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Uniqueness of a Polynomial and a Differential Monomial Sharing a Small Function

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Abstract. In this paper taking a question in [1] into background we investigate the uniqueness of a non-constant polynomial with the differential monomial generated by a non-constant mermorphic function f. Our result will also extend a result of *Banerjee-Majumder* [2] given earlier. An open question is also posed, in the paper, for future investigation.

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1 A first section

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [4].
If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities) and if we do not consider the multiplicities then f, g are said to share the value a IM (ignoring multiplicities). When $a = \infty$ the zeros of $f - a$ means the poles of f.

It will be convenient to let J denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function f, we denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o(T(r,f)) \qquad (r \longrightarrow \infty, r \notin J).$$

A meromorphic function $a = a(z)(\not\equiv \infty)$ is called a small function with respect to f provided that $T(r,a) = S(r,f)$ as $r \longrightarrow \infty, r \notin J$. If $a = a(z)$ is a small function we define that f and g share a IM or a CM according as $f - a$ and $g - a$ share 0 IM or 0 CM respectively. Also it is known to us that the hyper order of f, denoted by $\rho_2(f)$, is defined by

$$\rho_2(f) = \limsup_{r \longrightarrow \infty} \frac{\log\log T(r,f)}{\log r}.$$

We start our discussion on a well known result of *Rubel* and *Yang* ([10]), where they proved that if a non-constant entire function f and $f^{'}$ share two distinct finite numbers a, b CM, then $f = f^{'}$. This result is the starting point of the investigations about the relation between an entire or meromorphic function sharing some values with their derivatives.

In 1979, *Mues* and *Steinmetz* ([9]) obtained an analogous result for IM sharing. In this direction, in 1996, Brück ([3]) proposed his following famous conjecture.

Conjecture : *Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and $f^{'}$ share a finite value a CM, then $\frac{f^{'}-a}{f-a} = c$, where c is a non-zero constant.*

Brück himself proved the conjecture for $a = 0$. For $a \neq 0$, Brück ([3]) obtained the following result in which additional supposition was required.

Theorem A. ([3]) *Let f be a non-constant entire function. If f and $f^{'}$ share the value 1 CM and if $N(r,0;f^{'}) = S(r,f)$ then $\frac{f^{'}-1}{f-1}$ is a nonzero constant.*

Next we recall the following definitions.

Definition 1.1. ([12]) *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by*

$$N_p(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \ldots + \overline{N}(r,a;f \mid\geq p).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 1.2. ([12]) *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we put*

$$\delta_p(a,f) = 1 - \limsup_{r \to \infty} \frac{N_p(r,a;f)}{T(r,f)}.$$

Clearly

$0 \leq \delta(a,f) \leq \delta_p(a,f) \leq \delta_{p-1}(a,f) \leq \ldots \leq \delta_2(a,f) \leq \delta_1(a,f) = \Theta(a,f) \leq 1.$

Definition 1.3. ([1]) *For two positive integers n, p we define $\mu_p = min\{n,p\}$ and $\mu_p^* = p+1-\mu_p$. Then it is clear that*

$$N_p(r,0;f^n) \leq \mu_p N_{\mu_p^*}(r,0;f).$$

Definition 1.4. ([2]) *Let z_0 be a zero of $f-a$ of multiplicity p and a zero of $g-a$ of multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p > q \geq 1$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$.*

Definition 1.5. ([5]) *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

With the notion of weighted sharing of values Lahiri-Sarkar ([6]) improved the result of Zhang ([11]). In ([12]) Zhang extended the result of Lahiri-Sarkar ([6]) and replaced the concept of value sharing by small function sharing.

In 2008, Zhang and Lü ([13]) obtained the following result.

Theorem B. ([13]) *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $f^n - a$ and $f^{(k)} - a$ share $(0,l)$. If $l = \infty$ and*

$$(3+k)\Theta(\infty,f) + 2\Theta(0,f) + \delta_{2+k}(0,f) > 6+k-n \tag{1.1}$$

or $l = 0$ *and*

$$(6+2k)\Theta(\infty,f)+4\Theta(0,f)+2\delta_{2+k}(0,f) > 12+2k-n \tag{1.2}$$

then $f^n \equiv f^{(k)}$.

In ([13]), *Zhang* and *Lü* asked this question :

Question 1.1. *What will happen if* f^n *and* $[f^{(k)}]^m$ *share a small function ?*

In 2010 Banerjee and Majumder ([2]) answer the above open question affirmatively in the following manner.

Theorem C. ([2]) *Let* $k(\geq 1)$, $n(\geq 1)$, $m(\geq 2)$ *be integers and* f *be a non-constant meromorphic function. Also let* $a(z)(\not\equiv 0,\infty)$ *be a small function with respect to* f. *Suppose* $f^n - a$ *and* $[f^{(k)}]^m - a$ *share* $(0,l)$. *If* $l = 2$ *and*

$$(3+2k)\ \Theta(\infty,f)+2\ \Theta(0,f)+2\delta_{1+k}(0,f) > 7+2k-n \tag{1.3}$$

or $l = 1$ *and*

$$\left(\frac{7}{2}+2k\right)\ \Theta(\infty,f)+\frac{5}{2}\ \Theta(0,f)+2\delta_{1+k}(0,f) > 8+2k-n \tag{1.4}$$

or $l = 0$ *and*

$$(6+3k)\ \Theta(\infty,f)+4\ \Theta(0,f)+3\delta_{1+k}(0,f) > 13+3k-n \tag{1.5}$$

then $f^n \equiv [f^{(k)}]^m$.

Here we observe that in the conditions (1.3)-(1.5) there was no influence of m.

Next we recall the following definition.

Definition 1.6. ([4]) *Let* $n_{0j}, n_{1j}, \ldots, n_{kj}$ *be nonnegative integers.*
The expression $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}}\ldots(f^{(k)})^{n_{kj}}$ *is called a differential monomial generated by* f *of degree* $d_{M_j} = d(M_j) = \sum\limits_{i=0}^{k} n_{ij}$ *and weight* $\Gamma_{M_j} = \sum\limits_{i=0}^{k}(i+1)n_{ij}$.

The sum $P[f] = \sum\limits_{j=1}^{t} b_j M_j[f]$ *is called a differential polynomial generated by* f *of degree* $\overline{d}(P) = max\{d(M_j) : 1 \leq j \leq t\}$ *and weight* $\Gamma_P = max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, *where* $T(r,b_j) = S(r,f)$ *for* $j = 1,2,\ldots,t$.

The numbers $\underline{d}(P) = min\{d(M_j) : 1 \leq j \leq t\}$ *and* k *(the highest order of the derivative of* f *in* $P[f]$*) are called respectively the lower degree and order of* $P[f]$.

$P[f]$ *is said to be homogeneous if* $\overline{d}(P)$=$\underline{d}(P)$.

$P[f]$ *is called a linear differential polynomial generated by* f *if* $\overline{d}(P) = 1$. *Otherwise* $P[f]$ *is called a non-linear differential polynomial.*

We denote by $Q = max\ \{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = max\ \{n_{1j} + 2n_{2j} + \ldots + kn_{kj} : 1 \leq j \leq t\}$.

Also for the sake of convenience for a differential monomial $M[f]$ *we denote by* $\lambda = \Gamma_M - d_M$.

Since the natural extension of $[f^{(k)}]^m$ is a differential monomial, it will be interesting to see whether *Theorem C* can remain true when $[f^{(k)}]^m$ is replaced by $M[f]$. In this direction, very recently *Banerjee - Chakraborty* ([1]) have improved *Theorem C* in the following way which in turn improve a recent result of Li-Huang [7] as well.

Theorem D. ([1]) *Let* $k(\geq 1)$, $n(\geq 1)$ *be integers and* f *be a non-constant meromorphic function and* $M[f]$ *be a differential monomial of degree* d_M *and weight* Γ_M *and* k *is the highest derivative in* $M[f]$. *Also* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *Suppose* $f^n - a$ *and* $M[f] - a$ *share* $(0, l)$. *If* $l \geq 2$ *and*

$$(3 + \lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0; f) > \Gamma_M + 3 + \mu_2 - n \tag{1.6}$$

or $l = 1$ *and*

$$\left(\frac{7}{2} + \lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(0; f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0; f) > \Gamma_M + 4 + \mu_2 - n \tag{1.7}$$

or $l = 0$ *and*

$$\begin{aligned}&(6 + 2\lambda)\Theta(\infty, f) + 2\Theta(0; f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0; f) + d_M\delta_{k+1}(0; f)\\ &> 2\Gamma_M + 8 + \mu_2 - n\end{aligned} \tag{1.8}$$

then $f^n \equiv M[f]$.

In [1, Example 1.13], Banerjee-Chakraborty ([1]) have shown that f^n can't be replaced by an arbitrary polynomial $P[f] = a_0f^n + a_1f^{n-1} + \ldots + a_n$ in *Theorem D* for IM sharing case.

Observing Example 1.13 in [1] we note that $f(z) = e^z$, $P(f) = f^2 + 2f$ and $M[f] = f^{(3)}$. So $P + 1 = (M + 1)^2$. Thus P and M share $(-1, 0)$. Also

$\Theta(\infty; f) = 1 = \Theta(0; f) = \delta_q(0; f)$. Here $p = n = 1$, $m = 1$, $w_p = 0$, $\mu_2 = 1$, $d_M = 1$, $\Gamma_M = 4$, $\lambda = 3$. Here

$$\begin{aligned}&(6+2\lambda)\Theta(\infty, f) + 2\Theta(0; f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0; f) + d_M\delta_{k+1}(0; f)\\ &= 17 > 16 = 2\Gamma_M + 8 + \mu_2 - n\end{aligned}$$

so that (1.8) is satisfied but $M[f] \not\equiv P[f]$. On the basis of this observation in ([1]) following question was asked in [1]

Question 1.2. *Is it possible to replace f^n by arbitrary polynomial $P[f] = a_0 f^n + a_1 f^{n-1} + \ldots + a_n$ in Theorem D for $l \geq 1$?*

In this paper, we will not only try to find the possible answer of the above Question 1.2, but also improve *Theorem D* to a large extent. We have observed that if we consider the general polynomial $P(f) = a_n f^n + \ldots + a_0$ in the place of f^n in the line of the proof of *Theorem D*, we will get a different inequality in comparison to (1.8) such that Example 1.13 is not violating the new condition (1.14) given later on.

Through the paper we shall assume the following notations. Let

$$\mathcal{P}(w) = a_{n+m}w^{n+m} + \ldots + a_n w^n + \ldots + a_0 = a_{n+m}\prod_{i=1}^{s}(w - w_{p_i})^{p_i}$$

where $a_j (j = 0, 1, 2, \ldots, n+m-1)$, $a_{n+m} \neq 0$ and $w_{p_i} (i = 1, 2, ..., s)$ are distinct finite complex numbers and $2 \leq s \leq n+m$ and $p_1, p_2, \ldots, p_s$, $s \geq 2$, n, m and k are all positive integers with $\sum_{i=1}^{s} p_i = n+m$. Also let $p > \max_{p \neq p_i, i=1,\ldots,r} \{p_i\}$, $r = s-1$, where s and r are two positive integers.

Let $P(w_1) = a_{n+m}\prod_{i=1}^{s-1}(w_1 + w_p - w_{p_i})^{p_i} = b_q w_1^q + b_{q-1}w_1^{q-1} + \ldots + b_0$, where $a_{n+m} = b_q, w_1 = w - w_p, q = n+m-p$. Therefore, $\mathcal{P}(w) = w_1^p P(w_1)$.

Next we assume $P(w_1) = b_q\prod_{i=1}^{r}(w_1 - \alpha_i)^{p_i}$, where $\alpha_i = w_{p_i} - w_p$, $(i = 1, 2\ldots, r)$, be distinct zeros of $P(w_1)$.

The following theorem is the main result of this paper which gives an affirmative answer of the questions of *Banerjee - Chakraborty* ([1]) and also the question posed by *Zhang-Lü* ([13]) in a more convenient way.

Theorem 1.1. *Let $k(\geq 1)$, $n(\geq 1)$, $p(\geq 1)$ and $m(\geq 0)$ be integers and f and $f_1 = f - w_p$ be two non-constant meromorphic functions and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and k is the highest* *derivative in* $M[f]$. *Let* $\mathcal{P}(z) = a_{m+n}z^{m+n} + \ldots + a_n z^n + \ldots + a_0, a_{m+n} \neq 0$, *be a polynomial in* z *of degree* $m+n$ *such that* $\mathcal{P}(f) = f_1^p P(f_1)$. *Also let* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *Suppose* $\mathcal{P}(f) - a$ *and* $M[f] - a$ *share* $(0, l)$. *If* $l \geq 2$ *and*

$$(3+\lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\delta_{2+k}(0, f) > \Gamma_M + \mu_2 + 3 - p \tag{1.9}$$

or $l = 1$ *and*

$$\begin{aligned}&\left(\frac{7}{2}+\lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\delta_{2+k}(0, f)\\ > \;& \Gamma_M + \mu_2 + 4 + \frac{(m+n) - 3p}{2}\end{aligned} \tag{1.10}$$

or $l = 0$ *and*

$$\begin{aligned}&(6+2\lambda)\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\left(\sum_{i=1}^{2}\delta_{k+i}(0, f)\right)\\ > \;& 2\Gamma_M + \mu_2 + 8 + 2(m+n) - 3p\end{aligned} \tag{1.11}$$

then $\mathcal{P}(f) \equiv M[f]$.

The following Corollary can easily be deduced from the above theorem which is an extension and improvement of the Theorem D. It is clear that for $P(z) = 1$ i.e., $m = 0$, we get exactly Theorem E from Corollary 1.1.

Corollary 1.1. *Let* $k(\geq 1)$, $n(\geq 1)$ *and* $m(\geq 0)$ *be integers and* f *be a non-constant meromorphic function and* $M[f]$ *be a differential monomial of degree* d_M *and weight* Γ_M *and* k *is the highest derivative in* $M[f]$. *Let* $P(z) = a_m z^m + \ldots + a_0, a_m \neq 0$, *be a polynomial in* z *of degree* m. *Also let* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *Suppose* $f^n P(f) - a$ *and* $M[f] - a$ *share* $(0, l)$. *If* $l \geq 2$ *and*

$$(3+\lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\delta_{2+k}(0, f) > \Gamma_M + \mu_2 + 3 - n \tag{1.12}$$

or $l = 1$ *and*

$$\begin{aligned}&\left(\frac{7}{2}+\lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\delta_{2+k}(0, f)\\ > \;& \Gamma_M + \mu_2 + 4 + \frac{m}{2} - n\end{aligned} \tag{1.13}$$

or $l = 0$ *and*

$$\begin{aligned}&(6+2\lambda)\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\left(\sum_{i=1}^{2}\delta_{k+i}(0, f)\right)\\ > \;& 2\Gamma_M + \mu_2 + 8 + 2m - n\end{aligned} \tag{1.14}$$

then $f^n P(f) \equiv M[f]$.

We see that in case of Example 1.13 in [1] we have $\Theta(\infty; f) = 1 = \Theta(0; f) = \delta_q(0; f)$, $p = n = 1$, $m = 1$, $w_p = 0$, $\mu_2 = 1$, $d_M = 1$, $\Gamma_M = 4$, $\lambda = 3$. So when $l = 0$, we get

$$\begin{aligned}&(6+2\lambda)\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\left(\sum_{i=1}^{2}\delta_{k+i}(0, f)\right)\\ = \;& 17 < 18 = 2\Gamma_M + \mu_2 + 8 + 2m - n,\end{aligned}$$

thus (1.14) ceases to hold and hence Example 1.13 [1] is not violating Corollary 1.1.

However the following question is still open.

Question 1.3. *Is it possible to extend Theorem 1.1 up to differential polynomial $P[f]$ instead of differential monomial $M[f]$?*

Following example shows that in Theorem 1.1 $a(z) \not\equiv 0$ is essential.

Example 1.1. Let us take $f(z) = e^{Nz}$ where $N \neq 0, \pm 1$ and $\mathcal{P}(f) = f^3$, $M[f] = f^{(2)}$. Then $\mathcal{P}(f)$ and $M[f]$ share $a = 0$ *(or,* ∞*)*. Here $m = 0, p = n = 1, w_p = 0, d_M = 1, \mu_2 = 1, \Gamma_M = 3$ and $\lambda = 2$. Also $\Theta(\infty; f) = 1 = \Theta(0; f)$ and $\delta_q(0; f) = 1, \forall q \in \mathbb{N}$. Thus we see that the deficiency conditions stated in Theorem 1.1 are satisfied but $\mathcal{P}(f) \not\equiv M[f]$.

The next example shows that the deficiency conditions stated in Theorem 1.1 are not necessary.

Example 1.2. Let $f(z) = A\cos z + B\sin z, AB \neq 0$. Then $\overline{N}(r, f) = S(r, f)$ and

$$\overline{N}(r, 0; f) = \overline{N}\left(r, \frac{A+iB}{A-iB}; e^{2iz}\right) \sim T(r, f).$$

Here $m = 0, p = n = 1, w_p = 0, d_M = 1, \mu_2 = 1, \Gamma_M = 4k+1$ and $\lambda = 4k$. Again $\Theta(\infty, f) = 1$ and $\Theta(0, f) = \delta_p(0, f) = 0$. Let $m = 0$, hence $\mathcal{P}(f) = f$.

Therefore it is clear that $M[f] = f^{(4k)}$, for $k \in \mathbb{N}$ and $\mathcal{P}(f)$ share $a(z)$ and the deficiency conditions in Theorem 1.1 are not satisfied, but $\mathcal{P}(f) \equiv M$.

The following three examples show that the conditions (1.9) - (1.11) in Theorem 1.1 can not be removed.

Example 1.3. Let $f(z) = e^{Nz}$, where N is a non-zero integer. For $n \geq 2$ let

$$\mathcal{P}(f) = -N^{2n}\sum_{r=0}^{2n-1}(-1)^r\binom{2n}{r}f^{2n-r} \text{ and } M[f] = f^{(2n)}.$$

Then it is clear that

$$\mathcal{P}(f) - N^{2n} = -N^{2n}(e^{Nz}-1)^{2n} \text{ and } M[f] - N^{2n} = N^{2n}(e^{Nz}-1).$$

Thus we see that $\mathcal{P}(f)$ and $M[f]$ share $(N^{2n},0)$. Here $n+m=2n$, $p=1$, $w_p=0$, $d_M=1$, $\Gamma_M=2n+1$, $\mu_2=1$ and $\lambda=2n$. Also $\Theta(\infty;f)=1=\Theta(0;f)$ and $\delta_q(0;f)=1, \forall q\in\mathbb{N}$.
So for $l=0$

$$\begin{aligned} & (6+2\lambda)\Theta(\infty,f)+2\Theta(w_p,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+d_M\left(\sum_{i=1}^{2}\delta_{k+i}(0,f)\right) \\ = \; & 4n+11\leq 8n+8=2\Gamma_M+\mu_2+8+2(m+n)-3p \end{aligned}$$

and we see that $\mathcal{P}(f)\not\equiv M[f]$.

Example 1.4. Let $f(z)=-\sin(\alpha z)+a-\frac{a}{\alpha^{4k}}$, $k\in\mathbb{N}$; where $\alpha\neq 0, \alpha^{4k}\neq 1$ and $a\in\mathbb{C}-\{0\}$. Let $p=n=1$, $w_p=0$ and $m=0$. Then let $\mathcal{P}(f)=f$. Again let $M[f]=f^{(4k)}$. Then $M[f]=-\alpha^{4k}\sin(\alpha z)$. Here $m=0, \mu_2=1, \Gamma_M=4k+1, d_M=1$ and $\lambda=4k$. Again $\Theta(\infty;f)=1$ and

$$\overline{N}(r,0;f)=\overline{N}\left(r,a-\frac{a}{\alpha^{4k}};\sin(\alpha z)\right)\sim T(r,f).$$

Therefore,

$$\Theta(0;f)=0=\delta_q(0;f), \forall q\in\mathbb{N}.$$

Also it is clear that $\mathcal{P}(f)$ and $M[f]$ share (a,l) $(l\geq 0)$ but none of the inequalities *(1.9)*, *(1.10)* and *(1.11)* of Theorem 1.1 is satisfied and $\mathcal{P}(f)\not\equiv M[f]$.

Example 1.5. Let $f(z)=e^{\beta z}+a-\frac{a}{\beta^2}$; where $a\neq 0,\infty$ and $\beta\neq 0,\pm 1$. Let $p=n=1$, $w_p=0$ and $m=0$. Then let $\mathcal{P}(f)=f$. Again let $M[f]=f^{(2)}$. Then $M[f]=\beta^2e^{\beta z}$. Here $m=0, \mu_2=1, \Gamma_M=3, d_M=1$ and $\lambda=2$. Again $\Theta(\infty;f)=1$ and

$$\overline{N}(r,0;f)=\overline{N}\left(r,\frac{a}{\beta^2}-a;e^{\beta z}\right)\sim T(r,f).$$

Therefore,

$$\Theta(0;f)=0=\delta_q(0;f), \forall q\in\mathbb{N}.$$

Also it is clear that $\mathcal{P}(f)$ and $M[f]$ share (a,l) $(l\geq 0)$ but none of the inequalities *(1.9)*, *(1.10)* and *(1.11)* of Theorem 1.1 is satisfied and $\mathcal{P}(f)\not\equiv M[f]$.

2 Lemmas

In this section we present some Lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1. [8] *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{i=0}^{n} a_i f^i}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_i\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 2.2. ([1]) *For any non-constant meromorphic function f,*

$$N\left(r, \infty; \frac{M}{f^{d_M}}\right) \leq d_M N(r, 0; f) + \lambda \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.3. ([1]) *For the differential monomial $M[f]$,*

$$N_p(r, 0; M[f]) \leq d_M N_{p+k}(r, 0; f) + \lambda \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.4. *Let f be a non-constant meromorphic function and $a(z)$ be a small function in f. Let us define $F = \dfrac{\mathcal{P}(f)}{a} = \dfrac{f_1^p P(f_1)}{a}$ and $G = \dfrac{M[f]}{a}$. Then $FG \not\equiv 1$.*

Proof. On contrary suppose $FG \equiv 1$ i.e

$$f_1^p P(f_1) M[f] = a^2.$$

From above it is clear that the function f can't have any zeros and poles. Therefore

$$\overline{N}(r, 0; f) = S(r, f) = \overline{N}(r, \infty; f).$$

So by the First Fundamental Theorem and Lemma 2.1, we have

$$\begin{aligned}(m+n+d_M)T(r,f) &= T\left(r,\frac{a^2}{f_1^pP(f_1)f^{d_M}}\right)+S(r,f)\\ &\le T\left(r,\frac{M[f]}{f^{d_M}}\right)+S(r,f)\\ &\le m\left(r,\frac{M[f]}{f^{d_M}}\right)+N\left(r,\frac{M[f]}{f^{d_M}}\right)+S(r,f)\\ &\le N\left(r,\frac{M[f]}{f^{d_M}}\right)+S(r,f).\end{aligned}$$

Then using Lemma 2.2 and from above inequality, we get

$$(m+n+d_M)T(r,f)\le d_MN(r,0;f)+\lambda\overline{N}(r,f)+S(r,f)\le S(r,f),$$

which is not possible. $\square$

Lemma 2.5. ([2]) *Let F and G share $(1,l)$ and $\overline{N}(r,F)=\overline{N}(r,G)$ and $H\not\equiv 0$, where F, G and H are defined as earlier. Then*

$$\begin{aligned}&N(r,\infty;H)\\ \le\ &\overline{N}(r,\infty;F)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+\overline{N}_0(r,0;F')\\ &+\overline{N}_0(r,0;G')+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+S(r,f).\end{aligned}$$

Lemma 2.6. ([1]) *Let F and G share $(1,l)$. Then*

$$\overline{N}_L(r,1;F)\le\frac{1}{l+1}\overline{N}(r,\infty;F)+\frac{1}{l+1}\overline{N}(r,0;F)+S(r,F)\quad \textit{if}\ l\ge 1,$$

and

$$\overline{N}_L(r,1;F)\le\overline{N}(r,\infty;F)+\overline{N}(r,0;F)+S(r,F)\quad \textit{if}\ l=0.$$

Lemma 2.7. ([1]) *Let F and G share $(1,l)$ and $H\not\equiv 0$. Then*

$$\begin{aligned}&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\ \le\ &N(r,\infty;H)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\ &+\overline{N}(r,1;G)+S(r,f).\end{aligned}$$

The following lemma can be proved in the line of proof of Lemma 2.14 [1].

Lemma 2.8. *Let f be a non-constant meromorphic function and $a(z)$ be a small function of f. Let $F=\dfrac{\mathcal{P}(f)}{a}=\dfrac{f_1^pP(f_1)}{a}$ and $G=\dfrac{M[f]}{a}$ such that F and G shares $(1,\infty)$. Then one of the following cases holds:*

1. $T(r) \leq N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_L(r,\infty;F)$
$+\overline{N}_L(r,\infty;G)+S(r)$,

2. $F \equiv G$,

3. $FG \equiv 1$.

where $T(r)=\max\{T(r,F),T(r,G)\}$ *and* $S(r)=o(T(r))$, $r\in I$, I *is a set of infinite linear measure of* $r\in(0,\infty)$.

3 Proof of the theorem

Proof of Theorem 1.1. Let $F=\dfrac{\mathcal{P}(f)}{a}=\dfrac{f_1^pP(f_1)}{a}$ and $G=\dfrac{M[f]}{a}$. Then $F-1=\dfrac{f_1^pP(f_1)-a}{a}$ and $G-1=\dfrac{M[f]-a}{a}$. Since $\mathcal{P}(f)$ and $M[f]$ share (a,l), it follows that F and G share $(1,l)$ except the zeros and poles of $a(z)$. Now we consider the following cases.
Case 1 Let $H\not\equiv 0$.
Subcase 1.1. Let $l\geq 1$.
Using the Second Fundamental Theorem and Lemmas 2.7, 2.5 we get

$$\begin{aligned}
& T(r,F)+T(r,G)\\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,1;F)\\
& +\overline{N}(r,1;G)-\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+N(r,H)\\
& +\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)\\
& -\overline{N}_0(r,0;F^{'})-\overline{N}_0(r,0;G^{'})+S(r,f)\\
\leq\ & 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)+\overline{N}_E^{(2}(r,1;F)\\
& +2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,f).
\end{aligned}$$

Subsubcase 1.1.1. Suppose $l = 1$.
Then from the above inequality and using Lemmas 2.6, 2.3 we get

$$\begin{aligned}
& T(r,F) + T(r,G) \\
\leq\ & 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G) + \overline{N}_E^{(2}(r,1;F) \\
& +2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,f) \\
\leq\ & 3\overline{N}(r,\infty;F) + N_2(r,0;f_1^pP(f_1)) + N_2(r,0;M[f]) + \frac{1}{2}\overline{N}(r,\infty;F) \\
& +\frac{1}{2}\overline{N}(r,0;F) + N(r,1;G) + S(r,f) \\
\leq\ & 3\overline{N}(r,\infty;f) + \mu_2 N_{\mu_2^*}(r,w_p;f) + (m+n-p)T(r,f) + d_M N_{k+2}(r,0;f) \\
& +\lambda\overline{N}(r,\infty;f) + \frac{1}{2}\overline{N}(r,\infty;F) + \frac{1}{2}\overline{N}(r,0;F) + N(r,1;G) + S(r,f) \\
\leq\ & \left(\frac{7}{2}+\lambda\right)\overline{N}(r,\infty;f) + \mu_2 N_{\mu_2^*}(r,w_p;f) + (m+n-p)T(r,f) \\
& +d_M N_{k+2}(r,0;f) + \frac{1}{2}\overline{N}(r,w_p;f) + \frac{1}{2}(m+n-p)T(r,f) + N(r,1;G) \\
& +S(r,f). \\
\leq\ & \left(\frac{7}{2}+\lambda\right)\overline{N}(r,\infty;f) + \mu_2 N_{\mu_2^*}(r,w_p;f) + \frac{3}{2}(m+n-p)T(r,f) \\
& +d_M N_{k+2}(r,0;f) + \frac{1}{2}\overline{N}(r,w_p;f) + T(r,G) + S(r,f).
\end{aligned}$$

i.e., in view of Lemma 2.1, for any given $\varepsilon > 0$

$$\begin{aligned}
& (m+n)T(r,f) \\
\leq\ & \left(\frac{7}{2}+\lambda\right)\overline{N}(r,\infty;f) + \mu_2 N_{\mu_2^*}(r,w_p;f) + \frac{3}{2}(m+n-p)T(r,f) \\
& +d_M N_{k+2}(r,0;f) + \frac{1}{2}\overline{N}(r,w_p;f) + S(r,f). \\
\leq\ & \left\{\left(\frac{7}{2}+\lambda\right) - \left(\frac{7}{2}+\lambda\right)\Theta(\infty,f) + \frac{1}{2} - \frac{1}{2}\Theta(w_p,f) + \mu_2 - \mu_2\delta_{\mu_2^*}(w_p,f)\right. \\
& \left. +d_M - d_M\delta_{2+k}(0,f) + \frac{3}{2}(m+n-p) + \varepsilon\right\}T(r,f) + S(r,f).
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \left\{\left(\frac{7}{2}+\lambda\right)\Theta(\infty,f) + \frac{1}{2}\Theta(w_p,f) + \mu_2\delta_{\mu_2^*}(w_p,f) + d_M\delta_{2+k}(0,f) - \varepsilon\right\}T(r,f) \\
\leq\ & \left(\Gamma_M + \mu_2 + 4 + \frac{(m+n)-3p}{2}\right)T(r,f) + S(r,f),
\end{aligned}$$

which is a contradiction.
Subsubcase 1.1.2. Suppose $l \geq 2$.
Here by using Lemma 2.3, we obtained

$$\begin{aligned}
& T(r,F)+T(r,G) \\
\leq\ & 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)+\overline{N}_E^{(2}(r,1;F) \\
& +2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,f) \\
\leq\ & 3\overline{N}(r,\infty;f)+\mu_2 N_{\mu_2^*}(r,w_p;f)+(m+n-p)T(r,f)+d_M N_{k+2}(r,0;f) \\
& +\lambda\overline{N}(r,\infty;f)+N(r,1;G)+S(r,f) \\
\leq\ & (3+\lambda)\overline{N}(r,\infty;f)+\mu_2 N_{\mu_2^*}(r,w_p;f)+(m+n-p)T(r,f) \\
+\ & d_M N_{k+2}(r,0;f)+T(r,G)+S(r,f).
\end{aligned}$$

So, in view of Lemma 2.1, for any given $\varepsilon > 0$ we have

$$\begin{aligned}
& (m+n)T(r,f) \\
\leq\ & (3+\lambda)\overline{N}(r,\infty;f)+\mu_2 N_{\mu_2^*}(r,w_p;f)+(m+n-p)T(r,f) \\
+\ & d_M N_{2+k}(r,0;f)+S(r,f) \\
\leq\ & \{(3+\lambda)-(3+\lambda)\Theta(\infty,f)+\mu_2-\mu_2\delta_{\mu_2^*}(w_p,f)+d_M-d_M\delta_{2+k}(0,f) \\
+\ & (m+n-p)+\varepsilon\}T(r,f)+S(r,f)
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \left\{(3+\lambda)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+d_M\delta_{2+k}(0,f)-\varepsilon\right\}T(r,f) \\
\leq\ & (\Gamma_M+\mu_2+3-p)T(r,f)+S(r,f),
\end{aligned}$$

which is a contradiction.
Subcase 1.2. Suppose $l = 0$.
Then in view of the Second Fundamental Theorem and Lemmas 2.7, 2.5 we get

$$\begin{aligned}
& T(r,F)+T(r,G) \\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;G) \\
& +\overline{N}(r,1;G)-\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G) \\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;G)+N(r,\infty;H) \\
& +\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G) \\
& -\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G) \\
\leq\ & 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)+\overline{N}_E^{(2}(r,1;F) \\
& +2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,f)
\end{aligned}$$

Again by Lemmas 2.6, 2.3 we get from above

$$\begin{aligned}
& T(r,F)+T(r,G) \\
\leq\ & 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\mu_2 N_{\mu_2^*}(r,w_p,f)+(m+n-p)T(r,f) \\
& +N_2(r,0;G)+2(\overline{N}(r,\infty;F)+\overline{N}(r,0;F))+\overline{N}(r,\infty;G)+\overline{N}(r,0;G) \\
& +\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,f) \\
\leq\ & 4\overline{N}(r,\infty;F)+\mu_2 N_{\mu_2^*}(r,w_p,f)+(m+n-p)T(r,f)+N_2(r,0;G) \\
& +2\overline{N}(r,\infty;G)+\overline{N}(r,0;G)+2\overline{N}(r,0;F)+T(r,G)+S(r,f)
\end{aligned}$$

i.e., for any given $\varepsilon>0$

$$\begin{aligned}
& (m+n)T(r,f) \\
\leq\ & (6+2\lambda)\overline{N}(r,\infty;f)+2\overline{N}(r,w_p;f)+2(m+n-p)T(r,f) \\
+\ & \mu_2 N_{\mu_2^*}(r,w_p,f)+(m+n-p)T(r,f)+d_M N_{1+k}(r,0;f) \\
+\ & d_M N_{2+k}(r,0;f)+S(r,f) \\
\leq\ & \Bigg\{(6+2\lambda)-(6+2\lambda)\Theta(\infty,f)+2-2\Theta(w_p,f)+\mu_2-\mu_2\delta_{\mu_2^*}(w_p,f) \\
& +2d_M-d_M\left(\sum_{i=1}^{2}\delta_{k+i}(0,f)\right)+3(m+n-p)+\varepsilon\Bigg\}T(r,f)+S(r,f).
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \Bigg\{(6+2\lambda)\Theta(\infty,f)+2\Theta(w_p,f)+\mu_2\delta_{\mu_2^*}(w_p,f) \\
& +d_M\left(\sum_{i=1}^{2}\delta_{k+i}(0,f)\right)-\varepsilon\Bigg\}T(r,f) \\
\leq\ & (2\Gamma_M+\mu_2+8+2(m+n)-3p)T(r,f)+S(r,f),
\end{aligned}$$

which is a contradiction.
Case 2. Let $H\equiv 0$.
On Integration we get,

$$\frac{1}{G-1}\equiv\frac{A}{F-1}+B, \tag{3.1}$$

where $A(\neq 0), B$ are complex constants.

It is clear that F and G share $(1,\infty)$. Also by construction of F and G we see that F and G share $(\infty,0)$ also.

So using Lemma 2.3 and condition (1.9), we obtain

$$\begin{aligned}
& N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_L(r,\infty;F)\\
+ \;& \overline{N}_L(r,\infty;G)+S(r)\\
\leq \;& \mu_2 N_{\mu_2^*}(r,w_p;f)+N_2(r,0;P(f))+d_M N_{2+k}(r,0;f)+\lambda\overline{N}(r,\infty;f)\\
+ \;& 3\overline{N}(r,\infty;f)+S(r)\\
\leq \;& \mu_2 N_{\mu_2^*}(r,w_p;f)+(m+n-p)T(r,f)+d_M N_{2+k}(r,0;f)\\
+ \;& (3+\lambda)\overline{N}(r,\infty;f)+S(r)\\
\leq \;& \{(3+\lambda+d_M+\mu_2+m+n-p)-((\lambda+3)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(w_p,f)\\
+ \;& d_M\delta_{2+k}(0,f))\\
& T(r,f)+S(r)\\
\leq \;& \{(3+\Gamma_M+\mu_2+m+n-p)-(3+\Gamma_M+\mu_2-p)\}T(r,f)+S(r)\\
\leq \;& (m+n)T(r,f)+S(r)\\
< \;& T(r,F)+S(r).
\end{aligned}$$

Hence inequality (1) of Lemma 2.8 does not hold. Again in view of Lemma 2.4, we get $FG \not\equiv 1$. Therefore $F \equiv G$ i.e, $\mathcal{P}(f) \equiv M[f]$. □

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