Unipotent elements, tilting modules, and saturation.

Gary M. Seitz
University of Oregon

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0. Introduction.

The results in this paper were motivated by work of Serre [19] where semisimplicity results for representations of arbitrary groups in positive characteristic were established. A key ingredient in the arguments was the notion of *saturation* where one embeds a unipotent element of prime order of a simple algebraic group in a 1-dimensional unipotent subgroup. One of our goals here is to show that saturation can be achieved rather generally and to establish a uniqueness result for the resulting unipotent subgroups.

As a by product we show for a simple algebraic group in good characteristic a unipotent element of prime order is contained in a particularly nice subgroup of type A_1 and that the Lie algebra of the ambient algebraic group usually affords a tilting module for this subgroup. The tilting decompositions are applied to show that the centralizers of the unipotent element, a corresponding 1-dimensional unipotent group, and its Lie algebra all coincide and are closely related to the centralizer of the A_1 subgroup. We also establish a convenient factorization of the centralizer of the unipotent element. Results for finite groups of Lie type are also obtained.

Let G be a simple algebraic group over an algebraically closed field K of finite characteristic p. Assume p is a good prime for G and let $u \in G$ be a unipotent element of order p. By a group of type A_1 we mean a closed subgroup of G isomorphic to $SL_2(K)$ or $PSL_2(K)$. Throughout A will denote a group of type A_1 and T_A a maximal torus of A. We say that A is *good* if the weights of T_A on $L(G)$ are at most $2p-2$. If A is chosen

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to contain u, then $U < A$ will denote the unique 1-dimensional unipotent subgroup of A containing u, and $T_A < A$ is chosen to normalize U.

We recall the following notion introduced by Serre. A subgroup $X < G$ is said to be *G-completely reducible*, G-cr for short, if for every parabolic subgroup P of G containing X, there is a Levi subgroup of P containing X.

Our first result establishes existence and conjugacy of good $A_1's$ containing unipotent elements of prime order. It further relates these to tilting decompositions of the Lie algebra and the above notion of complete reducibility.

Theorem 1. Let G be a simple algebraic group over an algebraically closed field K of characteristic p, a good prime. Let $u \in G$ be a unipotent element of order p.

(i). There exists a good A_1 containing u.

(ii). Any two good $A_1's$ containing u are conjugate by an element of $R_u(C_G(u))$.

(iii). If A is a good A_1, then $L(G)|A$ is a tilting module, unless G has type A_n with $p|n+1$. In the exceptional case, if $G = SL_{n+1}$, then $L(GL_{n+1})|A$ is a tilting module.

(iv). If A is a good A_1 containing u, then A is G-cr.

The construction of the good $A_1's$ yields further information for which we refer the reader to Propositions 3.1 and 3.2. In particular, information is obtained on labelled diagrams which link these subgroups to corresponding groups in characteristic 0.

The next result is our main result on centralizers.

Theorem 2. Assume $u \in G$ has order p and $u \in A$, a good A_1. Let $U < A$ be a 1-dimensional unipotent group containing u.

(i). $C_G(u) = C_G(U) = C_G(L(U))$.

(ii). $C_G(u) = QC_G(A)$, a semidirect product, where $Q = R_u(C_G(u))$ and $C_G(A)$ is reductive.

(iii). Q/U acts regularly on the family of good $A_1's$ containing u.

We note that (iii) is a generalization of a well known result for long root elements.

One consequence of Theorems 1 and 2 is the following result on saturation.

Theorem 3. Let $u \in G$ have order p. There is a unique 1-dimensional unipotent group U containing u such that U is contained in a good A_1.

Theorem 3 implies that there is a unique monomorphism $\phi_u : G_a \to G$ with image contained in a good A_1 and satisfying $\phi_u(1) = u$. It is convenient to introduce the suggestive notation $u^t = \phi_u(t)$. We note that if G is of classical type and $u = 1 + e$ in the action on the natural module, then it follows from our results that $u^t = 1 + te + (t(t-1)/2)e^2 + \cdots = (1+e)^t$. This also holds for arbitrary G in the adjoint action, provided p is sufficiently large so that good $A_1's$ containing u have each composition factor restricted.

Finally, we consider the situation for finite groups of Lie type. Let σ be a Frobenius morphism of G, so that $G_\sigma = G(q)$ is a finite group of Lie type over a field of size q, a power of p. If A is a σ invariant group of type A_1, then we set $A(q) = A_\sigma$.

Theorem 4. Let $u \in G(q)$ have order p.

(i). There exists a good A_1 containing u which is σ-stable.

(ii). Two σ-stable good $A_1's$ containing u are conjugate by an element of $G(q)$; in fact by an element of $O_p(C_{G(q)}(u))$.

(iii). Assume $A = A^\sigma$ is a good A_1. Then $C_G(A) = C_G(A(q))$ and $C_{G(q)}(u) = QC_{G(q)}(A(q))$, a semidirect product, where $Q = O_p(C_{G(q)}(u))$.

(iv). Assume A is a σ-stable good A_1, containing u and that $q > 7$ in case G is of exceptional type. Then A_σ is G_σ-cr.

The organization of the paper is as follows. The first two sections are preliminary covering material on tilting modules and unipotent classes, respectively. The existence of good $A_1's$ preserving tilting decompositions of $L(G)$ is established in Section 3. Section 4 is concerned with a theory of exponentiation in positive characteristic and Section 5 is a key section on centralizers. Conjugacy results for good $A_1's$ are established in Section 6. Theorems 1-3 are then established in Section 7, while Section 8 establishes results on finite groups. In Section 9 we present examples showing that the hypothesis of a good prime is necessary and the last section is devoted to some directions for future work.

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1. Tilting Modules

Let X be a simple algebraic group over an algebraically closed field K of positive characteristic p. A rational X-module V is a *tilting module* provided V has a filtration by Weyl modules and also a filtration by dual Weyl modules.

General properties of tilting modules are given in the following lemma.

Lemma 1.1. (a). For each dominant weight λ there is an indecomposable tilting module $T_X(\lambda)$, unique up to isomorphism, with highest weight λ.

(b). Any tilting module is the direct sum of modules of form $T_X(\lambda)$.

(c). A direct summand of a tilting module is again a tilting module.

(d). The tensor product of tilting modules is a tilting module.

Proof. (a) and (b) are in Theorem 1.1 of [9]. (d) is Proposition 1.2 of [9] and (c) is Corollary 1.3 of [8].

We will be particularly interested in tilting modules for groups of type A_1, where high weights are represented by nonnegative integers. The modules of importance here have high weights at most $2p-2$.

We record the following notation. For c an integer let $L(c)$ denote the irreducible module for $SL_2(K)$ of highest weight c and $W(c)$ the corresponding Weyl module.

The following is a well-known result on extensions of simple modules for $SL_2(K)$.

Lemma 1.2. ([1]) Let $c, d \leq 2p-2$.

(i). $W(c) = L(c)$ if $0 \leq c < p$.

(ii). If $c \geq p$, write $c = r + p$. Then $W(c)$ has a unique maximal submodule, M, such that $M \cong L(p-r-2)$ and $W(c)/M \cong L(c)$.

(iii). $\mathrm{Ext}^1(L(c), L(d)) = 0$ unless $c \neq d$ with $c + d = 2p-2$, in which case the dimension is 1.

The next result describes the tilting modules $T(c)$ for $0 \leq c \leq 2p-2$, for $SL_2(K)$. They are particularly simple and behave well with respect to fixed points of unipotent elements.

Lemma 1.3. Assume $c \leq 2p-2$ and let $T(c)$ be the tilting module of high weight c. Then $T(c)$ satisfies the following conditions:

(a). If $0 \leq c < p$, then $T(c) = L(c)$ is irreducible and restricted.

(b). If $p \leq c \leq 2p-2$, write $c = r + p$, where $r \leq p-2$. Then $T(c) = L(p-r-2)/L(r+p)/L(p-r-2)$, a uniserial, self-dual module with simple factors as indicated.

(c). If $p \leq c \leq 2p-2$, then $\dim(T(c)) = 2p$ and $T(c)$ is projective for $A_1(p)$.

(d). $C_{T(c)}(u) = C_{T(c)}(U) = C_{T(c)}(L(U))$. This space has dimension 1 or 2, according to whether or not $c < p$. In the latter case the fixed points lie in the unique maximal submodule of $T(c)$.

Proof. If $c < p$, then $L(c) = W(c) \cong W(c)^*$ satisfies the definition of a tilting module, so by uniqueness of the tilting module of high weight c, $T(c) = L(c)$. So (a) holds and (d) is clear.

Suppose $c \geq p$ and write $c = r + p$. We claim that $T(c)$ is projective for $A_1(p) < A_1$. Start with $L(r + 1) \otimes L(p - 1)$. This is projective for $A_1(p)$ as $L(p - 1)$ affords the Steinberg module. Viewed as a module for A_1 it is the tensor product of two restricted irreducible tilting modules. So (1.1)(d) implies this is also a tilting module and, by (1.1)(c), the direct sum of indecomposable tilting modules corresponding to certain high weights. The highest weight is c, so $T(c)$ must be a direct summand and is hence projective for $A_1(p)$. Note that the high weight c occurs with multiplicity 1.

It follows from 1.2 that $W(c)$ has socle $L(p - r - 2)$ with quotient $L(c)$. Further, $Ext^1(L(c), L(p - r - 2))$ has dimension 1 while $Ext^1(L(d), L(c)) = Ext^1(L(d), L(p-r-2)) = 0$ for $d < 2p-2, d \notin \{c, p-r-2\}$. Hence, $T(c) = E \oplus F$, where E has all composition factors with high weights in $\{c, p-r-2\}$ and F has no such composition factor. As $T(c)$ is indecomposable having both a Weyl filtration and a dual Weyl filtration, we conclude that $T(c) = E$ with the structure indicated in (b). A dimension count yields (c).

As $dim(T(c)) = 2p$, $A_1(p)$-projectivity implies $dim(C_{T(c)}(u)) = 2$. Now consider the fixed points of U on $T(c)$. Clearly U fixes a weight vector of weight c (the high weight), but also one of weight $p - r - 2$, as this is the high weight of the socle. Since u also fixes these vectors we have $C_{T(c)}(u) = C_{T(c)}(U)$, a 2-space. Note that this 2-space lies in the unique maximal submodule of $T(c)$.

Finally, set $L(U) = \langle e \rangle$ and note that e acts on $L(r + 1) \otimes L(p - 1)$ as the sum of Jordan blocks of length p, so this also holds for $T(c)$. Of course, e is trivial on $C_{T(c)}(U)$, which gives (d).

Lemma 1.4. Assume $X = A_1$ and $0 \leq r \leq p - 2$. Set $c = r + p, d = p - r - 2$ and let V be a module with all composition factors isomorphic to either $L(c)$ or $L(d)$. Then

$$V = T(c)^r \oplus W(c)^s \oplus (W(c)^*)^t \oplus L(c)^u \oplus L(d)^v.$$

Moreover, V is self-dual if and only if $s = t$.

Proof. First note that by 1.2

(i). $Ext^1(L(c), L(c)) = Ext^1(L(d), L(d)) = 0$ and (ii). $Ext^1(L(c), L(d))$ has dimension 1,

where a nontrivial extension is realized by the Weyl module $W(c)$. So from (i) and the existence of $T(c)$ we have

(iii). $Ext^1(W(c)^*, L(d))$ has dimension 1,

a nontrivial extension realized by $T(c)$. Further, (i) and (ii) imply

(iv). $Ext^1(L(d), T(c)) = 0$.

Also, $W(c)$ is universal among cyclic modules generated by weight vectors of weight c. That is, if $v \in V$ is a weight vector of weight c, then Xv is an image of $W(c)$, so that $Xv \cong W(c)$ or $L(c)$. This and (ii) imply

(v). $Ext^1(L(c), T(c)) = 0$.

The irreducible modules for X are self dual, so the hypothesis also applies to V^*. Let S be the sum of all spaces $Xv \leq V^*$ where v is a weight vector of weight c. As above each of these cyclic modules is isomorphic to either $W(c)$ or to $L(c)$ and hence $S \cong W(c)^e \oplus L(c)^f$. Also, $V^*/S \cong L(d)^l$. Taking duals there is a submodule R of V such that

$$V/R \cong (W(c)^*)^e \oplus L(c)^f$$

$$R \cong L(d)^{l.}$$

Let J be the preimage over R of the summand $(W(c)^*)^e$. It follows from (iii) and the unicity of $T(c)$ that

$$J \cong T(c)^r \oplus (W(c)^*)^{e-r} \oplus L(d)^{l-r}.$$

Now (v) and the above implies that we can write

$$V \cong T(c)^r \oplus D,$$

where

$$D/J \cap D \cong L(c)^f$$

and

$$J \cap D \cong (W(c)^*)^{e-r} \oplus L(d)^{l-r}.$$

Now D is the sum of $J \cap D$ and all cyclic modules Xv for weight vectors $v \in D - (J \cap D)$ of weight c. The first assertion now follows using (ii). The second statement is an easy consequence.

2. Unipotent Classes and Labelled Diagrams.

In this section we present information on unipotent classes in simple algebraic groups and labelled Dynkin diagrams. We begin with general results, in particular results which link characteristic 0 and characteristic p. Later we separate the discussions of unipotent elements in classical and exceptional groups.

Let G be a simple algebraic group over an algebraically closed field K, where $\mathrm{char} K = p$, a good prime for G.

Conjugacy Classes

Let $P = QL$ be a parabolic subgroup of G, with $Q = R_u(P)$ and L a Levi subgroup. It is well known P has a dense orbit on $R_u(P)$ (see 5.2.3 of [5]), called the Richardson orbit. We say that P is *distinguished* if $\dim L = \dim(Q/Q')$. A complete list of distinguished parabolic subgroups of G is recorded in Carter [5], 174-177.

If L is a Levi subgroup of G and $P_{L'}$ a distinguished parabolic subgroup of L' we call the pair $(L, P_{L'})$ a *distinguished pair*. To each distinguished pair $(L, P_{L'})$ we associate the unipotent class of G containing the Richardson orbit of $R_u(P_{L'})$. Conjugating the pair by an element of G does not alter the resulting class, so the above map induces a map $\phi : \mathcal{D} \to \mathcal{U}$, where $\mathcal{D}$ is the set of G-classes of distinguished pairs and $\mathcal{U}$ is the set of the unipotent classes of G. Both $\mathcal{D}$ and $\mathcal{U}$ are finite sets.

Proposition 2.1 (Bala-Carter [4], Pommerening [17]). $\phi : \mathcal{D} \to \mathcal{U}$ is a bijection.

It follows from 2.1 that the classification of unipotent elements for groups of good characteristic is the same as that in characteristic 0. This is because the set of Levi subgroups and the distinguished parabolic subgroups within these Levi subgroups is the same in either characteristic.

In fact, more is true. Using results of Mizuno and others, Lawther [10] shows that one may choose representatives of the conjugacy classes of unipotent elements as explicit products of root elements and these representatives are the same in characteristic p and 0.

Special cases of 2.1 occur when $(L, P_{L'}) = (G, P)$, with P a distinguished parabolic subgroup of G. It is easily seen that the unipotent elements corresponding to such pairs are precisely those for which $C_G(u)$ does not contain a nontrivial torus. Such unipotent classes are called *distinguished*. That is, u is distinguished if and only if $C_G(u)^o$ is unipotent.

If u is an arbitrary unipotent element of G, let T be a maximal torus of $C_G(u)$. Then $L = C_G(T)$ is a Levi subgroup and u is a distinguished unipotent element of L'. So u is in the Richardson orbit of a distinguished parabolic subgroup of L' and it follows from 2.1 that $(L, P_{L'})^G$ and u^G correspond under ϕ.

Another special case occurs when $(L, P_{L'}) = (G, B)$, where B is a Borel subgroup. The resulting unipotent class is the class of *regular* unipotent elements.

On occasion we will require one further piece of terminology. A unipotent element is called *semiregular* if $C_G(u)$ contains no noncentral semisimple element. These unipotent elements cannot be embedded in proper maximal rank subgroups, hence form the base case of inductive arguments.

Labelled Diagrams

We next discuss labelled Dynkin diagrams, starting with the situation in characteristic 0.

Let $\hat{G}$ be a simple algebraic group over a closed field of characteristic 0 of the same type as G and $\hat{u} \in \hat{G}$ a unipotent element. Then $\hat{u}$ arises by exponentiation from a nilpotent element $\hat{e}$ of $L(\hat{G})$ which can be embedded in a subalgebra of type sl_2, using the Jacobson-Morozov theorem. Exponentiation yields a group $\hat{A} \leq \hat{G}$ of type A_1 containing $\hat{u}$. Choose a maximal torus $T_{\hat{A}}$ of $\hat{A}$ and embed this in a maximal torus $T_{\hat{G}}$ of $\hat{G}$. Each $T_{\hat{G}}$ root element of $L(\hat{G})$ affords an integral weight for $T_{\hat{A}}$. A fundamental system of root groups can be chosen such that the corresponding weights are all nonnegative. A result due to Dynkin (see 5.6.6 of [5]) shows these weights are $0, 1, 2$ and when $\hat{u}$ is distinguished only the weights $0, 2$ occur. The labelled diagram is then the Dynkin diagram with corresponding labels, $0, 1, 2$.

The labelled diagram described above is an invariant which determines the unipotent class. To see why this might be true consider the fact that by linearity the labelled diagram determines all weights of $T_{\hat{A}}$ on $L(\hat{G})$ and hence determines the composition factors of $\hat{A}$ on $L(\hat{G})$. As the action is completely reducible this, in turn, determines the Jordan decomposition of $\hat{u}$ on $L(\hat{G})$, which turns out to determine the class.

Now consider the situation for p a good prime. Testerman [20] shows that a unipotent element u of order p can be embedded in a group A of type A_1 and constructs it explicitly for semiregular elements. Other types of unipotent elements are contained in maximal rank subgroups so that induction can be applied. In particular, if T is a maximal torus of $C_G(U)$, then $A < C_G(T)$, so that the construction respects the Bala-Carter classification of 2.1. These turn out to be good $A_1's$.

Given u and A, as in the preceding paragraph, we can obtain another labelled diagram following the procedure mentioned above, where we consider the weights of a maximal torus of A, choose an appropriate positive system, etc. For exceptional groups it is shown in 6.3 of [12] that the nonnegative labelled diagram is uniquely determined (up to a graph automorphism) by the collection of weights. For classical groups the same holds. Indeed, here A acts on the natural module as a direct sum of restricted irreducible modules. Thus the weights of a maximal torus of A on this module are determined by the Jordan form of u. At this point it is easy to argue as in 6.3 of [12] to see that the labeling is uniquely determined, up to a graph automorphism.

Now let $u \in G$ of order p and $\hat{u} \in \hat{G}$ correspond via the Bala Carter classification. From each of these elements we obtain a labelled Dynkin diagram, using groups $A, \hat{A}$ of type A_1, where these labelled diagrams depend only on the conjugacy class of the unipotent elements.

Proposition 2.2. ([11], Theorem 4.2). Let $u \in G$ have order p and $\hat{u}$ a corresponding element of $\hat{G}$. The labelled Dynkin diagrams associated to u and $\hat{u}$ are equal. Consequently, the weights of A on $L(G)$ and those of $\hat{A}$ on $L(\hat{G})$ coincide.

Centralizers

We will require information on centralizers of unipotent elements, starting with information on component groups of centralizers. For classical results we refer to Springer-Steinberg [21], IV, 2.26 and for exceptional groups we rely on work of Mizuno ([15], [16]), Shoji ([20]), and Chang ([6]).

Proposition 2.3. Assume G is of adjoint type. Then $C_G(u)/C_G(u)^o \cong C_{\hat{G}}(\hat{u})/C_{\hat{G}}(\hat{u})^o \cong (Z_2)^r, S_3, S_4, S_5$. If G is of exceptional type, then $r = 1$.

We next record a result on dimensions of centralizers. Of course, $C_G(u)$ and $L(C_G(u))$ have the same dimension. For good primes $L(C_G(u)) = C_{L(G)}(u)$ (Carter [5], 1.15), although the case $G = A_{n-1}$ with $p|n$ must be excluded. In the exceptional case we can assert $L(C_{GL_n(K)}(u)) = C_{L(GL_n(K))}(u)$. This follows as in the proof of [21], III, 3.22, noting that $GL_n(K)$ is the set of invertible elements in $M_n(K) = L(GL_n(K))$.

For exceptional groups Lawther [10] gives the precise Jordan block structure for the action of u on $L(G)$ and $\hat{u}$ on $L(\hat{G})$. The number of Jordan blocks is of course the dimension of the fixed point space. For classical groups the reader is referred to [21], IV, 1.8, 2.25, and 2.28 for centralizer dimensions. Combining these results with the observations of the previous paragraph we have

Proposition 2.4. Assume that either G is simple but $G \neq A_n$ with

$p|n+1$ or that $G = GL_n$. Let $u \in G$ and $\hat{u} \in \hat{G}$ correspond as above. Then

$$\dim(C_G(u)) = \dim(C_{L(G)}(u)) = \dim(C_{L(\hat{G})}(\hat{u})) = \dim(C_{\hat{G}}(\hat{u})).$$

Elements of order p

It will be important to know when unipotent elements have order p. We record the following result of Testerman covering the case of distinguished elements. If P is a parabolic subgroup let $cl(R_u(P))$ denote the nilpotence class of $R_u(P)$. Lemma 4 of [3] shows that $cl(R_u(P))$ can be expressed in terms of heights of certain roots. Indeed, let $\Pi = \{\alpha_1, ..., \alpha_r\}$ be a fundamental set of roots and write $P = P_J$, where $J \subset \Pi$. Notation is chosen such that a Levi subgroup of P has J as fundamental set of roots. If $\delta = \sum_{\alpha \in \Pi} m_\alpha \alpha$ is a root, write $ht_J(\delta) = \sum_{\alpha \in \Pi - J} m_\alpha$. If α_o denotes the root of highest height, then $ht_J(\alpha_o) = cl(R_u(P))$.

Proposition 2.5 Let $P = P_J$ be a distinguished parabolic subgroup of G and u a unipotent element in the dense orbit of P on $R_u(P)$. Then $|u| = min\{p^a : p^a > ht_J(\alpha_o) = cl(R_u(P))\}$.

In the following we give separate discussions of unipotent elements for the classical and exceptional groups. For classical groups we link the above discussion with the Jordan decomposition of the underlying classical module.

Classical groups

Let V be a finite dimensional space over K, equipped with either the 0 form or a nondegenerate symmetric or skew-symmetric form. It will be convenient to work with the full classical group of V, hence $G = I(V) = GL(V), Sp(V)$, or $O(V)$.

If V is a space of dimension i, then $GL(V)$ contains a (regular) unipotent element acting on V as a single Jordan block. Now suppose that i is even and V has a nondegenerate symplectic form or that i is odd and V has a nondegenerate orthogonal form. Then the corresponding classical group $Sp(V)$ or $O(V)$ has such a unipotent element. Indeed, in a given system of root subgroups, choose a nontrivial root element from each root group for fundamental roots. The product will induce a single Jordan block on V in a suitable basis.

Conjugacy classes of unipotent elements in $GL(V)$ are, of course, determined by their Jordan decompositions. The same holds for orthogonal and symplectic groups although there are restrictions on the types of Jordan decompositions that can occur. Let J_i denote a Jordan block of size i, with 1's on the diagonal.

If u is a unipotent element of $GL(V)$, decompose V under the action of u into Jordan blocks

$$V = \bigoplus_i V_i = \bigoplus_i (J_i)^{r_i},$$

so that each V_i is the sum of r_i Jordan blocks of size i.

Proposition 2.6. ([21], IV, 2.19) (i) A conjugate of u is contained in $Sp(V)$ if and only if r_i is even whenever i is odd.

(ii). A conjugate of u is contained in $O(V)$ if and only if r_i is even whenever i is even.

(iii). Two unipotent elements of $Sp(V)$ or $O(V)$ are conjugate if and only if they are conjugate in $GL(V)$.

We next discuss a method for constructing unipotent elements of $Sp(V)$ and $SO(V)$ which also provides information on centralizers. In the above notation, write

$$V = \bigoplus_i V_i$$

and for each i

$$V_i = W_i \otimes Z_{r_i},$$

a tensor product of spaces of dimensions i and r_i, respectively.

Consider embeddings of classical groups of the form

$$I(W_i) \circ I(Z_{r_i}) < I(V_i),$$

where the product is a central product. When working in $GL(V)$ the containment is just $GL(W_i) \circ GL(Z_{r_i}) < GL(V_i)$. When working in $Sp(V)$ or $O(V)$ the groups $I(W_i)$ and $I(Z_{r_i})$ are always taken as orthogonal groups in odd dimension and symplectic groups in even dimension. The embeddings are obtained via the product form. For instance $SO(W_i) \circ Sp(Z_{r_i}) < Sp(V_i)$ when i is odd and r_i is even, and $Sp(W_i) \circ Sp(Z_{r_i}) < SO(V_i)$ when i and r_i are both even. Notice that these embeddings are consistent with the parity requirements of 2.6.

Choose a unipotent element $v_i \in I(W_i)$ acting on W_i as a single Jordan block. Then v_i acts homogeneously on V_i as the sum of r_i copies of J_i. Given a decomposition $V = \bigoplus_i V_i$ as above for which i odd (resp. even) in the symplectic (resp. orthogonal) case implies r_i even, set $u = \prod u_i$, and obtain a unipotent element in $I(V)$, where the form is chosen such that the spaces V_i are pairwise orthogonal. It is clear from the construction that

$C_{I(V)}(u) > \prod_i I(Z_{r_i})$. It follows from [21], IV, 2.26 that this product is the reductive part of $C_{I(V)}(u)$.

Proposition 2.7. Let $u = \prod u_i$ be as above.
(i). $|u| = p$ if and only if $p \geq i$ for each i.
(ii). $C_{I(V)}(u) = QR$, a semidirect product, where $Q = R_u(C_{I(V)}(u))$ and $R = \prod_i I(Z_{r_i})$ is as follows:
$GL(V): \quad R = \prod_i GL_{r_i}(K)$.
$Sp(V): \quad R = \prod_{i\ odd} Sp_{r_i}(K) \times \prod_{i\ even} O_{r_i}(K)$.
$O(V): \quad R = \prod_{i\ even} Sp_{r_i}(K) \times \prod_{i\ odd} O_{r_i}(K)$.

The component group of $C_{I(V)}(u)$ is clear from 2.7. It is trivial for $G = GL(V)$. For $G = Sp(V)$ or $O(V)$ it is the just the product of groups of type Z_2, one for each even i in $Sp(V)$ and each odd i in $SO(V)$.

It is easy to connect the analysis here with the earlier discussion of distinguished unipotent elements in simple algebraic groups. We now take G simple, $G = SL(V), Sp(V)$, or $SO(V)$.

For $SL(V)$ only the regular class is distinguished. The distinguished unipotent elements are described in $G = Sp(V)$ or $G = SO(V)$ as follows. Write $V = V_1 \perp ... \perp V_r$ as a direct sum of nondegenerate spaces of different dimensions, all of which have odd dimension in case V is an orthogonal group and even dimension for G symplectic. Choose a regular unipotent element in each of the corresponding classical groups. The product is a distinguished unipotent element of G. Notice that this element has order p if and only if $p \geq \dim(V_i)$ for each i.

We can also determine the semiregular unipotents, where $C_G(u)$ contains no nonidentity semisimple elements. It follows from 2.7 that in $SL(V), Sp(V)$, and $SO(V)$ for $\dim(V)$ odd, there is only one semiregular class, the class of regular elements (having a single Jordan block). However, for even dimensional orthogonal groups there are several semiregular classes and we record standard notation for these. Write $\dim(V) = 2n$ so that $G = D_n$ and for $0 \leq r \leq (n-2)/2$ consider a unipotent element $u = D_n(a_r)$ with Jordan decomposition $V = J_{2r+1} \perp J_{2n-2r-1}$. Then $u \in B_r \times B_{n-r-1} < D_n$. So $D_n(a_0)$ (notation usually just D_n) is a regular unipotent element of D_n, lying in B_{n-1}. Notice that for each of the elements $u = D_n(a_r)$ there is a noncentral involution in $O(V)$ centralizing u (inducing -1 on precisely one of the Jordan blocks). But this element does not lie in $SO(V)$, so that u is indeed semiregular in $SO(V)$.

Exceptional groups

Assume G is a simple algebraic group of exceptional type and u a unipotent element of G taken as in Lawther [10]. Let $\hat{u} \in \hat{G}$ be the corresponding unipotent element in characteristic 0. Carter [5], 401-407 has useful tables which list the unipotent classes in $\hat{G}$, and these tables include labelled diagrams, centralizer dimensions, and component groups of centralizers. In view of of 2.1-2.4 this information applies equally to unipotent classes in G.

Proposition 2.2 was established for a particular choice of A. For unipotent elements that are not semiregular, this subgroup was constructed within certain maximal rank subgroups. We will require extra information regarding maximal rank subgroups containing A.

We may take G to be adjoint. Fix a unipotent element, $u \in G$ of order p. If u is not distinguished let T be a maximal torus of $C_G(u)$. The group A used for 2.2 was constructed in $L = C_G(T)$, a Levi subgroup as in the Bala-Carter classification of 2.1.

Now assume u is distinguished. Then 2.3 implies $C_G(u)/C_G(u)^o \cong 1, Z_2, S_3, S_4$, or S_5 (the last two cases occur for just one class in F_4, E_8, respectively). The assumption that p is a good prime implies that p is greater than any prime divisor of the component group and hence $C_G(u) = QS$, a semi-direct product, where $Q = R_u(C_G(u))$ is unipotent and S is finite of the indicated type.

If $s \in S$ is any nonidentity semisimple element, then $C_G(s)$ is reductive of maximal rank and contains u. The verification of 2.2 involved a particular choice of s.

Let $E < S$ with $E \cong 1, Z_2, Z_3, Z_2 \times Z_2$, (the Klein four group) or Z_5, respectively. Note that E corresponds to a normal subgroup of the component group, unless the component group is S_5. We will have occasion to work with $C_G(E)$, a maximal rank subgroup containing u and for which u is semiregular.

Proposition 2.8. If u is distinguished, then $u \in A$, where A is a good A_1 satisfying 2.2 and contained in $C_G(E)$.

Proof. In the regular case the result is obvious, so assume u is distinguished but not semiregular. The component group of $C_G(u)$ is typically Z_2, so $C_G(E)$ agrees with the maximal rank subgroup $C_G(s)$ used to establish 2.2. And in some of the cases where $S = S_3$, s was taken as an element of order 3, hence generating E. However, there are 4 cases where the maximal rank group $C_G(s)$ used to establish 2.2 differs from $C_G(E)$. These are the semiregular classes of types $F_4(a_3), E_7(a_5), E_8(b_5), E_8(b_6)$, where notation is in the tables of Carter [5].

For these cases the result can be easily verified by direct check and we will do so for the case of $E_7(a_5)$. In the other cases, to conserve space we refer to the tables of [11], which describe all $A_1's$ containing u and their labelled diagrams, up to conjugacy, assuming that p is suitably large. The required characteristic restriction is a consequence of 2.4.

So consider the case where u has type $E_7(a_5)$ (the characteristic restriction of [11] may fail to hold here), where $E = Z_3$. Set $D = C_G(E)$. Since u centralizes no nontrivial torus, D must be semisimple of maximal rank, centralizing an element of order 3. A consideration of subsystems shows that $D = A_2A_5$. Then u projects to a regular element in each factor and we embed u in a group A of type A_1 which induces an irreducible restricted representation on the natural module for each factor. It is shown in 2.1 of [12] that

$$L(G)|D = L(D) \oplus (L_{A_2}(\lambda_1) \otimes L_{A_5}(\lambda_2)) \oplus (L_{A_2}(\lambda_2) \otimes L_{A_5}(\lambda_4)).$$

From here it is easy to compute the weights of a maximal torus of A. Comparing with the weights obtained from the labelled diagram of the class, we obtain equality.

3. Existence of good $\mathbf{A_1'}$s and tilting decompositions

In this section we show that if G is a simple algebraic group with p a good prime, then each unipotent element of order p is contained in a good A_1 and that the Lie algebra of G restricts to this A_1 as a tilting module, provided the Lie algebra is self-dual.

We first consider classical types. The isogeny type of G is unimportant for the result to be established, so in the classical types it will be convenient to take $G = SL(V), Sp(V)$, or $SO(V)$. We also consider $G = GL(V)$, even though this is not a simple group.

Proposition 3.1. Let u be a unipotent element of order p in $G = GL(V), SL(V), Sp(V)$, or $SO(V)$. There is a closed connected subgroup A containing u such that the following conditions hold

(i). A is a good A_1 and, in the notation preceding 2.6, $A \leq \prod_i I(W_i)$ with irreducible restricted projection to each factor.

(ii). $V|A$ is completely reducible with each composition factor restricted.

(iii). $L(G)|A$ is a tilting module unless $G = SL_n$, u has a Jordan block on V of size p, and $p|n$.

(iv). The weights of A on $L(G)$ determine the same labelled diagram as u, so that A can be taken as the group of type A_1 described in 2.2.

Proof. For the moment we will work within the full classical group $GL(V), Sp(V)$, or $O(V)$. Decompose V into Jordan blocks for u as in the discussion following 2.6 and for each i consider the corresponding embedding of classical groups $I(W_i) \circ I(Z_{r_i}) \leq I(V_i)$. Then $I(W_i)$ is one of the following groups: $GL_i(K), Sp_i(K)$ (i even), or $O_i(K)$ (i odd). As in the discussion we may take $u \in \prod_i I(W_i)$, such that each projection is a regular unipotent element.

The group $SL_2(K)$ has irreducible restricted representations of degrees $1, ..., p$. For $0 \leq c \leq p-1$, the irreducible restricted representation, $L(c)$ has degree $c+1$. This representation is self dual, symplectic for c odd and orthogonal for c even.

It follows that for each i we may embed the projection of u to $I(W_i)$ in a subgroup of type A_1 which acts irreducibly on W_i as $L(i-1)$. In this way we have embedded u in a group $A \leq \prod_i I(W_i)$ of type A_1 which respects the Jordan decomposition $V|u$ and such that $A \leq G$.

The argument for 2.2 in [11] was based on the Bala-Carter classification of unipotent elements. It follows from (i) and 2.7 that $A < C_G(T)$ where T is a maximal torus of $C_G(u)$, so verification of (iv) reduces to the distinguished case. For u distinguished, then the group used in [11] to establish 2.2 in the classical cases is exactly as above.

Write $V|A = \bigoplus L(c_i)$, a direct sum of restricted irreducible representations. Hence $L(GL(V))|A = (V \otimes V^*)|A = \sum (L(c_i) \otimes L(c_j))$. For each i and j, $L(c_i)$ and $L(c_j)$ are tilting modules, hence by Lemma (1.1)(d) so is their tensor product. Moreover, $c_i + c_j \leq 2p-2$. This establishes (i)-(iii) for $G = GL(V)$.

Now consider $SL(V)$. If $L(SL(V))$ is an A - direct summand of $L(GL(V))$, then by Lemma (1.1)(c) it is also a tilting module for A. Otherwise, $p|n$, and as $L(SL(V))$ has codimension 1 in $L(GL(V))$, (1.3)(b) implies that $T(2p-2)$ must be a direct summand of $L(GL(V))$. This forces $c_i = p-1$ for some i. Thus (iii) holds, while (i) and (ii) are inherited from $GL(V)$.

Next consider symplectic and orthogonal groups. Here we are assuming that $p > 2$, so $V \otimes V^* = \wedge^2(V) \oplus S^2(V)$. Therefore, the Lie algebras of $Sp(V)$ and $O(V)$ are each a direct summand of $V \otimes V^* = L(GL(V))$, hence by Lemma 1.1(c) they are also tilting modules for A.

Remark. Let A be a good A_1 containing u as in 3.1 and view A as the image of $SL_2(K)$ with natural module having basis $\{x, y\}$. Let U be the root group of A containing u and write $u = U(1)$, corresponding to the 2×2 matrix with 1's below the main diagonal and 0 above the main diagonal. Now consider a restricted irreducible module for A, say of high weight c which we can identify with the homogeneous polynomials of degree c. If we write $u = 1+e$, it follows that $U(t)$ acts as $1+te+(t(t-1)/2)e^2+\cdots = (1+e)^t$, thereby justifying the exponential notation $U(t) = u^t$ mentioned in the introduction.

We now consider exceptional groups.

Proposition 3.2. Assume G is an exceptional group and $u \in G$ is a unipotent element of order p. There is a closed connected subgroup A containing u such that the following conditions hold

(i). A is a good A_1 with labelled diagram satisfying 2.2.

(ii). $L(G)|A$ is a tilting module.

Proof. The first step is to establish the existence of A, a good A_1 containing u.

Choose u in one of the conjugacy classes given by 2.1. By 2.2 there is a group A of type A_1 affording the same labelled diagram as indicated in p.401 - 407 in [5]. The labelled diagram determines the weights of a maximal torus T_A of A afforded by root vectors for fundamental roots. By linearity this determines the weight on the root of highest height, which affords the largest weight. It remains to verify that this weight is at most $2p-2$.

This is easy in the distinguished case. For if u is distinguished with $P = P_J$ the corresponding distinguished parabolic subgroup, then 2.5 gives $p > ht_J(\alpha_o)$. The labelling of the Dynkin diagram is such that the label is 0 for elements of J and 2 for elements of $\Pi - J$. Hence, the highest weight of T_A on $L(G)$ is precisely $2ht_J(\alpha_o) \leq 2(p-1)$, as required.

Now suppose u is not distinguished, so that $u \in A < L'$, where L is a Levi subgroup and L' is specified in the tables of [5]. Moreover, u is distinguished in L'. At this point a routine check yields the result. One simply computes the largest weight on $L(G)$ (afforded by α_o) using the labelled diagram and bounds on $|u|$ as indicated in 2.5 and 2.7. The result also follows from Theorem 1 of [10] (the proof of which involves a similar check) which gives the largest Jordan block of $\hat{u}$ on $L(\hat{G})$, thus the largest weight of $\hat{A}$ on $L(\hat{G})$.

It remains to show that $L(G)|A$ is a tilting module. As usual we let $\hat{G}$ be the corresponding exceptional group in characteristic 0 and $\hat{u}$ a unipotent element in $\hat{G}$ corresponding to u. By 2.2 we can embed $\hat{u}$ in $\hat{A} \cong A_1$ so that A and $\hat{A}$ determine the same labelled diagram. Consequently they have precisely the same weights on $L(G), L(\hat{G})$, respectively.

Compare the restrictions $L(G)|A$ and $L(\hat{G})|\hat{A}$. All weights are at most $2p-2$. Let $c \leq 2p-2$ be a nonnegative weight. Then $L_{\hat{A}}(c)$ and $W_A(c)$ have the precisely the same weights. If $c \leq p$, then $W_A(c) = L_A(c)$, so that $L_{\hat{A}}(c)$ and $L_A(c)$ have the same weights. On the other hand, if $c > p$, write $c = r + p$. Then $W_A(c) = L(p-r-2)/L(r+p)$.

The weights determine the composition factors of $L(\hat{G})|\hat{A}$ and $L(G)|A$. So from the above remarks it is clear that each composition factor $L_{\hat{A}}(c)$ in the first restriction corresponds to either one composition factor $L_A(c)$ (if $c < p$) or two composition factors $L_A(r+p), L_A(p-r-2)$, (if $c = r+p$), in the second restriction.

For nonnegative integers $c, d \leq 2p-2$, 1.2(iii) shows $Ext^1(L(c), L(d)) = 0$, unless $\{c, d\} = \{r+p, p-r-2\}$ for some $0 \leq r \leq p-2$. It follows that there is a corresponding block decomposition of $L(G)|A$. Indeed, for $0 \leq r \leq p-2$, there is a submodule $V(r+p) < L(G)$ having all composition factors of the form $L(r+p), L(p-r-2)$ and having a complement involving neither of these composition factors. Similarly if $d < p$ is a nonnegative integer not of the form $p-r-2$ for some weight $c = r+p$, there is a submodule $V(d)$ which is completely reducible, homogeneous of type $L(d)$, and having a complement with no composition factors of this type. We then have

$$L(G) \ = \bigoplus_{0 \leq r \leq p-2} V(r+p) \quad \oplus \quad \bigoplus_{d} V(d).$$

There is a similar decomposition for $L(\hat{G})$ under the action of $\hat{A}$. For $0 \leq r \leq p-2$, let $\hat{V}(r+p)$ be the sum of all irreducible modules of $L(\hat{G})|\hat{A}$ of high weights $r+p$ or $p-r-2$. And if $0 \leq d < p$ but d is not of the form $p-r-2$, let $\hat{V}(d)$ be the sum of all submodules $L_{\hat{A}}(d)$. We then have

$$L(\hat{G}) \ = \bigoplus_{0 \leq r \leq p-2} \hat{V}(r+p) \quad \oplus \quad \bigoplus_{d} \hat{V}(d).$$

The weights in corresponding summands of each expression coincide. In particular, for fixed $0 \leq r \leq p-2$, the weights appearing in $V(r+p)$ and $\hat{V}(r+p)$ are the same.

As p is a good prime, $L(G)$ is self-dual, hence so is $V(r+p)$. Therefore, Lemma 1.4 implies

$$V(r+p) = L(r+p)^i \oplus L(p-r-2)^j \oplus W(r+p)^k \oplus (W(r+p)^*)^k \oplus T(r+p)^{l}.$$

We also have

$$\hat{V}(r+p) = L(r+p)^s \oplus L(p-r-2)^t.$$

Comparing multiplicities of the weight $p-r-2$ in $\hat{V}(r+p)$ and $V(r+p)$ we have $s+t = j+k+k+2l$. Now $\dim C_{\hat{V}(r+p)}(\hat{u}) = s+t$, while $\dim C_{V(r+p)}(u) = 2i+j+2k+2k+2l$ (see (1.3)(d) and its proof). The latter number is thus greater than or equal to the former, with equality only if $i = k = 0$.

Notice also that the summands $V(d)$ and $\hat{V}(d)$ for $0 \leq d < p$ are each homogenous of the same dimension, so that $\dim C_{\hat{V}(d)}(\hat{u}) = \dim C_{V(d)}(u)$.

By 2.4, the fixed point space of u on $L(G)$ must have the same dimension as that of $\hat{u}$ on $L(\hat{G})$. Hence, equality must hold summand by summand in the displayed decompositions. In the case of $V(r+p)$, this shows that $V(r+p) = L(p-r-2)^j \oplus T(r+p)^l$, a tilting module. This gives the required tilting decomposition for $L(G)$.

4. Exponentiation

In this section we establish exponentiation results linking the unipotent radical of a parabolic subgroup with its Lie algebra. This will be useful in obtaining information on centralizers of distinguished unipotent elements.

Let X be a split semisimple group over $\mathbb{Z}$, T its standard maximal torus and Σ its root system.

Let Γ be a subset of Σ which is closed and for which $\Gamma \cap -\Gamma = \emptyset$. Such a set defines a subgroup scheme X_Γ of X, which is a solvable group, containing T, and of the form $U_\Gamma T$ where U_Γ is unipotent and generatd by the root subgroups corresponding to Γ (see [7], p. 212).

Let u_Γ denote the Lie algebra of U_Γ. Then u_Γ has a $\mathbb{Z}$-basis of root elements e_α, for $\alpha \in \Gamma$. On the other hand, U_Γ can be written as the product of the root subgroups $U_\alpha, \alpha \in \Gamma$, in any fixed order.

Now consider the same groups over $\mathbb{Q}$, instead of over $\mathbb{Z}$. We have the exponential map $exp : u_\Gamma \to U_\Gamma$, which is an isomorphism, viewing u_Γ as an algebraic group, via the Hausdorff formula. Define an integer $h(\Gamma) = cl(U_\Gamma) + 1$, where $cl(\Gamma)$ denotes the nilpotence class of U_Γ, over $\mathbb{Q}$.

The argument in 2.2 of [19] yields the following result.

Proposition 4.1. Assume $p \geq h(\Gamma)$. The exponential isomorphism $u_\Gamma \to U_\Gamma$ is defined over the local ring $\mathbb{Z}_{(p)}$, and so is its inverse (the "logarithm").

A corollary of the proposition is that the power map morphism

$$U_\Gamma \times Aff^1 \to U_\Gamma, \quad (x,t) \to x^t$$

is also defined over $\mathbb{Z}_{(p)}$. We note that these maps and isomorphisms are compatible with the action of X_Γ on U_Γ by conjugation:

$$exp(Ad(g)l) = g(exp(l))g^{-1},$$

where l and g are points of u_Γ and X_Γ. Similarly, $gx^tg^{-1} = (gxg^{-1})^t$, for $x \in U$. This follows from the fact that such formulae are true over $\mathbb{Q}$, hence also over $\mathbb{Z}_{(p)}$.

Now pass to characteristic $p > 0$. Let k be a field of characteristic p and X a semisimple group over k (not necessarily split), and R a connected solvable subgroup of X containing a maximal torus T of X. Over suitable extension of k, one may split T and R is then given by a subset Γ of the corresponding root system as above. Moreover, Γ is essentially independent of the choice of T, hence it makes sense to speak of $h(\Gamma)$. Indeed if one splits R as a semidirect product UT, then the nilpotence class of U is $h(\Gamma) - 1$, except when the Dynkin diagram involves double bonds with $p = 2$ or triple bonds with $p \leq 3$. Let u be the Lie algebra of U, which is also of nilpotence class $h(\Gamma) - 1$.

Proposition 4.2. Assume $p \geq h(\Gamma)$. There exists one and only one isomorphism of k-algebraic groups $exp : u \to U$ with the following properties:
(i). Its tangent map is the identity.
(ii). It is compatible with the action of R by conjugation.

Proof. (a). Existence, assuming k separably closed. Here, T is split and the pair (X, R) comes from the pair (X, X_Γ) in characteristic 0 by the base change $\mathbb{Z}_{(p)} \to \mathbb{F}_p \to k$. Proposition 4.1 yields the assertion here.

(b). Unicity, assuming k separably closed. Choose T, which is split. Let e_α, U_α be as before and let T_α be the connected component of the kernel of $\alpha : T \to G_m$, a subtorus of T of codimension 1. If $\phi : u \to U$ is any isomorphism with properties (i) and (ii), then T_α centralizes ke_α, hence its image under ϕ. This implies that $\phi(ke_\alpha) = U_\alpha$. But there is only one automorphism of G_a with tangent map the identity, namely the identity. Hence ϕ maps te_α to $U_\alpha(t)$. Thus ϕ and exp coincide on each line ke_α. As these lines generate u, we see that ϕ and exp coincide.

(c). General case. Unicity follows from (b), while existence follows from (a) and (b) by Galois descent.

As a corollary we again get a power map

$$U \times Aff^1 \to U, \quad (x, t) \to x^t,$$

defined by $exp(l)^t = exp(tl)$

We now consider unipotent radicals of parabolic subgroups. Let P be a parabolic subgroup of X, and $U = R_u(P)$. It is well-known that $N_X(U) = P$, so that P is determined by U. If T is a maximal torus, then $R = UT$ is of the type considered in 4.2. There is a set Γ of roots as considered earlier and we set $h(P) = h(\Gamma)$.

Proposition 4.3.. Assume $p > cl(U)$. Then there is one and only one isomorphism $exp : u \to U$ whose tangent map is the identity and which is compatible with P-conjugation and with conjugation by any automorphism of X normalizing P.

Proof. Unicity. Choose a maximal torus T of P and put $R = UT$. Since R is contained in P, any isomorphism $\phi : u \to U$ satifying the conclusion of the proposition also satisfies the conditions of Proposition 4.2 and hence is unique.

Existence. As in the proof of Proposition 4.2 we may assume that k is separably closed, hence that X is split and P is standard. The map exp then comes by reduction mod p from characteristic 0, and its compatibility with P-conjugation follows from the corresponding fact in characteristic 0, where it is obvious.

The next result describes one parameter unipotent groups in U and resembles the Steinberg tensor product theorem for irreducible representations.

Proposition 4.4. Assume $p > cl(U)$ and let $f : G_a \to U$ be a one parameter subgroup of U. Then there are commuting elements $e_0, e_1, e_2 ...$ of $L(U)$ with $e_n = 0$ for large n such that

$$f(t) = exp(e_0 t) \cdot exp(e_1 t^p) \cdot exp(e_2 t^{p^2}) \cdots$$

Proof. Set $u = L(U)$ as before. We proceed in a series of steps.

(i). If $x, y \in L(U)$, then x, y commute in the group structure of u (given by the Hausdorff formula) if and only they commute in the Lie algebra structure.

To see this note that the group commutator has the form $[x, y]+$ higher terms, and that $[x, y] = 0$ implies all the higher terms are 0. The converse may be proved via an "inverse Hausdorff formula", expressing $[x, y]$ as the group commutator times higher terms, where the higher terms are in the descending central series of the group generated by x, y. The terms are all 0 if x, y commute.

(ii). Let f, f' be two polynomial maps of the affine line into u. Write $f(t) = \Sigma a_i t^i, f'(t) = \Sigma a'_i t^i$, with $a_i, a'_i \in u$. Then $f(t)$ and $f'(t')$ commute for every t, t' (in every extension of k in case k is finite) if and only if $[a_i, a'_j] = 0$ for every i, j.

To see this fix $x \in U$. Use (i) to see that x commutes with $f(t)$ for all t if and only if $[x, a_i] = 0$ for all i. Now take $x = f'(t')$, letting t' vary, and repeat the argument.

(iii). Let $f(t) = \Sigma a_i t^i$. Then $f(t + t') = f(t)f(t')$ for all t, t', if and only if $[a_i, a_j] = 0$ for all i, j and $a_i = 0$ except when i is a power of p.

If $f(t+t') = f(t)f(t')$, then $f(t), f(t')$ commute for all t, t', so (ii) gives $[a_i, a_j] = 0$ for all i, j. Hence f takes values in an abelian subalgebra. Choosing a basis we see that the coordinates of f are additive polynomials. It is well known that such a polynomial is a linear combination of $t, t^p, t^{p^2}, \cdots$. The converse is clear.

The result follows from (iii), using $exp : u \to U$.

We now apply the above to obtain information on centralizers. Let u be a distinguished unipotent element of order p and A a good A_1 containing u, as given by 3.1 or 3.2. Let $u \in UT_A < A$, where U is a 1-dimensional unipotent group and T_A a 1-dimensional torus normalizing U.

Then T_A determines a labelling of the Dynkin diagram satisfying 2.2. It follows from 2.2 that the corresponding parabolic subgroup P is distinguished (in characteristic 0 this follows from 5.7.4 of [5]). Moreover, u is in the Richardson orbit of $Q = R_u(P)$, while $L = C_X(T_A)$ is a Levi subgroup of P.

It follows from 2.5 that Q has nilpotent class at most $p-1$, hence $p \geq h(P)$ and 4.3 applies.

Corollary 4.5. With notation as above, $C_P(u) = C_P(U) = C_P(L(U))$.

Proof. Let $f : G_a \to U$ as in 4.4, so $U(t) = exp(a_0 t)exp(a_1 t^p) \cdots$. Let $t(\gamma) \in T_A$. From the equation $U(t)^{t(\gamma)} = U(\gamma^2 t)$ and the P-equivariance of exp, a computation shows each a_i is a weight vector of T_A of weight $2p^i$. However, the high weight of T_A on $L(G)$ is at most $2p-2$. Hence, $a_i = 0$ for all $i > 0$, showing that $U = exp(\langle a_0 \rangle)$.

So if $1 \neq u' \in U$, then $log(u') = ca_0$ for some nonzero scalar c. Hence, the P-equivariance of 4.3 yields $C_P(u') = C_P(ca_0) = C_P(a_0) = C_P(u)$, showing $C_P(u) = C_P(U)$.

Set $L(U) = \langle e \rangle$. Then

$$C_P(u) = C_P(U) \leq C_P(L(U)) = C_P(e). \quad (*)$$

First assume $X \neq A_n$. Then 3.1 and 3.2 show that $L(X)|A$ is a tilting module, so by 1.3(d) $C_{L(X)}(u) = C_{L(X)}(e)$. Hence, $\dim(C_X(u)) = \dim(C_X(e))$. By 5.2.2 of [5], $C_X(u)^o = C_P(u)^o$, so $(*)$ implies $C_X(u)^o = C_X(e)^o$. Hence u and e are in the Richardson orbits of P on Q and $L(Q)$, respectively. As the Richardson orbits correspond under exp, we conclude $C_P(u) = C_P(e)$, completing the argument.

Finally, assume $X = A_n$. Here the only distinguished unipotents are regular. Now u (respectively e) lies in a unique Borel subgroup (respectively, subalgebra), namely P (respectively, $L(P)$). So the centralizer in X is contained in P and is $Z(X)$ times a unipotent group of dimension n. Hence the result follows from $(*)$.

Remark: The proof of (4.5) also shows that the group P in the statement can be replaced by $N_{Aut(X)}(P)$.

5. Centralizers.

Let G be a simple algebraic group over K, an algebraically closed field of characteristic p, a good prime. Let u be a unipotent element of order p, a good prime for G, and let A be a good A_1 containing u, as indicated in 3.1 or 3.2. Embed u in a Borel subgroup $UT_A < A$. The results in this section compare the centralizers in G of $u, U, L(U)$, and A.

On occasion it will be convenient to replace G by its simply connected cover, $\tilde{G}$. The following lemma will be useful in this regard.

Lemma 5.1. Let $\phi : \tilde{G} \to G$ be the natural surjection.
(i). $d\phi(\mathrm{Ad}(\tilde{g})(\tilde{e})) = \mathrm{Ad}(\phi(\tilde{g}))(d\phi(\tilde{e}))$, for all $\tilde{g} \in \tilde{G}$.
(ii). If $\tilde{e} \in L(\tilde{G})$ is nilpotent and $d\phi(\tilde{e}) = e$, then $C_G(e) = \phi(C_{\tilde{G}}(\tilde{e}))$.

Proof. (i). Fix $f \in K[G]$. Regarding $L(G)$ as the tangent space at the identity we have

$$d\phi(\mathrm{Ad}(\tilde{g})(\tilde{e}))(f) = \mathrm{Ad}(\tilde{g})(\tilde{e}))(f \circ \phi) = \tilde{e}(f \circ \phi \circ \mathrm{int}(\tilde{g})).$$

On the other hand,

$$\mathrm{Ad}(\phi(\tilde{g})(d\phi(\tilde{e}))(f) = d\phi(\tilde{e}))(f \circ \mathrm{int}(\phi(\tilde{g}))) = \tilde{e}(f \circ \mathrm{int}(\phi(\tilde{g})) \circ \phi).$$

So it suffices to show $\phi \circ \mathrm{int}(\tilde{g}) = \mathrm{int}(\phi(\tilde{g})) \circ \phi$, which is clear.
(ii). As ϕ is surjective we have $C_G(e) = \{\phi(\tilde{g}) : \mathrm{Ad}(\phi(\tilde{g}))(e) = e\}$. So (i) and the equality $e = d\phi(\tilde{e})$ yield

$$C_G(e) = \{\phi(\tilde{g}) : d\phi(\mathrm{Ad}(\tilde{g})(\tilde{e})) = d\phi(\tilde{e})\} = \{\phi(\tilde{g}) : \mathrm{Ad}(\tilde{g})(\tilde{e}) = \tilde{e}\},$$

the last equality holding as $d\phi$ is an isomorphism when restricted to the nilpotent variety. It follows that $C_G(e) = \phi(C_{\tilde{G}}(\tilde{e}))$.

Lemma 5.2. Assume G is not of type A_n with $p|n+1$. If T is a maximal torus of $C_G(u)$, then $C_G(T)' = C_G(L(T))'$. The same result holds for $GL_n(K)$.

Proof. We first establish the result for G. If $T = 1$, then the assertion is clear, so assume $T > 1$. The centralizer of a torus is a Levi subgroup so

$D = C_G(T)'$ is the semisimple part of a Levi subgroup. As $u \in D$ and T is a maximal torus of $C_G(u)$, T is also a maximal torus of $C_G(D)$.

Let T_G denote a maximal torus of DT. By the above, T_G is a maximal torus of G. Certainly $C_G(T) \leq C_G(L(T))$, so if the assertion is false there is a root $\beta \in \Sigma(G) - \Sigma(D)$ such that $L(T)$ centralizes the T_G-root subgroup corresponding to β. Let $\Pi(G) - \Pi(D) = \{\beta_1, ..., \beta_r\}$, so that $\dim(T) = r$. Write $\beta = \alpha + \Sigma c_i\beta_i$, where α is in the linear span of $\Sigma(D)$. Choose i such that $c_i \neq 0$.

Since p is a good prime and since we have excluded the case of A_n with $p|n+1$, we have $Z(L(G)) = 0$. Hence, the annihilator in $L(T)$ of any subset of $\{\beta_1, ..., \beta_r\}$ has codimension equal to the size of the subset. In particular there is an element $l \in L(T)$ such that $\beta_j(l) = 0$ for each $j \neq i$ and $\beta_i(l) = 1$. But then $\beta(l) = c_i$. As p is good this is nonzero, contradicting the choice of β.

Finally, assume $G = GL_n$. Letting Z denote the group of scalars, we have $T \geq Z$. The same argument works, although here $Z(L(G)) = L(Z)$ and $\dim(T) = r + 1$.

Lemma 5.3. $C_G(u)^o = C_G(U)^o = C_G(L(U))^o$.

Proof. By 5.1 we may assume G is simply connected. Also, replace $G = SL_n$ with GL_n, noting that $C_{SL_n}(u)^o = (SL_n \cap C_{GL_n}(u)^o)^o$, and similarly for the other centralizers.

Then 3.1 and 3.2 show that $L(G)|A$ is a tilting module, so that 1.3(d) implies

$$C_{L(G)}(u) = C_{L(G)}(U) = C_{L(G)}(e), \quad (*)$$

where $L(U) = \langle e \rangle$.

We have $L(C_G(u)) \leq C_{L(G)}(u)$ and $L(C_G(e)) \leq C_{L(G)}(e)$. Since p is a good prime, 2.4 (and a version of this for e) imply $L(C_G(u)) = C_{L(G)}(u)$ and $C_{L(G)}(e) = L(C_G(e))$, so we conclude from $(*)$ that $C_G(u)$ and $C_G(e)$ have the same dimension. As $C_G(U) \leq C_G(e)$, it will suffice to show $C_G(u)^o = C_G(U)^o$.

If u is distinguished, then 5.2.2 of [5] gives $C_G(U)^o \leq C_G(u)^o \leq P$, where P is the corresponding distinguished parabolic subgroup. Here the result follows from 4.5.

Suppose u is not distinguished and fix a maximal torus T of $C_G(u)$. We may choose $A \leq C_G(T)$ (see the comments preceding 2.2). Let $g \in C_G(u)$, so that $u \in U^g$. From the above and $(*)$ applied to U^g we have $L(C_G(u)) = C_{L(G)}(u) = C_{L(G)}(U^g)$, so that

$$U^g \leq C_G(L(C_G(u)) \leq C_G(L(T)) \qquad (**)$$

Let $D = C_G(T)'$. Lemma 5.2 implies $C_G(T)' = C_G(L(T))'$, hence (**) gives $U^g \leq C_G(L(T))' = C_G(T)'$ for all $g \in C_G(u)$. Thus $U \leq C_G(T^x)$ for all $x \in C_G(u)$. Set $R = \langle T^{C_G(u)} \rangle$, the normal closure of T in $C_G(u)$. Then U centralizes R. A Frattini argument shows that $C_G(u)^o = RC_{C_G(u)^o}(T) = RC_D(u)^o$. Now u is distinguished in D, so from the distinguished case we have $C_D(u)^o = C_D(U)^o$. Hence $C_G(u)^o = C_G(U)^o$, completing the proof of the lemma.

We will require the following lemma for distinguished unipotent elements.

Lemma 5.4. Assume $u \in G$ is a distinguished unipotent element of order p and let A be the good A_1 containing u as described in 3.1 and 3.2. Then $C_G(u) = C_G(u)^o C_G(A)$.

Proof. First assume G is of classical type, where we may assume $G = SL(V), Sp(V)$, or $SO(V)$. Then $G \leq I(V) = GL(V), Sp(V), O(V)$, respectively. By 2.7 we have $C_{I(V)}(u) = QR$, where $Q = R_u(C_{I(V)}(u)) \leq C_G(u)^o$ and $R = \prod_i I(Z_{r_i})$. By 3.1(i), $A \leq \prod_i I(W_i)$. The groups $I(W_i)$ and $I(Z_{r_j})$ commute pairwise, so $C_{I(V)}(u) = QC_{I(V)}(A)$. Intersect with G to get the assertion.

Next suppose G is an exceptional group. Here we assume G is of adjoint type. By 2.3 $C_G(u) = QS$, where $Q = R_u(C_G(u)) = C_G(u)^o$ and $S = Z_2, S_3, S_4$, or S_5. Also, by 2.8 we may take $A < C_G(E)$, where $E = 1, Z_2, Z_3, K_4$, or Z_5, respectively.

If $S = 1$ or Z_2, the assertion is obvious as $E = S$. Suppose $S = S_3$ so that $E = Z_3$. Involutions of S inverting E induce graph automorphisms of $C_G(E)$ and it will suffice to show that A can be constructed within the centralizer of some graph automorphism. First note that as u centralizes no torus, $C_G(E)$ is semisimple with center of order 3. A check of subsystems shows that with the exception of the case $G = E_8$ and $C_G(E) = A_2E_6$, $C_G(E)$ is a product of groups of type A_r with $3|r+1$. In these cases A projects to an irreducible A_1 in each simple factor and hence the projection is contained in an appropriate symplectic or orthogonal group and so centralizes a graph automorphism. In the exceptional case $A < B_1F_4$ or B_1C_4, depending on the class of u, (this follows from Testerman's construction in [22]. Alternatively one can use Table 3 of [11]). Since F_4 and C_4 each centralize a graph automorphism of E_6, the result holds in this case as well.

Next assume $S = S_4$, which occurs only in F_4. We have $C_G(E)$ semisimple of maximal rank and it admits the action of $S_3 = S/E$. It follows that $C_G(E) = D_4$ (simply connected) with S/E inducing the full group of graph automorphisms. The discussion of section 2 shows that D_4 has two classes of distinguished unipotent elements contained in subgroups B_3 and B_2B_1. But in F_4, $B_3 < B_4$, so that the corresponding unipotent element centralizes a torus. Hence, $u \in B_2B_1$ and a direct check shows that the corresponding good A_1 has the correct weights on $L(G)$ and hence the correct labelled diagram (alternatively use Table 2 of [11]). There exists a triality morphism of $D_4 = C_G(E)$ with centralizer A_2 (irreducible action in 8-dimensions). The SO_3 subgroup of this A_2 has the proper embedding in a B_2B_1 subgroup. Thus we may choose A in such an A_2 centralized by an involutory graph automorphism of A_2 and hence $C_G(A) \geq S_4$, as required.

Finally assume $S \cong S_5$. This occurs only in E_8 where $C_G(E) = A_4A_4$. Here A projects to an irreducible A_1 in each factor. It is shown on p.365 of [13] that A centralizes S_5, so this completes the proof.

We are now in position to establish

Proposition 5.5. Let u be a unipotent element of order p and set $Q = R_u(C_G(u))$. There exists a good A_1, say A, containing u such that A satisfies the conditions of 3.1 and 3.2 and satisfying

$$C_G(u) = C_G(U) = C_G(L(U)) = QC_G(A).$$

Proof. By 5.1 we may take G simply connected. Lemma 5.3 gives $C_G(u)^o = C_G(U)^o = C_G(L(U))^o$. Also $Q = R_u(C_G(u)) \leq C_G(u)^o = C_G(U)^o$.

We first settle the case where u is distinguished, where $Q = C_G(u)^o$. Lemma 5.4 gives $C_G(u) = QC_G(A)$, so that $C_G(u) = C_G(U)$. To complete the distinguished case we must show $C_G(u) = C_G(e)$, where $L(U) = \langle e \rangle$. From what has been established so far we have $C_G(u) = C_G(U) \leq C_G(e)$ and $C_G(u)^o = C_G(e)^o$.

By using a Springer map (see [21], III, 3.12) we see that $C_G(e) = C_G(v)$ for some unipotent element $v \in G$. Then $C_G(u)^o = C_G(v)^o$. Now 5.2.2 of [5] shows that $C_G(u)^o < P$, where P is a distinguished parabolic corresponding to u. So a dimension argument shows that both u, v are in the dense orbit of P on $R_u(P)$, (note that III, 3.15 of [21] gives $v \in C_G(v)^o < P$). Then u, v are conjugate in P, so that the component groups of their centralizers have the same order. But then $C_G(u) = C_G(e)$, as required.

Now consider unipotent elements of order p which are not distinguished. For this part of the argument it will be convenient to replace $G = SL_n$ by GL_n. We do this so as to apply 5.2 at some point.

As $C_G(u)^o = C_G(U)^o$, this group is T_A-invariant. We will show that T_A centralizes $C_G(u)^o/Q = R$.

Fix a maximal torus, T, of $C_G(u)$. It follows from 3.1, 3.2 and the discussion prior to 2.2, that we may take $A \leq C_G(T)$. Also, $C_G(T) = DT$, with D semisimple, so $A \leq D$. Now $C_G(u) \cap N_G(T)$ acts on D.

We will apply 6.1, to follow, the proof of which uses only results from sections 2 and 3. This result establishes conjugacy of good $A_1's$ containing a given unipotent element for groups of classical type. Note that D has at most one simple factor of exceptional type and only exceptional factors of type E_6 admit outer automorphisms. Further, the proof of 5.4 showed that good $A_1's$ in E_6 containing distinguished unipotent elements can be chosen to centralize a graph automorphism.

It follows from the above paragraph that $C_G(u) \cap N_G(T) \leq D(N_G(T) \cap N_G(A)) = D(N_G(T) \cap C_G(A))$. Hence,

$$C_G(u) \cap N_G(T) \leq D(N_G(T) \cap C_G(A)). \qquad (1)$$

(1) immediately implies

$$C_G(u) \cap N_G(T) \leq D(N_G(T) \cap C_G(T_A)) \qquad (2)$$

and intersecting the right side of (1) with $C_G(u)$ gives

$$C_G(u) \cap N_G(T) \leq C_D(u) C_{N_G(T)}(A). \qquad (3)$$

Set $E = [N_G(T) \cap C_G(u)^o, T_A]$. Then (2) implies $E \leq D \cap C_G(u)^o$. However, E is connected and u is distinguished in D, so $E \leq \mathrm{C}_D(u)^o$, a unipotent group. But $C_D(u)^o \leq C_G(u)^o \cap C_G(T) \leq QT$, so

$$[N_G(T) \cap C_G(u)^o, T_A] \leq Q. \qquad (4)$$

Now T_A acts on $R = C_G(u)^o/Q$ and by (4) it centralizes a maximal torus and corresponding Weyl group. This is only possible if $[T_A, R] = 1$, so that $QT_A \triangleleft C_G(u)^o T_A$. A Frattini argument implies

$$C_G(u)^o = Q(C_G(T_A) \cap C_G(u)^o). \qquad (5)$$

Fix a maximal torus $\tilde{T}$ of $C_G(T_A) \cap C_G(u)^o$. Then $L(\tilde{T}) \leq C_{L(G)}(T_A) \cap C_{L(G)}(u)$. However, from the tilting decomposition of $L(G)|A$ it is clear that $C_{L(G)}(T_A) \cap C_{L(G)}(u) = C_{L(G)}(A)$. Hence, A centralizes $L(\tilde{T})$. However, $\tilde{T}$ is a conjugate of T and so 5.2 gives $C_G(L(\tilde{T}))' = C_G(\tilde{T})'$. Therefore, $\tilde{T} \leq C_G(A)$. It follows from (5) that

$$C_G(u)^o = QC_G(A)^o. \qquad (6)$$

A Frattini argument gives $C_G(u) = C_G(u)^o(C_G(u) \cap N_G(T))$, so from (3) we have $C_G(u) = C_G(u)^oC_D(u)C_{N_G(T)}(A)$. Also, u is distinguished in D, so from the distinguished case, $C_D(u) = C_D(u)^oC_D(A)$. Hence, $C_G(u) = C_G(u)^oC_G(A)$ and so (6) gives $C_G(u) = QC_G(A)$. At the start of the proof we showed $Q \leq C_G(U)$, so

$$C_G(u) = C_G(U) = QC_G(A). \qquad (7)$$

It remains to show $C_G(u) = C_G(e)$, when u is not distinguished. By 5.3, $C_G(e)^o = C_G(u)^o$, so a Frattini argument gives

$$C_G(e) = C_G(u)^o(C_G(e) \cap N_G(T)). \qquad (8)$$

Now u is distinguished in $D \triangleleft N_G(T)$. We will apply 4.5, allowing for automorphisms (see the remark following 4.5), to obtain $C_{N_G(T)}(e) = C_{N_G(T)}(u)$, which together with (8) yields the result. However, in order to apply 4.5 it is first necessary to show that $C_{N_G(T)}(e)$ normalizes the distinguished parabolic P of D described prior to 4.5.

Let $s \in C_{N_G(T)}(e)$. Then e is in the Richardson orbit of $L(R_u(P^s))$. The classes of distinguished parabolics of D are preserved under conjugation by s (see [5], pp.174-177), so $P^s = P^d$ for some $d \in D$ and we may adjust d if necessary so that $d \in C_D(e)$. However, from the distinguished case, $C_D(e) = C_D(u) = R_u(C_G(u))C_D(A)$. Now, $R_u(C_D(u)) = C_D(u)^o \leq R_u(P)$ (see 5.2.2 of [5]), while $C_D(A) \leq C_D(T_A)$, which is a Levi subgroup of P. Hence, $d \in C_D(e) \leq P$, so $P^s = P^d = P$, as required.

This completes the proof of 5.5.

6. Conjugacy of good $\mathbf{A_1's}$

In this section we show that there is just one conjugacy class of good $A_1's$ containing a given unipotent element of order p. We also verify that good $A_1's$ are G-cr.

We begin with the groups of classical type, where it is convenient to work in $G = SL(V), Sp(V)$, or $SO(V)$.

Proposition 6.1. Assume G is one of the above classical groups and $u \in G$ has order p.

(i). Any two good $A_1's$ containing u are conjugate by an element of $C_G(u)$.

(ii). If A is a good A_1 containing u, then A is G-cr.

Proof. Let A be the good A_1 as described in the proof of 3.1.

(i). It will suffice to show G-conjugacy, since we can then adjust by an element of a good A_1 to get conjugacy within $C_G(u)$. Let J be another good A_1 containing u

First assume $G = SL(V)$ and let c be the highest weight among composition factors of J on V. Then $2c$ is a weight of $L(G)$. Thus $2c \leq 2p - 2$, showing that $c \leq p - 1$. Hence all composition factors of J on V are restricted and $V|J$ is completely reducible. The composition factors of J on V correspond to Jordan blocks of u, so $V|A$ and $V|J$ have the same decomposition into irreducibles, and thus A and J are conjugate under the action of G.

From now on assume $G = SO(V)$ or $Sp(V)$. We first claim that all composition factors of J on V are restricted. As above, this is clear for $Sp(V)$, since $L(G) = S^2(V)$. But for $SO(V)$, where $L(G) = \bigwedge^2(V)$, it is conceivable that J has a composition factor on V of high weight p. The corresponding irreducible module, $L(p) = L(1)^p$, is symplectic, not orthogonal, so if this occurs, then there must be two such composition factors and hence $2p$ will be a weight of $L(G)$, violating the weight hypothesis. So the claim holds and as before, A and J have similar actions on V.

In the remainder of the proof we will show (this requires $p > 2$) that any two A_1 subgroups in G are conjugate, provided they have the same, restricted, composition factors. In the orthogonal case we work at first in the full orthogonal group, $O(V)$.

Decompose V into homogeneous components under the action of both A and J. All composition factors are self dual, so this is an orthogonal decomposition. By Witt's theorem we can reduce to the case where V is homogeneous under the action of both groups, with irreducibles of the same high weight. Say the dimension of the irreducible summands is n and $\dim(V) = sn$. Write $s = 2t$ or $2t + 1$.

If the action is irreducible, the assertion is quite easy. The groups are conjugate within $GL(V)$ and stabilize a unique (up to scalars) nondegenerate form, so the conjugating element preserves the form.

In the reducible case decompose the space for each A_1 into an orthogonal sum, of t spaces of dimension $2n$ (each the sum of two irreducibles) and, if s is odd, one irreducible of dimension n. Then from Witt's theorem and induction one reduces to the case where $\dim(V) = 2n$.

So both A and J stabilize a pair of n-spaces, with irreducible action on each. Moreover, one can assume the n-spaces are either both nondegenerate or both totally isotropic. In the nondegenerate case n must be odd for the orthogonal case and even for the symplectic case.

For n odd, $SO_{2n} > SO_n \circ SO_2$ and for n even $Sp_{2n} > Sp_n \circ SO_2$, as in the discussion preceding 2.7. This yields the existence of $A_1's$ of the correct type preserving both singular and nondegenerate decompositions. It follows from Witt's theorem that both A and J preserve a singular decomposition. Given any A_1 preserving such a singular decomposition, one can define a torus inducing scalars on each summand. The A_1 is then contained in the centralizer of this torus, which is a group of type GL_n. Still another application of Witt's theorem shows that any two such tori are conjugate in $O(V)$ or $Sp(V)$, respectively. Conjugacy now follows from the conjugacy of GL_n's and the unicity within GL_n.

However, in the orthogonal case, conjugacy has so far been established only within the full orthogonal group and we now correct this. Assume $G = SO(V)$ and $A^g = J$ with $g \in O(V)$. Adjusting by an element of A we may assume $g \in C(u)$. As in the remarks following 2.6 write $V|A = \bigoplus V_i$, where each V_i is a sum of irreducible modules of dimension i and for each i and write $V_i = W_i \otimes Z_{r_i}$, with $\dim(W_i) = i$. There are containments $O(V_i) \geq I(W_i) \circ I(Z_{r_i})$ and 3.1(i) shows that $A < \prod_i I(W_i)$. Hence, $C_{O(V)}(A) \geq R = \prod_i I(Z_{r_i})$. Therefore, 2.7 implies $g = rq$, where $r \in R$ and $q \in Q = R_u(C_{I(V)}(u))$. Then $J = A^{rq} = A^q$. As $O(V)/SO(V) \cong Z_2, Q \leq SO(V)$, establishing G-conjugacy.

(ii). By (i) we may assume the good A_1 is A, as described in 3.1. In particular, $V|A$ is completely reducible. Suppose $A < P$, a parabolic subgroup. Then P is the stabilizer of a flag of singular spaces and by refinement we may assume that A is irreducible on successive quotients. Inductively, it will suffice to establish the result when P is the stabilizer of a singular space, E, irreducible under the action of A.

At this point a description of Levi factors implies that we need only verify that there is a decomposition of V into A-invariant subspaces of the form $V = E \oplus W$ (if $G = SL(V)$) or $V = E \oplus F \perp W$, with F singular (if $G = Sp(V)$ or $SO(V)$).

For $G = SL(V)$ the requirement is immediate from complete reducibility. For the other types, first write $E^\perp = E \oplus W$ for some subspace W and then choose an irreducible F such that $V = (E \oplus F) \oplus W$. At this point the conjugacy from (i) applied to $I(E \oplus F)$, shows that A leaves invariant precisely two maximal singular spaces, E and a complement and that A is contained in the appropriate Levi subgroup.

The following result establishes conjugacy for exceptional groups.

Proposition 6.2. Let G be a simple exceptional group and $u \in G$ of order p.

(i). Any two good $A_1's$ containing u are conjugate by an element of

$C_G(u)$.

(ii). If A is a good A_1 containing u, then A is G-cr.

Proof. Assume G is of exceptional type. Let A be the good A_1 as indicated in Proposition 3.2 and suppose J is another good A_1 containing u. To establish (i) it will suffice to show that A and J are conjugate in G. In the course of the proof we will also establish (ii).

First suppose $J < C_G(s)$, where s is a noncentral semisimple element of G. Then $s \in C_G(u)$. By 5.5, replacing s by a $C_G(u)$ conjugate we may assume $A, J \leq C_G(s)$. The result now follows by induction, since both A and J are good within the reductive group $C_G(s)$.

Next assume that J is contained in a parabolic subgroup P of G. Write $P = QL$, with $Q = R_u(P)$ and L a Levi factor. Then $QJ = QR$ where R is a good A_1 of L, with respect to some unipotent element, say v, of order p. As R centralizes the central torus of L, the preceding paragraph shows that, R is uniquely determined, up to $C_G(v)$ conjugacy. Indeed, we can take R to be the good A_1 as given in 3.1 or 3.2. In particular, R has a tilting decomposition on $L(G)$.

It follows from [3] that Q has a filtration by normal subgroups of P, such that successive quotients afford modules for L. Indeed, these quotients are each isomorphic to weight spaces of $Z(L)^o$. Therefore R has a tilting decomposition on successive quotients in the filtration. Hence, the same holds for J. It follows (see (1.2)(iii) and (iv) in the proof of 1.4) that if V is any one of the quotients, then $H^1(J, V) = 0$. Using this repeatedly we conclude that J is conjugate to R. But then J centralizes a conjugate of the central torus of L, so the assertion follows from the previous paragraph. Note that we have also established (ii).

Now assume that J is contained in no proper parabolic subgroup and does not centralize a noncentral semisimple element. In particular, $C_G(J) = Z(G)$ and J induces an adjoint group on $L(G)$. It follows from 1.3 of [18] that $C_{L(G)}(J) = 0$. As J is a good A_1 this implies that $C_{L(G)}(L(J)) = 0$, which of course implies that $C_G(L(J))$ is finite. Hence, $N_G(L(J))^o = J$.

In the previous paragraph we verified the hypotheses of Theorem 2 of [18], except that the theorem requires a characteristic restriction slightly stronger than p a good prime.

Assume for the moment that this characteristic requirement also holds. Then Theorem 2 of [18] shows that either J is one of the maximal A_1's described in [18] or is in the list on pp.65-66 of [18]. In the first case the A_1's are unique up to conjugacy in G and are just those constructed by Testerman, containing semiregular unipotent elements. In the latter situation, with three exceptions J is shown to be in a maximal rank subgroup, hence centralizes a semisimple element, contradicting the above. In the exceptional cases, J is contained in a maximal subgroup M of G, where $(M, G) = (F_4, E_6), (C_4, E_6)$, or (A_1F_4, E_7), and u is a regular unipotent in M. There are several ways to complete the proof, the fastest being to apply the results of [11], where all classes of $A_1's$ containing a given unipotent element are determined. In all but the (A_1F_4, E_7) case, it is immediate from the tables in [11] that there is a unique class of $A_1's$ containing u. In the latter case there are several (corresponding to field twists on one of the simple factors of M), but only one class of good $A_1's$. Hence A and J are conjugate in G.

What remains are those situations where the prime restriction of Theorem 2 of [11] does not hold. These cases are as follows: $(G, p) = (E_8, 7), (E_7, 5$ or $7), (E_6, 5)$. Using what has already been established, we can argue exactly as in the proof of Lemma 2.3 of [18] to show that J determines a labelling of the Dynkin diagram by 0's and 2's.

We now argue that J has a fixed point on $L(G)$, thereby obtaining a contradiction. The computer program described in [11] calculates the composition factors of J on $L(G)$, given the labelled diagram. This is quite easy, since the labelling determines all weights and it is just necessary to sort them into weights of irreducibles. The program determines that in each case there are more composition factors of high weight 0 then weight $2p-2$. Now 1.2(c) shows that for $0 \leq c < 2p-2, Ext^1(L(0), L(c)) = 0$, while $Ext^1(L(0), L(2p-2))$ has dimension 1. Using the fact that $L(G)$ is a self-dual module we obtain a fixed point. This completes the proof.

7. Theorems 1-3

We can now establish the main results stated in the introduction. To a large extent this is just a matter of combining results established in previous sections.

As before, G is a simple algebraic group over an algebraically closed field of characteristic p, a good prime for G. Let $u \in G$ be a unipotent element of order p.

Proof of Theorem 1. The existence of good $A_1's$ containing u and the tilting decompositions are established in 3.1 and 3.2. Then 6.1(i) and 6.2(i) show that any two good $A_1's$ containing u are conjugate within $C_G(u)$. Therefore, if A is any good A_1 containing u, then 5.5 shows that $C_G(u) = QC_G(A)$, where $Q = R_u(C_G(u))$. It follows that Q is transitive on the class of good $A_1's$ containing u. It remains to show that good $A_1's$ containing u

are G-cr. This was established in 6.1(ii) for classical groups and 6.2(ii) for exceptional groups.

Proof of Theorem 2. Fix A a good A_1 containing u and set $Q = R_u(C_G(u))$. Proposition 5.5 settles most of Theorem 2. It remains to show $C_G(u) = QC_G(A)$, a semidirect product of algebraic groups and that $C_G(A)$ is reductive.

We first show that $C_G(A)$ is reductive, using the argument prior to Lemma 4.2 in [14]. Suppose $Q_o = R_u(C_G(A)) > 1$ and embed Q_oA in a parabolic subgroup P with $Q_o \leq R_u(P)$. Take P minimal for this. By Theorem 1, A is G-cr, so $A \leq L$, a Levi subgroup of P. By minimality, A is contained in no proper parabolic of L.

Let $\bar{w}_o$ denote either the long word of the Weyl group of G or the long word adjusted by a graph automorphism, so that $\bar{w}_o$ induces -1 on the root system. We may assume $\bar{w}_o$ normalizes each simple factor of L.

We claim that A is normalized by $\bar{w}_o l$, for some $l \in L$. Consider the action of $\bar{w}_o$ on simple factors of L. As A is contained in no proper parabolic of L, it can centralize no torus of L', so 5.5 implies that u is distinguished in L. Now the classes of distinguished parabolic subgroups of a simple factor of L' are invariant under outer automorphisms (see pp. 174-177 of [5]). Hence the classes of distinguished unipotents are also invariant and the claim follows from conjugacy of good $A_1's$ containing a given unipotent.

With l as above, $\bar{w}_o l$ normalizes $C_G(A)$ and hence Q_o. But, this is impossible, as $\bar{w}_o l$ conjugates P into its opposite parabolic. This shows that $Q \cap C_G(A) = 1$.

To have a semidirect product the Lie algebras of Q and $C_G(A)$ must intersect trivially, as well. However if this is not the case, then by the above $p = 2$ and $C_G(A)$ has factors of type B or C. As p is good, this forces $G = SL(V)$. But 3.1(i) shows that $C_{GL(V)}(A)$ is a product of groups of type GL. Hence $L(Q) \cap L(C_G(A)) = 0$, completing the proof of Theorem 2.

Proof of Theorem 3. To establish this result we must show that there is a unique 1-dimensional unipotent group U, such that $u \in U$ and U is contained in a good A_1. Existence of U follows from Theorem 1(i), taking U to be the 1-dimensional unipotent group containing u in a good A_1. Theorem 1(ii), shows that good $A_1's$ containing u are conjugate within $C_G(u)$, while Theorem 2(i) gives $C_G(u) = C_G(U)$. It follows that U is a maximal unipotent subgroup in each good A_1 containing u.

8. Finite Groups

In this section we use previous results for algebraic groups to establish results for finite groups, including Theorem 4. Fix notation as above, with G a simple algebraic group over an algebraically closed field of characteristic p, a good prime. Let σ be a Frobenius morphism of G such that $G_\sigma = G(q)$, a finite group of Lie type.

Proposition 8.1. Let $u \in G(q)$ have order p.

(i). There exists a good A_1 containing u which is σ-stable.

(ii). Two σ-stable good $A_1's$ containing u are conjugate by an element of $G(q)$; in fact by an element of $O_p(C_{G(q)}(u))$.

(iii). Assume $A = A^\sigma$ is a good A_1. Then $C_G(A) = C_G(A(q))$ and $C_{G(q)}(u) = O_p(C_{G(q)}(u))C_{G(q)}(A(q))$, a semidirect product.

Proof. (i). Let A be a good A_1 containing u. Then A^σ is also a good A_1 containing u. Let $\mathcal{E}$ denote the family of good $A_1's$ containing u. Theorem 1(ii) shows that $Q = R_u(C_G(u))$ is transitive on $\mathcal{E}$. An application of Lang's theorem (see [21], I, 2.7) shows that σ fixes an element of $\mathcal{E}$.

(ii). We claim that Q_σ is transitive on $\mathcal{E}_\sigma$. It follows from [20], I, 2.7 that $\mathcal{E}_\sigma$ decomposes into conjugacy classes under the action of Q_σ and the number of classes equals the the number of conjugacy classes in the coset $C\sigma$, where $C = N_Q(A)/N_Q(A)^o$. Hence it will suffice to show that $N_Q(A)$ is connected. Now $N_G(A) = AC_G(A)$ so that $N_G(A) \cap C_G(u) = C_G(A)(A \cap C_G(u)) = C_G(A) \times U$. By Theorem 2(ii), $C_G(u) = QC_G(A)$, a semidirect product, so it follows that $N_Q(A) = U$, a connected group. This proves the claim, which establishes (ii).

(iii). Let $A = A^\sigma$ be a good A_1. Let u, u' be noncommuting unipotent elements in $A_\sigma = A(q)$ and let U, U' be the corresponding 1-dimensional unipotent groups of A. Then $A = \langle U, U' \rangle$ and we have $C_G(A_\sigma) \geq C_G(A) = C_G(U) \cap C_G(U') = C_G(u) \cap C_G(u') \geq C_G(A_\sigma)$ (the second equality from Theorem 2(i)). Hence, $C_G(A) = C_G(A(q))$.

It remains to establish the factorization of $C_{G(q)}(u)$. By the last paragraph $C_G(A) = C_G(A(q))$. Moreover, $C_G(A)$ is σ-invariant, since A is, and reductive by Theorem 2(ii). As $C_G(u)/Q$ is naturally isomorphic to $C_G(A)$, it follows that $(C_G(u)/Q)_\sigma$ has no nontrivial normal p-subgroups (for this we note that since p is a good prime all p-elements lie in $C_G(u)^o$). Therefore, $O_p(C_{G(q)}(u)) < Q$.

Assume $x = qc \in C_{G(q)}(u)$ with $q \in Q$ and $c \in C_G(A)$. Then $xQ = cQ = (cQ)^\sigma$ and since $C_G(A)$ is σ-stable, this yields $c \in G(q)$. Then $x = x^\sigma$ implies $q = q^\sigma$ which establishes (iii).

Proposition 8.2. Let A be a σ-stable, good A_1. Assume $q > 7$ in case G is an exceptional group. Then

(i). For each parabolic subgroup P of G, $A_\sigma < P$ if and only if $A < P$.
(ii). A_σ is G_σ-cr.

Proof. We will separate the arguments for classical and exceptional groups, taking advantage of the natural module in the former case. First assume that G is of classical type. We may work with $G = I(V)$, with V the natural module.

In view of the conjugacy assertion of Theorem 1 we may take A as in the proof of 3.1. Hence, $V|A$ is completely reducible and each irreducible summand is restricted. It follows that A and A_σ stabilize precisely the same subspaces of V. So (i) is immediate, since parabolic subgroups are the stabilizers of certain flags of V. Theorem 1(iv) shows that A is G-cr, so the same holds for A_σ as Levi subgroups are stabilizers of certain decompositions of V.

To establish (ii) we must show that it is possible to pick a σ-invariant Levi subgroup of P containing A_σ, whenever P is a σ-invariant parabolic. An argument for this is given at the end of the proof of Theorem 7 of [14].

Now assume G is of exceptional type and let $P < G$ be a parabolic subgroup. If $A < P$, then certainly $A_\sigma < P$. Conversely, suppose $A_\sigma < P$.

We first establish the result when $q > p$. Then all weights of A on $L(G)$ are less than q, so Proposition 1.4 of [14] shows that A_σ and A leave invariant precisely the same subspaces of $L(G)$. At this point the rest of the argument (assuming $q > p$) is precisely as for the classical groups, since $P = N_G(L(P))$ and similarly for a Levi subgroup of P (see Lemma 5.1).

It remains to consider the case G exceptional and $q = p$. Recall that we are assuming $q > 7$ here, so that A_σ is G-cr by Theorem 7 of [14]. As above the argument at the end of the proof of Theorem 7 of [14] gives (ii).

Finally, we establish (i). Suppose $A_\sigma < P$. Then by the previous paragraph $A_\sigma \leq L$, a Levi subgroup of P. Let Z be the connected center of L, a nontrivial torus. Theorem 1(iii) shows that $L(G)|A$ is a tilting module, so from 1.2(d) we conclude $C_{L(G)}(A_\sigma) = C_{L(G)}(A)$. Therefore, A centralizes $L(Z)$ and the argument of 6.1 shows that $A < C_G(Z) = L \leq P$. This establishes (i), completing the proof.

Theorem 4 is immediate from Propositions 8.1 and 8.2.

9. Bad primes

In this section we show by way of examples that our assumption that p is a good prime is essential. In all cases G will be an exceptional group and $u \in G$ an element of order p.

For G_2 and $p = 3$ we will provide an example where u is not contained in any A_1. For $G = E_6$ and $p = 3$ we give an example where u is contained in $A_1's$, but not good $A_1's$. Similarly, for E_8 with $p = 5$ we produce an example where u is contained in an A_1 but not a good A_1 for which $L(G)$ affords a tilting module.

$\mathbf{G = G_2}$. Assume $p = 3$. We show that there is a an element of order 3 which is contained in no group of type A_1. In the notation of Lawther [10], let u have type $\tilde{A}_1^{(3)}$. Let V be the usual 7-dimensional orthogonal module for G. Then by [10],

$$V|u = J_3 \oplus J_2^2 \qquad (*).$$

Suppose $u \in A$, with $A = A_1$ and consider the action of A on V. If $A = SL_2$, then decompose V under the action of the central involution, say z, noting that this element must have determinant 1. It follows from $(*)$ that $V|A = V_4 \perp V_3$ an orthogonal sum of spaces of the indicated dimensions. Also, $V_4|A$ must be the sum of 2 singular spaces of dimension 2.

Now G_2 has just one class of involutions, so $A \leq C_{G_2}(z) = X \circ Y$, where X is a long A_1 and Y is a short A_1. Also, $X \circ Y$ induces SO_4 on V_4 and SO_3 on V_3. Given the action of u, we see that this is only possible if $A = Y$. But then u has class $\tilde{A}_1$, a contradiction.

Therefore, $A = PSL_2$ and A has composition factors on V of dimensions among $1, 3, 4$, where in the last case the module is a tensor product of twists of the natural 2-dimensional module. If there exists a submodule of dimension 4, then it must be nondegenerate and u acts on this module as $J_3 \oplus J_0$, contradicting $(*)$. Hence, there is no such submodule.

It follows from 1.2 that 3-dimensional irreducibles for A cannot extend the trivial module. From this, the fact that V is self-dual, and the above paragraph we conclulde that A leaves invariant a nondegenerate 1-space of V. This contradicts $(*)$.

$\mathbf{G = E_6}$. Here we take $p = 3$ and u of type $A_2A_2A_1$. We will show that u is not contained in a good A_1. By way of contradiction, assume that $u \in A$, a good A_1. From [10] we have

$$L(G)|u = J_3^{24} \oplus J_2^3. \quad (*)$$

First suppose $A = SL_2$ and let $\langle z \rangle = Z(A)$. Then $D = C_G(z) = A_1A_5$ or T_1D_5 as these are the involution centralizers in E_6. In either case p is a good prime for D so Theorem 1 implies that A is determined up to conjugacy within D by the D class of u. Since u has order 3, it is not distinguished in D (see the discussion following 2.7) and so it is distinguished within a proper

Levi subgroup of D. Using Lawther [10] to compare Jordan blocks we see that this Levi must have type $A_2A_2A_1$. However, such a Levi subgroup has a composition factor which is the tensor product of natural representations for the factors. Restricting to A, the highest weight is at least 5, contradicting the assumption that A is a good A_1.

Now suppose $A = PSL_2$. Using 1.3(iii) together with 1.4 we can write

$$L(G)|A = L(4)^a \oplus W(4)^b \oplus (W(4)^*)^c \oplus T(4)^d \oplus L(2)^e \oplus L(0)^f.$$

Now $L(4)|u = J_3 \oplus J_1, W(4)|u = W(4)^*|u = J_3 \oplus J_2$, and $T(4)|u = J_3^2$ (use 1.3(iii) for the latter cases). So from $(*)$ we have $a = f = 0$. A count of the number of Jordan blocks of size 3 yields the equation $b + c + 2d + e = 24$. Also, we find that the 0-weight space of T_A has the same dimension. But then $C_G(T_A)$ is a Levi factor of dimension 24 which is easily checked to be impossible.

$\mathbf{G = E_8}$. Assume $p = 5$ with $u \in G$ of type A_4A_3. We will show that u is not contained in a good A_1 for which $L(G)$ affords a tilting module. By way of contradiction, suppose such a group, say A, exists.

According to [10],

$$L(G)|u = J_5^{48} \oplus J_4^2. \quad (*)$$

Write $L(G)|A = \bigoplus T(c)$, a tilting decomposition with $0 \le c \le 2p - 2$ for each c. From $(*)$ and 1.2 we see that the possibilities for c are $8, 7, 6, 4$, and 3 and that $T(3) = L(3)$ occurs with multiplicity 2.

Write

$$L(G)|A = T(8)^a \oplus T(7)^b \oplus T(6)^c \oplus L(4)^d \oplus L(3)^2.$$

The existence of the last summand implies that $A \cong SL_2$. Set $\langle z \rangle = Z(A)$. Then $D = C_G(z) = D_8$ or A_1E_7 and $L(D) = C_{L(G)}(z)$. Hence

$$L(D)|A = T(8)^a \oplus T(6)^c \oplus L(4)^d. \quad (**)$$

Each summand has dimension a multiple of 5, so $C_G(z) = D = D_8$.

Consider $A < D_8$. Then A is also good in D_8. Let V denote the usual 16 dimensional orthogonal module for a cover, $\hat{D}$, of D and let $\hat{A}$ denote the derived group of the preimage of A. Then 3.1 and 6.1 imply $V|\hat{A}$ is a direct sum of restricted modules so we write

$$V|\hat{A} = L(4)^r \oplus L(3)^s \oplus L(2)^t \oplus L(1)^u \oplus L(0)^v$$

On the other hand, $(**)$ is a decomposition of $\bigwedge^2(V)|\hat{A}$.

If $L(i), L(j)$ are distinct direct summands of $V|\hat{A}$, then $L(i) \otimes L(j)$ is a summand of $\bigwedge^2(V)|\hat{A}$. Consequently i and j must have the same parity.

Assume $V|\hat{A} = L(3)^s \oplus L(1)^u$. We have $L(1) \otimes L(1) = L(2) \oplus L(0)$, $L(1) \otimes L(3) = L(4) \oplus L(2)$, and $L(3) \otimes L(3) = T(6) \oplus L(4) \oplus L(0)$. Hence $(**)$ implies $s = u = 0$, a contradiction.

Hence, $V|\hat{A} = L(4)^r \oplus L(2)^t \oplus L(0)^v$. Now $L(i) \otimes L(0) = L(i)$ and $L(2) \otimes L(2) = L(4) \oplus L(2) \oplus L(0)$, so we conclude from $(**)$ that $t + v \leq 1$. The only possibility is that

$$V|\hat{A} = L(4)^3 \oplus L(0).$$

By 3.1 and 6.1, $A < B_2 \times B_2 \times B_2 < B_7 < D$. Now $L(G)|D = L(D) \oplus E$, where E is a an irreducible spin module. It follows that $E|A = (L(3) \otimes L(3) \otimes L(3))^2$, so that A has high weight $9 = 2p - 1$ on $L(G)$, contradicting the assumption that A is good.

10. Open problems.

In this section we discuss some directions for future work. Let G be a simple algebraic group defined over an algebraically closed field K of characteristic $p > 0$.

Saturation. As mentioned in the introduction, the saturation problem motivated the results of this paper and remains an important issue. The problem is to associate to each unipotent element of order p a 1-dimensional unipotent group that is in some sense canonical. Theorem 3 is such a result for p a good prime.

Serre has carried out preliminary investigations for $p = 2$. It appears that satisfactory results may be available in many cases, although the arguments are based on case by case analysis using detailed information of centralizers of involutions. In particular, for exceptional groups G other than F_4 the results in [2] show that $Z(C_G(u))$ is a 1-dimensional unipotent group.

Tensor product theorems. The results of this paper and those of [12] suggest the possiblity of establishing a tensor product theorem for groups of type A_1, based on good $A_1's$.

Assume $\phi : SL_2(K) \to G$ is a homomorphism of algebraic groups such that the image is G-cr. One would like to show that under suitable hypotheses there are unique homomorphisms $\phi_i : SL_2(K) \to G$ for $1 \leq i \leq r$ and distinct powers q_i of p satisfying the following conditions

(i). $\phi_j(SL_2(K))$ is a good A_1 for each j.

(ii). $[\phi_j(SL_2(K)), \phi_k(SL_2(K))] = 1$ for $j \neq k$.
(iii). $\phi(x) = \prod_j \phi_j(x)^{(q_j)}$, for each $x \in SL_2(K)$.

It may also be worthwhile to investigate analogs of this for homomorphisms $\phi : G_a \to G$, generalizing Proposition 4.4. Here one would need a suitable notion of "good" for a 1-dimensional unipotent group.

Subgroups other than A_1. It is natural to ask if there is an analog of the notion of good A_1 for simple closed subgroups of G of rank greater than one and if tilting modules play a role. The simple subgroups of exceptional groups are determined in [12], subject to mild characteristic restrictions. Using these results one could also hope to establish tensor product theorems as above.

Elements of order larger than p. It would be useful to establish results for unipotent elements u of order greater than p. In particular, is it possible to establish a factorization of $C_G(u)$ similar to Theorem 2(ii)? What group would take the place of $C_G(A)$?

References

1. Andersen, H., Jørgensen, J., and Landrock, P., The projective indecomposable modules of $SL(2, p^n)$, Proc. London Math Soc., 46, (1983), 38-52.

2. Aschbacher, M. and Seitz, G.,Involutions in Chevalley groups over fields of even order, Nagoya Math. J., 63 (1976) 1-91.

3. Azad, H., Barry, M., and Seitz, G., On the structure of parabolic subgroups, Communications in Algebra, 18, (1990), 551-562.

4. Bala, P. and Carter, R. W., Classes of unipotent elements in simple algebraic groups, I,II, Proc. Cam. Phil. Soc., 79, 80, (1976), 401-425, 1-17.

5. Carter, R., *Finite groups of Lie type: conjugacy classes and complex characters*, Wiley-Interscience, New York, 1985.

6. Chang, B., The conjugate classes of Chevalley groups of type (G_2), J. Alg., 9, (1968), 190-211.

7. Demazure, M. and Grothendieck, A., *Schémas en Groupes*, Lecture Notes in Math., 153, Springer-Verlag, Berlin.

8. Donkin, S., A filtration for rational modules, Math. Z., 177, (1981), 1-8.

9. Donkin, S., On tilting modules for algebraic groups, Math. Z., 212, (1993), 39-60.

10. Lawther, R., Jordan block sizes of unipotent elements in exceptional algebraic groups, Comm. in Alg., 23, (1995), 4125-4156.

11. Lawther, R. and Testerman, D., A_1 subgroups of exceptional algebraic groups, Transactions AMS, (to appear Memoirs AMS).

12. Liebeck, M. and Seitz, G., *Reductive subgroups of exceptional algebraic groups*, Memoirs AMS, 580, (1996), 1-111.

13. Liebeck, M. and Seitz, G., Maximal subgroups of exceptional groups of Lie type, finite and algebraic, Geom. Ded., 36, (1990), 353-387.

14. Liebeck, M. and Seitz, G. On the subgroup structure of exceptional groups of Lie type, Trans. AMS, 350, (1998), 3409-3482.

15. Mizuno, K., The conjugate classes of Chevalley groups of type E_6, J. Fac. Sci. Univ. of Tokyo, 24, (1977), 525-563.

16. Mizuno, K., The conjugate classes of the Chevalley groups of type E_7 and E_8, Tokyo J. Math., 3, (1980), 391-461.

17. Pommerening, K., Über die unipotenten Klassen reduktiver Gruppen I, II, J. Alg., 49, 65, (1977), (1980), 525-536, 373-398.

18. Seitz, G., *Maximal subgroups of exceptional algebraic groups*, Memoirs AMS, 441, (1991), 1-197.

19. Serre, J-P., Sur la semi-simplicité des produits tensoriels de représentations de groupes, Invent. Math., 116, (1994), 513-530.

20. Shoji, T., The conjugacy classes of Chevalley groups of type (F_4) over finite fields of characteristic $p \neq 2$, J. Fac. Sci. Univ. Tokyo, 21, (1974), 1-17.

21. Springer, T.A. and Steinberg, R., Conjugacy Classes, in *Seminar on algebraic groups and related finite groups*, (ed. A. Borel, et al), Lecture Notes in Math., 131, Springer, Berlin, 168-266.

22. Testerman, D., A_1-type overgroups of elements of order p in semisimple algebraic groups and the associated finite groups, J. Alg., 177, (1995), 34-76.