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Uniqueness of Cournot Equilibrium: New Results from Old Methods

Gerard Gaudet
Stephen Salant

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DEPARTMENT OF ECONOMICS
University of Michigan
Ann Arbor, Michigan 48109

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by

Gérard Gaudet
Département d'économique
Université Laval
Québec, Québec G1K 7P4

and

Stephen W. Salant
Department of Economics
University of Michigan
Ann Arbor, Michigan 48109

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ABSTRACT

This paper provides a proof of a condition for uniqueness of Cournot equilibrium. Existing proofs of the same condition have shown it to imply a unique element within a limited class of Cournot equilibria, but leave open the possibility of other pure-strategy equilibria outside this class. A simpler approach permits us to derive the condition and to rule out the possibility of these other equilibria. The approach used also provides new insight into the conditions for existence of Cournot equilibrium.

Consider an industry composed of N firms producing a homogeneous good. Firm i produces the good in quantity $q_i \geq 0$. Its cost function is $C_i(q_i)$, defined for all $q_i \geq 0$. The inverse demand function for the good is $p(Q)$, where $Q = \sum_{i=1}^{N} q_i$ is the total industry output.

Assume that:

A.1 There exists a $\xi \in (0, \infty)$ such that $p(Q) > 0$ for $Q \in [0, \xi)$ and $p(Q) = 0$ for $Q \geq \xi$.

A.2 $p(Q)$ is twice continuously differentiable and $p'(Q) \leq 0$ for $Q \in [0, \xi)$.

A.3 $C_i(q_i)$ is twice continuously differentiable and, for any $q_i > 0$, $C_i(q_i) > C_i(0)$, $i = 1,\ldots,N$.

A.4 For all $Q \in [0, \xi)$ and $i = 1,\ldots,N$, there exists some $\alpha < 0$ (possibly dependent on Q and i) such that $p'(Q) - C_i''(q_i) \leq \alpha < 0$.

For any cost function satisfying A.3 and A.4, define the extended function, $\Gamma_i(x_i)$, as follows:

$$\Gamma_i(x_i) = \begin{cases} C_i(x_i) & \text{if } x_i \geq 0 \\ C_i(0) + C_i'(0)x_i + \frac{1}{2}C_i''(0)x_i^2 & \text{if } x_i < 0 \end{cases}$$

Note that $\Gamma_i(x_i)$ is twice continuously differentiable for any $x_i \in (-\infty, \infty)$. Moreover, since $\Gamma_i''(x_i) = C_i''(0)$ for $x_i < 0$ and A.4 holds for $q_i \geq 0$, then for all $Q \in [0, \xi)$ and $x_i \in (-\infty, \infty)$, $i = 1,\ldots,N$, there exists some $\alpha < 0$ (possibly dependent on Q and i) such that $p'(Q) - \Gamma_i''(x_i) \leq \alpha$. Thus the appropriately modified A.3 and A.4 must hold for the extended function.

Define $g_i(x_i, Z) = p(Z) + x_i p'(Z) - \Gamma_i'(x_i)$. Then Cournot equilibria must satisfy:

$$g_i(q_i,Z) \le 0, \quad q_i \ge 0, \quad g_i(q_i,Z)q_i = 0, \quad i = 1,\ldots,N \tag{1}$$

and

$$Z = Q \tag{2}$$

It is straightforward to verify that there exists at least one solution to (1) and (2). Consider first condition (1). By A.1 through A.3, the function $g_i(x_i,Z)$ has continuous partial derivatives for all $Z \in [0, \xi)$ and $x_i \in (-\infty, \infty)$. By A.4, its partial derivative with respect to x_i is negative, bounded away from zero. There must therefore exist a unique $x_i(Z)$ such that $g_i(x_i(Z), Z) = 0$. This implicit solution being unique, it must be continuous at all $Z \in [0, \xi)$. For suppose it is discontinuous at some $Z^0 \in [0, \xi)$. Then, in the neighborhood of Z^0, there exists no continuous solution to $g_i(x_i, Z) = 0$ since $x_i(Z^0)$ is unique. But this violates the implicit function theorem at that point. It follows that condition (1) has a unique solution for all $Z \in [0, \xi)$, given by $q_i(Z) = \text{MAX}[0, x_i(Z)]$.

Now since $q_i(Z)$ is a continuous function of Z, so is $Q(Z) = \sum_{i=1}^{N} q_i(Z)$. We also know that $Q(0) \ge 0$, since $q_i(0) \ge 0$ for all $i = 1,\ldots,N$. Furthermore, $Q(Z) = 0$ for sufficiently large Z since, by A.1, $p(Z) = 0$ for $Z \ge \xi$ and hence, by A.3, $q_i(Z) = 0$. There must therefore exist a Z^E which solves $Q(Z^E) = Z^E$ and the corresponding $q_i^E(Z^E)$, $i = 1,\ldots,N$, constitutes a solution to (1) and (2).

We now provide a condition for the solution of (1) and (2) to be unique:

<u>Theorem</u>: Assume that A.1 through A.4 hold. Then, if (only if) at all q_i^E, i = 1,...,N, we have:

$$\sum_{i \in M(Q^E)} \frac{p'(Q^E) + q_i^E p''(Q^E)}{C_i''(q_i^E) - p'(Q^E)} < 1 \quad (\leq 1) \tag{3}$$

where $M(Q^E) = \{i \mid q_i(Q^E) > 0\}$, there exists exactly one solution to (1) and (2).

<u>Proof</u>: Let $q_i'(Z)^-$ and $q_i'(Z)^+$ denote the left-hand and right-hand derivatives of $q_i(Z)$. The initial step of the proof is to establish that:

(i) $\quad q_i'(Z)^+ \geq q_i'(Z)^-$

(ii) $\quad q_i'(Z)^+ = \begin{cases} 0 & \text{if } q_i(Z) = 0 \\ x_i'(Z) & \text{if } q_i(Z) > 0 \end{cases}$

Since $x_i(Z)$ is unique for all $Z \in [0, \xi)$ and given A.4, we can again invoke the implicit function theorem to show that $x_i(Z)$ has a continuous derivative for $Z \in [0, \xi)$ and it is given by:

$$x_i'(Z) = \frac{p'(Z) + x_i(Z)p''(Z)}{C_i''(x_i(Z)) - p'(Z)}$$

If $x_i(Z) > 0$ then $q_i(Z) = x_i(Z) > 0$ and $q_i'(Z)^+ = q_i'(Z)^- = x_i'(Z)$. Thus (i) and (ii) both hold. If instead $x_i(Z') < 0$ for some $Z' \in [0, \xi)$ then $q_i(Z) = 0$ at Z' and in a neighborhood of Z', since $x_i(Z)$ is continuous at Z' and hence $x_i(Z) < 0$ in some neighborhood of Z'. Therefore $q_i'(Z')^+ = q_i'(Z')^- = 0$ and (i) and (ii) hold. Finally, if $x_i(Z') = 0$ for some $Z' \in [0, \xi)$ then $q_i(Z') = x_i(Z') = 0$. By continuity and nonnegativity of $q_i(Z)$, $q_i'(Z')^- \leq 0$ and $q_i'(Z')^+ \geq 0$. But A.2 and A.4 imply $x_i'(Z) \leq 0$ when $x_i(Z) = 0$; $q_i'(Z')^+ > 0$ is therefore impossible.

Hence $q_i'(Z')^- \leq q_i'(Z')^+ = 0$ and again (i) and (ii) hold.

Now let $Q'(Z)^+$ denote the right-hand derivative of Q(Z). Then $Q'(Z)^+ \equiv \sum_{i=1}^{N} q_i'(Z)^+ = \sum_{i\in M(Z)} q_i'(Z)^+ = \sum_{i\in M(Z)} x_i'(Z)$. The first inequality follows from the definition of Q(Z), while the second and third follow from (ii). Moreover, (i) implies that

$Q'(Z)^- \leq Q'(Z)^+$, and an upper bound on $Q'(Z)^+$ also bounds $Q'(Z)^-$. It follows that if (only if) we have $Q'(Z^E)^+ < 1$ (≤ 1), there must be only one Z^E (= Q^E) and hence only one $q_i^E(Z^E)$, i = 1,...,N. But:

$$Q'(Z^E)^+ = \sum_{i\in M(Z^E)} x_i'(Z^E) = \sum_{i\in M(Z^E)} \frac{p'(Z^E) + q_i(Z^E)p''(Z^E)}{C_i''(q_i(Z^E)) - p'(Z^E)}$$

and the condition stated in the theorem follows directly.[1] ■

Since any Cournot equilibrium must satisfy (1) and (2), A.1 through A.4 and (3) therefore insure that there exists at most one Cournot equilibrium. In addition, they identify the solution to (1) and (2) as the only candidate for the Cournot equilibrium.[2] Uniqueness follows if, for independent reasons, a Cournot equilibrium is known to exist. This could be insured, for example, by further assuming that the profit function, $p(q_i+Y)q_i - C_i(q_i)$, is pseudoconcave in q_i for i = 1,...,N and any $Y \in [0, \xi - q_i^E)$. Conditions (1) and (2) then become both necessary and sufficient for Cournot equilibrium.

Notice that if we require, in addition to A.1 through A.4, that a firm's marginal revenue be a non-increasing function of the output of its rivals, so that:

A.5 $p'(Q) + q_i p''(Q) \leq 0$ for all $Q \in [0, \xi)$, $q_i \leq Q$, $i = 1,\ldots,N$

then condition (3) holds with strict inequality. Assumptions A.4 and A.5 combined also insure that each firm's profit function is strictly concave in its own output. Thus A.1 through A.5 insure existence of a unique Cournot equilibrium.

Assumption A.5 is frequently involved in discussions of existence of Cournot equilibrium (see in particular Novshek, 1985 and Shapiro, 1988). Given that $p'(Q) \leq 0$ by A.2, it in fact is equivalent to the Novshek assumption that $p'(Q) + Qp''(Q) \leq 0$ for $Q \in [0, \xi)$. However, this assumption is unnecessarily strong for existence of equilibrium, given A.1 through A.4. For, any solution to (1) and (2) is a Cournot equilibrium if and only if q_i^E yields a global maximum of $p(q_i + Q^E_{-i})q_i - C_i(q_i)$, where $Q^E_{-i} = Q^E - q_i$ and $i = 1,\ldots,N$. As long as this condition holds at some solution to (1) and (2), existence of a Cournot equilibrium is assured. It is unnecessary for existence of Cournot equilibrium to assume that this condition holds at every solution to (1) and (2). Moreover, at the designated solution to (1) and (2), it is unnecessary that q_i^E be globally optimal in response to every aggregate output of the firms, but merely to Q^E_{-i}. Finally, it is unnecessary that each firm's profit function be pseudoconcave, much less everywhere strictly concave.

The uniqueness condition (3) is the same as that derived by Kolstad and Mathiesen (1987) (equation (16), Corollary 3.1, p. 687) and by Kolstad (1988) (equation (5), Theorem 2, p. 4). The approach taken here is however much simpler and the proof much shorter than theirs. Moreover, their papers do not in fact establish conditions for uniqueness of Cournot equilibrium. They only establish conditions for the uniqueness of one class of Cournot equilibria, which they label "nondegenerate". In neither of those papers can the authors rule out the existence of one or more other Cournot equilibria where some of the N firms are just at the margin of becoming active or not, i.e., equilibria where $q_i = 0$ and $g_i(q_i, Q) = 0$ for some or all $i \notin M(Q)$ (see Definition 2, p. 683 in Kolstad and Mathiesen, 1987, and pp. 2-3 in Kolstad, 1988). Hence, our results fill an important gap.

Our approach is closest to that of Szidarovszky and Yakowitz (1977). They use it to show the existence of a unique Cournot equilibrium when marginal cost is increasing and inverse demand is downward sloping and concave. Their assumptions imply A.1 through A.5, but are, of course, unnecessarily strong.

NOTES

1. Given A.1 through A.4 and (3), we will necessarily have $Z^E \in [0, \xi]$ and $Q(Z) - Z \gtreqless 0$ for all $Z \lesseqgtr Z^E$, and the method of interval bisection (see for example Conrad and Clark, 1987, p. 40) would in such a case rapidly converge to the unique solution to (1) and (2).

2. Not every solution to (1) and (2) need be a Cournot equilibrium. Although (1) and (2) insure that each firm selects a point in reply to the outputs of the other firms which satisfies the necessary conditions for a local maximum of its profit function, this need not be a local best reply, much less a global best reply, for every firm.

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