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UNIQUE FIXED POINT THEOREM FOR WEAKLY C-CONTRACTIVE MAPPINGS

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ABSTRACT
In this work we introduce the class of weakly c-contractive mappings. We establish that these mappings necessarily have unique fixed points in complete metric spaces. We support our result by an example. Our result also generalises an existing result in metric spaces.

Key words : Metric space, Fixed point, Weak C-contraction.

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INTRODUCTION
It is well known that Banach's contraction mapping theorem is one of the pivotal results of functional analysis. A mapping $T : X \to X$ where (X, d) is a metric space, is said to be a contraction if there exists $0 < k < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \ \leq \ kd(x, y). \tag{1.1}$$

If the metric space (X, d) is complete then the mapping satisfying (1.1) has a unique fixed point. Inequality (1.1) implies continuity of T. A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity.

Kannan [10,11] established the following result in which the above question has been answered in the affirmative.

If $T : X \to X$ where (X, d) is a complete metric space, satisfies the inequality

$$d(Tx, Ty) \leq k \ [d(x, Tx) + d(y, Ty)] \tag{1.2}$$

where $0 < k < \frac{1}{2}$ and $x, y \in X$, then T has a unique fixed point.

The mapping T need not be continuous as has been established through examples[11]. The mappings satisfying (1.2) are called Kannan type mappings. There is a large literature dealing with Kannan type mappings and their generalization some of which are noted in [8], [17] and [19].

A similar contractive condition has been introduced by Chatterjee [6]. We call this contraction a C-contraction (borrowing the name from the name of its author).

Definition 1.1 C-contraction [6].

Let $T : X \to X$ where (X, d) is a metric space is called a C-contraction if there exists $0 < k < \frac{1}{2}$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leq k\ [d(x, Ty) + d(y, Tx)]. \tag{1.3}$$

Theorem 1.1 [6] A C-contraction defined on a complete metric space has a unique fixed point.

In establishing theorem 1.1 there is no requirement of continuity of the C-contraction.

It has been established in [15] that inequalities (1.1), (1.2) and (1.3) are independent of one another. C-contraction and its generalizations have been discussed in a number of works some of which are noted in [4], [8], [9] and [19].

Banach's contraction mapping theorem has been generalized in a number of recent papers. As for example, asymptotic contraction has been introduced by Kirk [12] and generalized Banach contraction conjecture has been proved in [1] and [14].

Particularly a weaker contraction has been introduced in Hilbert spaces in [2]. The following is the corresponding definition in metric space.

Definition 1.2 Weakly contractive mapping

A mapping $T : X \to X$ where (X, d) is a complete metric space is said to be weakly

contractive if $\quad d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)),$ (1.4)

where $x, y \in X,\ \psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing,
$\psi(x) = 0$ if and only if $x = 0$ and $\lim_{x \to \infty} \psi(x) = \infty$.

If we take $\psi(x) = kx$ where $0 < k < 1$ then (1.4) reduces to (1.1).

There are a number of works in which weakly contractive mappings have been considered. Some of these works are noted in [3], [7], [13], and [16].

In the present work in the same spirit we introduce a generalization of C-contraction.

Definition 1.3 Weak C-contraction :

A mapping $T : X \to X$, where (X, d) is a metric space is said to be weakly C-contractive or a weak C-contraction if for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}\,[d(x, Ty) + d(y, Tx)] - \psi\,(d(x, Ty), d(y, Tx)) \qquad (1.5)$$

where $\psi : [0, \infty)^2 \to [0, \infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

If we take $\psi(x, y) = k\,(x + y)$ where $0 < k < \frac{1}{2}$ then (1.5) reduces to (1.4), that is weak C-contractions are generalisations of C-contractions. At the end of the next section we discuss an example which shows that weak C-contractions constitute a strictly larger class of mappings than C-contractions.

In the next section we establish that in a complete metric space a weak C-contraction has a unique fixed point.

MAIN RESULTS

Theorem 2.1 Let $T : X \to X$, where (X, d) is a complete metric space be a weak C-contraction. Then T has a unique fixed point.

Proof

Let $x_0 \in X$ and for all $n \geqq 1$, $\quad x_{n+1} = Tx_n$. (2.1)

If $x_n = x_{n+1} = Tx_n$ then x_n is a fixed point of T.
So we assume $x_n \neq x_{n+1}$.

Putting $x = x_{n-1}$ and $y = x_n$ in (1.5) we have for all n = 0, 1, 2, ...

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \frac{1}{2}\,(d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})) - \psi\,(d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}))$$

$$= \frac{1}{2}\,d(x_{n-1}, x_{n+1}) - \psi\,(d(x_{n-1}, x_{n+1}), 0)$$

$$\leq \frac{1}{2}\,(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) - \psi\,(d(x_{n-1}, x_{n+1}), 0) \qquad (2.2)$$

From (2.2), for all $n = 1, 2, ...$

$$d(x_n, x_{n+1}) \leq \ d\ (x_{n-1}, x_n) \tag{2.3}$$

Thus $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers and hence is convergent.

Let $d(x_n, x_{n+1}) \to r$ as $n \to \infty$. (2.4)

We next prove that $r = 0$.

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$\leq \frac{1}{2}\ (d(x_{n-1}, Tx_n) + d\ (x_n, Tx_{n-1})) - \psi\ (d(x_{n-1}, x_{n+1}), d(x_n, x_n))$ (by (1.5))

$\leq \frac{1}{2}d\ (x_{n-1}, x_{n+1})$

$\leq \frac{1}{2}\ (d(x_{n-1}, x_n) + d(x_n, x_{n+1}))$.

Making $n \to \infty$ we have by (2.4),

$$r \leq \lim_{n\to\infty} \frac{1}{2}d\ (x_{n-1}, x_{n+1})\ \leq\ \frac{1}{2}r + \frac{1}{2}r,$$

$$\text{or, } \lim_{n\to\infty}\ d\ (x_{n-1}, x_{n+1}) = 2r. \tag{2.5}$$

Making $n \to \infty$ in (2.2) and using (2.4), (2.5) and the continuity of ψ we have
$r \leq r - \psi(2r, 0)$.

or, $\psi(2r, 0) \leq 0$, which is a contradiction unless $r = 0$.

Thus we have established that
$d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. (2.6)

Next we show that $\{x_n\}$ is a cauchy sequence. If otherwise, then there exist $\epsilon > 0$ and increasing sequences of integers $\{m(k)\}$ and $\{n(k)\}$ such that for all integers k,

$$n(k) > m(k) > k, \tag{2.7}$$

$$d(x_{m(k)}, x_{n(k)}) \geq\ \epsilon, \text{ and} \tag{2.8}$$

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{2.9}$$

Then,

$$\begin{aligned}
\epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\
&= d(Tx_{m(k)-1}, Tx_{n(k)-1}) \\
&\leq \tfrac{1}{2}\,(d(x_{m(k)-1}, Tx_{n(k)-1}) + d\,(x_{n(k)-1}, Tx_{m(k)-1})) \\
&\quad -\psi\,(d(x_{m(k)-1}, Tx_{n(k)-1}), d(x_{n(k)-1}, Tx_{m(k)-1})) \qquad \text{(by(1.5))} \\
&= \tfrac{1}{2}\,(d(x_{m(k)-1}, x_{n(k)}) + d\,(x_{n(k)-1}, x_{m(k)})) \\
&\quad -\psi\,(d(x_{m(k)-1}, x_{n(k)}), d(x_{n(k)-1}, x_{m(k)})).
\end{aligned} \tag{2.10}$$

Again,

$$\begin{aligned}
\epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\
&\leq d(x_{m(k)}, x_{n(k)-1}) + d\,(x_{n(k)-1}, x_{n(k)})) \\
&\leq \epsilon + d(x_{n(k)-1}, x_{n(k)}). \quad \text{(by (2.9))}
\end{aligned}$$

Making $k \to \infty$ in the above inequality and using (2.6)we obtain

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon \tag{2.11}$$

and

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \epsilon. \tag{2.12}$$

Again,

$$\begin{aligned}
d(x_{m(k)}, x_{n(k)-1}) &\leq d(x_{m(k)}, x_{m(k)-1}) \\
&+ d(x_{m(k)-1}, x_{n(k)}) \\
&+ d(x_{n(k)}, x_{n(k)-1}).
\end{aligned}$$

Also, $$d(x_{m(k)-1}, x_{n(k)}) \leq d(x_{m(k)-1}, x_{m(k)}) + d\,(x_{m(k)}, x_{n(k)}).$$

Making $k \to \infty$ in the above two inequalities and using (2.6), (2.11) and (2.12) we get,

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon. \tag{2.13}$$

Next making $k \to \infty$ in (2.10) and using (2.6), (2.12) and (2.13) we obtain

$$\epsilon \leq \frac{1}{2}(\epsilon + \epsilon) - \psi(\epsilon, \epsilon).$$

or, $\psi(\epsilon, \epsilon) \leq 0$, which is a contradiction since $\epsilon > 0$. Hence $\{x_n\}$ is a cauchy sequence and therefore is convergent in the complete metric space (X, d).

Let $x_n \to z$ as $n \to \infty$. (2.14)

Then, $d(z, Tz) \leq d(z, x_{n+1}) + d(x_{n+1}, Tz)$

$$\leq d(z, x_{n+1}) + d(Tx_n, Tz)$$

$$\leq d(z, x_{n+1}) + \frac{1}{2}(d(z, Tx_n) + d(x_n, Tz)) - \psi(d(z, Tx_n), d(x_n, Tz))$$

$$= d(z, x_{n+1}) + \frac{1}{2}(d(z, x_{n+1}) + d(x_n, Tz)) - \psi(d(z, x_{n+1}), d(x_n, Tz))$$

Making $n \to \infty$, using (2.14) and continuity of ψ we obtain

$$d(z, Tz) \leq \frac{1}{2} d(z, Tz) - \psi(0, d(z, Tz)) \leq \frac{1}{2} d(z, Tz))$$

which is a contradiction unless $d(z, Tz) = 0$.
Hence $z = Tz$.
Next we establish that the fixed point z is unique.

If z_1 and z_2 are two fixed points of T, then

$$d(z_1, z_2) = d(Tz_1, Tz_2) \leq \frac{1}{2}(d(z_1, Tz_2) + d(z_2, Tz_1)) - \psi(d(z_1, Tz_2), d(z_2, Tz_1)).$$

that is, $d(z_1, z_2) \leq d(z_1, z_2) - \psi(d(z_1, z_2), d(z_1, z_2))$,

which by property of ψ is a contradiction unless $d(z_1, z_2) = 0$, that is $z_1 = z_2$. This completes the proof of the uniqueness of the fixed point.

We next consider the following example.

Example 2.1: Let $X = \{\alpha, \beta, \gamma\}$ and d is a metric defined on X as follows.
$d(\alpha, \beta) = 1$, $d(\beta, \gamma) = 2$, $d(\gamma, \alpha) = 1.5$

Then (X, d) is a complete metric space. Let $\psi(a, b) = \frac{1}{2}\min\{a, b\}$.

Let $T : X \to X$ be a mapping defined as follows.

$T\alpha = \beta,\ T\beta = \beta,\ T\gamma = \alpha.$

Then T is a weak C-contraction and conditions of theorem 2.1 are satisfied. Hence T must have a unique fixed point. Here β is the unique fixed point of T.

The mapping T in the above example is not a C-contraction. This is seen by noting that when $x = \alpha$ and $y = \gamma$ the inequality (1.3)will not be satisfied by T. Again as has been noted in the discussion following definition 1.3, every C-contraction is a weak C-contraction. Thus the class of weakly C-contractive mappings is actually a strictly larger class of mappings than the class of C-contractions.

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