ | 336 |
| 21 | $\langle L^{(2)}, y_1+2\sqrt{-3}y_2+y_3\rangle$<br>$y_1=\frac{1}{10}(-9,1,\dots,1)$,<br>$y_2=\frac{1}{5}(-4,1,\dots,1)$, $y_3=\frac{1}{2}(-1,1)$ | $A_9A_4A_1$ | $2^{13}.3^6.5^3.7$ | 336 |
| 22 | $\langle L^{(2)}\oplus\langle x\rangle$,<br>$(2+\sqrt{-3})y_1+\sqrt{-3}y_2+\frac{1}{36}x\rangle$,<br>$y_1=\frac{1}{9}(-8,1,\dots,1)$,<br>$y_2=\frac{1}{2}(1,\dots,1)$, $(x,x)=36$ | $A_8D_5(2)$ | $2^{15}.3^6.5^2.7$ | 336 |
| 23 | Complex conjugate of 22 | | | |
| 24 | $\langle L^{(2)}\oplus\langle x\rangle$,<br>$\sqrt{-3}y_1+\sqrt{-3}y_2+y_3+\frac{1}{84}x\rangle$,<br>$y_1=\frac{1}{7}(-6,1,\dots,1)$, $y_2=\frac{1}{\sqrt{-3}}(1,\dots,1)$,<br>$y_3=\frac{1}{3}(-2,1,1)$, $(x,x)=84$ | $A_6D_5(2)A_2$ | $2^{13}.3^5.5^2.7$ | 264 |
| 25 | $\langle L^{(2)}\oplus\langle x\rangle$,<br>$y_1+y_3+\frac{2+\sqrt{-3}}{8}x$, $y_2+y_3+\frac{2-\sqrt{-3}}{6}x\rangle$,<br>$y_1=\frac{1}{8}(-7,1,\dots,1)$, $y_2=\frac{1}{6}(-5,1,\dots,1)$,<br>$y_3=\frac{1}{2}(-1,1)$, $(x,x)=24$ | $A_7A_5A_1$ | $2^{13}.3^5.5^2.7$ | 264 |
| 26 | Complex conjugate of 25 | | | |

Table 2. Unimodular lattices of rank 14 (continued)

| No. | $L$ | $L^{(2)}$ | $\lvert G(L)\rvert$ | $\mu_2(L)$ |
|---|---|---|---|---|
| 27 | $\langle L^{(2)} \oplus \langle x\rangle$, $y_1+y_3+\frac{2+\sqrt{-3}}{8}x, y_3+y_4+\frac{1}{2}x, y_2+\frac{1}{3}x\rangle$, $y_1=\frac{1}{8}(-7,1,\ldots,1)$, $y_2=\frac{1}{\sqrt{-3}}(1,1,1,1)$, $y_{3,4}=\frac{1}{2}(-1,1)$, $(x,x)=24$ | $A_7D_4(\sqrt{-3})A_1{}^2$ | $2^{13}.3^7.5.7$ | 288 |
| 28 | $\langle L^{(2)}$, $y_1+y_2+y_3, 2\sqrt{-3}(y_1-y_2)\rangle$, $y_{1,2}=\frac{1}{6}(-5,1,\ldots,1)$, $y_3=\frac{1}{\sqrt{-3}}(1,1,1,1)$ | $A_5{}^2D_4(\sqrt{-3})$ | $2^{13}.3^9.5^2$ | 288 |
| 29 | $\langle L^{(2)} \oplus \langle x\rangle$, $y_1+\sqrt{-3}y_2+y_3+\frac{1}{60}x\rangle$, $y_1=\frac{1}{2}(1,1,1,1,1)$, $y_2=\frac{1}{5}(-4,1,1,1,1)$, $y_3=\frac{1}{\sqrt{-3}}(1,1,1,1)$, $(x,x)=60$ | $D_5(2)A_4D_4(\sqrt{-3})$ | $2^{14}.3^7.5^2$ | 288 |
| 30 | $\langle L^{(2)} \oplus \langle x\rangle \oplus \langle z\rangle$, $y_1+\sqrt{-3}y_2+\frac{1}{3}x$, $\frac{1}{2}(\sqrt{-3},1,1,1)+\frac{1}{2}x$, $\sqrt{-3}y_1+y_2+\frac{1}{3}z$, $\frac{1}{2}(-\sqrt{-3},1,1,1)+\frac{1}{2}z\rangle$, $y_{1,2}=\frac{1}{\sqrt{-3}}(1,1,1,1)$, $(x,x)=6$, $(z,z)=6$ | $D_4(\sqrt{-3})^2D_4(2)$ | $2^{14}.3^{11}$ | 288 |
| 31 | $\langle L^{(2)} \oplus \langle x\rangle \oplus \langle z\rangle$, $y_1+\sqrt{-3}y_2+\frac{2}{7}x, \sqrt{-3}y_1+y_2+\frac{2}{7}z\rangle$, $y_i=\frac{1}{7}(-6,1,\ldots,1)$, $(x,x)=7$, $(z,z)=7$ | $A_6{}^2$ | $2^{10}.3^5.5^2.7^2$ | 252 |
| 32 | $\langle L^{(2)} \oplus \langle x\rangle \oplus \langle z\rangle$, $3\sqrt{-3}y_1+y_2+\frac{1}{2}(\sqrt{-3},1,1,1)+\frac{1}{42}x+\frac{1}{3}z$, $\frac{1}{2}(-\sqrt{-3},1,1,1)+\frac{\sqrt{-3}}{3}x-\frac{\sqrt{-3}}{6}z\rangle$, $y_1=\frac{1}{7}(-6,1,\ldots,1)$, $y_3=\frac{1}{3}(-2,1,1)$ $(x,x)=42$, $(z,z)=6$ | $A_6A_2D_4(2)$ | $2^{12}.3^6.5.7$ | 216 |
| 33 | $\langle L^{(2)}$, $y_1+y_2+y_3+y_4, 2y_2+\omega y_3-\sqrt{-3}\omega y_4\rangle$ $y_{1,2}=\frac{1}{6}(-5,1,\ldots,1), y_{3,4}=\frac{1}{3}(-2,1,1)$, | $A_5{}^2A_2{}^2$ | $2^{12}.3^7.5^2$ | 216 |
| 34 | $\langle L^{(2)} \oplus \langle x\rangle$, $y_1+y_2+\frac{1}{3}x, y_3+\frac{\sqrt{-3}}{4}x, 2\sqrt{-3}(y_1-y_2)\rangle$ $y_{1,2}=\frac{1}{6}(-5,1,\ldots,1)$, $y_3=\frac{1}{4}(-3,1,1,1)$, $(x,x)=12$ | $A_5{}^2A_3$ | $2^{13}.3^6.5^2$ | 216 |
| 35 | Complex conjugate of 34 | | | |
| 36 | $\langle L^{(2)} \oplus \langle x\rangle \oplus \langle z\rangle$, $2\sqrt{-3}y_1+y_2+\frac{2}{5}x$, $\frac{1}{2}(\sqrt{-3},1,1,1)+\frac{1}{2}x$, $y_1+2\sqrt{-3}y_2+\frac{2}{5}z$, $\frac{1}{2}(1,-1,1,\sqrt{-3})+\frac{1}{2}z\rangle$, $y_{1,2}=\frac{1}{5}(-4,1,\ldots,1)$, $(x,x)=10$, $(z,z)=10$ | $A_4{}^2D_4(2)$ | $2^{14}.3^4.5^2$ | 192 |

Table 2. Unimodular lattices of rank 14 (continued)

| No. | $L$ | $L^{(2)}$ | $\|G(L)\|$ | $\mu_2(L)$ |
|---|---|---|---|---|
| 37 | $\langle L^{(2)} \oplus \langle x \rangle,$ $2y_1 + \sqrt{-3}y_2 + y_3 + y_4 + \frac{1}{60}x,$ $3\omega y_1 + 2y_3 + y_4\rangle,\ (x,x) = 60,$ $y_1 = \frac{1}{6}(-5,1,\dots,1),\ y_3 = \frac{1}{4}(-3,1,1,1),$ $y_2 = \frac{1}{5}(-4,1,1,1,1),\ y_4 = \frac{1}{2}(-1,1)$ | $A_5A_4A_3A_1$ | $2^{12}.3^5.5^2$ | 192 |
| 38 | Complex conjugate of 37 | | | |
| 39 | | ${A_5}^2{A_1}^2$ | $2^{12}.3^5.5^2$ | 192 |
| 40 | | $D_3(\sqrt{-3})^2{A_2}^4$ | $2^{10}.3^{12}$ | 180 |
| 41 | | $A_5D_3(\sqrt{-3}){A_2}^2$ | $2^9.3^9.5$ | 180 |
| 42 | | ${A_4}^2D_3(\sqrt{-3})A_1$ | $2^{10}.3^6.5^2$ | 180 |
| 43 | | $D_4(2)D_3(\sqrt{-3})^2$ | $2^{13}.3^{10}$ | 180 |
| 44 | $\langle L^{(2)} \oplus \langle x \rangle \oplus \langle z \rangle,$ $y_1 + 2\sqrt{-3}y_2 + \frac{1}{5}x, y_3 + y_4 + \frac{1}{3}x,$ $2\sqrt{-3}y_1 + y_2 + \frac{1}{15}z, y_3 - y_4 + \frac{1}{3}z\rangle,$ $y_{1,2} = \frac{1}{5}(-4,1,\dots,1),\ (x,x) = 15,$ $y_{3,4} = \frac{1}{3}(-2,1,1),\ (z,z) = 15$ | ${A_4}^2{A_2}^2$ | $2^{10}.3^5.5^2$ | 156 |
| 45 | $\langle L^{(2)} \oplus \langle x \rangle \oplus \langle z \rangle \oplus \langle u \rangle,$ $2\sqrt{-3}(y_1 - y_2) + y_3 + \frac{1}{2}u + \frac{1}{20}x,$ $2\sqrt{-3}(y_1 + y_2) + 2y_3 + \frac{\sqrt{-3}}{4}u + \frac{1}{20}z\rangle,$ $y_{1,2} = \frac{1}{5}(-4,1,\dots,1),\ (x,x) = 20,$ $y_3 = \frac{1}{4}(-3,1,1,1),\ (z,z) = 20,\ (u,u) = 4$ | ${A_4}^2A_3$ | $2^{11}.3^4.5^2$ | 156 |
| 46 | | $A_4{A_3}^2{A_1}^2$ | $2^{13}.3^4.5$ | 144 |
| 47 | | ${A_3}^4$ | $2^{15}.3^5$ | 144 |
| 48 | | ${A_3}^2{A_2}^2{A_1}^2$ | $2^{12}.3^5$ | 120 |
| 49 | | ${A_3}^3{A_1}^2$ | $2^{13}.3^5$ | 120 |
| 50 | Complex conjugate of 49 | | | |
| 51 | | ${A_2}^6$ | $2^9.3^9$ | 108 |
| 52 | | $A_3{A_2}^4$ | $2^{10}.3^7$ | 108 |
| 53 | | ${A_3}^2{A_2}^2$ | $2^{11}.3^6$ | 108 |
| 54 | | ${A_2}^4{A_1}^2$ | $2^{10}.3^5$ | 84 |
| 55 | | ${A_2}^2{A_1}^6$ | $2^{11}.3^5$ | 72 |
| 56 | | ${A_2}^4$ | $2^8.3^9$ | 72 |
| 57 | | ${A_1}^8$ | $2^{13}.3^2.7$ | 48 |
| 58 | | ${A_1}^6$ | $2^{13}.3^4.5$ | 36 |

Table 3. Unimodular lattices of rank 15

| No. | $L^{(2)}$ | $\|G(L)\|$ | $\mu_2(L)$ |
|---|---|---|---|
| 1 | $\varnothing$ | $2^9.3^7.5.7$ | 0 |
| 2 | $\varnothing$ | $2^5.3^7$ | 0 |
| 3 | $D_{15}(\sqrt{-3})$ | $2^{12}.3^{20}.5^3.7^2.11.13$ | 1890 |
| 4 | $D_9(\sqrt{-3})D_6(\sqrt{-3})$ | $2^{12}.3^{19}.5^2.7$ | 918 |
| 5 | $E_7D_7(\sqrt{-3})$ | $2^{15}.3^{13}.5^2.7^2$ | 756 |
| 6 | $A_{15}$ | $2^{16}.3^7.5^3.7^2.11.13$ | 720 |
| 7 | $A_{14}$ | $2^{12}.3^7.5^3.7^2.11.13$ | 630 |
| 8 | complex conjugate of 7 | | |
| 9 | $D_9(2)D_5(\sqrt{-3})$ | $2^{19}.3^{10}.5^2.7$ | 612 |
| 10 | $A_8D_7(\sqrt{-3})$ | $2^{12}.3^{13}.5^2.7^2$ | 594 |
| 11 | $D_6(\sqrt{-3})D_6(\sqrt{-3})D_3(\sqrt{-3})$ | $2^{11}.3^{17}.5^2$ | 594 |
| 12 | $A_9U_5$ | $2^{16}.3^9.5^3.7$ | 540 |
| 13 | $A_5D_5(\sqrt{-3})U_5$ | $2^{15}.3^{12}.5^3$ | 540 |
| 14 | $E_6U_5D_3(\sqrt{-3})$ | $2^{16}.3^{13}.5^2$ | 540 |
| 15 | $A_{11}D_4(\sqrt{-3})$ | $2^{14}.3^{10}.5^2.7.11$ | 504 |
| 16 | $A_8D_6(\sqrt{-3})$ | $2^{12}.3^{12}.5^2.7$ | 486 |
| 17 | $D_7(2)E_6$ | $2^{18}.3^8.5^2.7$ | 468 |
| 18 | $A_{11}D_3(\sqrt{-3})$ | $2^{12}.3^9.5^2.7.11$ | 450 |
| 19 | $A_9D_5(\sqrt{-3})$ | $2^{12}.3^{10}.5^3.7$ | 450 |
| 20 | complex conjugate of 20 | | |
| 21 | $A_8E_6$ | $2^{15}.3^9.5^2.7$ | 432 |
| 22 | $D_6(\sqrt{-3})D_3^3(\sqrt{-3})$ | $2^9.3^{18}.5$ | 432 |
| 23 | $U_5D_4(\sqrt{-3})D_4(\sqrt{-3})$ | $2^{15}.3^{13}.5$ | 432 |
| 24 | $D_7(2)A_7$ | $2^{18}.3^5.5^2.7^2$ | 420 |
| 25 | $D_6(2)D_5(\sqrt{-3})A_3$ | $2^{16}.3^9.5^2$ | 396 |
| 26 | $A_{10}A_4$ | $2^{12}.3^6.5^3.7.11$ | 390 |
| 27 | $A_5D_5(\sqrt{-3})D_4(\sqrt{-3})$ | $2^{11}.3^{12}.5^2$ | 378 |
| 28 | $A_9A_5$ | $2^{13}.3^7.5^3.7$ | 360 |
| 29 | complex conjugate of 28 | | |
| 30 | $A_7D_6(2)$ | $2^{17}.3^5.5^2.7$ | 348 |
| 31 | $A_6D_5(\sqrt{-3})A_3$ | $2^{11}.3^9.5^2.7$ | 342 |
| 32 | $A_8A_6$ | $2^{12}.3^7.5^2.7^2$ | 342 |
| 33 | complex conjugate of 32 | | |
| 34 | $A_8D_4(\sqrt{-3})A_2$ | $2^{12}.3^{10}.5.7$ | 342 |
| 35 | complex conjugate of 34 | | |
| 36 | $A_8D_5(2)$ | $2^{15}.3^6.5^2.7$ | 336 |
| 37 | $D_5(\sqrt{-3})D_4(2)D_4(2)$ | $2^{17}.3^{10}.5$ | 324 |
| 38 | $D_6(2)A_5D_3(\sqrt{-3})$ | $2^{15}.3^8.5^2$ | 324 |
| 39 | $A_8D_3(\sqrt{-3})A_3$ | $2^{12}.3^9.5.7$ | 306 |
| 40 | $A_8A_5$ | $2^{12}.3^7.5^2.7$ | 306 |
| 41 | complex conjugate of 40 | | |
| 42 | $A_7A_6$ | $2^{12}.3^5.5^2.7^2$ | 294 |
| 43 | complex conjugate of 42 | | |

Table 3. Unimodular lattices of rank 15 (continued)

| No. | $L^{(2)}$ | $\|G(L)\|$ | $\mu_2(L)$ |
|---|---|---|---|
| 44 | $D_5(2)D_4(\sqrt{-3})A_4$ | $2^{14}.3^7.5^2$ | 288 |
| 45 | $D_4(\sqrt{-3})A_5A_5$ | $2^{12}.3^9.5^2$ | 288 |
| 46 | $D_5(2)D_5(2)A_3$ | $2^{19}.3^4.5^2$ | 276 |
| 47 | $A_6A_5D_3(\sqrt{-3})$ | $2^{10}.3^8.5^2.7$ | 270 |
| 48 | $A_6D_4(\sqrt{-3})A_3$ | $2^{11}.3^8.5.7$ | 270 |
| 49 | $A_5D_4(\sqrt{-3})D_3(\sqrt{-3})A_2$ | $2^{10}.3^{11}.5$ | 270 |
| 50 | $D_4(\sqrt{-3})D_4(\sqrt{-3})A_2A_2A_2$ | $2^{11}.3^{13}$ | 270 |
| 51 | $D_3(\sqrt{-3})^5$ | $2^8.3^{16}.5$ | 270 |
| 52 | $A_6A_6A_2$ | $2^{11}.3^6.5^2.7^2$ | 270 |
| 53 | complex conjugate of 52 | | |
| 54 | $A_7A_4A_3$ | $2^{14}.3^5.5^2.7$ | 264 |
| 55 | $A_6D_5(2)A_2$ | $2^{13}.3^5.5^2.7$ | 264 |
| 56 | complex conjugate of 55 | | |
| 57 | $A_6A_6A_1$ | $2^{10}.3^5.5^2.7^2$ | 258 |
| 58 | $D_5(2)D_4(2)D_3(\sqrt{-3})A_1$ | $2^{16}.3^7.5$ | 252 |
| 59 | $A_5A_5D_4(2)$ | $2^{16}.3^7.5^2$ | 252 |
| 60 | $A_7A_4A_2$ | $2^{12}.3^5.5^2.7$ | 246 |
| 61 | $D_5(2)A_4A_4$ | $2^{14}.3^4.5^3$ | 240 |
| 62 | $D_5(2)A_4A_4$ | $2^{15}.3^4.5^3$ | 240 |
| 63 | $A_7A_3A_3$ | $2^{15}.3^5.5.7$ | 240 |
| 64 | $A_5D_4(\sqrt{-3})A_2A_2$ | $2^{10}.3^9.5$ | 234 |
| 65 | $A_4A_4D_4(\sqrt{-3})A_1$ | $2^{11}.3^7.5^2$ | 234 |
| 66 | $A_6A_5A_2$ | $2^{10}.3^6.5^2.7$ | 234 |
| 67 | $A_6A_5A_2$ | $2^{10}.3^6.5^2.7$ | 234 |
| 68 | complex conjugate of 67 | | |
| 69 | $A_5A_5D_3(\sqrt{-3})$ | $2^{11}.3^8.5^2$ | 234 |
| 70 | complex conjugate of 69 | | |
| 71 | $D_5(2)A_3A_3A_3$ | $2^{17}.3^6.5$ | 228 |
| 72 | $A_6A_4A_3$ | $2^{11}.3^5.5^2.7$ | 222 |
| 73 | complex conjugate of 72 | | |
| 74 | $D_4(\sqrt{-3})A_3A_3A_3$ | $2^{14}.3^9$ | 216 |
| 75 | $D_4(2)D_4(\sqrt{-3})A_2A_2$ | $2^{13}.3^{10}$ | 216 |
| 76 | $D_4(\sqrt{-3})D_3(\sqrt{-3})A_2A_2A_2$ | $2^9.3^{13}$ | 216 |
| 77 | $A_5D_3(\sqrt{-3})^2A_2$ | $2^9.3^{11}.5$ | 216 |
| 78 | $A_6D_3(\sqrt{-3})A_2A_2$ | $2^9.3^8.5.7$ | 216 |
| 79 | $A_6D_4(2)A_2$ | $2^{12}.3^6.5.7$ | 216 |
| 80 | $A_5A_5A_3$ | $2^{12}.3^6.5^2$ | 216 |
| 81 | $A_5A_5A_3$ | $2^{13}.3^6.5^2$ | 216 |
| 82 | complex conjugate of 81 | | |
| 83 | $A_6A_4A_2A_1$ | $2^{10}.3^5.5^2.7$ | 210 |
| 84 | complex conjugate of 83 | | |
| 85 | $A_5A_4A_4$ | $2^{11}.3^5.5^3$ | 210 |
| 86 | complex conjugate of 85 | | |
| 87 | $A_5D_4(2)A_3A_1$ | $2^{15}.3^5.5$ | 204 |

Table 3. Unimodular lattices of rank 15 (continued)

| No. | $L^{(2)}$ | $\lvert G(L)\rvert$ | $\mu_2(L)$ |
|---|---|---|---|
| 88 | $A_6A_3A_2A_2$ | $2^{10}.3^6.5.7$ | 198 |
| 89 | $A_5A_5A_2$ | $2^{11}.3^7.5^2$ | 198 |
| 90 | $A_5A_5A_2$ | $2^{11}.3^6.5^2$ | 198 |
| 91 | complex conjugate of 90 | | |
| 92 | $A_5A_3D_3(\sqrt{-3})A_2$ | $2^{10}.3^8.5$ | 198 |
| 93 | complex conjugate of 92 | | |
| 94 | $A_5A_4A_3A_1$ | $2^{12}.3^5.5^2$ | 192 |
| 95 | complex conjugate of 94 | | |
| 96 | $A_4A_4D_4(2)$ | $2^{14}.3^4.5^2$ | 192 |
| 97 | complex conjugate of 96 | | |
| 98 | $A_5A_4A_3$ | $2^{11}.3^5.5^2$ | 186 |
| 99 | $A_5A_4A_3$ | $2^{11}.3^5.5^2$ | 186 |
| 100 | $A_5D_4(2)A_1A_1A_1$ | $2^{14}.3^6.5$ | 180 |
| 101 | $D_4(2)D_3(\sqrt{-3})A_3A_1A_1A_1$ | $2^{14}.3^7$ | 180 |
| 102 | $D_4D_4(2)A_3$ | $2^{17}.3^6$ | 180 |
| 103 | $A_3A_3A_3A_3A_3$ | $2^{18}.3^7.5$ | 180 |
| 104 | $A_4A_4A_4$ | $2^{10}.3^5.5^3$ | 180 |
| 105 | complex conjugate of 104 | | |
| 106 | $A_5A_4A_2A_1$ | $2^{10}.3^5.5^2$ | 174 |
| 107 | $A_5A_4A_2A_1$ | $2^{10}.3^5.5^2$ | 174 |
| 108 | complex conjugate of 107 | | |
| 109 | $A_4A_4A_3A_2$ | $2^{11}.3^5.5^2$ | 174 |
| 110 | complex conjugate of 109 | | |
| 111 | $A_4D_4(2)A_2A_2$ | $2^{12}.3^5.5$ | 168 |
| 112 | $A_4D_4(2)A_2A_2$ | $2^{13}.3^5.5$ | 168 |
| 113 | $A_4A_3A_3A_3$ | $2^{14}.3^5.5$ | 168 |
| 114 | $A_5A_3A_3A_1$ | $2^{13}.3^5.5$ | 168 |
| 115 | complex conjugate of 114 | | |
| 116 | $A_4D_3(\sqrt{-3})A_3A_1A_1$ | $2^{10}.3^6.5$ | 162 |
| 117 | $A_3A_3A_3D_3(\sqrt{-3})$ | $2^{11}.3^8$ | 162 |
| 118 | $A_3A_3D_3(\sqrt{-3})A_2A_2$ | $2^{11}.3^8$ | 162 |
| 119 | $A_5A_2A_2A_2A_2$ | $2^9.3^8.5$ | 162 |
| 120 | $D_3(\sqrt{-3})D_3(\sqrt{-3})A_2A_2A_2$ | $2^7.3^{12}$ | 162 |
| 121 | $A_5A_3A_2A_2$ | $2^{10}.3^6.5$ | 162 |
| 122 | complex conjugate of 121 | | |
| 123 | $A_4A_4A_3A_1$ | $2^{11}.3^4.5^2$ | 162 |
| 124 | complex conjugate of 123 | | |
| 125 | $A_4A_4A_2A_2$ | $2^{10}.3^5.5^2$ | 156 |
| 126 | $D_4(2)A_3A_3A_1A_1$ | $2^{16}.3^4$ | 156 |
| 127 | $A_4A_4A_2A_1A_1$ | $2^{10}.3^4.5^2$ | 150 |
| 128 | $A_4A_4A_2A_1A_1$ | $2^{11}.3^4.5^2$ | 150 |
| 129 | $A_4A_3A_3A_2$ | $2^{11}.3^5.5$ | 150 |
| 130 | $A_4A_3A_3A_2$ | $2^{11}.3^5.5$ | 150 |
| 131 | $A_4A_3A_3A_2$ | $2^{11}.3^5.5$ | 150 |
| 132 | complex conjugate of 131 | | |

Table 3. Unimodular lattices of rank 15 (continued)

| No. | $L^{(2)}$ | $\|G(L)\|$ | $\mu_2(L)$ |
|---|---|---|---|
| 133 | $A_4A_3A_3A_1A_1$ | $2^{12}.3^4.5$ | 144 |
| 134 | $A_3D_3(\sqrt{-3})A_3A_2A_2A_2$ | $2^8.3^9$ | 144 |
| 135 | $A_5A_2A_2A_1A_1A_1$ | $2^{11}.3^6.5$ | 144 |
| 136 | $A_4D_3(\sqrt{-3})A_2A_1A_1$ | $2^9.3^7.5$ | 144 |
| 137 | $A_5A_2A_2A_2$ | $2^9.3^8.5$ | 144 |
| 138 | $A_3A_3A_3A_3$ | $2^{16}.3^6$ | 144 |
| 139 | $D_4(2)A_2A_2A_2A_2$ | $2^{12}.3^8$ | 144 |
| 140 | $D_4(2)A_2A_2A_2A_2$ | $2^{13}.3^7$ | 144 |
| 141 | $A_3A_3A_3A_2A_2$ | $2^{13}.3^7$ | 144 |
| 142 | complex conjugate of 141 | | |
| 143 | $A_4A_4A_1A_1A_1$ | $2^{11}.3^3.5^2$ | 138 |
| 144 | $A_4A_3A_2A_2A_1$ | $2^{10}.3^5.5$ | 138 |
| 145 | $A_4A_3A_2A_2A_1$ | $2^{10}.3^5.5$ | 138 |
| 146 | complex conjugate of 145 | | |
| 147 | $A_4A_3A_2A_2A_1$ | $2^{10}.3^5.5$ | 138 |
| 148 | complex conjugate of 147 | | |
| 149 | $A_4A_3A_3A_1$ | $2^{11}.3^4.5$ | 138 |
| 150 | complex conjugate of 149 | | |
| 151 | $A_4A_3A_2A_2$ | $2^9.3^5.5$ | 132 |
| 152 | $A_3D_3(\sqrt{-3})A_2A_1A_1A_1$ | $2^9.3^7$ | 126 |
| 153 | $D_3(\sqrt{-3})A_2A_2A_2A_2$ | $2^8.3^9$ | 126 |
| 154 | $A_3A_3A_3A_2$ | $2^{11}.3^6$ | 126 |
| 155 | $A_3A_3A_2A_2A_2$ | $2^{11}.3^6$ | 126 |
| 156 | $A_3A_3A_2A_2A_2$ | $2^{11}.3^6$ | 126 |
| 157 | complex conjugate of 156 | | |
| 158 | $A_3A_3A_2A_2A_2$ | $2^{11}.3^7$ | 126 |
| 159 | complex conjugate of 158 | | |
| 160 | $A_4A_3A_2A_1A_1$ | $2^{10}.3^4.5$ | 126 |
| 161 | complex conjugate of 160 | | |
| 162 | $A_3A_3A_2A_2A_1A_1$ | $2^{11}.3^5$ | 120 |
| 163 | $A_4A_3A_1A_1A_1A_1$ | $2^{12}.3^3.5$ | 120 |
| 164 | $A_4A_3A_3A_3A_1$ | $2^8.3^6.5$ | 120 |
| 165 | $A_3A_3A_3A_1A_1$ | $2^{12}.3^5$ | 120 |
| 166 | $A_3A_3A_3A_1A_1$ | $2^{13}.3^4$ | 120 |
| 167 | complex conjugate of 166 | | |
| 168 | $A_4A_2A_2A_1A_1A_1$ | $2^9.3^4.5$ | 114 |
| 169 | $A_3A_3A_2A_2A_1$ | $2^{10}.3^5$ | 114 |
| 170 | $A_3A_3A_2A_2A_1$ | $2^{10}.3^5$ | 114 |
| 171 | complex conjugate of 170 | | |
| 172 | $A_3A_3A_2A_2A_1$ | $2^{11}.3^5$ | 114 |
| 173 | complex conjugate of 172 | | |

Table 3. Unimodular lattices of rank 15 (continued)

| No. | $L^{(2)}$ | $\|G(L)\|$ | $\mu_2(L)$ |
|---|---|---|---|
| 174 | $A_4A_2A_2A_1A_1$ | $2^9.3^5.5$ | 108 |
| 175 | $A_3A_3A_2A_2$ | $2^{10}.3^6$ | 108 |
| 176 | $A_3A_2A_2A_2A_2$ | $2^9.3^6$ | 108 |
| 177 | $A_3A_3A_1A_1A_1A_1A_1A_1$ | $2^{14}.3^4$ | 108 |
| 178 | $D_3(\sqrt{-3})A_2A_2A_2$ | $2^6.3^{11}$ | 108 |
| 179 | $D_3(\sqrt{-3})D_3(\sqrt{-3})$ | $2^{10}.3^{13}$ | 108 |
| 180 | $D_3(\sqrt{-3})A_1A_1A_1A_1A_1A_1A_1A_1A_1$ | $2^{14}.3^7$ | 108 |
| 181 | $A_2A_2A_2A_2A_2A_2$ | $2^8.3^9$ | 108 |
| 182 | $A_2A_2A_2A_2A_2A_2$ | $2^{10}.3^{10}$ | 108 |
| 183 | $A_3A_3A_2A_1A_1$ | $2^{10}.3^4$ | 102 |
| 184 | $A_3A_2A_2A_2A_1A_1$ | $2^9.3^5$ | 102 |
| 185 | $A_3A_2A_2A_2A_1A_1$ | $2^9.3^5$ | 102 |
| 186 | $A_3A_2A_2A_2A_1A_1$ | $2^9.3^5$ | 102 |
| 187 | complex conjugate of 186 | | |
| 188 | $A_3A_3A_2A_1A_1$ | $2^{11}.3^4$ | 102 |
| 189 | complex conjugate of 188 | | |
| 190 | $A_3A_3A_2A_1A_1$ | $2^{11}.3^4$ | 102 |
| 191 | complex conjugate of 190 | | |
| 192 | $A_3A_2A_2A_1A_1A_1A_1$ | $2^{11}.3^4$ | 96 |
| 193 | $A_3A_3A_1A_1A_1A_1$ | $2^{13}.3^3$ | 96 |
| 194 | $A_3A_2A_2A_2A_1$ | $2^9.3^5$ | 96 |
| 195 | $A_3A_2A_2A_1A_1A_1A_1$ | $2^{12}.3^4$ | 96 |
| 196 | $A_3A_3A_1A_1A_1$ | $2^{11}.3^4$ | 90 |
| 197 | $A_2A_2A_2A_2A_1A_1A_1$ | $2^9.3^6$ | 90 |
| 198 | $A_3A_2A_2A_1A_1A_1$ | $2^9.3^5$ | 90 |
| 199 | $A_3A_2A_2A_1A_1A_1$ | $2^9.3^4$ | 90 |
| 200 | $A_3A_2A_2A_1A_1A_1$ | $2^9.3^4$ | 90 |
| 201 | complex conjugate of 200 | | |
| 202 | $A_2A_2A_2A_2A_2$ | $2^8.3^7$ | 90 |
| 203 | $A_2A_2A_2A_2A_2$ | $2^7.3^7$ | 90 |
| 204 | complex conjugate of 203 | | |
| 205 | $A_2A_2A_2A_2A_2$ | $2^8.3^7.5$ | 90 |
| 206 | complex conjugate of 205 | | |
| 207 | $A_3A_2A_2A_1A_1$ | $2^9.3^4$ | 84 |
| 208 | $A_3A_1A_1A_1A_1A_1A_1A_1A_1$ | $2^{15}.3^3$ | 84 |
| 209 | $A_2A_2A_2A_2A_1A_1$ | $2^8.3^5$ | 84 |
| 210 | complex conjugate of 209 | | |
| 211 | $A_3A_2A_1A_1A_1A_1$ | $2^9.3^3$ | 78 |
| 212 | $A_2A_2A_2A_2A_1$ | $2^7.3^5$ | 78 |
| 213 | $A_2A_2A_2A_1A_1A_1A_1$ | $2^9.3^4$ | 78 |
| 214 | $A_2A_2A_2A_1A_1A_1A_1$ | $2^9.3^4$ | 78 |
| 215 | complex conjugate of 214 | | |

Table 3. Unimodular lattices of rank 15 (continued)

| No. | $L^{(2)}$ | $\|G(L)\|$ | $\mu_2(L)$ |
|---|---|---|---|
| 216 | $A_3A_2A_2$ | $2^8.3^6$ | 72 |
| 217 | $A_3A_2A_1A_1A_1$ | $2^9.3^5$ | 72 |
| 218 | $A_2A_2A_2A_1A_1A_1$ | $2^7.3^5$ | 72 |
| 219 | $A_3A_3A_3A_1A_1A_1$ | $2^8.3^6$ | 72 |
| 220 | $A_2A_2A_1A_1A_1A_1A_1A_1$ | $2^{10}.3^5$ | 72 |
| 221 | $A_2A_2A_1A_1A_1A_1A_1A_1$ | $2^{10}.3^4$ | 72 |
| 222 | complex conjugate of 221 | | |
| 223 | $A_3A_1A_1A_1A_1A_1A_1$ | $2^{12}.3^3$ | 72 |
| 224 | complex conjugate of 223 | | |
| 225 | $A_2A_2A_2A_2A_1A_1$ | $2^7.3^4$ | 66 |
| 226 | $A_2A_2A_2A_2A_1A_1$ | $2^7.3^5$ | 66 |
| 227 | $A_2A_2A_1A_1A_1A_1A_1$ | $2^8.3^3$ | 66 |
| 228 | $A_2A_2A_1A_1A_1A_1A_1$ | $2^9.3^3$ | 66 |
| 229 | complex conjugate of 228 | | |
| 230 | $A_2A_2A_1A_1A_1A_1$ | $2^8.3^3$ | 60 |
| 231 | $A_2A_2A_1A_1A_1A_1$ | $2^{10}.3^3$ | 60 |
| 232 | $A_1A_1A_1A_1A_1A_1A_1A_1A_1A_1$ | $2^{13}.3^2.5$ | 60 |
| 233 | $A_2A_2A_2$ | $2^5.3^8$ | 54 |
| 234 | $A_2A_2A_1A_1A_1$ | $2^7.3^5$ | 54 |
| 235 | $A_2A_2A_1A_1A_1$ | $2^7.3^4$ | 54 |
| 236 | complex conjugate of 235 | | |
| 237 | $A_2A_1A_1A_1A_1A_1A_1$ | $2^9.3^2$ | 54 |
| 238 | $A_2A_1A_1A_1A_1A_1A_1$ | $2^9.3^3$ | 54 |
| 239 | complex conjugate of 238 | | |
| 240 | $A_2A_1A_1A_1A_1A_1$ | $2^8.3^3$ | 48 |
| 241 | $A_1A_1A_1A_1A_1A_1A_1A_1$ | $2^{14}.3$ | 48 |
| 242 | $A_1A_1A_1A_1A_1A_1A_1A_1$ | $2^{10}.3^2.7$ | 48 |
| 243 | complex conjugate of 242 | | |
| 244 | $A_2A_1A_1A_1A_1$ | $2^8.3^2$ | 42 |
| 245 | $A_1A_1A_1A_1A_1A_1A_1$ | $2^9.3^2$ | 42 |
| 246 | $A_1A_1A_1A_1A_1A_1A_1$ | $2^9.3.7$ | 42 |
| 247 | complex conjugate of 246 | | |
| 248 | $A_2A_1A_1A_1$ | $2^6.3^5$ | 36 |
| 249 | $A_1A_1A_1A_1A_1A_1$ | $2^9.3^2$ | 36 |
| 250 | $A_2A_2$ | $2^6.3^7$ | 36 |
| 251 | $A_3$ | $2^{13}.3^6$ | 36 |
| 252 | $A_1A_1A_1A_1A_1$ | $2^7.3.5$ | 30 |
| 253 | $A_1A_1A_1A_1A_1$ | $2^8.3^2.5$ | 30 |
| 254 | complex conjugate of 253 | | |
| 255 | $A_1A_1A_1A_1$ | $2^9.3^2$ | 24 |
| 256 | $A_1A_1A_1$ | $2^5.3^4$ | 18 |
| 257 | $A_2$ | $2^4.3^6.7.13$ | 18 |
| 258 | $A_1A_1$ | $2^{10}.3^3$ | 12 |
| 259 | $A_1$ | $2^4.3^2.7.13$ | 6 |

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