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UNIQUENESS OF HAAR SERIES WHICH ARE $(C, 1)$ SUMMABLE TO DENJOY INTEGRABLE FUNCTIONS

BY

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ABSTRACT. A Haar series $\Sigma\, a_k\chi_k$ satisfies *Condition H* if $a_k\chi_k/k \to 0$ uniformly as $k \to \infty$. We show that if such a series is $(C, 1)$ summable to a Denjoy integrable function f, except perhaps on a countable subset of $[0, 1]$, then that series must be the Denjoy-Haar Fourier series of f.

1. **Introduction.** The *Haar functions* $\chi_0, \chi_1, \cdots$ are a complete orthonormal system in the Hilbert space $L^2[0, 1]$. For the purposes of this paper we need only recall that χ_0 is identically 1 and that, given any positive integer $n = 2^m + k$, where $0 \le k < 2^m$, the corresponding Haar function χ_n takes on the value $+\sqrt{2^m}$ on the open interval

$$\Delta(1, n) \equiv (2k/2^{m+1}, [2k+1]/2^{m+1}), \tag{1}$$

and takes on the value $-\sqrt{2^m}$ on the open interval

$$\Delta(2, n) \equiv ([2k+1]/2^{m+1}, [2k+2]/2^{m+1}). \tag{2}$$

Furthermore, the support of that nth Haar function is precisely the closure of the union of intervals (1) and (2):

$$\operatorname{Supp}[\chi_n] = [k/2^m, [k+1]/2^m].$$

The *Denjoy integral* (see [5, pp. 84–85]), $(D)\int_a^b$, is more general than either Lebesgue's integral or the improper Riemann integral.

The *D-Haar Fourier series* of a Denjoy integrable function f is a Haar series $S(x) = \Sigma_{k=0}^\infty a_k\chi_k(x)$ which is related to f by the following formula:

$$a_k = (D)\int_0^1 f(x)\chi_k(x)\,dx. \tag{3}$$

If (3) holds and f is also Lebesgue integrable, then S is simply the *Haar Fourier series* of f.

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The harmonic analysis of Haar series is greatly simplified by the following property of Haar Fourier series:

> The Haar Fourier series of any Lebesgue integrable function f converges to f almost everywhere on $[0, 1]$.

The standard proof of this fact [1, pp. 47–50] uses only one consequence of the Lebesgue integrability of f: the derivative of an indefinite Lebesgue integral is equal to the integrand almost everywhere. Since this property is also shared by Denjoy's integral, that same proof will establish that

(4) *The D-Haar Fourier series of f converges to f a.e.*

2. **The uniqueness theorem.** In this section we state the uniqueness theorem proved in §4 and relate it to the research presented in [2] and [4].

A Haar series $\sum_{k=0}^{\infty} a_k\chi_k(x)$ satisfies *Condition G* if, given any $t_0 \in [0, 1]$,

$$\lim_{j\to\infty} a_{K_j}/\chi_{K_j}(t_0) = 0, \tag{5}$$

where $K_1, K_2, \cdots$ are all those indices p for which $\chi_p(t_0) \neq 0$. Lemma 3 of [2] shows that a D-Haar Fourier series always satisfies *Condition G*.

A Haar series $\sum_{k=0}^{\infty} a_k\chi_k(x)$ satisfies *Condition H* if

$$\lim_{n\to\infty} a_n\chi_n(x)/n = 0 \quad \text{uniformly for } x \in [0, 1]. \tag{6}$$

Condition G is equivalent to supposing that the limit in (6) exists pointwise, since if $\chi_p(t_0) \neq 0$ then $p \geq |\chi_p^2(t_0)| \geq p/8$. Thus *Condition H* can be viewed as the uniform analogue of *Condition G*.

F. G. Arutjunjan [2] has shown that if a Haar series S satisfying *Condition G* converges, except perhaps on a countable subset of $[0, 1]$, to a Denjoy integrable function f, then S must be the D-Haar Fourier series of f.

We shall denote the *nth partial* $(C, 1)$ *sum* of a Haar series $S(x) = \sum_{k=0}^{\infty} a_k\chi_k(x)$ by

$$\sigma_n(S; x) \equiv \frac{S_1(x) + \cdots + S_{n+1}(x)}{n+1} \equiv \sum_{k=0}^{n}\left(1 - \frac{k}{n+1}\right)a_k\chi_k(x) \tag{7}$$

where $S_{n+1}(x) = \sum_{k=0}^{n} a_k\chi_k(x)$. If the sequence displayed in (7) converges at x then S is said to be $(C, 1)$ summable at x. If S converges at x it is automatically $(C, 1)$ summable at x; the converse of this statement is false.

The question motivating this research was: Does Arutjunjan's result still hold if S is only $(C, 1)$ summable off that countable set?

If S also satisfies *Condition H* (loosely, if S satisfies *Condition G* uniformly) the answer to this question is yes and is obtained as a corollary to the following result, which is proved in §4.

Theorem. *Let*

$$S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x) \tag{8}$$

be a Haar series satisfying Condition H such that

$$\limsup_{n\to\infty} |\sigma_n(S; x)| < \infty \tag{9}$$

for all but countably many x's in $[0, 1]$. *Suppose further that f is a Denjoy integrable function such that*

$$\lim_{n\to\infty} \sigma_n(S; x) = f(x) \quad a.e. \ in \ [0, 1]. \tag{10}$$

Then S is the D-Haar Fourier series of f.

This result is already known in the case that f is Lebesgue integrable (see [4]), but the techniques used here to establish Lemma 4 in the next section are radically different and substantially more complex due to the fact that a function can be conditionally Denjoy integrable.

3. Fundamental lemmas.

Lemma 1. *Let S be a Haar series satisfying Condition G and $\Delta(i_0, k_0)$ be an interval of the form* (1) *or* (2) *such that*

$$S_{k_0+1}(x) \not\equiv 0 \quad for \ x \in \Delta(i_0, k_0). \tag{12}$$

Then given any $t_0 \in [0, 1]$ there is an interval $\Delta(i_0', k_0')$ of the form (1) *or* (2) *such that t_0 does not lie in the closure of $\Delta(i_0', k_0')$ and such that*

$$S_{k_0'+1}(x) \not\equiv 0 \quad for \ x \in \Delta(i_0', k_0'). \tag{13}$$

This lemma was proved on pp. 225–226 of [4].

Lemma 2. *Let f be a Denjoy integrable function and $\mathfrak{D}$ be any collection of nonoverlapping subintervals of a fixed interval $\Delta(i_0, k_0)$ of the form* (1) *or* (2). *Define the function f^* by*

$$\begin{aligned} f^*(x) &= f(x) \quad if \ x \in \Delta(i_0, k_0) \sim \bigcup \mathfrak{D}, \\ &= \frac{1}{|\Delta|} (D)\int_\Delta f(t)\, dt \quad if \ x \in \Delta \in \mathfrak{D}. \end{aligned} \tag{14}$$

Then the $(k_0 + 1)$st partial sum of the D-Haar Fourier series T of f can be written as

$$T_{k_0+1}(x) = \frac{1}{|\Delta(i_0, k_0)|} (D)\int_{\Delta(i_0,k_0)} f^*(t)\, dt,$$

for any $x \in \Delta(i_0, k_0)$.

This lemma was proved on pp. 338–339 of [2].

Lemma 3. *Let f be a Denjoy integrable function, T be its D-Haar Fourier series and $\mathfrak{D}$ be any collection of nonoverlapping subintervals of a fixed interval $\Delta(i_0, k_0)$ of the form* (1) *or* (2). *Suppose further that S^* is any Haar series whose partial* $(C, 1)$ *sums satisfy*

$$\lim_{n\to\infty} \int_{\Delta(i_0,k_0)} \sigma_n(S^*; t)\,dt = (D)\int_{\Delta(i_0,k_0)} f^*(t)\,dt, \tag{15}$$

where f^* is defined by (14). *Then $S^*_{k_0+1}(x) \equiv T_{k_0+1}(x)$ for $x \in \Delta(i_0, k_0)$.*

Proof. The integral of a Haar function is zero, and the support of each Haar function $\chi_{k_0+1}, \chi_{k_0+2}, \cdots$ is either a subset of $\Delta(i_0, k_0)$ or disjoint from it. Consequently, if $n > k_0$,

$$\begin{aligned}\int_{\Delta(i_0,k_0)} &\sigma_n(S^*; t)\,dt \\ &\equiv \frac{k_0+1}{n+1}\int_{\Delta(i_0,k_0)} \sigma_{k_0}(S^*; t)\,dt + \frac{n-k_0+1}{n+1}\int_{\Delta(i_0,k_0)} S^*_{k_0+1}(t)\,dt.\end{aligned} \tag{16}$$

But $\chi_0, \chi_1, \cdots, \chi_{k_0}$ are all constant on $\Delta(i_0, k_0)$ so the last summand of (16) is simply

$$\frac{n-k_0+1}{n+1} \quad |\Delta(i_0, k_0)| S^*_{k_0+1}(x)$$

for any $x \in \Delta(i_0, k_0)$. Solving (16) for $S^*_{k_0+1}(x)$ we then obtain

$$\begin{aligned}S^*_{k_0+1}(x) = {} & \frac{n+1}{n-k_0+1}\cdot\frac{1}{|\Delta(i_0, k_0)|}\int_{\Delta(i_0,k_0)} \sigma_n(S^*; t)\,dt \\ & - \frac{k_0+1}{n-k_0+1}\cdot\frac{1}{|\Delta(i_0, k_0)|}\int_{\Delta(i_0,k_0)} \sigma_{k_0}(S^*; t)\,dt.\end{aligned}$$

Taking the limit of both sides of this equation as $n \to \infty$ and applying (15) results in

$$S^*_{k_0+1}(x) = \frac{1}{|\Delta(i_0, k_0)|}(D)\int_{\Delta(i_0,k_0)} f^*(t)\,dt,$$

for any $x \in \Delta(i_0, k_0)$. According to Lemma 2, this is the value of T_{k_0+1} over $\Delta(i_0, k_0)$.

Lemma 4. *Let f be Denjoy integrable, T be its Haar Fourier series and suppose that $S(x) = \Sigma\, a_k\chi_k(x)$ is a Haar series satisfying Condition H which is* $(C, 1)$ *summable to f almost everywhere. Suppose further that on some fixed interval $\Delta(i_0, k_0)$ of the form* (1) *or* (2) *that*

$$S_{k_0+1}(x) \not\equiv T_{k_0+1}(x) \quad \text{for } x \in \Delta(i_0, k_0). \tag{17}$$

Then given any $M > 0$ *there is an interval* $\Delta(i'_k, k'_0)$ *contained in* $\Delta(i_0, k_0)$ *such that* $|\sigma_{k'_0}(S; x)| > M$ *for* $x \in \Delta(i'_0, k'_0)$ *and such that* $T_{k'_0+1}$ *and* $S_{k'_0+1}$ *satisfy* (17) *on* $\Delta(i'_0, k'_0)$.

Proof. We shall prove this lemma in six steps:

I. We begin by supposing the lemma is false; i.e., that there is an M_0 such that if $\Delta(j, n)$ is any subinterval of $\Delta(i_0, k_0)$ of the form (1) or (2) then

$$|\sigma_n(S; x)| > M_0 \quad \text{for } x \in \Delta(j, n) \qquad \text{implies} \quad S_{n+1}(x) \equiv T_{n+1}(x) \quad \text{for } x \in \Delta(j, n). \tag{18}$$

II. We next shall use Lemma 1 to show that it is no loss of generality to suppose that $|\alpha_n\chi_n(x)/n| < M_0$ for $n \geq k_0$.

III. We shall then construct a "maximal" class of intervals

$$\mathfrak{D} \equiv \{\Delta(i_1, \rho_1), \Delta(i_2, \rho_2), \cdots\}$$

on which S_{ρ_k+1} and T_{ρ_k+1} are identically equal.

IV. Next we shall construct a subseries S^* of S such that

(a) S^* is $(C, 1)$ summable to the function f^* almost everywhere, where f^* is defined with respect to the class $\mathfrak{D}$ chosen in III by (14);

(b) S^* and S are identical in $\Delta(i_0, k_0) \sim \mathfrak{D}$;

(c) $S^*_{k_0+1}(x) \equiv S_{k_0+1}(x)$ for $x \in \Delta(i_0, k_0)$.

V. We shall then show that the partial $(C, 1)$ sums of this series S^* are bounded by $13M_0$ by showing that if they are not, we are lead to a contradiction of the construction of S^* in step IV.

VI. Finally, we shall use IV and V to lead to the ultimate contradiction. Indeed, since any $(C, 1)$ partial sum of a Haar series is Lebesgue integrable, we can use IV(a) and V to conclude that f^* is Lebesgue integrable and that (15) holds. Consequently, by Lemma 3, $S^*_{k_0+1} \equiv T_{k_0+1}$ on $\Delta(i_0, k_0)$. But, by IV(c), this identity also implies $S_{k_0+1} \equiv T_{k_0+1}$ on $\Delta(i_0, k_0)$. Since this and hypothesis (17) of this lemma are incompatible, assumption I was false; i.e., the proof of the lemma is complete by contradiction.

What remains, then, is to execute steps II through V:

II. By (6) we choose an integer 2^Q so large that

$$|\alpha_n\chi_n(x)/n| < M_0 \quad \text{whenever } n \geq 2^Q \tag{19}$$

and for any $x \in [0, 1]$.

Using Lemma 1 successively on the points $t_0 = 1/2^Q$, $t_0 = 2/2^Q, \cdots, t_0 = (2^Q - 1)/2^Q$ and the series $S - T$, we may suppose with no loss of generality that $|\Delta(i_0, k_0)| < 1/2^Q$. This fact together with (19) will assure us that

$$|\alpha_n\chi_n(x)/n| < M_0 \quad \text{for } n \geq k_0 \tag{20}$$

and for any $x \in [0, 1]$.

III. If there is no subinterval $\Delta(i_1, \rho_1)$ of $\Delta(i_0, k_0)$ such that $S_{\rho_1+1} \equiv T_{\rho_1+1}$ on $\Delta(i_1, \rho_1)$ then set $\mathfrak{D} = \emptyset$. Otherwise let $\Delta(i_1, \rho_1)$ be the first such interval of the form (1) or (2).

Suppose that we have either terminated this process or have managed to choose $N-1$ intervals $\Delta(i_1, \rho_1), \cdots, \Delta(i_{N-1}, \rho_{N-1})$. If there is no subinterval $\Delta(i_n, \rho_n)$ of $\Delta(i_0, k_0)$ disjoint from $\bigcup_{l=1}^{N-1} \Delta(i_l, \rho_l)$ such that $S_{\rho_n+1} \equiv T_{\rho_n+1}$ then set

$$\mathfrak{D} = \{\Delta(i_l, \rho_l) \colon l = 1, 2, \cdots, N-1\}.$$

Otherwise let $\Delta(i_N, \rho_N)$ be the first such interval of the form (1) or (2).

If this process can be continued indefinitely, set

$$\mathfrak{D} = \{\Delta(i_l, \rho_l) \colon l = 1, 2, \cdots\}.$$

Clearly $\mathfrak{D}$ is a collection of nonoverlapping intervals of the form (1) or (2). Define the function f^* by (14). Notice that if $\mathfrak{D}$ is empty then f^* is identically equal to f.

IV. We shall construct the series S^* by choosing a particular sequence of integers $n_1 < n_2 < \cdots$ which are indices of Haar functions whose support lies in the closure of $\Delta(i_0, k_0)$.

Let n_1 be that integer such that $\Delta(1, n_1) \cup \Delta(2, n_1)$ is the interval $\Delta(i_0, k_0)$ without its midpoint. For instance, if $\Delta(i_0, k_0) = (\frac{1}{4}, \frac{1}{2})$ then $n_1 = 5$. By (17), n_1 is the first integer such that $S_{k_0+1} \not\equiv T_{k_0+1}$ on $\Delta(i_0, k_0)$.

Let n_2 be the very next integer such that the support of χ_{n_2} lies in the closure of $\Delta(i_0, k_0)$ and such that, if i_1 and k_1 are chosen so that $\Delta(1, n_2) \cup \Delta(2, n_2)$ is the interval $\Delta(i_1, k_1)$ without its midpoint, then $S_{k_1+1} \not\equiv T_{k_1+1}$ on $\Delta(i_1, k_1)$. Throughout the following pages we shall denote the interval $\Delta(j, n)$ without its midpoint as $\Delta^*(j, n)$.

We continue this process as long as possible, thereby generating subintervals $\Delta(i_j, k_j)$ of $\Delta(i_0, k_0)$ and integers n_j $(j = 1, 2, \cdots)$ such that

$$\Delta(1, n_{j+1}) \cup \Delta(2, n_{j+1}) = \Delta^*(i_j, k_j) \tag{21}$$

and

$$S_{k_j+1} \not\equiv T_{k_j+1} \quad \text{on} \quad \Delta(i_j, k_j) \text{ for } j = 0, 1, 2, \cdots. \tag{22}$$

Finally, using the sequence $n_1, n_2, \cdots$ just generated we set

$$S^*(x) \equiv S_{k_0+1}(x) + \sum_{j=1}^{\infty} a_{n_j} \chi_{n_j}(x). \tag{23}$$

In case the process for selecting the n_j's terminates after a finite number of steps, S^* is just a finite series. Note also that IV(c) is trivially satisfied.

We shall now show IV(a) and (b) are also satisfied by this S^*.

Indeed, by the disjointness of the collection $\mathfrak{D}$ and by the choice of the sequence $\{n_j\}$, if the support of χ_n is contained in $\Delta(i_0, k_0)$ but χ_n does not appear in the sum (23), then it must be the case that Supp $(\chi_n) \subseteq \Delta(i_l, \rho_l)$ for some l. In particular, the series S and S^* are the same series in the set $\Delta(i_0, k_0) \sim \bigcup \mathfrak{D}$. By the hypotheses of this lemma, then

$$(24)\quad \lim_{n\to\infty} \sigma_n(S^*; x) \equiv f(x) = f^*(x) \quad \text{for almost every } x \text{ in } \Delta(i_0, k_0) \sim \bigcup \mathfrak{D}.$$

On the other hand, if $x_0 \in \Delta(i_l, \rho_l)$ for some l used to define $\mathfrak{D}$, then the disjointness of $\mathfrak{D}$ means that the series (23) must be truncated at ρ_l. Since S and S^* were identical up to that point, $S^*(x_0) \equiv S_{\rho_l+1}(x_0)$. But by the choice of ρ_l, $S_{\rho_l+1} \equiv T_{\rho_l+1}$ on $\Delta(i_0, \rho_0)$. Lemma 2 and the fact f^* is constant on $\Delta(i_l, \rho_l)$ now imply that $T_{\rho_l+1} \equiv f^*$ on $\Delta(i_l, \rho_l)$. Combining these three facts we conclude that $S^*(x_0) = f^*(x_0)$. Since x_0 was any point in any interval of $\mathfrak{D}$, we can now conclude that

$$(25)\qquad \lim_{n\to\infty} \sigma_n(S^*; x) = f^*(x) \quad \text{on each } \Delta \in \mathfrak{D}.$$

Combining (24) and (25) we have IV(a).

V. Suppose that the partial $(C, 1)$ sums of S^* are not bounded by $13M_0$ on $\Delta(i_0, k_0)$ and let L be the smallest index greater than or equal to n_1 such that $|\sigma_L(S^*; t_0)| > 13M_0$ for some $t_0 \in \Delta(i_0, k_0)$.

If we let n_p be the largest number in the sequence $n_1, n_2, \cdots$ which is less than or equal to L, then for some choice of $j_p = 1$ or 2,

$$(26)\qquad |\sigma_L(S^*; x)| > 13M_0 \quad \text{for } x \in \Delta(j_p, n_p).$$

Indeed, if $L = n_1$ then (26) is trivial by (21). Otherwise we use the least property of L.

We first begin by noting that

$$(27)\qquad S_{n_p+1} \not\equiv T_{n_p+1} \quad \text{on } \Delta^*(i_{p-1}, k_{p-1}).$$

Indeed, if (27) were false, then $S_{n_p+1} = S_{k_{p-1}+1} + \alpha_{n_p}\chi_{n_p}$, and a corresponding equation involving T and its n_pth coefficient, say β_{n_p}, implies $S_{k_{p-1}+1} - T_{k_{p-1}+1} \equiv (\beta_{n_p} - \alpha_{n_p})\chi_{n_p}$ on $\Delta^*(i_{p-1}, k_{p-1})$. But χ_{n_p} changes signs in that punctured interval while the left-hand side of the above identity is constant in that punctured interval. The only possibility, then, is that $\beta_{n_p} = \alpha_{n_p}$ which in turn forces $S_{k_{p-1}+1} - T_{k_{p-1}+1} \equiv 0$ on $\Delta^*(i_{p-1}, k_{p-1})$. Since both partial sums are constant throughout $\Delta(i_{p-1}, k_{p-1})$, this statement and (22) are incompatible; consequently (27) does hold.

Let $j_p' \neq j_p$ with $j_p' = 1$ or 2. Then, by (21), $\Delta(j_p, n_p) \cup \Delta(j_p', n_p) = \Delta^*(i_{p-1}, k_{p-1})$; and, by (22), $S_{k_{p-1}+1} \not\equiv T_{k_{p-1}+1}$ on $\Delta(i_{p-1}, k_{p-1})$. Hence by the contrapositive of (18),

$$|\sigma_{k_{p-1}}(S; x)| \leq M_0 \quad \text{for } x \in \Delta(i_{p-1}, k_{p-1}).$$

Note also by (22) and the fact that S and S^* are identical outside $\mathfrak{D}$, we have

$$\sigma_n(S; x) \equiv \sigma_n(S^*; x) \quad \text{for } x \in \Delta(i_{p-1}, k_{p-1})$$

whenever $n = k_{p-1}, k_{p-1} + 1, \cdots, n_{p+1} - 1$. Consequently, the above inequality becomes

$$|\sigma_{k_{p-1}}(S^*; x)| \leq M_0 \quad \text{for } x \in \Delta(i_{p-1}, k_{p-1}). \tag{28}$$

Using (27) and the contrapositive of (18) we can also conclude that on at least one of the intervals $\Delta(j_p, n_p)$, $\Delta(j_p', n_p)$,

$$|\sigma_{n_p}(S^*; x)| \leq M_0.$$

But

$$\sigma_{n_p}(S^*; x) = \sum_{k=0}^{n_p - 1} \left(1 - \frac{k}{n_p + 1}\right) a_k \chi_k(x) + \frac{a_{n_p} \chi_{n_p}(x)}{n_{p+1}}$$

$$\equiv \Sigma_1(x) + \frac{a_{n_p} \chi_{n_p}(x)}{n_{p+1}}.$$

Hence by (20) and the triangle inequality,

$$|\Sigma_1(x)| \leq |\sigma_{n_p}(S^*; x)| + |a_{n_p} \chi_{n_p}(x)/(n_p + 1)| \leq M_0 + M_0 = 2M_0$$

on at least one of the intervals $\Delta(j_p, n_p)$, $\Delta(j_p', n_p)$. But Σ_1 is constant throughout the union of both these intervals, so $|\Sigma_1(x)| \leq 2M_0$ for $x \in \Delta(i_{p-1}, k_{p-1})$. Applying (20) and the triangle inequality again, we conclude that

$$|\sigma_{n_p}(S^*; x)| \leq 3M_0 \quad \text{for } x \in \Delta(i_{p-1}, k_{p-1}). \tag{29}$$

We shall now complete the proof of step V by showing that (28), (29) and (26) are incompatible with (27).

The definition of $(C, 1)$ sums and the construction of S^* allow us to write the following equations for any $x \in \Delta(i_{p-1}, k_{p-1})$.

$$\sigma_L(S^*; x) = \frac{n_p + 1}{L + 1} \sigma_{n_p}(S^*; x) + \frac{L - n_p + 1}{L + 1} S^*_{n_p+1}(x); \tag{30}$$

$$\sigma_L(S^*; x) = \frac{k_{p-1}+1}{L+1}\sigma_{k_{p-1}}(S^*; x) + \frac{L-k_{p-1}+1}{L+1} S^*_{n_p}(x)$$

$$+\frac{L-n_p+1}{L+1}\alpha_{n_p}\chi_{n_p}(x); \tag{31}$$

$$\sigma_{n_p}(S^*; x) = \frac{k_{p-1}+1}{n_p+1}\sigma_{k_{p-1}}(S^*; x) + \frac{n_p-k_{p-1}+1}{n_p+1} S^*_{n_p}(x) + \frac{\alpha_{n_p}\chi_{n_p}(x)}{n_p+1}. \tag{32}$$

Since χ_{n_p} is constant on $\Delta(j_p, n_p)$ there are two possible cases:

$$\frac{L-n_p+1}{L+1}|\alpha_{n_p}\chi_{n_p}(x)| > M_0 \quad \text{for } x \in \Delta(j_p, n_p) \tag{34}$$

or

$$\frac{L-n_p+1}{L+1}|\alpha_{n_p}\chi_{n_p}(x)| \le M_0 \quad \text{for } x \in \Delta(j_p, n_p). \tag{35}$$

If (34) holds then we use (29) and (26) on (30) to conclude that

$$\frac{L-n_p+1}{L+1}|S^*_{n_p+1}(x)| > 10M_0 \quad \text{on } \Delta(j_p, n_p).$$

But $S^*_{n_p+1} = S^*_{n_p} + \alpha_{n_p}\chi_{n_p}$, so (34) implies

$$\frac{L-n_p+1}{L+1}|S^*_{n_p}| > 10M_0 - M_0 = 9M_0 \quad \text{on } \Delta(j_p, n_p).$$

Since $S^*_{n_p}$ is constant on $\Delta(i_{p-1}, k_{p-1})$ this inequality must hold throughout the larger interval:

$$\frac{L-n_p+1}{L+1}|S^*_{n_p}(x)| > 9M_0 \quad \text{for } x \in \Delta(i_{p-1}, k_{p-1}). \tag{36}$$

On the other hand, if (35) holds, then using (28) on (31) implies

$$\left|\frac{L-k_{p-1}+1}{L+1}S^*_{n_p}(x) + \frac{L-n_p+1}{L+1}\alpha_{n_p}\chi_{n_p}(x)\right| > 12M_0.$$

on $\Delta(i_{p-1}, k_{p-1})$, which by (35) guarantees

$$\frac{L-k_{p-1}+1}{L+1}|S^*_{n_p}| > 11M_0 > 9M_0 \quad \text{on } \Delta(j_p, n_p).$$

This show that (36) holds in any case.

Now, $2n_p$ is an element of the sequence $n_1, n_2, \cdots$ by (27). Hence by the choice of n_p relative to L, $L < 2n_p$. The construction of the sequence k_j shows that $2k_{p-1} \le n_p$. These two inequalities together yield $3(n_p - k_{p-1} + 1) > L - k_{p-1} + 1$. Consequently, applying (20), (28), and (36) to (32), we conclude $|\sigma_{n_p}(S^*; x)| > (1/3)9M_0 - M_0 - M_0 = M_0$ for $x \in \Delta(i_{p-1}, k_{p-1})$. This, together with (18) and (21), implies $S_{n_p+1} \equiv T_{n_p+1}$ on $\Delta^*(i_{p-1}, k_{p-1})$. By (27) this is impossible.

This final contradiction completes the proof of step V which in turn completes the proof of this lemma.

4. **The proof of the theorem.** Let $\{Z_1, Z_2, \cdots\}$ be the set of points in $[0, 1]$ where

$$\limsup_{n\to\infty} |\sigma_n(S; x)| = +\infty.$$

Suppose that T is the D-Haar Fourier series of f, but that the theorem is false. Choose k_0 least so that the k_0th Fourier coefficient of f is different from α_{k_0}. Clearly, then, $S_{k_0+1} \neq T_{k_0+1}$. Since T satisfies *Condition G* (Lemma 3 of [2]) the series $S - T$ satisfies the hypotheses of Lemma 1. The series S and T also satisfy the hypotheses of Lemma 4.

Applying Lemmas 1 and 4 countably many times, we can thus choose a sequence of intervals $\Delta(i_1, k_1), \cdots, \Delta(i_N, k_N), \cdots$ of the form (1) or (2) such that

(37) $$Z_n \text{ does not lie in the closure of } \Delta(i_N, k_N),$$

(38) $$\text{the closure of } \Delta(i_N, k_N) \text{ is a subset of } \Delta(i_{N+1}, k_{N+1}),$$

and

(39) $$|\sigma_{k_N}(S; x)| > N \quad \text{for } x \in \Delta(i_N, k_N), \text{ for } N = 1, 2, \cdots.$$

By (38), $\bigcap_{N=1}^{\infty} \Delta(i_N, k_N)$ is not empty; let ξ be in this intersection.

By (37), $\xi \notin \{Z_1, Z_2, \cdots\}$ which, by the definition of this sequence, implies

(40) $$\limsup_{n\to\infty} |\sigma_n(S; \xi)| < \infty.$$

Yet by (39), since $\xi \in \Delta(i_N, k_N)$ for all N, $\limsup_{n\to\infty} |\sigma_n(S; \xi)| = \infty$. This being incompatible with (40) completes the proof of the theorem by contradiction.

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