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Unique Trajectory Method in Migdal Renormalization Group Approach and Crossover Phenomena

Masahiro IMACHI, Sachiko KAWABE and Hiroshi YONEYAMA

Department of Physics, Kyushu University 33, Fukuoka 812

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Migdal renormalization group approach, combined with Wilson-Kogut topological argument, is applied to four dimensional lattice gauge theory of finite subgroup $\tilde{I}(120)$ of $SU(2)$. i) A slight (compared with the Monte Carlo results) but clear crossover from strong coupling regime to weak coupling regime is observed for the Wilson action. ii) For mixed action, of the fundamental and the adjoint representation, a clearer stepwise transition, suggesting first order phase transition, is found at $1 \lesssim \beta_a{}^b \lesssim 3$ (where $\beta_a{}^b$ denotes the bare inverse coupling constant of the adjoint representation). This stepwise transition changes into crossover for smaller $\beta_a{}^b$. iii) There are four critical lines in $(\beta_f{}^b, \beta_a{}^b)$ plane starting from a quadruple point ($\beta_f{}^b \sim 0.75$, $\beta_a{}^b \sim 3.2$) where $\beta_f{}^b$ denotes the bare inverse coupling constant of the fundamental representation; 1) $SO(3)$ critical line, 2) $Z(2)$ critical line, 3) a critical line due to the discreteness of $\tilde{I}(120)$, 4) a critical line related to crossover. In this investigation, the unique trajectory of renormalization group is very important and plays a powerful role in finding crossover and stepwise transition.

§ 1. Introduction

In a previous paper,[1] it is pointed out that the point $\beta_{\text{t.o.}}$, which we will call "turnover point,"*) situates very close to (i) the crossover[2),3)] point in the case of LGT's of non abelian finite subgroups of $SU(2)$ and continuous group $SU(2)$ itself, and to (ii) the deconfining phase transition in the case of LGT's of abelian subgroups $Z(N)$ of $U(1)$ and the group $U(1)$ itself.[4] The turnover point $\beta_{\text{t.o.}}$ is defined as[1] the point

$$d\beta_{i_0}/d\ln\lambda=0 \quad \text{at} \quad \lambda=1\,, \quad \beta_{i_0}=\beta_{\text{t.o.}} \quad \text{and} \quad \beta_{i\neq i_0}=0 \tag{1·1}$$

for all $i(\neq i_0)$, where i and i_0 denote the irreducible and the fundamental representation respectively and λ denotes the amount of scale change of single iteration in the Migdal transformation.[5] The turnover point $\beta_{\text{t.o.}}$ is, qualitatively speaking, considered to be the point which separates the β_{i_0}-axis (Wilson axis) into two regions, 1) $\beta_{i_0}<\beta_{\text{t.o.}}$, where the Migdal renormalization group trajectory rapidly converges to strong coupling regime, i.e., $\beta_{i\neq i_0} \ll \beta_{i_0} \ll 1$, namely without leaving the Wilson axis too much, and 2) $\beta_{i_0}>\beta_{\text{t.o.}}$, where the trajectory deviates from the original Wilson axis and moves to the Gaussian action. The fact, that $\beta_{\text{t.o.}}$ situates quite close to the crossover point in $\tilde{I}(120)$ and $SU(2)$ cases,

*) $\beta_{\text{t.o.}}$ was called critical point in Ref. 1), but it will be called "turnover point" in this paper in order to avoid confusion, because $\beta_{\text{t.o.}}$ does not situate on the critical surface.

suggests it deserves further investigation on the relation between the Migdal renormalization group and the crossover phenomena.

In this paper, we will extensively investigate whether the Migdal renormalization group approach can really probe the crossover phenomena or not.[6] For this purpose, flow diagram of the renormalization group trajectory of coupling constants is studied. In this procedure, the fact that every trajectory starting from various bare actions converges to *a unique trajectory*[7] in confining phase plays an essential role. Using this property and the Wilson-Kogut topological argument,[8] we can derive the relation between bare coupling constants and the lattice constant a. Namely, we fix a renormalized coupling constant β_G on the unique trajectory, which is called "gate",[8] and calculate the amount of change of scale between the gate β_G and some bare coupling constant $\beta^b(a)$, and we obtain the functional relation between the bare coupling constant and the lattice constant a, which gives the coupling constant renormalization procedure. We will call this process "unique trajectory method". The unique trajectory method, which combines the Migdal recursion equation with the Wilson-Kogut topological argument, makes it possible to study phase structures of various lattice gauge theories.

In the Migdal renormalization group approach, the change of scale is given by a parameter λ and the total amount of change of scale between bare theory at a and the renormalized theory, whose scale is denoted as ξ_G, is given by λ^{t_G},

$$\xi_G=\lambda^{t_G}a\,, \tag{1·2}$$

where t_G denotes the number of steps of the Migdal transformation necessary for a trajectory starting from a bare theory to reach the gate. The beta function (Gell-Mann-Low function) is given by

$$\psi(\beta^b)=1/(dt_G/d\beta^b)|_{\beta_G}\,. \tag{1·3}$$

Main results are as follows:

i) The relation between t_G and β^b is calculated for the LGT of $\tilde{I}(120)$, icosahedral subgroup of $SU(2)$. We will observe a slight but clear crossover from strong coupling regime to weak coupling regime in the case of the Wilson action.[9]

ii) The same procedure, the unique trajectory method, is applied to the case of mixed action.[10),11)] It is shown that the renormalization group trajectories starting from various bare actions approach (i) the same trajectory found in the bare Wilson action case if $(\beta_f{}^b, \beta_a{}^b)$ lies far from the β_a-axis and (ii) another unique trajectory dominated by adjoint representation if $(\beta_f{}^b, \beta_a{}^b)$ lies close to the β_a-axis, where β_f and β_a denotes the inverse coupling constant of the fundamental and adjoint representation, respectively. Since these two unique trajectories lead to the same fixed point $\beta_i=0$ (i=all), we define a gate on the second unique trajectory so as to give the same amount of T (isospin)=1 contribution as that given by the coupling constant at the gate on the first unique trajectory. By this procedure, the unique trajectory method leads to the functional relation between the bare coupling constants and the lattice constant a. For $\beta_a^b \lesssim 1.0$, crossover phenomena, similar to the Wilson action case, are observed and at $1 \lesssim \beta_a{}^b \lesssim 3$ region we find a *clearer stepwise* transition from strong coupling regime to weak coupling regime suggesting the first order phase transition.

iii) There is a quadruple point where four critrical lines meet together in the fundamental-adjoint coupling constant plane. Four critical lines are 1) $SO(3)$ critical line, 2) $Z(2)$ critical line, 3) critical line due to the discreteness of $\tilde{I}(120)$ and 4) critical line related to crossover. The first critical line represents the phase transition of the orthogonal subgroup $I(60)$ $(=\tilde{I}(120)/Z(2))$ of $SO(3)$. The second critical line is due to $Z(2)$ subgroup of $\tilde{I}(120)$ at $\beta_a \to \infty$, namely the freedom of $I(60)$ group is frozen in $\tilde{I}(120)$ and only $Z(2)$ freedom survives. The third critical line is due to the finiteness of $\tilde{I}(120)$ group. In $SU(2)$ group, this critical line is expected to disappear.

In § 2, we will briefly list the formula necessary to apply the Migdal recursion in LGT of finite group.

In § 3, the unique trajectory method is performed for LGT of $\tilde{I}(120)$. In § 4, some conclusions and discussion are presented.

§ 2. Migdal recursion equation for finite group LGT

The Migdal renormalization group transformation[5] is written down for LGT of finite group G.[1] The partition function at scale L is given by the function $F(L, v_{\mu\nu})$ as

$$Z_L = \prod_{\mu<\nu} \sum_{\{v_{\mu\nu}\}} F(L, v_{\mu\nu}), \tag{2·1}$$

where $v_{\mu\nu}$ denotes the group element of the loop with size L and situated in $\mu\nu$-plane (μ, $\nu=1, 2, \cdots, D$). In general the bare theory is defined at lattice constant a as,

$$F(a, v) = \exp[\beta_0{}^b - \sum_{i=2}^{s} \beta_i{}^b (1 - \chi_i(v)/n_i)], \tag{2·2}$$

where $\chi_i(v)$ denotes the character of the irreducible representation i of dimension n_i of the group G, and $\beta_i{}^b$ denotes the bare coupling constants and $\beta_0{}^b$ is the normalization constant. Since $\chi_i(v)$ depends only on class of the group, we denote it as $\chi_i(\alpha)$, where $\alpha(1, \cdots, s)$ specifies the class and s is the number of irreducible representations or the number of classes of G. The recursion equation is[1]

$$F(\lambda L, \alpha)=[\sum_{i=1}^{s}(n_i/h)\chi_i(\alpha)(\tilde{F}_i(L))^{\lambda^2}]^{\lambda^{D-2}}, \tag{2·3}$$

where D is the euclidean space time dimension and h denotes the total number of elements of the group G and $\tilde{F}_i$ is defined as

$$\tilde{F}_i(L)=\sum_{\alpha=1}^{s}(p_\alpha/n_i)\chi_i^*(\alpha)F(L, \alpha), \tag{2·4}$$

where p_α is the number of elements contained in the class α. Using this relation we are able to obtain coupling constants at scale λL from those at scale L;

$$\beta_i(\lambda L)=\lambda^{D-2}[\sum_{\alpha=1}^{s}(p_\alpha n_i\chi_i^*(\alpha)/h)\ln\{\sum_{j=1}^{s}(\chi_j(\alpha)n_j/h)\times(\tilde{F}_j(L)^{\lambda^2}\}]. \tag{2·5}$$

The characters should satisfy,

$$\sum_{\alpha=1}^{s}p_\alpha\chi_i(\alpha)\chi_j^*(\alpha)=h\delta_{ij}, \tag{2·6}$$

$$\sum_{i=1}^{s}\chi_i(\alpha)\chi_i^*(\beta)=h\delta_{\alpha\beta}/p_\alpha, \tag{2·7}$$

$$p_\alpha p_\beta\chi_i(\alpha)\chi_i(\beta)=n_i\sum_{\gamma=1}^{s}c_{\alpha\beta}^{\gamma}p_\gamma\chi_i(\gamma), \tag{2·8}$$

where $c_{\alpha\beta}^{\gamma}$ denotes the class constants.[12] The characters $\chi_i(\alpha)$ for the 120 element icosahedral subgroup of $SU(2)$, $\tilde{I}(120)$, are listed in the Appendix.

By the successive transformation (2·3)~(2·5), the renormalization group trajectories are given in $(s-1)$-dimensional parameter space of $(\beta_2, \beta_3, \cdots, \beta_s)$.

The merit of studying the finite subgroup is that renormalization transformation is given in a closed form about the finite set of coupling constants and makes it quite easier to perform renormalization transformation numerically.

§ 3. The unique trajectory method in the Migdal renormalization group

In this section, we present a method of analysis which is based on the Migdal renormalization group transformation and the Wilson-Kogut topological argument. We will show that the Migdal-Wilson-Kogut combined method provides useful information on the phase structures including crossover phenomena. When it is applied to finite subgroup of $SU(2)$ it gives not only the critical surfaces but also the crossover and the stepwise structure suggesting the first order phase transition. In this analysis the role of the unique trajectory is essential, so we call this method "unique trajectory method". The existence of the unique trajectory means that trajectories starting from various bare theories with different lattice constant, a, and bare coupling constants, β_i^b, converge to a unique renormalized theory. In other words, we can derive the relation between the lattice constant a and the bare coupling constant β_i^b by fixing *any* renormalized coupling constant β^r on the unique trajectory.

3.1. *Unique trajectory*

Starting from a bare action with bare coupling constants β_i^b $(i=i_0, \cdots, s)$ on the Wilson axis, $\beta_{i_0}^b \neq 0$ and $\beta_{i \neq i_0}^b = 0$ for all the other i's, where i_0 denotes the fundamental representation, we obtain a trajectory in $(s-1)$ dimensional parameter space of coupling constants by successive renormalization transformations. For different initial values $\beta_{i_0}^b$ we obtain a set of trajectories. An important feature of these trajectories is that every trajectory converges to a single trajectory irrespective of $\beta_{i_0}^b$, which is thus called "unique trajectory", after sufficient iterations. A flow diagram of renormalization group trajectories of $\tilde{I}(120)$ subgroup for the Wilson action projected on the β_f-β_a plane, is shown in Fig. 1, where $\beta_f = \beta_{i_0} = \beta_2$ and $\beta_a = \beta_4$ in $\tilde{I}(120)$ denote the coupling constant of the fundamental and the adjoint representation respectively.

We have two fixed points A and C. The point A denotes the extreme of the strong coupling limit. A critical line absorbed by the point*) C cuts across the Wilson axis at point B. Let the value of coupling constant $\beta_{i_0}^b$ of this critical point be $\beta_c{}^b$. For $\beta_{i_0}^b < \beta_c{}^b$, all trajectories approach a unique trajectory connecting two fixed points A and C. The region specified by the fixed point A is called the confining phase. The region $\beta_{i_0}^b > \beta_c$ is the non confining phase and the presence of this phase is due to the finiteness of $\tilde{I}(120)$ group. In continuous group $SU(2)$, $\beta_c \to \infty$ and this phase is expected to disappear.

The phase which we are interested in is the confining phase, i.e., $\beta_{i_0}^b < \beta_c{}^b$ in $\tilde{I}(120)$. In this confining phase there are still two qualitatively different regions, the strong coupling region and the weak coupling region. For smaller $\beta_{i_0}^b$, i.e., the strong coupling region, the renormalization group trajectory rapidly approaches the unique trajectory which lies close to the Wilson axis, namely, the ratio of renormalized coupling constants on this part of unique trajectory satisfies $\beta_{i \neq i_0}^r / \beta_{i_0}^r \to 0$ as $\beta_{i_0}^r \to 0$. For larger $\beta_{i_0}^b$ which is smaller than β_c, i.e., for the weak

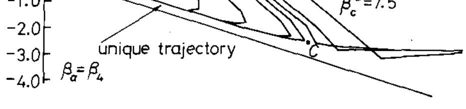

Fig. 1. Flow diagram of renormalization group trajectories of the bare Wilson action. Critical point is B ($\beta_2{}^b \cong 7.5$). A is the fixed point of confinement phase. C is the fixed point which separates confinement and non-confinement phase. All renormalization group trajectories for $\beta_2{}^b < 7.5$ flow into a unique trajectory.

*) The fixed point C is given by $\beta_2 = 10.2$, $\beta_3 = 3.53 \times 10^{-3}$, $\beta_4 = -2.59$, $\beta_5 = -9.23 \times 10^{-2}$, $\beta_6 = 1.31$, $\beta_7 = -2.02 \times 10^{-1}$, $\beta_8 = -7.10 \times 10^{-1}$ and $\beta_9 = 6.30 \times 10^{-1}$.

coupling region, the unique trajectory which the renormalization group trajectory approaches situates far from the Wilson axis and near the Gaussian model. The Gaussian action is represented by the irreducible characters as

$$\theta^2 = c_0 + \sum_{i=2}^{\infty} (\chi_i(\theta)/n_i - 1) c_i$$

with

$$c_i = (-1)^{2T} 2(2T+1)^2 / \{T^2(T+1)^2\}, \qquad i = 2T+1 \tag{3·1}$$

for $SU(2)$ group. In the weak coupling region, the ratio between the renormalized couplings approaches finite value,

$$\beta^T_{i+i_0}/\beta^T_{i_0} \to c_{2T+1}/c_2 . \tag{3·2}$$

After reaching the unique trajectory with $\beta^T_{i+i_0}/\beta^T_{i_0} \sim c_{2T+1}/c_2$, the renormalized coupling constants move along the unique trajectory from the weak coupling region to the strong coupling region and finally tend to $\beta^T_{i+i_0}/\beta^T_{i_0} \to 0$ as shown in Fig. 1. In Fig. 2, various trajectories starting from various bare coupling constants are shown. Convergence to the unique trajectory, namely, independence of the asymptotic form of the unique trajectory from the bare coupling constants, is outstandingly clear.

We are able to obtain the analytic form of the unique trajectory at strong coupling limit ($\beta_{i_0} = \beta_2 \ll 1$). First, in order to know the order of magnitude of coupling constants of each irreducible representation, we consider a simple example,

$$F(\alpha) = \exp(A_{\alpha 2}\beta_2(L)), \quad \beta_2(L) \ll 1 \quad \text{with} \quad A_{\alpha i} \equiv \chi_i(\alpha)/n_i - 1 . \tag{3·3}$$

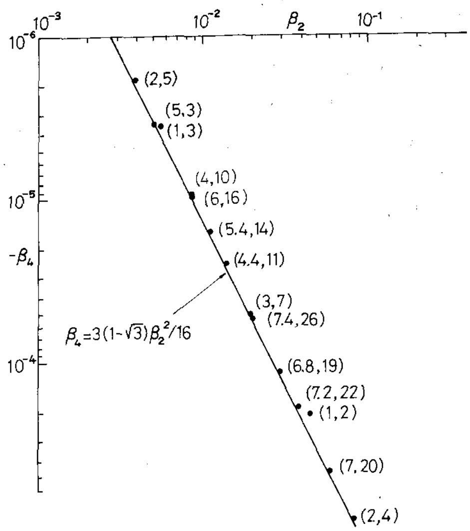

Fig. 2. The coupling constants β_2 and β_4 on the unique trajectory in the strong coupling region are expected to satisfy $\beta_4 = 3(1-\sqrt{3})\beta_2^2/16$, which is denoted by a solid line. The renormalized couplings (β_2, β_4) given by the Migdal recursion equation starting from various Wilson actions are plotted by the points (●), where the numbers in the parenthesis denote the bare coupling β_2^b and the number of iterations necessary to reach each point respectively.

Keeping up to $O((\beta_2)^2)$ term, we have

$$F(\alpha)\sim 1+A_{\alpha 2}\beta_2+\frac{1}{2}A_{\alpha 2}^2\beta_2{}^2=g_0+g_2\chi_2(\alpha)+g_4\chi_4(\alpha), \tag{3·4}$$

where

$$g_0=1-\beta_2+5\beta_2{}^2/4\,,\quad g_2=(\beta_2-\beta_2{}^2)/2 \quad \text{and} \quad g_4=\beta_2{}^2/8 \tag{3·5}$$

and we used $\chi_4(\alpha)=\chi_2{}^2(\alpha)-1$. We obtain

$$\tilde{F}_i=\sum_\alpha(\chi_i{}^*(\alpha)p_\alpha/n_i)F(\alpha)=h(g_0\delta_{i1}+(g_2/2)\delta_{i2}+(g_4/3)\delta_{i4}) \tag{3·6}$$

and $(D=4)$

$$\beta_i(\lambda L)=\lambda^2\sum_\alpha(p_\alpha n_i\chi_i{}^*(\alpha)/h)\ln G(\alpha)=(\lambda^2 n_i/h)\sum_\alpha \chi_i{}^*(\alpha)p_\alpha \ln G(\alpha)$$
$$\cong\lambda^2\{[\ln(h^{\lambda^2-1}g_0^{\lambda^2})-2(g_2/2g_0)^{2\lambda^2}]\delta_{i1}+4(g_2/2g_0)^{\lambda^2}\delta_{i2}$$
$$+[9(g_4/3g_0)^{\lambda^2}-6(g_2/2g_0)^{2\lambda^2}]\delta_{i4}\}, \tag{3·7}$$

where $G(\alpha)=\sum_j(\chi_j(\alpha)n_j/h)(\tilde{F}_j)^{\lambda^2}$ and $\ln G(\alpha)$ is expanded into power series. Now we derive,

$$\beta_2(\lambda L)\sim 4\lambda^2(g_2/2g_0)^{\lambda^2}\sim 4\lambda^2(\beta_2(L)/4)^{\lambda^2}\,, \tag{3·8}$$

$$\beta_4(\lambda L)\sim\lambda^2\{9(g_4/3g_0)^{\lambda^2}-6(g_2/2g_0)^{2\lambda^2}\}$$
$$\sim\lambda^2(9/(24)^{\lambda^2}-6/(16)^{\lambda^2})(\beta_2(L))^{2\lambda^2}\,, \tag{3·9}$$

so $\beta_4(\lambda L)\propto(\beta_2(\lambda L))^2$ in the strong coupling region.

When we slightly generalize to the case

$$F(\alpha)=\exp(A_{\alpha 2}\beta_2(L)+A_{\alpha 4}\beta_4(L)),\quad \beta_4(L)\ll\beta_2(L)\ll 1\,, \tag{3·10}$$

we have the result $(3·7)\sim(3·9)$ where g_0, g_2 and g_4 are replaced by

$$g_0=1-\beta_2(L)-\beta_4(L)+5\beta_2{}^2(L)/4\,,$$
$$g_2=(\beta_2(L)-\beta_2{}^2(L))/2$$

and

$$g_4=\beta_4(L)/3+\beta_2{}^2(L)/8\,. \tag{3·11}$$

There is a solution of renormalized couplings satisfying

$$\beta_4{}^r(\lambda L)=k(\beta_2{}^r(\lambda L))^2\,, \tag{3·12}$$

and λ-dependent constant k is obtained as the solution of

$$k=\frac{9}{16\lambda^2}\left(\frac{16}{3}\left(\frac{k}{3}+\frac{1}{8}\right)\right)^{\lambda^2}-\frac{3}{8\lambda^2}. \tag{3·13}$$

For the choice $\lambda^2=2$,

$$k=3(1-\sqrt{3})/16=-0.1373\cdots. \tag{3·14}$$

The solid line in Fig. 2 is given by Eq. (3·12) with $k=3(1-\sqrt{3})/16$. Other coupling constants $\beta^r_{i\neq 2,4}(\lambda L)$ are given by higher integer power of $\beta_2{}^r(\lambda L)$ as $\beta_3{}^r(\lambda L)\propto\beta_2{}^r(\lambda L)^7$, $\beta_5{}^r(\lambda L)\propto\beta_2{}^r(\lambda L)^6$, $\beta_6{}^r(\lambda L)\propto\beta_2{}^r(\lambda L)^3$, $\beta_7{}^r(\lambda L)\propto\beta_2{}^r(\lambda L)^6$, $\beta_8{}^r(\lambda L)\propto\beta_2{}^r(\lambda L)^4$ and $\beta_9{}^r(\lambda L)\propto\beta_2{}^r(\lambda L)^5$.

3.2. *Coupling constant renormalization*

Let us investigate the coupling constant renormalization according to the Wilson-Kogut topological argument.[8] The existence of the unique trajectory shows that there is a unique renormalized theory for various theories with different bare coupling constants. We will set a "gate" β_G on the unique trajectory. Let $t_G(\beta^b)$ be the number of iterations to reach the gate β_G from a bare coupling constant β^b. The amount of change of scale per iteration is taken to be λ. When we fix a renormalized coupling constant β_G at some physical length ξ_G, t_G changes depending on the bare coupling constant β^b, namely, we obtain t_G as a function of β^b. In the example shown in Fig. 1, t_G is small for small β^b and large for larger $\beta^b(<\beta_c)$. t_G tends to infinity as $\beta^b\to\beta_c$ from below. The trajectory starting from β_c (point B) is the critical line and $t_G=\infty$ means that the dimensionless correlation length is divergent. The beta function (Gell-Mann-Low function[13]) is given by

$$\psi(\beta^b)=1/(dt_G/d\beta^b)|_{\beta_G}, \tag{3·15}$$

and the point $\beta_c{}^b$ gives $\psi(\beta_c{}^b)=0$.

Now we consider the relation between the lattice constant a of bare theory and the bare coupling constant β^b. The lattice constant a is related to the physical length ξ_G as

$$a=\lambda^{-t_G}\xi_G. \tag{3·16}$$

t_G is chosen so as to fix renormalized coupling constant β_G at physical length ξ_G by adjusting β^b suitably, namely, t_G is given as a function of bare coupling constant β^b. So Eq. (3·16) provides the relation between β^b and a. The relation between a and β^b thus obtained corresponds to the a-β^b relation in Monte Carlo calculation obtained by fixing the string tension.[2] In the Monte Carlo calculation, a clear crossover from strong to weak coupling regime was observed.

3.3. *Wilson action and crossover phenomena*

It is an interesting question whether the Migdal renormalization group can detect the crossover phenomena or not. It is only method which is known to be applicable analytically, but due to the crudeness of the approximation, such a delicate structure as crossover, compared with ordinary phase transitions, may escape from the detection. In conclusion we find a slight but clear crossover in the Wilson action case, by the unique trajectory method. According to the unique trajectory method, we calculated $t_G(\beta^b)$. The obtained result is shown in Fig. 3 where β_G is set to be 10^{-3} and $\lambda=\sqrt{2}$.

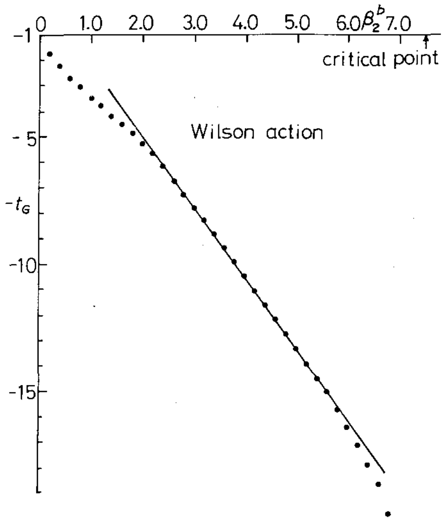

Fig. 3. The number of iterations, t_G, necessary to reach the gate is given as a function of $\beta_2{}^b$ (Wilson action). A slight but a clear crossover is seen in the t_G-$\beta_2{}^b$ relation.

1) A slight but clear sudden crossover from the strong coupling regime to the weak coupling regime is observed at $\beta^b_{i_0}=\beta_2{}^b\sim 2.3$ ($i_0=2$ denotes the fundamental representation).

2) Phase transition is observed at $\beta_2{}^b\sim 7.5=\beta_c{}^b$, which is seen as the zero point of the beta function ψ of (3·15). This phase transition corresponds to the phase transition observed in the Monte Carlo calculation at $\beta_2{}^b\sim 6$. The position of $\beta_c{}^b$ is shifted due to the value of $\lambda=\sqrt{2}$ which differs from unity. This phase transition is considered[3] to be due to the discreteness of $\tilde{I}(120)$ and is expected to disappear in the continuous $SU(2)$ group case.

3) Phase transition at $\beta_2{}^b\sim 7.5$ is second order in Migdal renormalization group, while it is first order like in the Monte Carlo calculation.

4) Between $\beta^b_{\text{c.o.}}\sim 2.3$ and $\beta_c{}^b\sim 7.5$, the curve is approximately linear,

$$t_G(\beta^b)=c\beta^b+d\,, \quad \text{namely,} \qquad \psi\sim(dt_G/d\beta^b)^{-1}=1/c\,, \tag{3·17}$$

where c and d are constants. It gives qualitatively the same results as perturbative QCD beta function. Since the "physical quantity" ξ_G which gives physical coupling constant β_G should be independent of lattice constant a, it satisfies

$$0=d\xi_G/da=(\partial\xi_G/\partial a)+(\partial\xi_G/\partial\beta^b)(d\beta^b/da). \tag{3·18}$$

From $\xi_G=a\lambda^{t_G(\beta_b)}$ and (3·18), we obtain

$$d\beta^b/da=-(\partial\xi_G/\partial a)/(\partial\xi_G/\partial\beta^b)=-\lambda^{t_G}/(ac\ln\lambda\cdot\lambda^{t_G})=-(1/ac\ln\lambda). \tag{3·19}$$

By $\beta^b\equiv 4/g_b{}^2$ (g_b denotes the bare coupling constant),

$$a\frac{dg_b}{da}=\frac{1}{8c\ln\sqrt{2}}g_b{}^3\,. \tag{3·20}$$

It is similar to the form of beta function obtained in perturbative QCD. Numerically, however, the coefficient of $g_b{}^3$ term is $1/8c\ln\sqrt{2}\sim 0.13$ in our calculation ($c\sim 2.7$, $\lambda=\sqrt{2}$) which is too much greater compared with the value $(11/24\pi^2)\sim 0.046$ by factor 2.9. According to the original Migdal's paper,[5] $SU(2)$ group gives $(\lambda^2-1)/24\ln\lambda\sim 0.12$ for $\lambda=\sqrt{2}$, which is close to our $\tilde{I}(120)$ result (0.13). The value of this coefficient depends strongly on λ. For example, the discrepancy becomes ~ 1.8 instead of 2.9 in $\lambda\to 1$ limit.

3.4. *Mixed action and stepwise transition*

In order to know more about the phase structure of $\tilde{I}(120)$ group, we next consider the mixed action model,[10),11)] where some members of $\beta_i{}^b$ other than $\beta_2{}^b$ (fundamental representation $i_0=2$) are set equal to non vanishing value. In this section we consider the fundamental + adjoint model, where $\beta_2{}^b$ and $\beta_4{}^b$ are non vanishing and others are zero. Renormalization group transformation drives various bare coupling constants to the unique trajectory as in the previous subsection (Fig. 4). For small $\beta_2{}^b=\beta_f{}^b$ and large $\beta_4{}^b=\beta_a{}^b$, the coupling constant $\beta_a{}^r$ of adjoint representation dominates, and the renormalization group trajectory approaches another unique trajectory, which we will call "unique trajectory II". Leading coupling constants on unique trajectory II are those of integer isospin T, $\beta_4{}^r(T=1)$, $\beta_8{}^r(T=2)$ etc.

Unique trajectory II runs along the Gaussian action of $SO(3)$ theory at $\beta_4{}^r$ = large (but below $\beta_{4c}^b\sim 3.5$) and goes along $\beta_8{}^r\cong k_{\rm II}(\beta_4{}^r)^2$ at strong coupling limit ($\beta_4{}^r\to 0$) and is finally absorbed by the fixed point A, $\beta_i=0$ ($i=2,\cdots,9$). The unique trajectory found in renormalization group transformation starting from bare Wislon action, which we will call "unique trajectory I", flows into the same fixed point A as unique trajectory II and the behavior of both trajectories at weak and strong coupling regime is described by the θ^2 type and the cosine type Lagrangian in the respective region. In order to adjust the gate on unique trajectory I and that on II, we choose the gate $\beta_{a,G}^{\rm II}$ on unique trajectory II in the strong coupling region as done in the case of unique trajectory I as,

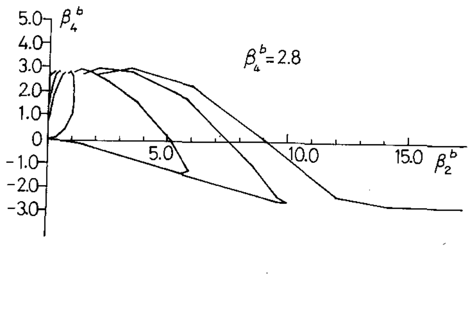

Fig. 4. The flow diagram of renormalization group trajectories starting from bare mixed action ($\beta_2{}^b$, $\beta_4{}^b$), where $\beta_4{}^b=2.8$. For larger $\beta_2{}^b$ (but smaller than β_c), the renormalization group trajectory is given by unique trajectory I and for smaller $\beta_2{}^b$, the renormalization group trajectory is given by unique trajectory II.

$$\beta^{\rm II}_{a,G}=3(\beta^{\rm I}_{f,G})^2/8+\beta^{\rm I}_{a,G}\,, \tag{3·21}$$

where $\beta^{\rm I}_{a,G}\simeq k_{\rm I}(\beta^{\rm I}_{f,G})^2$ (Eq. (3·12)) with $k_{\rm I}=3(1-\sqrt{3})/16$ (Eq. (3·14)). This choice leads to the same contribution of $\cos^2(\theta/2)$ term in the small β expansion of the partition function. For the choice $\beta^{\rm I}_{f,G}=1.0\times10^{-3}$ we have

$$\beta^{\rm II}_{a,G}=3(3-\sqrt{3})(\beta^{\rm I}_{f,G})^2/16\sim2.377\times10^{-7}\,. \tag{3·22}$$

Now we are able to obtain t_G-$(\beta_2{}^b, \beta_4{}^b)$ relation. When we start from some bare coupling $(\beta_2{}^b, \beta_4{}^b)$, we obtain the value of t_G by the gate $\beta^{\rm I}_{f,G}=10^{-3}$ if the renormalization group trajectory goes into unique trajectory I and by the gate $\beta^{\rm II}_{a,G}=2.377\times10^{-7}$ if renormalization group trajectory goes into unique trajectory II. We show the t_G-$\beta_2{}^b$ relation in the mixed action model in Fig. 5.

1) Surprisingly clear stepwise structure is found in the region $1\lesssim\beta_4{}^b\lesssim3$. Since this stepwise structure means the approximate *discontinuity of the dimensionless correlation length, it suggests first order phase transition.* This structure continuously tends to the crossover behavior as $\beta_4{}^b\to$small.

2) Phase transition with $\psi(\beta^b)=0$ appears along a line connecting a point $Q_0(=(\beta_2{}^b, \beta_4{}^b)=(0.75, 3.2))$ and the point $B((7.5, 0))$ (Fig. 6). This phase transition is expected to be due to the discreteness of $\tilde{I}(120)$.

3) Critical lines with $\psi(\beta^b)=0$ run from the point Q_0 to $(0, 3.5)$ and from Q_0 to

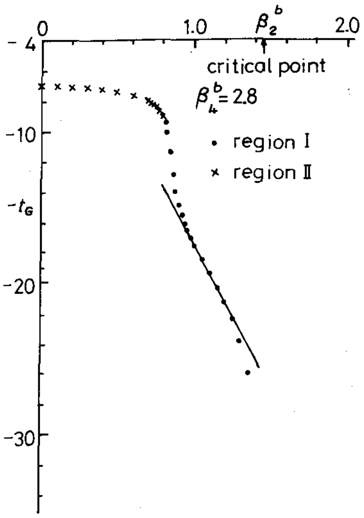

Fig. 5. The stepwise structure at $\beta_2{}^b\simeq0.85$ seen in the t_G-$\beta_2{}^b$ relation for the mixed action model at $\beta_4{}^b=2.8$. The points (●) and the crosspoints (×) denote the bare theories which lead to the unique trajectories I and II, respectively.

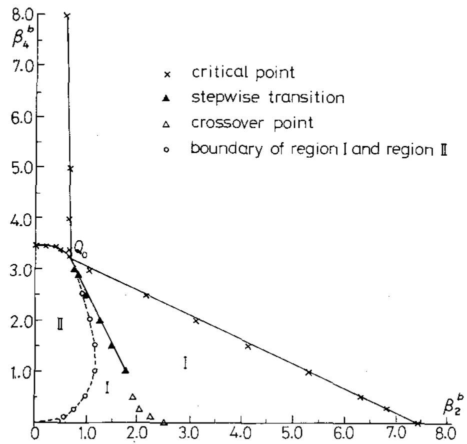

Fig. 6. The phase diagram of the fundamental-adjoint mixed action model. The crosspoints (×) denote the critical points of the second order phase transition ($t_G=\infty$) and the points (▲) and (△) show the stepwise transitions and the crossover structures, respectively. The regions I and II are distinguished by the points (○). The quadruple point Q_0 situates at (0.75, 3.2).

$(0.6, \infty)$. The former is due to the finite subgroup of $SO(3)$, $I(60)$ $(=\tilde{I}(120)/Z(2))$ and the latter is due to $Z(2)$. These phase transitions are of second order in the Migdal approximation.

4) The "stepwise transition" connected to crossover runs along a line connecting the point Q_0 and $(2.3, 0)$. In this way, the point Q_0 is a quadruple point of phase transition in $(\beta_2{}^b, \beta_4{}^b)$ plane.

5) Phase diagram thus obtained is shown in Fig. 6. The two regions of $(\beta_2{}^b, \beta_4{}^b)$ which lead to the unique trajectories I and II are denoted as I and II in this figure. The region whose renormalization group trajectory is given by unique trajectory II is $SO(3)$ like. Two regions I and II are separated by a line which is shown by a dotted curve.

6) The phase transition and possible crossover point in the Manton[14] action are also plotted in Fig. 6 for comparison. It should be noted that other $\beta^b_{l\neq 2,4}$ are not zero in this case so the points are the projected ones into $(\beta_2{}^b, \beta_4{}^b)$ plane. The crossover is very slight in the Manton action as shown in Fig. 7 ($\beta^b_{2c} \sim 10.5$, $\beta^b_{2c.o.} \sim 3.5$).

7) By fixing the renormalized coupling (e.g., that of the gate) at ξ_G, the "equi-lattice constant surface" can be derived. According to the relation $\xi_G = \lambda^{t_G} a$, the equi-lattice constant surface can be obtained by the condition $t_G(\beta_2{}^b, \beta_4{}^b) =$ constant. The equi-lattice constant surface in confinement phase is shown in Fig. 8.

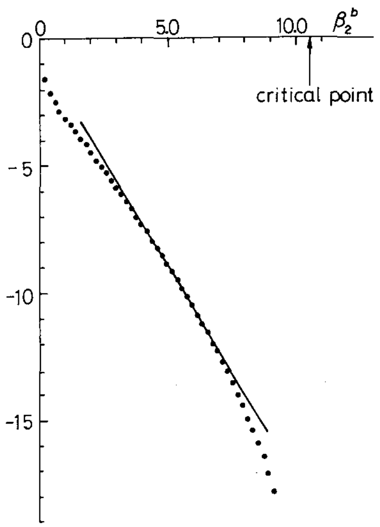

Fig. 7. The t_G-$\beta_2{}^b$ relation in the Manton action.

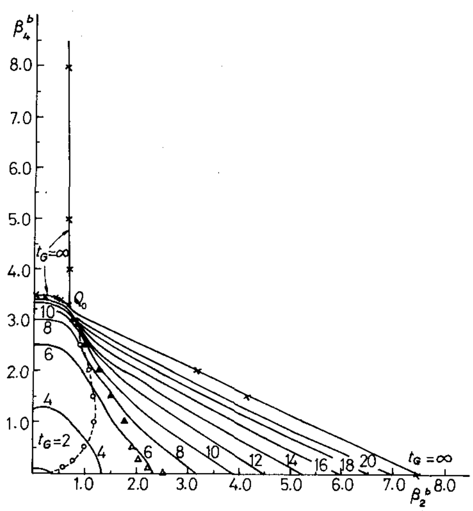

Fig. 8. The "equi-lattice constant surface" in the regions I and II for the mixed action model. The points (×, ▲, △, ○) are same as those in Fig. 6.

§ 4. Discussion

In this paper we presented the "unique trajectory method" in the Migdal renormalization group approach and applied it to $\tilde{I}(120)$ group in four dimensional lattice gauge theory. It probes various phase structures. In the Wilson action case, a slight but clear sudden crossover is observed and in the mixed action case approximately singular behavior like a discontinuity is observed. The latter singular stepwise behavior means the sudden change of the correlation length like a gap, then this critical line approximately indicates the first order phase transition. This feature is outstanding in the neighborhood of the quadruple point and becomes weaker as we approach the Wilson axis. On the way to reach the Wilson axis this stepwise singular behavior changes into crossover structure. Therefore, the crossover can be considered to be reminiscence of the first order phase transition.

There is a region $(\beta_f{}^b, \beta_a{}^b)$ whose renormalization trajectory is given by unique trajectory II (that of $SO(3)$). The boundary between two regions (one leads to unique trajectory I and another leads to unique trajectory II) was observed (Fig. 6).

The success of the unique trajectory method in probing crossover and stepwise singular behavior is very encouraging. It is worth while to apply the method to other gauge groups including the continuous group $SU(2)$,[15] $SU(3)$, $SU(4)$, $SU(5)$, etc.

There are, however, problems in the method: i) Quantitative results, e.g., the slope of beta function deviates by factor 3 from the value expected from continuum theory and ii) the order of phase transition cannot be reproduced correctly, namely, the first order phase transition is approximately well reproduced in one case but not in other (three) cases around the quadruple point.

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Appendix

We list the irreducible characters[1] $\chi_i(a)$ of $\tilde{I}(120)$ group in Table I. $i=i_0=2$, $i=4$, $i=6$, $i=8$ and $i=9$ correspond to the fundamental representation $(T=1/2)$, the adjoint representation $(T=1)$, $T=3/2$, $T=2$ and $T=5/2$ representation respectively.

Table I. Irreducible characters of $\tilde{I}$.

		χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
class	p_a \ n_i	1	2	2	3	3	4	4	5	6
C_1	1	1	2	2	3	3	4	4	5	6
C_2	20	1	1	1	0	0	-1	1	-1	0
C_3	20	1	-1	-1	0	0	1	1	-1	0
C_4	12	1	$(\sqrt{5}+1)/2$	$-(\sqrt{5}-1)/2$	$(\sqrt{5}+1)/2$	$-(\sqrt{5}-1)/2$	1	-1	0	-1
C_5	12	1	$(\sqrt{5}-1)/2$	$-(\sqrt{5}+1)/2$	$-(\sqrt{5}-1)/2$	$(\sqrt{5}+1)/2$	-1	-1	0	1
C_6	12	1	$-(\sqrt{5}-1)/2$	$(\sqrt{5}+1)/2$	$-(\sqrt{5}-1)/2$	$(\sqrt{5}+1)/2$	1	-1	0	-1
C_7	12	1	$-(\sqrt{5}+1)/2$	$(\sqrt{5}-1)/2$	$(\sqrt{5}+1)/2$	$-(\sqrt{5}-1)/2$	-1	-1	0	1
C_8	30	1	0	0	-1	-1	0	0	1	0
C_9	1	1	-2	-2	3	3	-4	4	5	-6

References

1) M. Imachi, S. Kawabe and H. Yoneyama, Prog. Theor. Phys. **66** (1981), 1025.
2) M. Creutz, Phys. Rev. **D21** (1980), 2308.
3) C. Rebbi, Phys. Rev. **D21** (1980), 3350.
D. Petcher and D. H. Weingarten, Phys. Rev. **D22** (1980), 2465.
4) M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. **D20** (1979), 1915.
5) A. A. Migdal, JETP **69** (1975), 810, 1457.
6) S. Caracciolo and P. Menotti, Nucl. Phys. **B180** (**FS2**) (1981), 428.
M. Nauenberg and D. Toussaint, Nucl. Phys. **B190**(**FS3**) (1981), 217.
7) J. V. Jose, L. P. Kadanoff, S. Kirkpatrick and D. R. Nelson, Phys. Rev. **B16** (1977), 1217.
8) K. Wilson and J. Kogut, Phys. Reports **12C** (1974), 75.
9) K. Wilson, Phys. Rev. **D10** (1974), 2445.
10) G. Bhanot and M. Creutz, Phys. Rev. **D24** (1981), 3212.
11) G. Bhanot and R. Dashen, Phys. Letters **113B** (1982), 299.
12) For example, K. Asano and H. Nagao, *Gun-ron* (Group Theory) (Iwanami-shoten publishers, 1965).
13) M. Gell-Mann and F. E. Low, Phys. Rev. **95** (1954), 1300.
14) N. S. Manton, Phys. Letters, **96B** (1980), 328.
15) M. Imachi, S. Kawabe and H. Yoneyama, KYUSHU-82-HE-8 (September, 1982), to be published in Prog. Theor. Phys. **69** (1983), No. 3.