Uniqueness and Estimation of Three-Dimensional Motion Parameters of Rigid Objects with Curved Surfaces — Source link

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UNIQUENESS AND ESTIMATION OF THREE-DIMENSIONAL MOTION PARAMETERS OF RIGID OBJECTS WITH CURVED SURFACES

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The processing of image sequences for efficient encoding, enhancement, and dynamic scene analysis has become increasingly important. A key issue in image sequence processing is motion estimation. Past work has concentrated on the approach of solving nonlinear equations iteratively, which is unsatisfactory because of convergence and uniqueness problems. In this report we prove some important theorems on uniqueness and present a totally new motion estimation algorithm which does not require the iterative solution of nonlinear equations.

We show that seven point correspondences are sufficient to uniquely determine from two time-sequential views the three-dimensional motion parameters (within a scale factor for the translations) of a rigid object with curved surfaces. The seven points should not be traversed by two planes with one plane containing the origin, nor by a cone containing the origin. A set of "essential paramenters" are introduced which uniquely determine the motion parameters up to a scale factor for the translations, and can be estimated by solving a set of eight linear equations which are derived from the correspondences of eight image points. The actual motion parameters can subsequently be determined by computing the singular value decomposition (SVD) of a 3x3 matrix containing the essential parameters. No nonlinear equations need be solved.

UNIQUENESS AND ESTIMATION OF THREE-DIMENSIONAL MOTION PARAMETERS OF RIGID OBJECTS WITH CURVED SURFACES

R. Y. Tsai and T. S. Huang*

August 14, 1981

ABSTRACT

We show that seven point correspondences are sufficient to uniquely determine from two perspective views the three-dimensional motion parameters (within a scale factor for the translations) of a rigid object with curved surfaces. The seven points should not be traversed by two planes with one plane containing the origin, nor by a cone containing the origin. A set of "essential parameters" are introduced which uniquely determine the motion parameters up to a scale factor for the translations, and can be estimated by solving a set of linear equations which are derived from the correspondences of eight image points. The actual motion parameters can subsequently be determined by computing the singular value decomposition (SVD) of a 3x3 matrix containing the essential parameters. No nonlinear equations need be solved.

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I. INTRODUCTION

The importance of motion estimation in image sequence analysis has long been recognized, particularly in such fields as image coding, tracking and robotic vision. Methods for two-dimensional motion estimation are relatively well known [11-18]. As for three-dimensional motion estimation from two image frames, [2-3,20] show that when the object surface is planar, there exist a set of eight pure parameters that can be estimated by solving a set of linear equations. The equations were derived using the Lie Group theory [2], and the uniqueness of the eight pure parameters given all the image point correspondences on the image plane is established either using Lie Group Theory [2] or using elementary Mathematics [21]. In [20], it is shown that only four image point correspondences (no three points colinear) are sufficient to ensure the uniqueness of the pure parameters. [3] shows that once these pure parameters are estimated, the motion parameters can be calculated by computing the SVD of a 3x3 matrix A consisting of the eight pure parameters, and the number of solutions is either one or two (usually two) depending on the multiplicity of the singular values of the matrix A. [20] shows that regardless of the multiplicity of the singular values, the motion parameters are always unique given three image frames.

For the case when the object surface is not restriced to be planar, existing theorectical analyses and estimation schemes were unsatisfactory in the sense that, theorectically, it was not known precisely how many image point correspondences are needed to ensure the uniqueness of the motion parameters (up to a scale factor for the translation parameters, of course), and practically, all estimation schemes rely on the solution of nonlinear equations using iterative search [4,10,19,23-25]. For example, it was stated in [10] that "in any case, the general method was not really practicable, nor was it designed for efficient use." [19] ended up with 18 nonlinear equations, and [4] 5 nonlinear equations. The results stated in [23] on the minimum number of image correspondences were not intended to be rigorous or exact since the author tried simply to equate the numbers of unknowns and equations. Another related problem is the stereo imaging problem in photogrammetry and computer vision without assuming the relative orientation of the two cameras since pictures taken at two time instances with one camera can be regarded as taken with two cameras at one instance. After the motion parameters are computed, the surface structure of the object can be determined by computing the z coordinates up to a scale factor using Eqs. (5a) or (5b) in this paper. Despite the fact that much work has been done in this area (e.g., [27,28]), no one has studied the problem of minimum information required to ensure unique solutions, nor was there any technique developed other than solving nonlinear equations iteratively or making severe approximatons to the unknowns. Another related problem is the so called "Location Determination Problem" as described in [26], where the distances between the observed points are assumed to be known a prior, which of course creates a different but simpler problem. In short, the results in the present paper should be of interest to many areas of research.

In this paper, a solution to the problem of estimating three-dimensional motion of a rigid body from two image frames is presented. Two major theorems, one lemma and six corollaries regarding the uniqueness and estimation of motion parameters are stated and proved. First, a 3x3 matrix E containing 8 essential parameters are introduced. It can be factored into a product of a skew-sysmetric matrix containing only the translation parameters, and an orthonormal matrix containing only the rotation parameters. Theorem I states that given the E matrix, the actual motion parameters are unique and can be determined simply by computing the SVD of the E matrix. The E matrix can be estimated by solving a set of linear equations given 8 image point correspondences. Lemma I shows that the actual motion parameters are unique if and only if a certain 4x4 matrix C is skew-symmetric. Theorem II shows that if all the observed points are not traversable by two planes with one plane containing the origin, nor by a cone containing the origin, then the matrix C has to be skew-symmetric. All other results follow from these two theorems and the lemma. For example: two planar patches determine the motion parameters uniquely; 4 points on a plane not passing through the origin and 2 other points not on this plane determine the motion parameters uniquely; 6 points with 4 on one plane, 4 on another, and 2 common to the above two groups of 4 points on the intersections of the two planes can insure unique solution; 7 points in general positions are sufficient to determine the motion parameters uniquely; etc. Note that Theorem II only gives a sufficient condition. Although 7 or more points in general positions are enough to ensure uniqueness, 6 or even 5 points are usually sufficient from our experience. (One should be cautious not to take solutions that yield z's that are positive before the motion and negative after the motion, or vice versa.) However, with 5, 6 or 7 points, one has to solve nonlinear equations with iterative search, while with 8 or more points, the simple method using SVD as stated in Theorem I can be used.

II. THE E MATRIX AND THE EIGHT ESSENTIAL PARAMETERS.

The basic geometry of the problem is sketched in Fig. 1. Consider a particular point P on an object. Let

(x,y,z) = object-space coordinates of a point P before motion.

(x',y',z') = object-space coordinates of P after motion.

(X,Y) = image-space coordinates of P before motion.

(X',Y') = image-space coordinates of P after motion.

The mapping $(X,Y) \longrightarrow (X',Y')$ for a particular point is called an image point correspondence. It is well known [22] that any 3-D rigid body motion is equivalent to a rotation by an angle θ around an axis through the origin with directional cosines n1,n2,n3, followed by a translation $(\Delta x, \Delta y, \Delta z)$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} + T \qquad (1)$$

where R is a 3 x 3 orthonormal matrix of the 1st kind (i.e. det(R)=1)

$$R = \begin{bmatrix} n1^2+(1-n1^2)\cos\theta & n1n2(1-\cos\theta)-n3\sin\theta & n1n3(1-\cos\theta)+n2\sin\theta \\ n1n2(1-\cos\theta)+n3\sin\theta & n2^2+(1-n2^2)\cos\theta & n2n3(1-\cos\theta)-n1\sin\theta \\ n1n3(1-\cos\theta)-n2\sin\theta & n2n3(1-\cos\theta)+n1\sin\theta & n3^2+(1-n3^2)\cos\theta \end{bmatrix} \qquad (2)$$

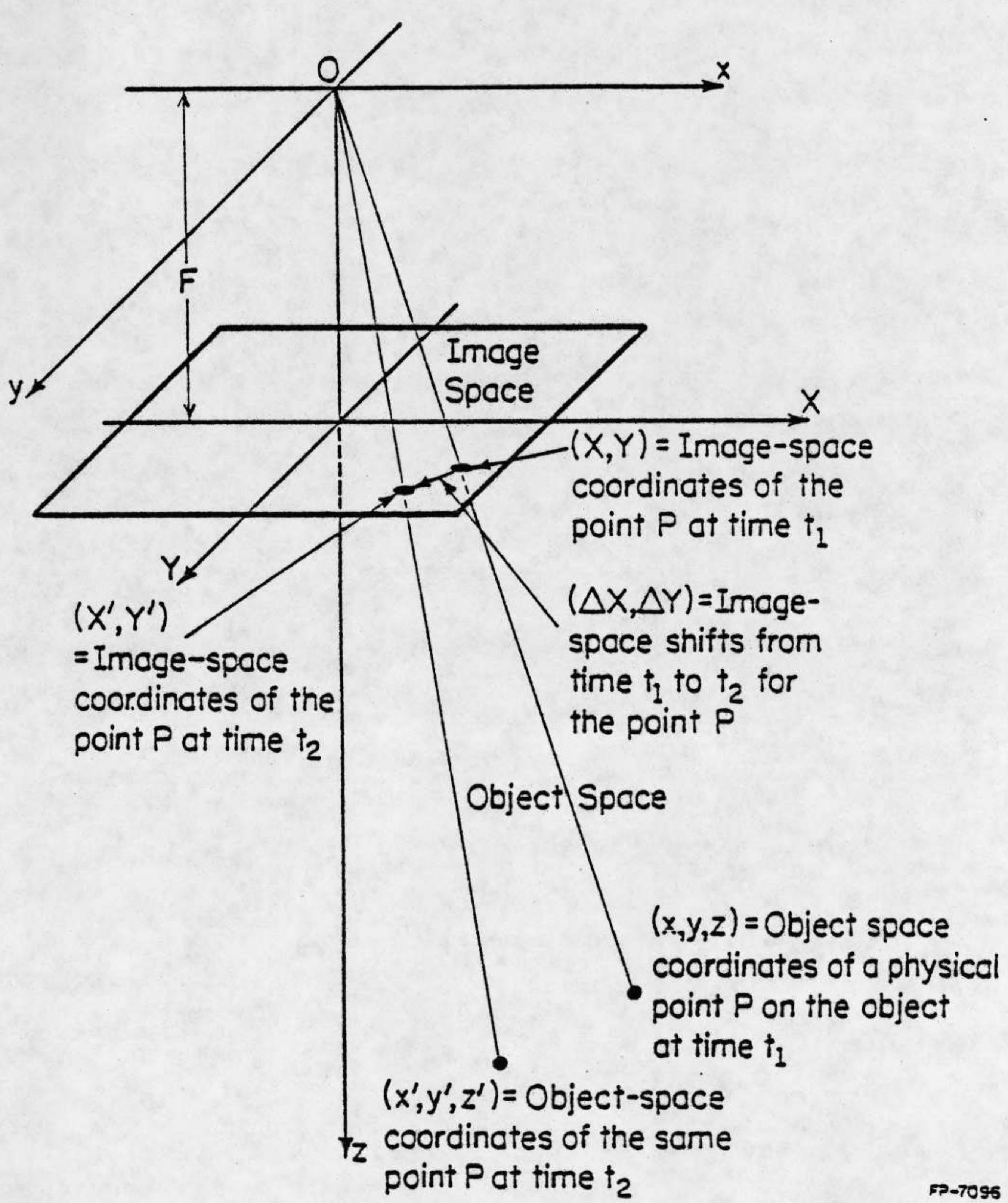

Fig. 1 Basic geometry for three-dimensional motion estimation.

$$\triangleq \begin{bmatrix} r1 & r2 & r3 \\ r4 & r5 & r6 \\ r7 & r8 & r9 \end{bmatrix} \quad \text{and} \quad T \triangleq \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

Although the elements in R, namely r1,r2,...,r9, are nonlinear functions of the rotation parameters n1,n2,n3 and θ, throughout this paper, the uniqueness and computation of R rather than n1,n2,n3 and θ are discussed. The reason is two fold. First, as will be seen later, to each possible R in (2), there corresponds exactly two sets of rotation parameters n1,n2,n3,θ with one set the negative of the other. Since these two solutions are physically indistinguishable, we may regard the relationship between R and the rotation parameters as one to one. The second reason is that once R is determined, the task of computing n1,n2,n3 and θ is trivial, as can be seen in the following:

From (2), we have

$$R = S + K$$

where

$$S = \begin{bmatrix} n1^2+(1-n1^2)\cos\theta & n1n2(1-\cos\theta) & n1n3(1-\cos\theta) \\ n1n2(1-\cos\theta) & n2^2+(1-n2^2)\cos\theta & n2n3(1-\cos\theta) \\ n1n3(1-\cos\theta) & n2n3(1-\cos\theta) & n3^2+(1-n3^2)\cos\theta \end{bmatrix} \quad \text{is symmetric}$$

and

$$K = \sin\theta \cdot \begin{bmatrix} 0 & -n3 & +n2 \\ +n3 & 0 & -n1 \\ -n2 & +n1 & 0 \end{bmatrix} \quad \text{is skew-symmetric.}$$

Since any matrix can be decomposed uniquely into a sum of a symmetric and a skew-symmetric matrix, we see that K is unique given R, and thus n1,n2, n3,θ are fixed up to a possible sign change. In fact, it is trvial to see that

$$K = 1/2 \begin{bmatrix} 0 & r2-r4 & r3-r7 \\ r4-r2 & 0 & r8-r6 \\ r7-r3 & r8-r7 & 0 \end{bmatrix}$$

or $n1 \cdot \sin\theta = (r8-r6)/2$, $n2 \cdot \sin\theta = (r3-r7)/2$, $n3 \cdot \sin\theta = (r4-r2)/2$,

which imply $\sin^2\theta(n1^2+n2^2+n3^2) = \sin^2\theta \cdot 1 = d/4$

or $\sin\theta = \pm d/2$, $n1 = \pm(r8-r6)/d$,

$n2 = \pm(r3-r7)/d$, $n3 = \pm(r4-r2)/d$,

where $d = (r8-r6)^2+(r3-r7)^2 + (r4-r2)^2$. (If d=0, then θ=0, R=I, and n1,n2,n3 can be anything since without rotation, the axis is meaningless.)

Since sinθ alone does not determine θ uniquely, we still need cosθ to fix θ. From (2), $n1^2 + (1 - n1^2)\cos\theta = r1$

$$\cos\theta = \frac{r1 - n1^2}{1-n1^2} = \frac{r1 - (\frac{r8-r6}{d})^2}{1 - (\frac{r8-r6}{d})^2} = \frac{d^2 r1 - (r8 - r6)^2}{d^2 - (r8-r6)^2} .$$

Therefore, θ, n1, n2 and n3 can be easily determined from R.

Just as in [2-7], we now combine (1) with the following equations relating the object and image space coordinates:

$$X = x/z \qquad X' = x'/z'$$
$$Y = y/z \qquad Y' = y'/z' \tag{3}$$

to obtain

$$X' = \frac{(r1\ X + r2\ Y + r3)z + \Delta x}{(r7\ X + r8\ Y + r9)z + \Delta z} \tag{4a}$$

$$Y' = \frac{(r4\ X + r5\ Y + r6)z + \Delta y}{(r7\ X + r8\ Y + r9)z + \Delta z} \tag{4b}$$

where the focal length F is normalized to 1 for simplicity. From (4),

$$z = \frac{\Delta x - \Delta z \cdot X'}{X'(r7\ X + r8\ Y + r9) - (r1\ X + r2\ Y + r3)} \tag{5a}$$

and

$$z = \frac{\Delta y - \Delta z \cdot Y'}{Y'(r7\ X + r8\ Y + r9) - (r4\ X + r5\ Y + r6)} \tag{5b}$$

Equating the right hand sides of (5a) and (5b) gives

$$\begin{bmatrix} X' & Y' & 1 \end{bmatrix} E \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = 0 \tag{6}$$

where

$$E \triangleq \begin{bmatrix} \Delta z \cdot r4 - \Delta y \cdot r7 & \Delta z \cdot r5 - \Delta y \cdot r8 & \Delta z \cdot r6 - \Delta y \cdot r9 \\ \Delta x \cdot r7 - \Delta z \cdot r1 & \Delta x \cdot r8 - \Delta z \cdot r2 & \Delta x \cdot r9 - \Delta z \cdot r3 \\ \Delta y \cdot r1 - \Delta x \cdot r4 & \Delta y \cdot r2 - \Delta x \cdot r5 & \Delta y \cdot r3 - \Delta x \cdot r6 \end{bmatrix} \tag{7}$$

$$\triangleq \begin{bmatrix} e1 & e2 & e3 \\ e4 & e5 & e6 \\ e7 & e8 & e9 \end{bmatrix} \tag{8}$$

Note that the equality of (6) will not be influenced by multiplying E by any scalar. Since each element of E is linear in Δx, Δy and Δz, this simply means that there is a common scale factor for the translation parameters that cannot be determined.(This scale factor also influences z in (5a) and (5b), but not the rotation parameters.) For this reason, we can always set e9 equal to some fixed number, say 1, without losing generality. We call the elements in E "essential parameters" for reasons that will be seen later.

It is obvious by observing (6) that the 3x3 matrix E contains all the information one can possibly obtain given any number of image point correspondences $(X,Y) \rightarrow (X',Y')$. Thus if the E matrix can be determined uniquely from a number of image point correspondences, then whether the motion parameters are unique or not depends soly on whether the motion parameters in (7) can be uniquely determined from E. This is one reason why we call the elements in E essential parameters. The second reason is that the actual 3-D motion parameters can be determined uniquely given E, and can be computed simply by taking the SVD of E without having to solve any nonlinear equations at all. The third reason is that given the image correspondences of eight object points in general positions, the E matrix can be determined uniquely by solving 8 linear equations.

Before giving Theorem I (which concerns the uniqueness and the computation of motion parameters given the matrix E), let us first analyze the matrix E. From (7), we have

$$E = \begin{bmatrix} \Delta z & & \\ & \Delta x & \\ & & \Delta y \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} R - \begin{bmatrix} \Delta y & & \\ & \Delta z & \\ & & \Delta x \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} R$$

$$= \begin{bmatrix} & \Delta z & \\ & & \Delta x \\ \Delta y & & \end{bmatrix} R - \begin{bmatrix} & & \Delta y \\ \Delta z & & \\ & \Delta x & \end{bmatrix} R = G\,R \qquad (9)$$

where

$$G \triangleq \begin{bmatrix} 0 & \Delta z & -\Delta y \\ -\Delta z & 0 & \Delta x \\ \Delta y & -\Delta x & 0 \end{bmatrix} \qquad (10)$$

is skew-symmetric and contains only the translation parameters and R is the rotation matrix. It is well known in matrix theory [1] that any skew-symmetric matrix K must have even rank, say 2n, and that K, if real, always assumes the following normal form:

$$K = Q^T \begin{bmatrix} 0 & \varphi_1 & & & & & & & & & \\ -\varphi_1 & 0 & & & & & & & & & \\ & & 0 & \varphi_2 & & & & & & & \\ & & -\varphi_2 & 0 & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & 0 & \varphi_n & & & & \\ & & & & & -\varphi_n & 0 & & & & \\ & & & & & & & 0 & & & \\ & & & & & & & & 0 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 0 \end{bmatrix} Q \qquad (11)$$

where Q is some orthonormal matrix, not necessarily unique and the φ's are real constants. Since G in (10) is 3x3 skew-symmetric, we can see from the above that G must be singular, and that there exist a 3x3 orthonormal matrix Q and a real number φ such that

$$G = Q^T \begin{bmatrix} 0 & \varphi & \\ -\varphi & 0 & \\ & & 0 \end{bmatrix} Q \qquad (12)$$

Eq.(12) will play an important role in the analysis hereafter.

Let $P = i\,E$ where $i=\sqrt{-1}$, then from (10), we have

$$P = i\cdot E = i\cdot G\cdot R = H\cdot R \qquad (13)$$

where

$$H \triangleq i\cdot G = \begin{bmatrix} 0 & i\cdot\Delta z & -i\cdot\Delta y \\ -i\cdot\Delta z & 0 & i\cdot\Delta x \\ i\cdot\Delta y & -i\cdot\Delta x & 0 \end{bmatrix}$$

Note that H is Hermitian. Therefore, (13) gives the polar decomposition [1] of P. Since the polar decomposition of any nonsingular matrix with distinct singular values is always unique [1], we can see that G and R would be unique if P should satisfy the conditions that it was nonsingular and that P^*P did not have multiple eigenvalues. (* denotes conjugate transpose) However, we have seen that G is always singular, which implies that P is always singular. Furthermore, P always contains multiple singular values since

$$P^*\cdot P = R^*\cdot H^*\cdot H\cdot R \quad (\text{* denotes conjugate transpose})$$

$$= R^*\cdot H^2\cdot R = R^*\cdot(iG)(iG)\cdot R = -R^*\cdot G^2\cdot R$$

$$= -R^*\left\{Q^T\begin{bmatrix}0 & \varphi & \\ -\varphi & 0 & \\ & & 0\end{bmatrix}Q\right\}\left\{Q^T\begin{bmatrix}0 & \varphi & \\ -\varphi & 0 & \\ & & 0\end{bmatrix}Q\right\}R$$

$$= -R^*\cdot Q\cdot\begin{bmatrix}-\varphi^2 & & \\ & -\varphi^2 & \\ & & 0\end{bmatrix}\cdot Q\cdot R = R\cdot Q^T\cdot\begin{bmatrix}\varphi^2 & & \\ & \varphi^2 & \\ & & 0\end{bmatrix}\cdot Q\cdot R \qquad (14)$$

and thus the eigenvalues of P^*P (or the square of the singular values of P) are φ^2, φ^2, and 0. However, we shall show in Theorem I that because of the special structure of G, once E is given, G and R are unique.

III.1 UNIQUENESS AND ESTIMATION OF MOTION PARAMETERS GIVEN E : THEOREM I.

THEOREM I

Let the SVD of E be given by

$$E = U\Lambda V^T \qquad (15)$$

then there are two solutions for the rotation matrix

$$R = U \begin{bmatrix} 0 & -1 & \\ 1 & 0 & \\ & & s \end{bmatrix} V^T \qquad (16)$$

or

$$= U \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & s \end{bmatrix} V^T \qquad (17)$$

where $s = \det(U)\cdot\det(V) = +1$ or -1

and one solution for the translation vector (up to a scale factor)

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \alpha \begin{bmatrix} \phi_1^T\phi_2 / \phi_2^T\phi_3 \\ \phi_1^T\phi_2 / \phi_1^T\phi_3 \\ 1 \end{bmatrix}$$

where ϕi is the ith row of E, $i = 1, 2, 3$, and α is some scale factor. Furthermore, although U and V are not unique given E, once a particular pair of U and V are selected, (16) and (17) include all the possible solutions. However, only one of the two solutions yield positive z in (5a) and (5b). Since the object must be in front of the camera, the solution is unique.

[Proof]

Let us first verify the uniqueness of $\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$ given E, and give the computational formula for it.

From (9), we have

$$E\,E^T = G\,R\,R^T G^T = G\,G^T = -G^2$$

$$= \begin{bmatrix} \Delta z^2 + \Delta y^2 & -\Delta x\cdot\Delta y & -\Delta x\cdot\Delta z \\ -\Delta x\cdot\Delta y & \Delta z^2 + \Delta x^2 & -\Delta y\cdot\Delta z \\ -\Delta x\cdot\Delta z & -\Delta y\cdot\Delta z & \Delta x^2 + \Delta y^2 \end{bmatrix} \qquad (18)$$

or

$$\Delta z^2 + \Delta y^2 = \phi_1^T \phi_1 \quad (19)$$
$$\Delta z^2 + \Delta x^2 = \phi_2^T \phi_2 \quad (20)$$
$$\Delta x^2 + \Delta y^2 = \phi_3^T \phi_3 \quad (21)$$
$$\Delta x \cdot \Delta y = -\phi_1^T \phi_2 \quad (22)$$
$$\Delta x \cdot \Delta z = -\phi_2^T \phi_3 \quad (23)$$
$$\Delta y \cdot \Delta z = -\phi_1^T \phi_3 \quad (24)$$

(19)+(20)-(21) gives

$$2 \cdot \Delta z^2 = \phi_1^T \phi_1 + \phi_2^T \phi_2 - \phi_3^T \phi_3$$

or

$$\Delta z = \pm 1/\sqrt{2} \, (\phi_1^T \phi_1 + \phi_2^T \phi_2 - \phi_3^T \phi_3)^{1/2} \quad (25)$$

Similarly,

$$\Delta x = + 1/\sqrt{2} \, (-\phi_1^T \phi_1 + \phi_2^T \phi_2 + \phi_3^T \phi_3)^{1/2} \quad (26)$$

$$\Delta y = + 1/\sqrt{2} \, (\phi_1^T \phi_1 - \phi_2^T \phi_2 + \phi_3^T \phi_3)^{1/2} \quad (27)$$

Therefore, given E, Δx, Δy and Δz are fixed except for the signs. When a particular sign for Δz is chosen, the signs for Δx and Δy are determined from (28) and (29). Thus the translation vector $\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix}$ is fixed except for the sign. Since, as mentioned twice before, multiplying E or G with any scalar does not alter the equality of (6), $\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix}$ is unique up to a scale factor. Alternatively, since there is a common scale factor among the translations, Δx, Δy and Δz, we have from (23) and (24),

$$\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix} = \alpha \begin{bmatrix}\phi_1^T \phi_2 / \phi_2^T \phi_3\\ \phi_1^T \phi_2 / \phi_1^T \phi_3\\ 1\end{bmatrix}$$

where α is a scale factor. We now proceed to prove that given E, there are two solutions given in (17) and (18) for the rotation matrix R with only one among the two yielding z in (5a) and (5b) with the same signs before and after the motion.

From (9), (12) and (15), we have

$$E = U \Lambda V^T = G\ R = Q^T \begin{bmatrix} 0 & \varphi & \\ -\varphi & 0 & \\ & & 0 \end{bmatrix} Q\ R \qquad (28)$$

Since $P^* P = (i\ E)^* (i\ E) = E^T E$, it follows from (14) that φ^2, φ^2 and 0, which are the squares of the singular values of $P^* P$, are also singular values of $E\ E^T$ and $E^T E$. Since, as mentioned earlier, multiplying E with any scalar will not influence the equality of (6) and will only scale the translation parameters in G, we can always, for the purpose of simplicity, set φ in (12) to -1 without losing generality. Thus (28) becomes

$$E = U \cdot \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \cdot V^T = Q^T \cdot \begin{bmatrix} 0 & -1 & \\ +1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q \cdot R \qquad (29)$$

By taking Q as U^T, and premultiplying (29) with U^T, we have

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \cdot V^T = \begin{bmatrix} 0 & -1 & \\ +1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q \cdot R \qquad (30)$$

Let the ith column of V be denoted by Vi, and the ith column of the product QR be denoted by Qi, where i=1,2,3. Then (30) gives

$$\begin{bmatrix} & V1^T & \\ & V2^T & \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} & -Q2^T & \\ & +Q1^T & \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $Q2 = -V1$, $Q1 = +V2$. Then it follows from the orthonormality of QR that $Q3 = \pm V3$. Thus

$$R = Q^T \cdot Q \cdot R = U \cdot Q \cdot R$$

$$= U\ [+V2\ -V1\ \pm V3\ ] = U \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & s \end{bmatrix} V^T \qquad (31)$$

where s = +1 or -1. Since R is orthonormal of the first kind, we have from (31),

$$\det(R) = 1 = \det(U) \cdot \det\left(\begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & s \end{bmatrix}\right) \cdot \det(V)$$

$$= \det(U) \cdot s \cdot \det(V)$$

Thus

$$\det(U)\ \det(V)\ 1 = s\ [\det(U)]^2 \cdot [\det(V)]^2$$

$$= s \cdot 1 \cdot 1$$

or s = det(U)·det(V). Although U and V are not unique given E since the multiplicity of the sngular values of E is 2, we shall show later that due to the special structure of G, the solution for R is either given by (31), or by

$$R = U \begin{bmatrix} 0 & -1 & \\ 1 & 0 & \\ & & s \end{bmatrix} V^T \qquad (32)$$

and no others. Furthermore, only one of (31) and (32) can be accepted.

Let R1 and R2 be two orthonormal matrices of the 1st kind (i.e., det(R1) = det(R2) = +1 and not -1) that satisfy (9), i.e.,

$$E = G \cdot R1 = \pm G \cdot R2 \qquad (33)$$

The "-" sign in (33) comes from the fact explained earlier that a sign change of E will not influence the equality of (6). From (33) and (12), there exist two orthonormal matrices Q1 and Q2, not necessarily equal, such that

$$Q1^T \cdot \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q1 \cdot R1 = Q2^T \cdot \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q2 \cdot R2 \qquad (34)$$

where $$G = Q1^T \cdot \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} Q1 = \pm Q2^T \cdot \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q2 \qquad (35)$$

First, we show that Q1 and Q2 have to be related by following

$$Q2 = \begin{bmatrix} W & \\ & \pm 1 \end{bmatrix} \cdot Q1$$

where W is a 2x2 orthonormal matrix. From (9) and (12) with φ set to 1 as explained earlier, we have

$$E \cdot E^T = G\ R\ R^T G^T = -G^2$$

$$= -\left\{ Qi^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} Qi \right\} \left\{ Qi^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} Qi \right\} = Qi^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} Qi \qquad (36)$$

Since $E \cdot E^T$ is fixed (including the sign) given $\pm$ E, we have from (36),

$$Q1^T \cdot \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \cdot Q1 = Q2^T \cdot \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \cdot Q2 \qquad (37)$$

Premultiplying (37) by Q2 and postmultiplying by Q1 give

$$Q2 \cdot Q1^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \cdot Q2 \cdot Q1^T \qquad (38)$$

Let $Q2 \cdot Q1^T \triangleq \begin{bmatrix} q1 & q2 & q3 \\ q4 & q5 & q6 \\ q7 & q8 & q9 \end{bmatrix}$, then from (38),

$$\begin{bmatrix} q1 & q2 & 0 \\ q4 & q5 & 0 \\ q7 & q8 & 0 \end{bmatrix} = \begin{bmatrix} q1 & q2 & q3 \\ q4 & q5 & q6 \\ 0 & 0 & 0 \end{bmatrix}$$

which implies that $q3 = q6 = q7 = 0$, or

$$Q2 \cdot Q1^T = \begin{bmatrix} q1 & q2 & 0 \\ q4 & q5 & 0 \\ 0 & 0 & q9 \end{bmatrix} = \begin{bmatrix} W & \\ & q9 \end{bmatrix} \qquad (39)$$

where $W \triangleq \begin{bmatrix} q1 & q2 \\ q4 & q5 \end{bmatrix}$ (40)

(alternatively, one can show from (35), after some similar derivation as above, that W has to be either $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$. But since (40) is sufficient and handy for all the later purposes, it is simpler just to maintain (40)). Since Q1 and Q2 are both orthonormal, it follows from (39) that W is orthonormal and $q9 = \pm 1$. Therefore,

$$Q2 = \begin{bmatrix} W & \\ & \pm 1 \end{bmatrix} Q1 \tag{41}$$

Next, we substitute (41) into (34) to obtain

$$Q1^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} Q1\ R1 = Q1^T \begin{bmatrix} W & \\ & \pm 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} W & \\ & \pm 1 \end{bmatrix} Q1\ R2 = Q1^T \begin{bmatrix} W^T K\ W & \\ & 0 \end{bmatrix} Q1\ R2 \tag{42}$$

where $K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since W is defined by (40), we have

$$W^T K\ W = \begin{bmatrix} q1 & q4 \\ q2 & q5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} q1 & q2 \\ q3 & q4 \end{bmatrix} = \begin{bmatrix} -q3\ q1 + q3\ q1 & -q3\ q2 + q1\ q4 \\ -q4\ q1 + q3\ q2 & -q2\ q4 + q2\ q4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \det(W) \\ -\det(W) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Thus (42) gives

$$Q1^T \cdot \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q1 \cdot R1 = Q1^T \cdot \begin{bmatrix} 0 & s & \\ -s & 0 & \\ & & 0 \end{bmatrix} \cdot Q1 \cdot R2 \tag{43}$$

where $s = +1$ or -1. Premultiplying (43) by Q1 and postmultiplying by $R1^T \cdot Q1^T$ give

$$\begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} s(Q1 \cdot R2 \cdot R1^T \cdot Q1^T) \tag{44}$$

Let $B \triangleq Q1 \cdot R2 \cdot R1^T \cdot Q1^T \triangleq \begin{bmatrix} B1^T \\ B2^T \\ B3^T \end{bmatrix}$ (45)

Thus

$$\begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} = s\begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix}\begin{bmatrix} B1^T \\ B2^T \\ B3^T \end{bmatrix} = \begin{bmatrix} +sB2^T \\ -sB1^T \\ 0\ 0\ 0 \end{bmatrix}$$

Hence

$$B2^T = s[0 \quad 1 \quad 0] \tag{46}$$

$$B1^T = -s[-1 \quad 0 \quad 0] = s[1 \quad 0 \quad 0] \tag{47}$$

Since Q1, R1 and R2 in (45) are orthonormal, and that

$$\det(B) = \det(Q1)\cdot\det(R2)\cdot\det(r1)\cdot\det(Q1) = (\det(Q1))^2 = (\pm 1)^2 = 1$$

we see that B is orthonormal is orthonormal of the 1st kind. This fact, together with (46) and (47), imply that

$$B3^T = [0 \quad 0 \quad 1] \quad \text{or} \quad B\,(= Q1\cdot R2\cdot R1^T\cdot Q1^T) = \begin{bmatrix} s & & \\ & s & \\ & & 1 \end{bmatrix}$$

Thus $\quad R2 = Q1^T\begin{bmatrix} s & & \\ & s & \\ & & 1 \end{bmatrix} Q1\ R1$

For $s = +1$, $\quad R2 = Q1^T\cdot I\cdot Q1\cdot R1 = R1 \qquad (48)$

For $s = -1$, $\quad R2 = Q1^T\begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} Q1\ R1 \qquad (49)$

Therefore, given E, if we regard R1 as a reference solution, then should there be any other solution for the rotation matrix, it must satisfy (49). We now show that although Q1 is not unique, (49) remains fixed for different choices of Q1.

Let Q2 be another orthonormal matrix that satisfies (35) or

(36), and let $R3 = Q2^T \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot Q2 \cdot R1$, then from (41),

$$R3 = Q1^T \cdot \begin{bmatrix} W & \\ & \pm 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} W & \\ & \pm 1 \end{bmatrix} \cdot Q1 \cdot R1 = Q1^T \cdot \begin{bmatrix} -W^T W & \\ & \pm 1 \end{bmatrix} \cdot Q1 \cdot R1$$

$$= Q1^T \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & \pm 1 \end{bmatrix} \cdot Q1 \cdot R1 \tag{50}$$

$$\text{or } = Q1^T \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \cdot Q1 \cdot R1 = -R1 \tag{51}$$

(51) is obviously not a solution since it implies that $\det(R3) = -1$, not +1. But (50) is exactly the same as (49). Note that (49) implies that

$$R1 = Q1^T \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot Q1 \cdot R2$$

i.e., no matter which solution is chosen as the reference, the other solution must be given be (49), of course.

It is now obvious that (16) and (17) are the only possible two solutions despite the fact that U and V are not unique, since if we regard (31) as the reference solution R1, then the only other solution must be given by

$$R2 = Q^T \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot Q \cdot R1 = Q^T \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot Q \cdot U \cdot \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & s \end{bmatrix} \cdot V^T$$

$$= U^T \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot I \cdot \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & s \end{bmatrix} \cdot V^T \text{(since } Q = U^T\text{)}$$

$$= U^T \cdot \begin{bmatrix} 0 & -1 & \\ 1 & 0 & \\ & & s \end{bmatrix} \cdot V^T$$ which is identical to (32). We now show that **among the two solutions, exactly one of them must yield z with opposite signs before and after the motion.**

Since the E matrix in (6) has nothing to do with the geometry of of the object surface, for a particular point with image correspondence $(X,Y) \longrightarrow (X',Y')$, we can imagine that there are two planes passing through this point neither of them containing the origin. In section IV, we shall show that given the image correspondences of the points on two planes, neither of which containing the origin, the E matrix is fixed. In [3], it was shown that there are two solutions for the rotation matrix given the image corrrespondences of one plane only :

$$R1 = O1^T \cdot \begin{bmatrix} \alpha & \beta & \\ -s\beta & s\alpha & \\ & & 1 \end{bmatrix} \cdot O2 \qquad (52)$$

$$R2 = O1^T \cdot \begin{bmatrix} \alpha & \beta & \\ s\beta & s\alpha & \\ & & 1 \end{bmatrix} \cdot O2 \qquad (53)$$

where O1 and O2 are some 3x3 orthonormal matrices.(Note that the rows of O1 and O2 are permutated for convenience.) There are two other solutions corresponding to $k < 0$ not stated in [3](see [3] for the definition of κ) because it was proved in [3] that when $k < 0$, the object **points move from the front to the back of the camera, or vice versa. It can be shown using exactly the same procedure as in Theorem II of [3] that these two other solutions are**

$$R1' = O1^T \cdot \begin{bmatrix} -\alpha & -\beta & \\ s\beta & -s\alpha & \\ & & 1 \end{bmatrix} \cdot O2 \qquad (54)$$

and $$R2' = O1^T \cdot \begin{bmatrix} -\alpha & \beta & \\ -s\beta & -s\alpha & \\ & & 1 \end{bmatrix} \cdot O2 \qquad (55)$$

Since the E matrix is fixed, it was proved earlier in this theorem that only two solutions among the four in (52)-(55) may exist, and they must be related precisely by (49). There are only two such possibilities, one is

$$R1' = O1^T \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot O1 \cdot R1 \qquad (56)$$

and the other is

$$R2' = O1^T \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot O1 \cdot R2 \qquad (57)$$

Therefore, the two solutions are either R1,R1′ or R2,R2′. In either case, one of the solution must be one among (56) and (57), which corresponds to the case when $k < 0$ and the object points must move from the front to the back, or from the back to the front of the camera, as was indicated above. We have thus proved that only one among (16) and (17), or equivalently (48) and (49) is acceptable.

* END OF PROOF FOR THEOREM I *

III.2 ESTIMATION OF E GIVEN 8 IMAGE POINT CORRESPONDENCES.

Given eight image point correspondences $(Xi, Yi) \rightarrow (Xi', Yi')$, for $i=1,\dots,8$, we have from (6),

$$\begin{bmatrix} X1'X1 & X1'Y1 & X1' & Y1'X1 & Y1'Y1 & Y1' & X1 & Y1 \\ X2'X2 & X2'Y2 & X2' & Y2'X2 & Y2'Y2 & Y2' & X2 & Y2 \\ X3'X3 & X3'Y3 & X3' & Y3'X3 & Y3'Y3 & Y3' & X3 & Y3 \\ X4'X4 & X4'Y4 & X4' & Y4'X4 & Y4'Y4 & Y4' & X4 & Y4 \\ X5'X5 & X5'Y5 & X5' & Y5'X5 & Y5'Y5 & Y5' & X5 & Y5 \\ X6'X6 & X6'Y6 & X6' & Y6'X6 & Y6'Y6 & Y6' & X6 & Y6 \\ X7'X7 & X7'Y7 & X7' & Y7'X7 & Y7'Y7 & Y7' & X7 & Y7 \\ X8'X8 & X8'Y8 & X8' & Y8'X8 & Y8'Y8 & Y8' & X8 & Y8 \end{bmatrix} \begin{bmatrix} e1 \\ e2 \\ e3 \\ e4 \\ e5 \\ e6 \\ e7 \\ e8 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \qquad (58)$$

Therefore, e1,e2,...,e8 can be estimated by solving a system of linear equations expressed in (58). The conditions when the ei's are unique (or equivalently when the 8x8 matrix in (58) is nonsingular) are stated and proved in Lemma I and Theorem II in Sec. IV. In practice, given eight image point correspondences, one first substitute the image point coordinates into the above 8x8 matrix and check its determinant. If it is nonzero, the matrix E can be determined by solving (58) for the ei's. Next the SVD of E is computed and used to calculate the actual motion parameters by the simple formula described in Theorem I.

IV. RESTRICTIONS ON THE SPATIAL DISTRIBUTION OF OBJECT POINTS TO ENSURE UNIQUENESS: LEMMA I AND THEOREM II.

Multiplying (6) by z and z′ gives

$$z' \, [X' \;\; Y' \;\; 1] \cdot E \cdot z \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = 0 \tag{59}$$

From (3) and (59),

$$[x' \;\; y' \;\; z'] \cdot E \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \text{or} \quad [x' \;\; y' \;\; z'] \cdot G \cdot R \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \tag{60}$$

Let $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ be transformed from $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with some reference rotation matrix

Ro and translation vector To $= \begin{bmatrix} \Delta xo \\ \Delta yo \\ \Delta zo \end{bmatrix}$, i.e.,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = Ro \begin{bmatrix} x \\ y \\ z \end{bmatrix} + To = Ro \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \Delta xo \\ \Delta yo \\ \Delta zo \end{bmatrix} \tag{61}$$

Let $Go = \begin{bmatrix} 0 & \Delta zo & -\Delta yo \\ -\Delta zo & 0 & \Delta xo \\ \Delta yo & -\Delta xo & 0 \end{bmatrix}$ and $Eo = Go \cdot Ro$.

The purpose of this section is to investigate **how many image** point correspondences are needed to ensure that there are no other solutions to G and R as factors of E in (9) than the reference Go and Ro that can satisfy (59) (or (60)), and to state the **conditions** or restrictions on the spatial distribution of the object points under observation in order to ensure unique solutions.

Substituting (61) into (60) gives

$$([x \;\; y \;\; z] \cdot Ro^T + To^T) \cdot E \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x \;\; y \;\; z] \cdot Ro^T \cdot E \begin{bmatrix} x \\ y \\ z \end{bmatrix} + To^T \cdot E \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x \; y \; z \; 1] \begin{bmatrix} & & & 0 \\ & Ro^T E & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \left\{ [x \; y \; z \; 1] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} [To^T E \;\; 0] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= [x\ y\ z\ 1] \begin{bmatrix} R^T o E & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ T^T o E & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = [x\ \ y\ \ z\ \ 1]\, C \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0 \qquad (62)$$

where

$$C = \begin{bmatrix} R^T o E & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ T^T o E & 0 \end{bmatrix} = \begin{bmatrix} R^T o GR & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ T^T o GR & 0 \end{bmatrix} \qquad (63)$$

Note that if C is skew-symmetric, then (62) is always satisfied regardless of what x,y,z or X,Y are, since

$$2\cdot[x\ \ y\ \ z\ \ 1]\cdot C\cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = [x\ \ y\ \ z\ \ 1]\cdot C\cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \left\{ [x\ \ y\ \ z\ \ 1]\, C \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \right\}^T$$

$$= [x\ \ y\ \ z\ \ 1]\cdot C\cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + [x\ \ y\ \ z\ \ 1]\cdot C^T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= [x\ \ y\ \ z\ \ 1]\cdot C\cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + [x\ \ y\ \ z\ \ 1]\cdot(-C)\cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

It is to be proved in Lemma I that C is skew-symmetric if and only if E = Eo (then according to Theorem I, the solution for the motion parameters is unique). The purpose of Theorem II is to prove that the matrix C in (63) has to be skew-symmetric if the object points under observation do not reside on two planes with one of the two planes containing the origin, nor do they lie on a cone containing the origin. We note that five or fewer points in space can always be traversed by two planes with one plane containing the origin, and that six or fewer points in space can always be traversed by a cone containing the origin. A minimum of seven points is needed to violate these two conditions. Therefore, it follows from Theorem II and Lemma I that seven points in general positions can ensure a unique solution for the motion parameters.

LEMMA I

The necessary and sufficient conditions for C defined by (63) to be skew-symmetric is that

$$R = Ro \qquad (64)$$

or

$$R = Q^T \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} Q\, Ro \qquad (65)$$

where Q is a 3x3 orthonormal matrix such that

$$G = Q^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} Q \qquad (66)$$

and

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \alpha \begin{bmatrix} \Delta xo \\ \Delta yo \\ \Delta zo \end{bmatrix} \qquad (67)$$

where α is some constant. (According to Theorem I, (65) and (67) are equivalent to $E = \alpha Eo$)

[Proof]

If C is skew-symmetric, then it is necessary from (63) that

$$Ro^T \cdot G \cdot R = -(Ro^T G\, R)^T \qquad (68)$$

and

$$To^T \cdot G \cdot R = [0 \quad 0 \quad 0]$$

or

$$R^T \cdot G^T \cdot To = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (69)$$

(68) gives

$$Ro^T \cdot G \cdot R = - R^T G^T \cdot Ro = R^T \cdot G \cdot Ro$$

Substituting (66) into the above gives

$$Ro^T Q^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} Q R = R^T Q^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} Q Ro \qquad (70)$$

Premultiplying (70) by QR and postmultiplying by $R^T \cdot Q^T$ give

$$Q R Ro^T Q^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} Q Ro R^T Q^T$$

or

$$L \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot L^T \qquad (71)$$

where

$$L \triangleq Q R Ro^T Q^T \triangleq \begin{bmatrix} j1 & j2 & j3 \\ j4 & j5 & j6 \\ j7 & j8 & j9 \end{bmatrix} \qquad (72)$$

From (71) and (72)

$$\begin{bmatrix} -j2 & j1 & 0 \\ -j5 & j4 & 0 \\ -j8 & j7 & 0 \end{bmatrix} = \begin{bmatrix} j2 & j5 & j8 \\ -j1 & -j4 & -j7 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\quad j7 = j8 = 0$

and $\quad j2 = -j2 \quad$ or $j2 = 0$

$j4 = -j4 \quad$ or $j4 = 0$

$j1 = j5$

Then L becomes

$$L = \begin{bmatrix} j1 & 0 & j3 \\ 0 & j1 & j6 \\ 0 & 0 & j9 \end{bmatrix} \qquad (73)$$

(72) implies that L is orthonormal of the 1st kind since R, Ro, Qo are orthonormal and that $\det(L) = \det(Q)\det(R)\det(Ro)\det(Q) = (\det(Q))^2 = (\pm 1)^2 = 1$. Taking the inner product of the 1st and 3rd rows of L in (73), and equating it to zero gives $j3 \cdot j9 = 0$. Since $j9 \neq 0$ (otherwise the 3rd row of L would be zero), j3=0. Similarly, j6 = 0. With these and the fact that det(L) = 1, we conclude taht L can assume only the following forms:

$$L = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \text{or} \quad L = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$$

From (72),

$$R = Q^T \cdot \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \cdot Q \cdot Ro = Ro$$

or

$$R = Q^T \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot Q \cdot Ro$$

Thus (64) and (65) are the necessary conditions for C to be skew-symmetric. The next thing is to verify (67).

Premultiplying (69) by R gives

$$G^T \cdot To = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & \Delta z & -\Delta y \\ -\Delta z & 0 & \Delta x \\ \Delta y & -\Delta x & 0 \end{bmatrix} \begin{bmatrix} \Delta xo \\ \Delta yo \\ \Delta zo \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives

$$\Delta z \cdot \Delta yo - \Delta y \cdot \Delta zo = 0$$

$$-\Delta z \cdot \Delta xo + \Delta x \cdot \Delta zo = 0$$

$$\Delta y \cdot \Delta xo - \Delta x \cdot \Delta yo = 0$$

Let $\alpha = \Delta z / \Delta zo$, then $\Delta y = \alpha \cdot \Delta yo$, $\Delta x = \alpha \cdot \Delta xo$. Hence

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \Delta xo \\ \Delta yo \\ \Delta zo \end{bmatrix}$$

which is the same as (67). The E matrix then is equal to Eo (i.e., unique) up to a scale factor since

$$E = G \cdot R = \begin{bmatrix} 0 & \Delta z & -\Delta y \\ -\Delta z & 0 & \Delta x \\ \Delta y & -\Delta x & 0 \end{bmatrix} R = \alpha \cdot Go \cdot Ro = \alpha \cdot Eo$$

if (64) is used, or

$$= Go\, Q^T \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} Q\, Ro = \alpha\, Q^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} Q \cdot Q^T \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} Q\, Ro$$

$$= \alpha Q^T \begin{bmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{bmatrix} Q\, Ro = -\alpha\, Eo$$

if (65) is used.

(Sufficiency part)

From the structure of C in (63), it is obvious that in order for C to be skew-symmetric, the row vector $To^T \cdot G \cdot R$ on the 4th row has to be equal to the negative of the transpose of the 4th column, which is a zero vector, and that the 3x3 matrix $Ro^T \cdot G \cdot R$ on the upper-left corner of C must be itself skew-symmetric. With (67), $To^T \cdot G \cdot R$ in (63) becomes

$$To^T \cdot G \cdot R = [\Delta xo \quad \Delta yo \quad \Delta zo] \begin{bmatrix} 0 & \Delta zo & -\Delta yo \\ -\Delta zo & 0 & \Delta xo \\ \Delta yo & -\Delta xo & 0 \end{bmatrix} R$$

$$= [-\Delta yo \cdot \Delta zo + \Delta zo \cdot \Delta yo \quad \Delta xo \cdot \Delta zo - \Delta zo \cdot \Delta xo \quad -\Delta xo \cdot \Delta yo + \Delta yo \cdot \Delta xo]\ R$$

$$= [0 \quad 0 \quad 0]\ R = [0 \quad 0 \quad 0] \tag{74}$$

We now proceed to show that with R either given by (64) or by (65), the 3x3 submatrix $To^T \cdot G \cdot R$ in C has to be skew-symmetric.

With (64), $Ro^T \cdot G \cdot R$ in (63) becomes

$$Ro^T \cdot G \cdot R = Ro^T \cdot G \cdot Ro = Ro \cdot (-G^T) \cdot Ro = -(Ro^T\ G\ Ro)^T \tag{75a}$$

On the other hand, with (65), $Ro^T \cdot G \cdot R$ in (63) becomes

$$Ro^T \cdot G \cdot R = Ro^T \cdot G \cdot Q \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot Q \cdot Ro = Ro^T \cdot Q \cdot \begin{bmatrix} 0 & -1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q \cdot Q \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \cdot Q \cdot Ro$$

$$= Ro^T \cdot Q^T \begin{bmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{bmatrix} Q\ Ro = -Ro^T \cdot G \cdot Ro$$

Thus

$$(Ro^T \cdot G \cdot R)^T = (-Ro^T \cdot G \cdot Ro)^T = Ro^T \cdot G \cdot Ro = -Ro^T \cdot G \cdot R \tag{75b}$$

(75a) and (75b) shows that either with (64) or (65), $Ro^T \cdot G \cdot R$ is skew-symmetric. This fact, together with (74), imply that C in (63) is skew-symmetric.

* END OF PROOF FOR LEMMA I *

THEOREM II

If $[X' \quad Y' \quad 1]\, E \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = 0$ is satisfied by the image point correspondences

of a group of object points not lying on two planes with one plane containing the origin, nor on a cone containing the origin, then the C matrix in (63) has to be skew-symmetric.

[Proof]

From (62), which is the necessary condition of (6), we have

$$[x \quad y \quad z \quad 1]\, C \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \left\{ [x \quad y \quad z \quad 1]\, C \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \right\}^{T} = 0$$

or

$$[x \quad y \quad z \quad 1](C + C^{T}) \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0 \qquad (76)$$

From (63)

$$C + C^{T} = \left[\begin{array}{c|c} Ro^{T}\cdot E + E^{T}\cdot Ro & E^{T}\cdot To \\ \hline To^{T}\cdot G\cdot R & 0 \end{array} \right]$$

$$C + C^{T} = \left[\begin{array}{c|c} Ro^{T}\cdot G\cdot R + R^{T}G^{T}Ro & R^{T}G^{T}\cdot To \\ \hline To^{T}\cdot G\cdot R & 0 \end{array} \right]$$

Substituting (12) into the above gives

$$C + C^T = \left[\begin{array}{c:c} Ro^T \cdot Q^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q \cdot R - R^T \cdot Q^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q \cdot Ro & -R^T \cdot Q^T \cdot \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q \cdot To \\ \hdashline To^T \cdot Q^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q \cdot R & 0 \end{array}\right]$$

$$= \left[\begin{array}{c:c} R^T \cdot Q^T & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hdashline 0 \quad 0 \quad 0 & 1 \end{array}\right] \left[\begin{array}{c:c} Q \cdot R \cdot Ro^T \cdot Q^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} Q \cdot Ro \cdot R^T \cdot Q^T & - \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot Q \cdot To \\ \hdashline To^T \cdot Q^T \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} & 0 \end{array}\right]$$

$$\cdot \left[\begin{array}{c:c} Q \cdot R & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hdashline 0 \quad 0 \quad 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c:c} Q \cdot R & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hdashline 0 \quad 0 \quad 0 & 1 \end{array}\right]^T \left[\begin{array}{c:c} M \cdot \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot M^T & - \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} \cdot To' \\ \hdashline To' \cdot \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix} & 0 \end{array}\right] \left[\begin{array}{c:c} Q \cdot R & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hdashline 0 \quad 0 \quad 0 & 1 \end{array}\right] \qquad (77)$$

where $M \triangleq Q \cdot R \cdot Ro^T \cdot Q^T$

$$To' \triangleq Q \cdot To \triangleq \begin{bmatrix} t1 \\ t2 \\ t3 \end{bmatrix}$$

Let $M \triangleq \begin{bmatrix} m1 & m2 & m3 \\ m4 & m5 & m6 \\ m7 & m8 & m9 \end{bmatrix}$

Then (77) becomes

$$C + C^T = \begin{bmatrix} Q\ R & \\ & 1 \end{bmatrix}^T \begin{bmatrix} 2\ m4 & m5\ m1 & m6 & -t2 \\ m5-m1 & -2\ m2 & -m3 & t1 \\ m6 & -m3 & 0 & 0 \\ -t2 & t1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q\ R & \\ & 1 \end{bmatrix} \quad (78)$$

Let the original cordnate system be rotated with R·Q such that

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = Q \cdot R \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (79)$$

then from (76) and (78),

$$[x_c \quad y_c \quad z_c \quad 1] \cdot J \cdot \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = 0 \quad (80)$$

where

$$J = \begin{bmatrix} 2\ m4 & m5-m1 & m6 & -t2 \\ m5-m1 & -2\ m2 & -m3 & t1 \\ m6 & -m3 & 0 & 0 \\ -t2 & t1 & 0 & 0 \end{bmatrix} \quad (81)$$

(80) gives

$$2[m4 \cdot x_c^2 + (m5-m1)x_c \cdot y_c - m2 \cdot y_c^2 - t2 \cdot x_c + t1 \cdot y_c + (m6 \cdot x_c - m3 \cdot y_c)z_c] = 0 \quad (82)$$

or

$$z = [m4 \cdot x_c^2 + (m5-m1)x_c \cdot y_c - m2 \cdot y_c^2 - t2 \cdot x_c + t1 \cdot y_c]/(m6 \cdot x_c - m3 \cdot y_c) \quad (83)$$

Unless J in (81) is identically zero, (82) indicate that all the points must lie on a quadric surface of some type containing the origin. However, (83) implies that z_c is a single-valued function of x_c and y_c unless $m6 \cdot x_c - m3 \cdot y_c = 0$. There are two cases to be discussed. The first is when $m6 \cdot x_c - m3 \cdot y_c$ devides the numerator. Then (83) must be a 1st order polynomial, say $a \cdot x_c + b \cdot y_c + c$. Thus (82) becomes $(z_c - a \cdot x_c - b \cdot y_c - c) \cdot (m6 \cdot x_c - m3 \cdot y_c) = 0$, which implies that in the new coordinate system, all the points must lie on two planes with one plane vertical and passing through the origin. Since, as in (79), the new coordinate system is obtained by rotating the old coordinate system around an axis through the origin, these two planes must still be two planes with one plane passing through the origin in the old coordinate system except that it is not necessarily vertical.

The second case is when $m6 \cdot x_c - m3 \cdot y_c$ does not devide the numerator in (83). In this case, z_c must be $\pm\infty$ or $\frac{0}{0}$ (i.e., indeterminate) along the line $m6 \cdot x_c - m3 \cdot y_c = 0$, while for other values of (x_c, y_c), z_c has to be single-valued. It is well known that any quadric surface must fall in one of the following categories [29]:

(1) imaginary quadric surface (e.g., $x_c^2 + y_c^2 + z_c^2 = -1$)

(2) ellipsoid

(3) hyperboloid of one sheet

(4) hyperboloid of two sheets

(5) elliptic paraboloid

(6) hyperbolic paraboloid

(7) elliptic cylinder

(8) hyperbolic cylinder

(9) parabolic cylinder

(10) a cone

(11) two planes

Since z_c is single-valued, the surface expressed in (83) cannot be ellipsoid or cylinder of any type. Paraboloid also is not possible since z_c is ∞ or indeterminate along the line $m6 \cdot x_c - m3 \cdot y_c = 0$ and as can be seen in Fig. 2 and 3, no such possibility can exist either for the elliptic paraboloid or hyperbolic paraboloid. Hyperboloid of one sheet should be excluded for consideration since, as is depicted in Fig. 4, this type of surface cannot be single-valued in z_c. It might seem that hyperboloid of two sheets in Fig. 5 with one of the separating hyperplanes vertical to the (x_c, y_c) plane and containing the z_c axis could be qualified since it is single-valued in z_c except along a line passing through origin, where z_c is $\pm\infty$. However, since the surface must contain the origin as was explained earlier, one sheet of the two in Fig. 5 must touch the vertical separating hyperplane. But it is well known in geometry that if a hyperboloid intercepts its separating plane, it must degenerate into a cone as depicted in Fig. 6, in which case the intersection must be the z_c axis. Therefore we conclude that unless J in (81) is a zero matrix, all the points must either lie on two planes with one plane containing the origin, or on a cone passing through the origin. But, as was defined in (78),

$$C + C^T = \left[\begin{array}{c|c} QR & \begin{array}{c}0\\0\\0\end{array} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right]^T \cdot J \cdot \left[\begin{array}{c|c} QR & \begin{array}{c}0\\0\\0\end{array} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right]$$

therefore,

$$C + C^T = 0$$

or

$$C = -C^T$$

which means that C has to be skew-symmetric.

* END OF PROOF FOR THEOREM II *

Fig. 2 Elliptic Paraboloid can be single-valued in z_c, but cannot diverge along a straight line.

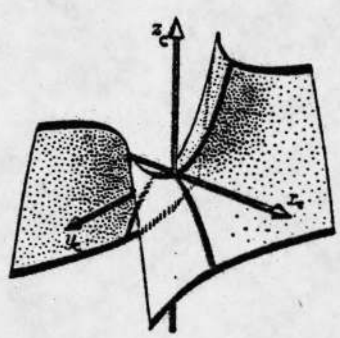

Fig. 3 Hyperbolic paraboloid can be single-valued in z_c, but it cannot diverge along a straight line.

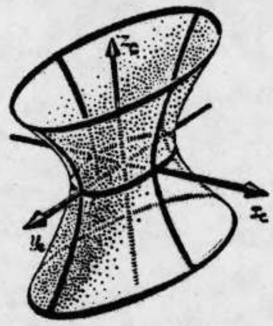

Fig. 4 Hyperboloid of one sheet cannot be single-valued in z_t.

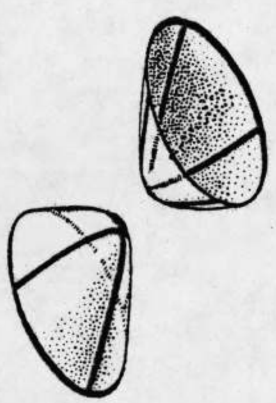

Fig. 5 Hyperboloid of two sheets with vertical separating hyperplane.

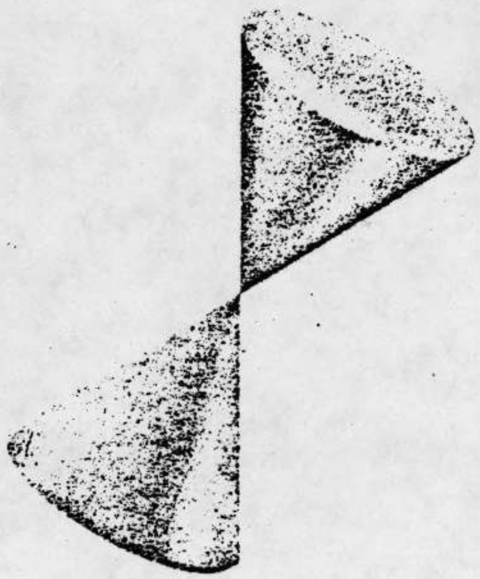

Fig. 6 If a hyperboloid intersepts its separating plane, it has to degenerate into a cone.

COROLLARY I

Given the image correspondences of two planes not passing through the origin, the motion is unique.

[Proof]

Since neither a cone nor two planes with one plane passing through the origin can contain two planes not passing through the origin, it follows from Theorem II that the C matrix in (63) has to be skew-symmetric. Then the uniqueness of the motion parameters follow directly from Lemma I. Q.E.D.

COROLLARY II

Given the image correspondences of six points with four points on one plane not containing the origin, four points on the other plane also not containing the origin, and two points common to the above two groups of four points on the intersection of the two planes can ensure unique solutions for the motion parameters.

[Proof]

Since as was proved in [20], the image correspondences of four points with none of the three points colinear determine the image motion of the whole plane, we can see that the six points with four points on one plane, four on the other plane can determine the image correspondences of two planes not containing the origin. Therefore, it follows from Corollary I that the motion parameters are unique. Q.E.D.

CORCLLARY III

The image correspondences of four points on a plane not passing through the origin and two other points not on this plane determine the motion parameters uniquely.

[Proof]

Obviously, on the very plane determined by the four points, whose image correspondences can be determined from these four points according to [20], there always exist two points that are coplanar with the other two points not on this plane. Therefore, it follows from Corollary II that the motion parameters are unique. Q.E.D.

COROLLARY IV

Given the image correspondences of seven or more points not traversable by two planes with one plane containing the origin, nor by a cone containing the origin, the motion parameters are unique.

[Proof]

If one of the image points before motion is chosen to be at the origin, which can always be done, then should there be a cone containing the origin passes through all the points, one of the separating hyperplane of the cone already passes through the z axis. Therefore, the rotation matrix QR in (79) need only rotate the original coordinate system around the z axis in order to arrive at (81), or

$$Q \cdot R = \begin{bmatrix} W & \\ & \pm 1 \end{bmatrix}$$

where W is some 2x2 orthonormal matrix. Then from (78)

$$C + C^T = \begin{bmatrix} W & & \\ & \pm 1 & \\ & & 1 \end{bmatrix}^T \cdot J \cdot \begin{bmatrix} W & & \\ & \pm 1 & \\ & & 1 \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$W^T \cdot N \cdot W$}} & +m6 & -t2 \\ & & +m3 & t1 \\ \hline +m6 & +m3 & 0 & 0 \\ -t2 & t1 & 0 & 0 \end{array}\right]$$

where

$$N \triangleq \begin{bmatrix} 2 \cdot m4 & m5-m1 \\ m5-m1 & -2\ m2 \end{bmatrix}$$

Therefore, even in the original coordinate system, the surface is still given by the equation in the form of (82). Since (82) contains seven terms with six effective coefficients, there is always a unique cone containing the origin that passes through six points in general positions, while no such cone exists that contain the origin and passes through seven points in general positions, nor can two planes with one plane containing the origin. Thus we conclude that given seven or more image point correspondences in general positions, the matrix C in (63) has to be skew-symmetric and the motion parameters can be uniquely determined. Q.E.D.

Since Corollary IV only gives the sufficient condition for uniqueness, even if the seven points are traversable by two planes with one plane passing through the origin, or by a cone containing the origin, the motion parameters might still be unique in some situations. For example, if six among the seven points satisfy the condition stated in Corollary III, then the motion parameters are unique even if there may be two planes passing through these seven points with one plane containing the origin.

From (82), the criteria for whether there exists a cone containing the origin that passes through n points is whether the following n by 7 rectangular matrix has full column rank or not.

$$\begin{bmatrix} x1^2 & x1y1 & y1^2 & x1 & y1 & z1x1 & z1y1 \\ x2^2 & x2y2 & y2^2 & x2 & y2 & z2x2 & z2y2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ xn^2 & xnyn & yn^2 & xn & yn & znxn & znyn \end{bmatrix}.$$

However, since only the image coordinates are given, the only useful criteria available is whether or not the 8x8 matrix in (58) is nonsingular or not. If it is nonsingular, one can solve for the E matrix, compute its SVD, and then use the formula in Theorem I to calculate the actual motion parameters. The following two corollaries state the necessary and sufficient conditions for the 8 x 8 matrix in (58) to be singular.

Corollary V

Given the image correspondences of eight points among which more than six points are coplanar, the 8 x 8 coefficient matrix in (58) is singular.

[Proof]

Let H be defined as the 8 x 8 coefficient matrix in (58), i.e.,

$$H = \begin{bmatrix} X1'X1 & X1'Y1 & X1' & Y1'X1 & Y1'Y1 & Y1' & X1 & Y1 \\ X2'X2 & X2'Y2 & X2' & Y2'X2 & Y2'Y2 & Y2' & X2 & Y2 \\ X3'X3 & X3'Y3 & X3' & Y3'X3 & Y3'Y3 & Y3' & X3 & Y3 \\ X4'X4 & X4'Y4 & X4' & Y4'X4 & Y4'Y4 & Y4' & X4 & Y4 \\ X5'X5 & X5'Y5 & X5' & Y5'X5 & Y5'Y5 & Y5' & X5 & Y5 \\ X6'X6 & X6'Y6 & X6' & Y6'X6 & Y6'Y6 & Y6' & X6 & Y6 \\ X7'X7 & X7'Y7 & X7' & Y7'X7 & Y7'Y7 & Y7' & X7 & Y7 \\ X8'X8 & X8'Y8 & X8' & Y8'X8 & Y8'Y8 & Y8' & X8 & Y8 \end{bmatrix} \qquad (84)$$

We shall prove that if at least seven among the eight points are coplanar in the object space, H is singular. Since interchanging the rows of H will not alter the singularity of H, we can assume without losing generality that the first seven object points corresponding to the first seven rows of H are coplanar. Let Hz be defined as

$$Hz = \begin{bmatrix} z1 \cdot z1' & & & & 0 \\ & z2 \cdot z2' & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \cdot \\ & & & & z8 \cdot z8' \end{bmatrix} \cdot H \qquad (85)$$

$$= \begin{bmatrix} x1'x1 & x1'y1 & x1'z1 & y1'x1 & y1'y1 & y1'z1 & z1'x1 & z1'y1 \\ x2'x2 & x2'y2 & x2'z2 & y2'x2 & y2'y2 & y2'z2 & y2'x2 & z2'y2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x8'x8 & x8'y8 & x8'z8 & y8'x8 & y8'y8 & y8'z8 & z8'x8 & z8'y8 \end{bmatrix}$$

From (85),

$$\det(Hz) = \left(\prod_{i=1}^{8} zi\ zi' \right) \cdot \det(H) \tag{86}$$

Since the object points must be in front of the camera lense in order to be imaged, zi and zi' are greater than 1 (the normalized focal length) for i = 1, ..., 8. Therefore, from (86), det (Hz) = 0 if and only if det (H) = 0, i.e., H is singular if and only if Hz is singular. We now prove that the first seven rows of Hz must be linearly dependent.

Let the 7 x 8 submatrix of Hz corresponding to the first seven rows be denoted by B. Since the first seven points are assumed to be coplanar, from [3], we have

$$\begin{bmatrix} xi' \\ yi' \\ zi' \end{bmatrix} = k\,A \begin{bmatrix} xi \\ yi \\ zi \end{bmatrix} \tag{87}$$

for i = 1, ..., 7, where

$$A = \begin{bmatrix} a1 & a2 & a3 \\ a4 & a5 & a6 \\ a7 & a8 & 1 \end{bmatrix}$$

ai's are the "pure parameters" defined in [2] and [3].

k is some constant.

Let D be defined as

$$D \triangleq k \begin{bmatrix} x1^2 & y1^2 & z1^2 & x1\cdot y1 & x1\cdot z1 & y1\cdot z1 \\ x2^2 & y2^2 & z2^2 & x2\cdot y2 & x2\cdot z2 & y2\cdot z2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x7^2 & y7^2 & z7^2 & x7\cdot y7 & x7\cdot z7 & y7\cdot z7 \end{bmatrix}$$

Then, with (87), the columns of B become

$$B1 = k \begin{bmatrix} a1x1^2 + a2x1y1 + a3x1z1 \\ a1x2^2 + a2x2y2 + a3x2z2 \\ \cdot \\ \cdot \\ \cdot \\ a1x7^2 + a2x7y7 + a3x7z7 \end{bmatrix} = D \begin{bmatrix} a1 \\ 0 \\ 0 \\ a2 \\ a3 \\ 0 \end{bmatrix}$$

$$B2 = k \begin{bmatrix} a1x1y1 + a2y1^2 + a3y1z1 \\ a1x2y2 + a2y2^2 + a3y2z2 \\ \cdot \\ \cdot \\ \cdot \\ a1x7y7 + a2y7^2 + a3y7z7 \end{bmatrix} = D \begin{bmatrix} 0 \\ a2 \\ 0 \\ a1 \\ 0 \\ a3 \end{bmatrix}$$

$$B3 = k \begin{bmatrix} a1x1z1 + a2y1z1 + a3z1^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a1x7z7 + a2y7z7 + z3z7^2 \end{bmatrix} = D \begin{bmatrix} 0 \\ 0 \\ a3 \\ 0 \\ a1 \\ a2 \end{bmatrix}$$

$$B4 = k \begin{bmatrix} a4x1^2 + a5x1y1 + a6x1z1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a4x7^2 + a5x7y7 + a6x7z7 \end{bmatrix} = D \begin{bmatrix} a4 \\ 0 \\ 0 \\ a5 \\ a6 \\ 0 \end{bmatrix}$$

$$B5 = k \begin{bmatrix} a4 \cdot x1 \cdot y1 + a5 \cdot y1^2 + a6 \cdot y1 \cdot z1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a4 \cdot x7 \cdot y7 + a5 \cdot y7^2 + a6 \cdot y7 \cdot z7 \end{bmatrix} = D \begin{bmatrix} 0 \\ a5 \\ 0 \\ a4 \\ 0 \\ a6 \end{bmatrix}$$

$$B6 = k \begin{bmatrix} a4 \cdot x1 \cdot z1 + a5 \cdot y1 \cdot z1 + a6 \cdot z1^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a4 \cdot x7 \cdot z7 + a5 \cdot y7 \cdot z7 + a6 \cdot z7^2 \end{bmatrix} = D \begin{bmatrix} 0 \\ 0 \\ a6 \\ 0 \\ a4 \\ a5 \end{bmatrix}$$

$$B7 = k \begin{bmatrix} a7 \cdot x1^2 + a8 \cdot x1 \cdot y1 + a9 \cdot x1 \cdot z1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a7 \cdot x7^2 + a8 \cdot x7 \cdot y7 + a9 \cdot x7 \cdot z7 \end{bmatrix} = D \begin{bmatrix} a7 \\ 0 \\ 0 \\ a8 \\ a9 \\ 0 \end{bmatrix}$$

$$B8 = k \begin{bmatrix} a7 \cdot x1 \cdot y1 + a8 \cdot y1^2 + a9 \cdot y1 \cdot z1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a7 \cdot x7 \cdot y7 + a8 \cdot y7^2 + a9 \cdot y7 \cdot z7 \end{bmatrix} = D \begin{bmatrix} 0 \\ a8 \\ 0 \\ a7 \\ 0 \\ a9 \end{bmatrix}$$

where Bi denotes the ith column of B.

Therefore,

$$B = D \begin{bmatrix} a1 & 0 & 0 & a4 & 0 & 0 & a7 & 0 \\ 0 & a2 & 0 & 0 & a5 & 0 & 0 & a8 \\ 0 & 0 & a3 & 0 & 0 & a6 & 0 & 0 \\ a2 & a1 & 0 & a5 & a4 & 0 & a8 & a7 \\ a3 & 0 & a1 & a6 & 0 & a4 & a9 & 0 \\ 0 & a3 & a2 & 0 & a6 & a5 & 0 & a9 \end{bmatrix} \triangleq D \cdot L \qquad (88)$$

Since, as can be seen in (88), B is the product of a 7 x 6 matrix D and a 6 x 8 matrix L, the column and row rank of B can be at most 6. To elaborate on this, since D is a 7 x 6 matrix, the SVD of D is given by

$$D = U_D \cdot \begin{bmatrix} \Lambda_D \\ 0\ 0\ \ldots.\ 0 \end{bmatrix} \cdot V_D^T$$

where

$$\Lambda_D = \begin{bmatrix} \lambda 1 & & & & & \\ & \lambda 2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \lambda 6 \end{bmatrix}$$

λi's are the singular values of D

U_D is a 7 x 7 orthonormal matrix

V_D is a 6 x 6 orthonormal matrix.

Then (88) becomes

$$B = U_D \cdot \begin{bmatrix} \Lambda_D \\ 0\ 0\ \ldots\ 0 \end{bmatrix} \cdot V_D^T \cdot L$$

$$= U_D \cdot \begin{bmatrix} \Lambda_D \cdot V_D^T \cdot L \\ 0\ 0\ \ldots..\ 0 \end{bmatrix}$$

or

$$U_D^T \cdot B = \begin{bmatrix} \Lambda_D \cdot V_D^T \cdot L \\ 0\ 0 \ldots\ldots 0 \end{bmatrix} \tag{89}$$

Since U_D is orthonormal, the row rank of B is the same as that of $U_D^T \cdot B$. But the last row of $U_D^T \cdot B$ is zero, as can be seen in (89). Therefore, the row rank of B can be at most 6. Since B is the 7 x 8 submatrix of Hz, one of the first seven rows of Hz can be expressed as a linear combination of the others. Therefore, Hz is singular, which implies that H is singular.

Q. E. D.

Corollary VI

If the 8 x 8 coefficient matrix H containing the image correspondences of eight points in (58) is singular, then either seven or eight points are coplanar in the object space, or the eight object points are on a cone containing the origin.

[Proof]

Corollary IV implies that if the motion parameters are not unique, or equivalently the E matrix is not unique and H in (58) is singular, the eight points are either traversable by two planes with one plane containing the origin, or by a cone containing the origin. This conclusion is certainly correct but can be made stronger since there are cases when the eight points are traversable by two planes with one plane containing the origin while the motion parameters are still unique. According to Corollary III, so long as four among the eight points are on a plane not containing the origin, and two other points not on this plane determine the motion parameters uniquely. Obviously there are only three possibilities for this to happen when the eight points are traversable by two planes with one plane containing the origin:

(1) Six points are on a plane not passing through the origin, and two points on another plane containing the origin.

(2) Five points are on a plane not passing through the origin, and three points on another plane containing the origin.

(3) Four points are on a plane not passing through the origin, and four points on another plane containing the origin.

This leaves only the following two cases which have been shown in Corollary V to be the sufficient conditions for H to be singular:

(1) Exactly seven points among the eight are on a plane not passing through the origin.

(2) All the eight points are on a plane not passing through the origin.

Therefore, the assertion of the corollary is justified.

Q. E. D.

The results developed in this paper can also be applied to the stereo imaging problems in photogrammetry and computer vision without assuming the relative orientation of the two cameras since pictures taken at two time instances can be regarded as taken by two cameras at one instance. After the motion parameters are computed using the formula in Theorem I, the surface structure of the object can be determined up to a common scale factor by computing the z coordinates using (5a) or (5b).

V. PURE ROTATION AND PLANAR PATCH MOTION

Note that when the object undergoes pure rotation around an axis through the origin, $\Delta x = \Delta y = \Delta z = 0$, and therefore, from (7), E is a zero matrix. The converse is also true since if $E = \underline{0}$ ($\underline{0}$ stands for a 3 x 3 zero matrix), then from (9), $G = ER^T = \underline{0}R^T = \underline{0}$, i.e., $\Delta x = \Delta y = \Delta z = 0$. In this case, the results described earlier in this paper cannot be applied since (5a) and (5b) become $z = 0/0$, and are no longer meaningful. However, it is to be seen in the following that the image motions for the case of three-dimensional pure rotation are equivalent to the image motions of any planar patch undergoing three-dimensional pure rotation with the same rotation parameters θ, n1, n2, and n3. This means that even if the object surface is nonplanar, the motion parameters can still be computed using the results described in [3] for the planar patch motion. Furthermore, since the motion parameters have been proved to be unique for a rigid planar patch undergoing three-dimensional pure rotation (see Theorem III in [3]), the motion parameters for any curved surface undergoing three-dimensional pure rotation are also unique. A simple test for detecting the presence of pure rotation and the planar patch motion will also be described.

By setting Δx, Δy and Δz in (4a) and (4b) to zero, we have

$$X' = \frac{r1 \cdot X + r2 \cdot Y + r3}{r7 \cdot X + r8 \cdot Y + r9}$$
$$Y' = \frac{r4 \cdot X + r5 \cdot Y + r6}{r7 \cdot X + r8 \cdot Y + r9} \qquad (90)$$

It can be seen from [2,3] that (90) gives the image mapping $(X,Y) \rightarrow (X',Y')$ of a rigid planar patch undergoing 3-D motion with the 3 x 3 A matrix containing the pure parameters in [2,3] being

$$A = r_9^{-1} R \qquad (91)$$

Let $U_a + R$, $V_a = I$, $\Lambda_a = r_9^{-1} I$. Then (91) becomes

$$A = U_a \cdot \Lambda_a \cdot V_a^T \qquad (92)$$

Since R and I (and thus U_a and V_a) are orthonormal, (92) is the singular value decomposition of A with three identical singular values. Therefore, according to Theorem III in [3], (90) gives the image point correspondences of any rigid planar patch undergoing 3-D pure rotation with rotation matrix R.

We now describe a simple procedure for detecting whether the object points are on a planar patch or are undergoing 3-D pure rotation (given eight or more image point correspondences), which are the cases when (58) are not to be applied, and the motion parameters have to be computed using the resutls in [2,3,4].

From [2] and [3], the following mapping characterizes image correspondences of n object points on a rigid planar patch undergoing 3-D motion:

$$Xi' = \frac{a1 \cdot Xi + a2 \cdot Yi + a3}{a7 \cdot Xi + a8 \cdot Yi + 1}$$

$$Yi' = \frac{a4 \cdot Xi + a5 \cdot Yi + a6}{a7 \cdot Xi + a8 \cdot Yi + 1} \qquad (93)$$

for i = 1, 2, ..., n, and a1, ..., a8 are some constants. Rewriting (93) as a matrix equation with the ai's as the unknowns gives

$$M \begin{bmatrix} a1 \\ a2 \\ \cdot \\ \cdot \\ \cdot \\ a8 \end{bmatrix} = B \qquad (94)$$

where the 2n x 8 matrix M is given by

$$M = \begin{bmatrix} X1 & Y1 & 1 & 0 & 0 & 0 & -X1\cdot X1' & -Y1\cdot X1' \\ 0 & 0 & 0 & X1 & Y1 & 1 & -X1\cdot Y1' & -Y1\cdot Y1' \\ X2 & Y2 & 1 & 0 & 0 & 0 & -X2\cdot X2' & -Y2\cdot X2' \\ 0 & 0 & 0 & X2 & Y2 & 1 & -X2\cdot Y2' & -Y2\cdot Y2' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Xn & Yn & 1 & 0 & 0 & 0 & -Xn\cdot Xn' & -Yn\cdot Xn' \\ 0 & 0 & 0 & Xn & Yn & 1 & -Xn\cdot Yn' & -Yn\cdot Yn' \end{bmatrix}$$

and $B \triangleq \begin{bmatrix} X1' & Y1' & X2' & Y2' & \ldots & Xn' & Yn' \end{bmatrix}^T$

Therefore, given eight image point correspondences, one first examines the consistency of the 16 x 8 matrix equation in (94). If

$$\text{rank}\ (M) = \text{rank}\ (M \vdots B)$$

then (94) is consistent. An efficient way of checking the consistency of (94) is to solve the following 8 x 8 normal equation of (94) for the least square solution of (94):

$$M^T M \begin{bmatrix} a1 \\ a2 \\ \cdot \\ \cdot \\ \cdot \\ a8 \end{bmatrix} = M^T B$$

The solution of the above normal equation is then substituted back to (94). If it is satisfied, (94) is consistent. The solution will then be used to form the 3 x 3 A matrix defined in [2,3]. If the singular values of A are all identical, the motion consists of pure rotation around an axis through the origin only. In this case, the rotation matrix R is equal to A multiplied by a constant (which is equal to the inverse of the norm of any column of A since R is orthonormal) (See Theorem III in [4]). If (94) is not consistent, one solves (58) for the E matrix, and then computes the actual motion parameters using the method described in Theorem I of Sec. III.1.

VI. NUMERICAL EXAMPLES FOR THE CASES WHEN FIVE AND SIX POINTS CAN YIELD TWO SOLUTIONS

Note that Theorem II only gives the sufficient conditions for uniqueness. Although there always exists a cone passing through six points in general position and the origin, this does not imply that there are two solutions, one for the case when C is skew-symmetric and the other not skew-symmetric. Experimental results show that six points are usually but not always sufficient to yield unique solution. In fact, even five points are sometimes sufficient. The following are two numerical examples for the cases when five and six points are not sufficient to ensure uniqueness of solutions for the motion parameters. In these two examples, the image point correspondences were obtained by simulation. First, the image coordinates at t1 of a number of object points with randomly chosen object space coordinates (xi, yi, zi), i = 1, 2, ..., n (n = 5 for Example 1, and 6 for Example 2), are obtained using (3). Next the object points are rotated with some reference rotation parameters θ_o, n_{o1}, n_{o2}, n_{o3}, n_{o3}, and translated with some reference translation parameters Δxo, Δyo, Δzo (=1), with computer simulation using (1) to obtain (xi', yi', zi'), i = 1, ..., n. Then the image coordinates of these n points at t2, i.e., (Xi', Yi'), i = 1, 2, ..., n, were computed using (3). These n simulated image point correspondences

$(Xi,Yi) \rightarrow (Xi',Yi')$, $i = 1, 2, \ldots, n$, were then substituted into (6) to obtain n simultaneous nonlinear equations, one for each image point correspondence $(Xi,Yi) \rightarrow (Xi',Yi')$. The motion parameters in E of (6) with Δz set to 1 were obtained by solving this system of nonlinear equations using global search. For each of the following two examples, two solutions were found.

[Example 1] Five point case.

The object coordinates of the five points at t1:

$(x1, y1, z1) = (3.0, 15.7, 5.0)$, $(x2, y2, z2) = (28.1, 15.0, 32.3)$

$(x3, y3, z3) = (5.0, 12.9, 7.0)$, $(x4, y4, z4) = (32.7, 24.7, 18.0)$

$(x5, y5, z5) = (13.1, 31.0, 22.2)$.

By using (3),(Xi, Yi), $i = 1, \ldots, 5$, were found to be:

$(X1, Y1) = (0.6, 3.14)$, $(X2, Y2) = (0.869969, 0.464396)$

$(X3, Y3) = (0.714286, 1.842857)$, $(X4, Y4) = (1.816667, 1.372222)$

$(X5, Y5) = (0.590090, 1.396394)$.

The reference rotation and translation parameters:

$\theta o = 78$, $n01 = 0.615661475$, $n02 = 0.258819045$, $n03 = 0.74429406$,

$\Delta xo = 23$, $\Delta yo = -10$, $\Delta zo = 1$

The object coordinates (xi', yi', zi'), and image coordinates (Xi', Yi') at t2 were then computed accordingly using (1) and (3). The following two solutions were found:

Solution 1: the same as the reference solution.

Solution 2: $\theta = 159.722148$, $n1 = 0.087422567$, $n2 = 0.36295928$

$n3 = -0.9276949$, $\Delta x = 5.97327196$, $\Delta y = 1.50137639$.

[Example 2] Six point case.

The object coordinates at t1:

$(x1, y1, z1) = (3, 15.7, 54.908)$, $(x2, y2, z2) = (28.1, 15, 166.111)$

$(x3, y3, z3) = (5, 12.9, 42.232)$, $(x4, y4, z4) = (32.7, 24.7, 309.716)$

$(x5, y5, z5) = (13.1, 31, 249.971)$, $(x6, y6, z6) = (15, 9.7, 55.868)$

The image coordinates at t1:

$(X1, Y1) = (0.0546368, 0.285933)$, $(X2, Y2) = (0.169164, 0.0903011)$

$(X3, Y3) = (0.1183936, 0.3054556)$, $(X4, Y4) = (0.1055806, 0.0797505)$

$(X5, Y5) = (0.0524061, 0.1240144)$, $(X6, Y6) = (0.26849, 0.1736235)$

The reference motion parameters:

$\theta o = 78$, $no1 = 0.615661475$, $no2 = 0.258819045$, $no3 = 0.74429406$,

$\Delta xo = 23$, $\Delta yo = -10$, $\Delta zo = 1$.

(xi', yi', zi') and (Xi', Yi') were then computed using (1) and (3) with the above reference motion parameters. The following two solutions were found:

Solution 1: Same as the reference solution.

Solution 2: $\theta = 47.65578$, $n1 = 0.6304461986$, $n2 = 0.06582693435$,
$n3 = 0.7735214391$, $\Delta x = -3.683375707$, $\Delta y = 0.6458049137$.

For each of the two solutions in the above two examples, the z coordinates for each point using (5a) and (5b) were all positive.

VII. CONCLUSIONS

Several theorems and corollaries have been stated and proved regarding the uniqueness and estimation of 3-D motion parameters of rigid bodies. In summary, the following results have been established:

(1) The fact that we can define 8 essential parameters e1,e2,...,e8, that contain all the information one can possibly obtain given any number of image correspondences, and are unique given the image correspondences of at least seven points not lying on two planes with one plane passing through the origin, nor on a cone containing the origin.

(2) The fact that given the E matrix consisting of the eight essential parameters, the actual motion parameters are unique, and can be computed simply by taking the singular value decomposition(SVD) of the 3x3 E matrix.

(3) A method of determining the E matrix given 8 image correspondences. This requires the solution of a set of linear equations only.

(4) An operational criteria for the uniqueness of motion parameters. If the determinant of a certain 8x8 matrix containing only the image coordinates of eight image correspondences does not vanish, the uniqueness is assured.

The results in this paper should be of interest to numerous areas of research, including image sequence analysis, tracking, image coding, stereo imaging, photogrammetry, and robotic vision.

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