# *Uniqueness of ground states for quasilinear elliptic equations in the exponential case*

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We consider ground states of the quasilinear equation

$$\text{div}(A(|Du|)Du) + f(u) = 0 \qquad \text{in } \mathbb{R}^n, \quad n \geq 2, \tag{1.1}$$

that is, $C^1$ solutions of (1.1) such that

$$u \geq 0, \quad u \not\equiv 0; \qquad u(x) \to 0 \quad \text{as } |x| \to \infty. \tag{1.2}$$

Two particular model operators $A$ motivate this work, first

$$A(t) \equiv 1, \qquad n = 2, \tag{1.3}$$

the Laplace case, and second

$$A(t) = t^{m-2}, \qquad t > 0, \qquad m \geq n, \tag{1.4}$$

the case of the degenerate Laplacian operator. When $n = m$ it is well known that radially symmetric ground states can exist in the case (1.4), even for functions $f(u)$ with exponential growth as $u \to \infty$, that is

$$f(u) = O(e^{\beta u^{n/(n-1)}}) \qquad \text{as } u \to \infty,$$

where $\beta$ is an appropriate positive constant, see [1, 2, 4, 6, 7].

The question of uniqueness of ground states for (1.1) when $f(u)$ has such exponential behavior has not previously been treated; the purpose of this paper is to make a beginning on this problem.

Appropriate assumptions on the operator $A$ will be given in the next section. On the other hand, for the nonlinearity $f$ we consider the specific family of functions

$$f(u) = -\alpha u + (e^u - 1 - u), \qquad \alpha > 0, \tag{1.5}$$

or, more generally,

$$f(u) = u^{p-1}\{-\alpha u^p + (e^{u^p} - 1 - u^p)\}, \qquad \alpha > 0, \qquad \tfrac{1}{2} < p \leq 1, \tag{1.6}$$

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of which (1.5) is the special case $p = 1$. For the family (1.6) it is not hard to show (see Section 2) that $f$ is in $C[0,\infty) \cap C^1(0,\infty)$ and that there exists $a > 0$ such that $f(a) = 0$ and

$$f(u) < 0 \quad \text{for } 0 < u < a, \qquad f(u) > 0 \quad \text{for } a < u < \infty.$$

In consequence we are able to apply the principal result of [9] to obtain uniqueness of the corresponding radial ground states of (1.1). Our main conclusions will be given in the next section.

In recent work, e.g. [8, 10], the question whether all ground states of (1.1) are radially symmetric has been considered. In particular, when the operator $A$ is regular on the entire interval $[0,\infty)$, as for example for the Laplacian case (1.3), it is known for wide classes of functions $f$, including the family (1.6), that ground states are radially symmetric with respect to an appropriate origin $O \in \mathbb{R}^n$, see [10]. In such cases the a priori assumption of radial symmetry is then automatically satisfied, and we find for $n = 2$ that ground states of (1.1), (1.6) are unique up to translations, see Theorem 2.

## §2. The main result.

We consider the quasilinear equation (1.1), in which the operator $A$ is of class $C^1(0,\infty)$. Let

$$\Omega(t) = tA(t), \qquad G(t) = \int_0^t \Omega(s)ds, \qquad t > 0,$$

and suppose that the following specific conditions are satisfied:

(1) $\Omega'(t) > 0$ for $t > 0$, $\quad \Omega(t) \to 0$ as $t \to 0$,
(2) $t\Omega(t)/G(t)$ is (non–strictly) increasing in $(0,\infty)$;

see [9, Section 1].

We define the critical constant

$$m = \lim_{t\to 0} \frac{t\Omega(t)}{G(t)}. \tag{2.1}$$

It is clear that $m \geq 1$, since $G(t) < t\Omega(t)$ for $t > 0$ by (1).

The operator $A$ in (1.4) obviously satisfies conditions (1), (2), where the exponent $m$ there coincides with the constant $m$ given in (2.1).

**Theorem 1.** *Suppose that the operator $A$ obeys conditions* (1), (2)*, and let the critical constant $m$ satisfy*

$$m \geq n.$$

*Assume that $f$ in* (1.1) *is of the form* (1.6)*. Then radially symmetric ground states of* (1.1) *are unique up to translations.*

*Proof.* By [9, Theorem 1 and Lemma 2.1] it is enough to prove for the family (1.6) that

$$\frac{d}{du}\left[\frac{F(u)}{f(u)}\right] \geq 0 \qquad \text{for } u > 0, \quad u \neq a, \tag{2.2}$$

where

$$F(u) = \int_0^u f(s)ds = \frac{1}{p}\left\{e^{u^p} - 1 - u^p - \frac{(\alpha+1)}{2}u^{2p}\right\}$$

and $a$ is the unique positive zero of $f$.

In fact, to see that $f$ has exactly one positive zero $a$, we note that, in the new variable $v = u^p$,

$$f = f(u(v)) = v^{(p-1)/p}[e^v - 1 - (\alpha+1)v].$$

It is thus enough to show that $e^v - 1 - (\alpha+1)v$ has only one zero in $(0, \infty)$, which follows by trivial calculus. Indeed one finds $a > [\log(\alpha+1)]^{1/p}$.

Since

$$\left[\frac{F}{f}\right]' = \frac{f^2 - f'F}{f^2}, \qquad ' = \frac{d}{du},$$

to show (2.2) it is enough to verify that

$$f^2 - f'F \geq 0 \qquad \text{for } u > 0. \tag{2.3}$$

Again in the $v$ variable, we find

$$f^2 - f'F = \frac{1}{p}v^{(p-2)/p}[(1-p)(e^v-1)^2 + v(e^v-1)\{\mathbb{A} + \mathbb{B}v + \mathbb{C}v^2\} + v^2\{\mathbb{D} + \mathbb{E}v\}],$$

where

$$\mathbb{A} = (2p-1)(\alpha+1) - 1, \qquad \mathbb{B} = -\tfrac{1}{2}(3p+1)\alpha - \tfrac{1}{2}(p+1),$$
$$\mathbb{C} = \tfrac{1}{2}p(\alpha+1), \qquad \mathbb{D} = (1-2p)\alpha + 1 - p, \qquad \mathbb{E} = \tfrac{1}{2}(\alpha+1)(\alpha+p+1).$$

Let $\varphi$ be the function in brackets in the previous formula, namely

$$\varphi(v, \alpha\,; p) = (1-p)(e^v-1)^2 + v(e^v-1)\{\mathbb{A} + \mathbb{B}v + \mathbb{C}v^2\} + v^2\{\mathbb{D} + \mathbb{E}v\}.$$

The required condition (2.3) then follows if we show that

$$\varphi(v, \alpha\,; p) > 0 \quad \text{for } v, \alpha \in (0, \infty) \text{ and } p \in [\tfrac{1}{2}, 1]. \tag{2.4}$$

Note moreover that $\varphi(v, \alpha\,; p)$ is *linear* in $p$, so that it is enough to prove (2.4) at the endpoints $p = 1/2$ and $p = 1$.

*Case 1.* $p = 1/2$. Here

$$\begin{aligned}&\varphi(v,\alpha\,;1/2) = \tfrac{1}{2}(e^v-1)^2 + v(e^v-1)\{\mathbb{A}+\mathbb{B}v+\mathbb{C}v^2\} + v^2\{\mathbb{D}+\mathbb{E}v\},\\ &\mathbb{A}=-1,\qquad \mathbb{B}=-\frac{1}{4}(5\alpha+3),\qquad \mathbb{C}=\frac{1}{4}(\alpha+1),\\ &\mathbb{D}=\frac{1}{2},\qquad \mathbb{E}=\frac{1}{4}(2\alpha^2+5\alpha+3).\end{aligned} \tag{2.5}$$

We first show

$$\varphi(v,\alpha\,;1/2) > 0 \quad \text{for } v \geq 5; \qquad \varphi(0,\alpha\,;1/2) = 0 \text{ for } \alpha > 0. \tag{2.6}$$

Indeed

$$\mathbb{A}+\mathbb{B}v+\mathbb{C}v^2 = \begin{cases} -1 & \text{when } v=0 \\ 3/2 & \text{when } v=5 \end{cases}.$$

Therefore, since $\mathbb{A}+\mathbb{B}v+\mathbb{C}v^2$ is quadratic in $v$, with $\mathbb{C}>0$, we get $\mathbb{A}+\mathbb{B}v+\mathbb{C}v^2 \geq 3/2$ for $v \geq 5$, $\alpha > 0$. Now (2.6) follows from (2.5).

Next, observe that

$$\tfrac{1}{2}(e^v-1)^2 + v(e^v-1)\mathbb{A} + \mathbb{D}v^2 = \tfrac{1}{2}(e^v-1-v)^2 > 0.$$

Therefore, since $\mathbb{B}$ is linear in $\alpha$ and $\mathbb{E}$ quadratic, and also $e^v - 1 \leq \frac{1}{5}(e^5-1)v$ for $v \in [0,5]$, it is clear from (2.5) that there is $\alpha_1 > 0$ such that

$$\varphi(v,\alpha\,;1/2) > 0 \quad \text{if } 0 < v \leq 5 \text{ and } \alpha \geq \alpha_1. \tag{2.7}$$

Also, by a straightforward calculation we obtain

$$\begin{aligned}\varphi(v,0\,;1/2) &= \frac{1}{8}v^5 + \left(e^v - 1 - v - \frac{v^2}{2}\right)\\ &\qquad\cdot\left\{-\frac{1}{4}v^2 + \frac{1}{3}v^3 + \frac{1}{2}(e^v - 1 - v - \frac{v^2}{2} - \frac{v^3}{6})\right\}.\\ &= \frac{1}{12}v^5 + \text{Positive function (of order } v^6 \text{ at } v=0).\end{aligned}$$

Therefore

$$\varphi(v,0\,;1/2) > 0 \qquad \text{for all } v > 0. \tag{2.8}$$

From (2.6)–(2.8) it follows that if $\varphi(v,\alpha\,;1/2)$ takes negative values in $(0,\infty)\times(0,\infty)$, then it attains a *negative minimum* at some point $(v_0,\alpha_0)$, with $0 < v_0 < 5$ and $0 < \alpha_0 < \alpha_1$. Moreover, by direct calculation we find

$$\begin{aligned}\frac{\partial\varphi}{\partial v}(v,\alpha\,;1/2) &= (e^v-1)^2 + v(e^v-1)\{(\mathbb{A}+2\mathbb{B}) + (\mathbb{B}+3\mathbb{C})v + \mathbb{C}v^2\}\\ &\qquad + (\mathbb{B}+3\mathbb{E})v^2 + \mathbb{C}v^3.\end{aligned} \tag{2.9}$$

$$\frac{\partial\varphi}{\partial\alpha}(v,\alpha\,;1/2) = \frac{1}{4}v^2\{(e^v-1)(v-5) + (4\alpha+5)v\}. \tag{2.10}$$

We claim that at the minimum point $(v_0, \alpha_0)$ there holds

$$\begin{aligned}\varphi(v_0, \alpha_0\,; 1/2) = \frac{v_0^2}{4(5-v_0)^2}\big[&32\alpha_0^2 - 2\alpha_0(25\alpha_0+22)v_0 \\ &+4(5\alpha_0^2+8\alpha_0+3)v_0^2 - 2(\alpha_0+1)^2 v_0^3\big].\end{aligned} \tag{2.11}$$

Indeed, since $(v_0, \alpha_0)$ is a critical point, by (2.10) and the fact that $0 < v_0 < 5$ there follows

$$e^{v_0} - 1 = \frac{4\alpha_0+5}{5-v_0}\, v_0. \tag{2.12}$$

The claim (2.11) now arises by eliminating $e^{v_0} - 1$ from the main formula (2.5).

By (2.9), we also have at $(v_0, \alpha_0)$

$$(e^{v_0}-1)^2 = -v_0(e^{v_0}-1)\{(\mathbb{A}+2\mathbb{B}) + (\mathbb{B}+3\mathbb{C})v_0 + \mathbb{C}v_0^2\} - (\mathbb{B}+3\mathbb{E})v_0^2 - \mathbb{C}v_0^3.$$

Eliminating $(e^{v_0}-1)^2$ from the main formula (2.5) and then using (2.12) once more, we get

$$\begin{aligned}\varphi(v_0, \alpha_0\,; 1/2) = \frac{v_0^2}{4}\,\big[&\big(70\alpha_0^2+108\alpha_0+75\big) - \big(28\alpha_0^2+44\alpha_0+23\big)\, v_0 \\ &+ \big(2\alpha_0^2+3\alpha_0+1\big)\, v_0^2\big]\,.\end{aligned} \tag{2.13}$$

Define

$$\psi(v,\alpha) = 70\alpha^2 + 108\alpha + 75 - \big(28\alpha^2+44\alpha+23\big)\, v + \big(2\alpha^2+3\alpha+1\big)\, v^2.$$

Clearly

$$\begin{aligned}\psi(v,0) &= 75 - 23v + v^2 > 0 &&\text{for } v \in [0,\ 3.9],\\ \frac{\partial\psi}{\partial\alpha}(v,0) &= 108 - 44v + 3v^2 > 0 &&\text{for } v \in [0,\ 3.1],\\ \frac{\partial^2\psi}{\partial\alpha^2}(v,\alpha) &= 4[35 - 14v + v^2] > 0 &&\text{for } \ \in [0,\ 3.2] \text{ and all } \alpha > 0.\end{aligned}$$

Consequently, by integration with respect to $\alpha$ from 0 to any $\alpha > 0$, we get $\psi(v,\alpha) > 0$ on $[0,3]\times[0,\infty)$. But $\psi(v_0,\alpha_0) = 2v_0^{-2}\varphi(v_0,\alpha_0\,; 1/2) < 0$, so that

$$3 < v_0 < 5. \tag{2.14}$$

Next define

$$\omega(v,\alpha) = 32\alpha^2 - 2\alpha(25\alpha+22)v + 4(5\alpha^2+8\alpha+3)v^2 - 2(\alpha+1)^2v^3.$$

Then

$$\begin{aligned} \omega(v,0) &= 2v^2(6-v) > 0 && \text{for } v \in (0,6), \\ \frac{\partial\omega}{\partial\alpha}(v,0) &= -4v(11-8v+v^2) > 0 && \text{for } v \in [1.8, 6.2], \\ \frac{\partial^2\omega}{\partial\alpha^2}(v,\alpha) &= 4(1-v)(v^2-9v+16) > 0 && \text{for } v \in [2.5, 6.5] \text{ and all } \alpha > 0. \end{aligned}$$

Hence, by integration with respect to $\alpha$ from 0 to $\alpha > 0$, we get $\omega(v,\alpha) > 0$ on $[2.5, 6) \times (0,\infty)$. Since $\omega(v_0,\alpha_0) = 4(5-v_0)^2 v_0^{-2}\varphi(v_0,\alpha_0\,;1/2) < 0$ by (2.11) and since $v_0 < 5$, this implies

$$0 < v_0 < 2.5,$$

which contradicts (2.14). Therefore $\varphi(v,\alpha\,;1/2)$ cannot take negative values. This completes the proof of Case 1.

*Case 2.* $p = 1$. Here $v = u$ and

$$\tag{2.15} \begin{gathered} \varphi(v,\alpha\,;1) = v(e^v-1)\{\mathbb{A}+\mathbb{B}v+\mathbb{C}v^2\} + v^2\{\mathbb{D}+\mathbb{E}v\}, \\ \mathbb{A} = \alpha, \qquad \mathbb{B} = -2\alpha+1, \qquad \mathbb{C} = \tfrac{1}{2}(\alpha+1), \\ \mathbb{D} = -\alpha, \qquad \mathbb{E} = \tfrac{1}{2}(\alpha+1)(\alpha+2). \end{gathered}$$

By Taylor's expansion and use of the formula (2.15) for the coefficients, $\mathbb{A}, \ldots, \mathbb{E}$, one finds

$$\begin{aligned} \varphi(v,\alpha\,;1) &= \frac{1}{2}\alpha^2v^3 - \frac{1}{3}\alpha v^4 + \sum_{k=2}^{\infty} v^{k+3}\left\{\frac{\alpha}{(k+2)!} - \frac{2\alpha+1}{(k+1)!} + \frac{\alpha+1}{2k!}\right\} \\ &= \frac{1}{2}v^3\left\{\alpha^2 - \frac{2}{3}\alpha v + \sum_{k=2}^{\infty}\frac{v^k}{(k+2)!}[(k^2-k-4)\alpha + k^2+k-2]\right\}. \end{aligned}$$

For all $k \geq 3$ the coefficients $(k^2-k-4)\alpha + k^2+k-2$ are non–negative. Hence, dropping all terms with $k \geq 4$, we obtain

$$\tag{2.16} \varphi(v,\alpha\,;1) \geq \frac{1}{2}v^3\left\{\alpha^2 - \frac{2}{3}\alpha v + \frac{1}{12}(2-\alpha)v^2 + \frac{1}{60}(\alpha+5)v^3\right\}.$$

By (2.16), to prove the assertion it is enough to show that

$$\tag{2.17} 60\alpha^2 - 40\alpha v + 5(2-\alpha)v^2 + (\alpha+5)v^3 > 0 \qquad \text{on } (0,\infty)\times(0,\infty).$$

By Cauchy's inequality $40\alpha v \leq 20\alpha^2 + 20v^2$. We are thus led to consider the function

$$\tag{2.18} \psi(v,\alpha) = 40\alpha^2 - 5(\alpha+2)v^2 + (\alpha+5)v^3 \qquad v > 0, \quad \alpha > 0.$$

For fixed $\alpha$ the minimum of $\psi(\cdot, \alpha)$ is attained at

$$v_\alpha = \frac{10}{3}\,\frac{\alpha+2}{\alpha+5}.$$

Hence

$$\psi(v,\alpha) \geq \psi(v_\alpha,\alpha) = \frac{20}{27}\,\frac{1}{(\alpha+5)^2}\{54\alpha^2(\alpha+5)^2 - 25(\alpha+2)^3\}.$$

The expression in braces is the quartic function

$$Q(\alpha) = 54\alpha^4 + 515\alpha^3 + 1200\alpha^2 - 300\alpha - 200.$$

Observe that $Q(2/3) > 0$, $Q'(2/3) > 0$ and $Q''(\alpha) > 0$ on $(0,\infty)$. Consequently, $Q(\alpha) > 0$ for all $\alpha \geq 2/3$. It is now evident that $\psi(v,\cdot) > 0$ for $\alpha \geq 2/3$, and so (2.17) holds in the subset $(0,\infty) \times [2/3,\infty)$.

On the other hand,

$$60\alpha^2 - 40\alpha v + 5(2-\alpha)v^2 + (\alpha+5)v^3 \geq 5\{12\alpha^2 - 8\alpha v + (2-\alpha)v^2\}.$$

The quadratic function on the right side has minimum value

$$\frac{20\alpha^2}{2-\alpha}(2-3\alpha) > 0$$

for $\alpha < 2/3$. Hence (2.17) holds on the entire set $(0,\infty)\times(0,\infty)$, as required.

This completes the proof of Case 2, and in turn of the theorem.

It is surprising that the proof of Case 1 is so tricky. On the other hand, the function $\varphi(\cdot,\cdot\,;1/2)$ in (2.5) is very close to zero when $(v,\alpha)$ is in $[0,1]^2$. Indeed

$$\varphi(0.01,\,0.01\,;1/2) = 21\times10^{-12}, \quad \varphi(0.1,\,0.1\,;1/2) = 22\times10^{-7},$$
$$\varphi(1,1/2\,;1/2) = 0.039, \quad \varphi(1/2,1\,;1/2) = 0.040, \quad \varphi(1,1\,;1/2) = 0.181.$$

Instead of (1.6), another natural choice for the nonlinearity $f(u)$ in (1.1) would be

$$f(u) = -\alpha u + (e^{u^p} - 1 - u^p), \qquad \alpha > 0, \quad \frac{1}{2} < p \leq 1.$$

For this family, however, the verification of (2.2) does not seem to be obvious.

Since in Theorem 1 we treat radially symmetric ground states $u = u(r)$ of (1.1), it is clear that the dimension $n$ may be taken as any real number greater than 1. In fact, it is precisely this point of view which was used in Theorem 1 of [9]. With this interpretation the condition $m \geq n$ in Theorem 1 then allows values $m < 2$ when $n < 2$.

When the operator $A$ is smooth on the entire interval $[0,\infty)$, the result of Theorem 1 can be improved. In particular we have the following

**Theorem 2.** *Let $n = 2$. Suppose that $A$ is of class $C^{1,1}[0,\infty)$, that $\Omega'(t) > 0$ for $t \geq 0$, and that condition* (2) *is satisfied. Also let $f$ in* (1.1) *have the form* (1.6)*. Then ground states of* (1.1) *having connected support are radially symmetric, and unique up to translations.*

*When $p = 1$ in* (1.6)*, the condition of connected support can be omitted.*

*Proof.* First observe that $A$ satisfies conditions (1), (2). Moreover $\Omega(t) = t\Omega'(0) + o(t)$ as $t \to 0$, with $\Omega'(0) > 0$. Hence from the definition (2.1) and use of condition (2) we get $m = 2$.

The family (1.6) has the property that $f'(u) < 0$ for $0 < u < \delta$ and some $\delta > 0$; indeed, it is not hard to see that (using the variable $v = u^p$)

$$f'(u) \leq v^{2(p-1)/p} e^v - 1 - \frac{2p-1}{p}\alpha) < 0$$

for $v \in (0,\eta)$, $\eta = \log(1 + \frac{2p-1}{p}\alpha) > 0$ since $p > 1/2$. Thus we can take $\delta = \eta^{1/p}$. Hence $f$ is nonincreasing for $u$ near zero (note also that $f'(0) = -\infty$ when $1/2 < p < 1$), and by Theorem 1 of [10] any corresponding ground state $u$ of (1.1) having connected support is radially symmetric about some origin $O = O(u) \in \mathbb{R}^n$. The first part of Theorem 2 now follows from Theorem 1, since for $n = 2$ the required condition $m \geq n$ is fulfilled.

To obtain the second part of the theorem, note that when $p = 1$ the function $f$ in (1.6), see also (1.5), is uniformly Lipschitz continuous on $[0,\delta]$ for any $\delta > 0$. Hence by Corollary 1.1 of [10] any ground state is radially symmetric and the result follows from Theorem 1 above.

Similar results can be proved for singular operators $A$ such as the $m$–Laplacian. We omit the details since the required conditions are somewhat complicated to state (see in particular [5, 10]).

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