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Uniqueness of iterative positive solutions for the singular fractional differential equations with integral boundary conditions

Limin Guo[1], Lishan Liu[1,2*] and Yonghong Wu[2]

*Correspondence: mathlls@163.com
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, People's Republic of China
[2]Department of Mathematics and Statistics, Curtin University, Perth, WA 6845, Australia

Abstract

The uniqueness of positive solution for a class of singular fractional differential system with integral boundary conditions is considered in this paper and many types of equation system are contained in this equation system because there are many parameters which can be changeable in this equation system. The fractional orders are involved in the nonlinearity of the boundary value problem and the nonlinearity is allowed to be singular in regard to not only time variable but also space variable. The existence of uniqueness of positive solution is mainly obtained by fixed point theorem of mixed monotone operator and the positive solution of equation system is dependent on λ. An iterative sequence and convergence rate are given which are important for practical application and an example is given to demonstrate the validity of our main results.

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Keywords: fractional differential system; integral boundary conditions; singular problem; iterative positive solution; mixed monotone operator

1 Introduction

In the past couple of decades, boundary value problems for nonlinear fractional differential equations arise from the studies of complex problems in many disciplinary areas such as aerodynamics, fluid flows, electrodynamics of complex medium, electrical networks, rheology, polymer rheology, economics, biology chemical physics, control theory, signal and image processing, blood flow phenomena, and so on. Fractional-order models have been shown to be more accurate and realistic than integer-order models, and with this advantage in the application of these models, it is important to theoretically establish the conditions for the existence of positive solutions because theoretical results can help people to get an in-depth understanding for the dynamic behavior in the practical process, so the study of abstract fractional models is important and relevant nowadays. In recent years, many authors investigated the existence of positive solutions for fractional equation boundary value problems (see [1–22] and the references therein), and a great deal of results have been developed for differential and integral boundary value problems. The authors in [23] studied the following system of singular fractional differential equations:

$$\begin{cases} -D_{0^+}^{\alpha}u(t) = f(t,u(t),D_{0^+}^{\beta}u(t),y(t)), \qquad -D_{0^+}^{\gamma}y(t) = g(t,u(t)), \quad 0<t<1, \\ D_{0^+}^{\beta}x(0)=0, \qquad D_{0^+}^{\mu}x(1)=\sum_{j=1}^{p-2}a_jD_{0^+}^{\mu}x(\xi_j), \\ y(0)=0, \qquad D_{0^+}^{\nu}y(1)=\sum_{j=1}^{p-2}b_jD_{0^+}^{\nu}y(\xi_j), \end{cases}$$

where $\alpha,\gamma,\beta,\nu,\mu \in \mathbb{R}_+^1 = [0,+\infty)$, $1<\gamma<\alpha\le 2$, $1<\alpha-\beta<\gamma$, $0<\beta\le\mu<1$, $0<\nu<1$, $0<\xi_1<\xi_2<\cdots<\xi_{p-2}<1$, $a_j,b_j\in\mathbb{R}_+^1$ with $\sum_{j=1}^{p-2}a_j\xi_j^{\alpha-\mu-1}<1$, $\sum_{j=1}^{p-2}b_j\xi_j^{\gamma-1}<1$, $f\in C([0,1]\times(0,+\infty)^3,\mathbb{R}_+^1)$ and may be singular at $x_i=0$ $(i=1,2,3)$, $g\in C([0,1]\times(0,+\infty),\mathbb{R}_+^1)$, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$, $D_{0^+}^{\gamma}$, $D_{0^+}^{\mu}$, $D_{0^+}^{\nu}$ are the standard Riemann-Liouville derivatives. By using the fixed point theorem of the mixed monotone operator, the authors obtained the uniqueness of the positive solution. In [24], the authors investigated the fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha}u(t)+f(t,u(t),D_{0^+}^{\nu}u(t),D_{0^+}^{\mu}u(t))=0, \quad 0<t<1, \\ u(0)=u'(0)=u''(0)=u''(1)=0, \end{cases}$$

where $\alpha,\nu,\mu\in\mathbb{R}_+^1$, $3<\alpha\le 4$, $0<\nu\le 1$, $0<\mu\le 1$ are real numbers, f is a Carathéodory function, $f(t,x,y,z)$ is singular at $x,y,z=0$, and $D_{0^+}^{\alpha}$, $D_{0^+}^{\nu}$, $D_{0^+}^{\mu}$ are the Riemann-Liouville fractional derivatives. The authors obtained the existence and multiplicity of positive solutions by means of Krasnosel'skii's fixed point theorem. In [25], the authors investigated the fractional-order model for turbulent flow in a porous medium,

$$\begin{cases} -D_t^{\beta}(\varphi_p(-D_t^{\alpha}x))(t)=f(x(t),D_t^{\gamma}x(t)), \quad 0<t<1, \\ D_t^{\alpha}x(0)=D_t^{\alpha+1}x(0)=D_t^{\alpha}x(1)=0, \qquad D_t^{\gamma}x(0)=0, \qquad D_t^{\gamma}x(1)=\int_0^1 D_t^{\gamma}x(s)\,dA(s), \end{cases}$$

where $\alpha,\beta,\gamma\in\mathbb{R}_+^1$, $0<\gamma\le 1<\alpha\le 2<\beta<3$, $\alpha-\gamma>1$, $\int_0^1 x(s)\,dA(s)$ denotes a Riemann-Stieltjes integral, A is a function of bounded variation, and dA can be a signed measure. The p-Laplacian operator is defined as $\varphi_p(s)=|s|^{p-2}s$, $p>2$, and the nonlinearity $f(t,u,v)$ may be singular at both $u=0$ and $v=0$, and D_t^{α}, D_t^{β}, D_t^{γ} are the standard Riemann-Liouville derivatives. The authors obtained the uniqueness of a positive solution by using the fixed point theorem of the mixed monotone operator. In [26], the authors investigated the following system of singular problems:

$$\begin{cases} -D_t^{\alpha}u(t)+\lambda f(t,u(t),D_t^{\beta}u(t),v(t))=0, \\ -D_t^{\gamma}v(t)+\lambda g(t,u(t))=0, \quad 0<t<1, \\ D_t^{\beta}u(0)=D_t^{\beta+1}u(0)=0, \qquad D_t^{\beta}u(1)=\int_0^1 D_t^{\beta}u(s)\,dA(s), \\ v(0)=v'(0)=0, \qquad v(1)=\int_0^1 v(s)\,dB(s), \end{cases}$$

where $\alpha,\beta,\gamma\in\mathbb{R}_+^1$, $2<\alpha,\gamma\le 3$, $0<\beta<1$, u denotes the number of uninfected CD4$^+$T cells and v denotes the number of infected cells, $\lambda>0$ is a parameter, $\alpha-\beta>2$, $\int_0^1 D_t^{\beta}u(s)\,dA(s)$, and $\int_0^1 v(s)\,dB(s)$ denote the Riemann-Stieltjes integrals of u with respect to A and B, respectively, A, B are bounded variations, $f:(0,1)\times\mathbb{R}_+^3\to(-\infty,+\infty)$, $g:(0,1)\times\mathbb{R}_+^1\to(-\infty,+\infty)$ are two continuous functions and may be singular at $t=0,1$, D_t^{α}, D_t^{β}, D_t^{γ} are the standard Riemann-Liouville derivatives. The authors obtained the existence of positive solution by fixed point theorem.

In present, many papers are devoted to the fractional differential equations in which the fractional orders are involved in the nonlinearity; see [1–4, 23–26]. On the other hand, there are some papers studying singular equations in regard to space variable, we refer the reader to [5–9]. Motivated by the results above, we utilize a fixed point theorem to investigate the existence results of positive solution of the following class of nonlinear singular fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t),D_{0^+}^{\mu_1}u(t),D_{0^+}^{\mu_2}u(t),\ldots,D_{0^+}^{\mu_{n-2}}u(t),v(t))=0, \quad 0<t<1,\\ D_{0+}^{\beta}v(t)+\mu g(t,u(t),D_{0^+}^{\eta_1}u(t),D_{0^+}^{\eta_2}u(t),\ldots,D_{0^+}^{\eta_{m-2}}u(t))=0, \quad 0<t<1,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \qquad u^{(n-2)}(1)=\chi\int_0^{\eta}h(s)u^{(n-2)}(s)\,dA(s),\\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0, \qquad v^{(m-2)}(1)=\iota\int_0^{\vartheta}a(s)v^{(m-2)}(s)\,dB(s), \end{cases} \tag{1.1}$$

where $\alpha,\beta,\mu_\kappa,\eta_\varrho\in\mathbb{R}_+^1$, $n,m,\kappa,\varrho\in\mathbb{N}$ (natural number set) and $n-1<\alpha\le n$, $m-1<\beta\le m$, $m<n\,(n,m\ge 2)$, $\kappa-1<\mu_\kappa\le\kappa$ $(\kappa=1,2,\ldots,n-2)$, $\varrho-1<\eta_\varrho\le\varrho$ $(\varrho=1,2,\ldots,m-2)$, and $0<\eta,\vartheta\le 1$, $\lambda,\mu,\chi,\iota>0$ are parameters, $f\in C((0,1)\times(0,+\infty)^n,\mathbb{R}_+^1)$, and $f(t,x_1,x_2,\ldots,x_n)$ has a singularity at $x_i=0$ $(i=1,2,\ldots,n)$ and $t=0,1$, $g\in C((0,1)\times(0,+\infty)^{m-1},\mathbb{R}_+^1)$, $h,a\in C(0,1)$ with $\int_0^{\eta}\chi t^{\alpha-n+1}h(t)\,dA(t)<1$, $\int_0^{\vartheta}\iota t^{\beta-m+1}a(t)\,dB(t)<1$, A,B are functions of bounded variation, $\int_0^{\eta}h(s)u^{(n-2)}(s)\,dA(s)$, $\int_0^{\vartheta}a(s)v^{(m-2)}(s)\,dB(s)$ denote the Riemann-Stieltjes integral with respect to A and B, $D_{0^+}^{\alpha}u$, $D_{0+}^{\beta}v$, $D_{0^+}^{\mu_\kappa}u$ $(\kappa=1,2,\ldots,n-2)$, $D_{0^+}^{\eta_\varrho}u$ $(\varrho=1,2,\ldots,m-2)$ are the standard Riemann-Liouville derivatives. The existence of positive solutions is obtained by means of a mixed monotone operator in cones in this paper.

In this paper, we study the existence of positive solutions to BVP (1.1), where $(u,v)\in C[0,1]\times C[0,1]$ is said to be a positive solution of BVP (1.1) if and only if u, v satisfies (1.1) and $u(t)>0$, $v(t)>0$ for any $t\in(0,1]$. We should address here that our work presented in this paper has various new system features. First of all, the system of equations in question include many types of system of equations as special cases because there are many parameters which can be changeable in this equation system. Second, our study is on singular nonlinear differential boundary value problems, that is, $f(t,x_1,x_2,\ldots,x_n)$ has singularity at $x_i=0$ $(i=1,2,\ldots,n)$ and $t=0,1$, and $g(t,x_1,x_2,\ldots,x_{m-1})$ may be singular at $t=0,1$. Third, fractional derivatives is involved in the nonlinear terms and boundary conditions of fractional differential (1.1). Fourth, the uniqueness of the positive solution of equation (1.1) is dependent on λ.

2 Preliminaries and lemmas

For the convenience of the reader, we first present some basic definitions and lemmas that are important and are to be used in the rest of the paper. The definitions can also be found in recent literature such as [27, 28].

Definition 2.1 [27, 28] The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $y:(0,\infty)\to\mathbb{R}^1$ is given by

$$I_{0^+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 [27, 28] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \to \mathbb{R}^1$ is given by

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 [27, 28] *Assume that $u \in C^n[0,1]$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

where n is the least integer greater than or equal to α, $C_i \in \mathbb{R}^1$ $(i = 1,2,\ldots,n)$.

Lemma 2.2 [29]

(1) *If $x \in L(0,1)$, $\nu > \sigma > 0$, then*

$$I_{0^+}^{\nu} I_{0^+}^{\sigma} x(t) = I_{0^+}^{\nu+\sigma} x(t), \qquad D_{0^+}^{\sigma} I_{0^+}^{\nu} x(t) = I_{0^+}^{\nu-\sigma} x(t), \qquad D_{0^+}^{\sigma} I_{0^+}^{\sigma} x(t) = x(t).$$

(2) *If $\nu > 0$, $\sigma > 0$, then*

$$D_{0^+}^{\nu} t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)} t^{\sigma-\nu-1}.$$

For convenience in our presentation, we here list some conditions to be used throughout the paper.

(S_1) $f(t,x_1,x_2,\ldots,x_n) = \phi(t,x_1,x_2,\ldots,x_n) + \psi(t,x_1,x_2,\ldots,x_n)$, where $\phi : (0,1) \times (0,+\infty)^n \to \mathbb{R}_+^1$ is continuous, $\phi(t,x_1,x_2,\ldots,x_n)$ may be singular at $t = 0,1$ and is nondecreasing on $x_i > 0$ $(i = 1,2,\ldots,n)$; $\psi : (0,1) \times (0,+\infty)^n \to \mathbb{R}_+^1$ is continuous, $\psi(t,x_1,x_2,\ldots,x_n)$ may be singular at $t = 0,1$ and $x_i = 0$ $(i = 1,2,\ldots,n)$, and is nonincreasing on $x_i > 0$ $(i = 1,2,\ldots,n)$.

(S_2) There exists $\sigma \in (0,1)$ such that, for all $x_i > 0$ $(i = 1,2,\ldots,n)$, $t \in (0,1)$, and $l \in (0,1)$,

$$\begin{aligned} &\phi(t,lx_1,lx_2,\ldots,lx_n) \geq l^{\sigma} \phi(t,x_1,x_2,\ldots,x_n), \\ &\psi\left(t,l^{-1}x_1,l^{-1}x_2,\ldots,l^{-1}x_n\right) \geq l^{\sigma} \psi(t,x_1,x_2,\ldots,x_n). \end{aligned}$$

(S_3) $g \in C((0,1) \times (0,+\infty)^{m-1},\mathbb{R}_+^1)$, $g(t,x_1,x_2,\ldots,x_{m-1})$ may be singular at $t = 0,1$ and is nondecreasing on $x_i > 0$ $(i = 1,2,\ldots,m-1)$. Moreover, there exists $\varsigma \in (0,1)$ such that

$$\begin{aligned} &g(t,lx_1,lx_2,\ldots,lx_{m-1}) \\ &\quad \geq l^{\varsigma} g(t,x_1,x_2,\ldots,x_{m-1}), \quad t,l \in (0,1), x_i > 0, i = 1,2,\ldots,m-1. \end{aligned}$$

(S_4)

$$\begin{aligned} &0 < \int_0^1 (1-s)^{\alpha-n+1} s^{-\sigma(\alpha-1)} \psi(s,1,1,\ldots,1)\, ds < +\infty, \\ &0 < \int_0^1 (1-s)^{\alpha-n+1} \phi(s,1,1,\ldots,1)\, ds < +\infty, \end{aligned}$$

$$0 < \int_0^1 (1-s)^{\beta-m+1} g(s,1,1,\ldots,1)\,ds < +\infty.$$

Remark 2.1 According to (S_2) and (S_3), we have

$$\begin{aligned}
&\phi(t,lx_1,lx_2,\ldots,lx_n)\\
&\quad\le l^{\sigma}\phi(t,x_1,x_2,\ldots,x_n),\quad t\in(0,1),l\ge1,x_i>0,i=1,2,\ldots,n,\\
&\psi\bigl(t,l^{-1}x_1,l^{-1}x_2,\ldots,l^{-1}x_n\bigr)\\
&\quad\le l^{\sigma}\psi(t,x_1,x_2,\ldots,x_n),\quad t\in(0,1),l\ge1,x_i>0,i=1,2,\ldots,n,\\
&g(t,lx_1,lx_2,\ldots,lx_{m-1})\\
&\quad\le l^{\varsigma}g(t,x_1,x_2,\ldots,x_{m-1}),\quad t\in(0,1),l\ge1,x_i>0,i=1,2,\ldots,m-1.
\end{aligned}$$

Lemma 2.3 *Let $\rho\in L^1(0,1)\cap C(0,1)$, then the equation of the BVPs*

$$\begin{cases} -D_{0+}^{\alpha-n+2}x(t)=\rho(t), & 0<t<1,\\ x(0)=0, & x(1)=\chi\int_0^{\eta}h(s)x(s)\,dA(s),\end{cases} \tag{2.1}$$

$$\begin{cases} -D_{0+}^{\beta-m+2}y(t)=\rho(t), & 0<t<1,\\ y(0)=0, & y(1)=\iota\int_0^{\vartheta}a(s)y(s)\,dB(s),\end{cases} \tag{2.2}$$

have the integral representation

$$\begin{aligned}
x(t)&=\int_0^1 G(t,s)\rho(s)\,ds=\int_0^1\bigl(G_1(t,s)+G_2(t,s)\bigr)\rho(s)\,ds,\\
y(t)&=\int_0^1 H(t,s)\rho(s)\,ds=\int_0^1\bigl(H_1(t,s)+H_2(t,s)\bigr)\rho(s)\,ds,
\end{aligned}\tag{2.3}$$

respectively, where

$$G(t,s)=G_1(t,s)+G_2(t,s),\qquad H(t,s)=H_1(t,s)+H_2(t,s),$$

in which

$$G_1(t,s)=\frac{1}{\Gamma(\alpha-n+2)}\begin{cases} t^{\alpha-n+1}(1-s)^{\alpha-n+1}-(t-s)^{\alpha-n+1}, & 0\le s\le t\le1,\\ t^{\alpha-n+1}(1-s)^{\alpha-n+1}, & 0\le t\le s\le 1,\end{cases}\tag{2.4}$$

$$G_2(t,s)=\frac{\chi t^{\alpha-n+1}}{1-\chi\int_0^{\eta}t^{\alpha-n+1}h(t)\,dA(t)}\int_0^{\eta}h(t)G_1(t,s)\,dA(t),\tag{2.5}$$

$$H_1(t,s)=\frac{1}{\Gamma(\beta-m+2)}\begin{cases} t^{\beta-m+1}(1-s)^{\beta-m+1}-(t-s)^{\beta-m+1}, & 0\le s\le t\le1,\\ t^{\beta-m+1}(1-s)^{\beta-m+1}, & 0\le t\le s\le 1,\end{cases}\tag{2.6}$$

$$H_2(t,s)=\frac{\iota t^{\beta-m+1}}{1-\iota\int_0^{\vartheta}t^{\beta-m+1}a(t)\,dB(t)}\int_0^{\vartheta}a(t)H_1(t,s)\,dB(t).\tag{2.7}$$

Proof The proof is similar to that for Lemma 2.3 in [30], we omit it here. □

Lemma 2.4 *Let $\theta_1(s)=m_1s(1-s)^{\alpha-n+1}$, $\theta_2(s)=m_2s(1-s)^{\beta-m+1}$, then the Green functions $G(t,s)$ and $H(t,s)$ satisfy*:

(1) $G(t,s) > 0$, $H(t,s) > 0$, $\forall t,s \in (0,1)$,
(2) $\forall t,s \in [0,1]$,

$$t^{\alpha-n+1}\theta_1(s) \le G(t,s) \le M_1 t^{\alpha-n+1}(1-s)^{\alpha-n+1},$$
$$t^{\beta-m+1}\theta_2(s) \le H(t,s) \le M_2 t^{\beta-m+1}(1-s)^{\beta-m+1},$$

where

$$m_1 = \frac{1}{\Gamma(\alpha-n+2)}\left(1 + \frac{\chi\int_0^\eta h(t)t^{\alpha-n+1}\,dA(t)}{1-\chi\int_0^\eta h(t)t^{\alpha-n+1}\,dA(t)}\right),$$
$$m_2 = \frac{1}{\Gamma(\beta-m+2)}\left(1 + \frac{\iota\int_0^\vartheta a(t)t^{\beta-m+1}\,dB(t)}{1-\iota\int_0^\vartheta a(t)t^{\beta-m+1}\,dB(t)}\right),$$
$$M_1 = \frac{2}{\Gamma(\alpha-n+2)}\left(1 + \frac{\chi\int_0^\eta h(t)t^{\alpha-n}\,dA(t)}{1-\chi\int_0^\eta h(t)t^{\alpha-n+1}\,dA(t)}\right),$$
$$M_2 = \frac{2}{\Gamma(\beta-m+2)}\left(1 + \frac{\iota\int_0^\vartheta a(t)t^{\beta-m}\,dB(t)}{1-\iota\int_0^\vartheta a(t)t^{\beta-m+1}\,dB(t)}\right).$$

Proof The proof is similar to that for Lemma 2.4 in [30], we omit it here. □

Let $u(t) = I_{0^+}^{n-2}x(t)$, $v(t) = I_{0^+}^{m-2}y(t)$, then $x(t) = D_{0^+}^{n-2}u(t)$, $y(t) = D_{0^+}^{m-2}v(t)$, the problem (1.1) can turn into the following modified problem of the BVP (2.8):

$$\begin{cases} D_{0^+}^{\alpha-n+2}x(t) + \lambda f(t, I_{0^+}^{n-2}x(t), I_{0^+}^{n-2-\mu_1}x(t),\dots,I_{0^+}^{n-2-\mu_{n-2}}x(t), I_{0^+}^{m-2}y(t)) = 0, \quad 0<t<1,\\ D_{0^+}^{\beta-m+2}y(t) + \mu g(t, I_{0^+}^{n-2}x(t), I_{0^+}^{n-2-\eta_1}x(t),\dots,I_{0^+}^{n-2-\eta_{m-2}}x(t)) = 0, \quad 0<t<1,\\ x(0)=0, \qquad x(1) = \chi\int_0^\eta h(s)x(s)\,dA(s),\\ y(0)=0, \qquad y(1) = \iota\int_0^\vartheta a(s)y(s)\,dB(s). \end{cases} \tag{2.8}$$

Lemma 2.5 *Let* $u(t) = I_{0^+}^{n-2}x(t)$, $v(t) = I_{0^+}^{m-2}y(t)$, $x(t),y(t) \in C[0,1]$. *Then* (1.1) *can be transformed into* (2.8). *Moreover, if* $(x,y) \in C[0,1]\times C[0,1]$ *is a positive solution of the problem* (2.8), *then* $(I_{0^+}^{n-2}x, I_{0^+}^{m-2}y)$ *is a positive solution of the problem* (1.1).

Proof Put $u(t) = I_{0^+}^{n-2}x(t)$, $v(t) = I_{0^+}^{m-2}y(t)$ into (1.1), by Lemma 2.2, we have

$$\begin{aligned} D_{0^+}^{\alpha}u(t) &= \frac{d^n}{dt^n}I_{0^+}^{n-\alpha}I_{0^+}^{n-2}x(t) = \frac{d^n}{dt^n}I_{0^+}^{2n-\alpha-2}x(t) = D_{0^+}^{\alpha-n+2}x(t),\\ D_{0^+}^{\mu_i}u(t) &= D_{0^+}^{\mu_i}I_{0^+}^{n-2}x(t) = I_{0^+}^{n-2-\mu_i}x(t), \quad i=1,2,\dots,n-2,\\ D_{0^+}^{\beta}v(t) &= \frac{d^m}{dt^m}I_{0^+}^{m-\beta}I_{0^+}^{m-2}y(t) = \frac{d^m}{dt^m}I_{0^+}^{2m-\beta-2}y(t) = D_{0^+}^{\beta-m+2}y(t),\\ D_{0^+}^{\eta_i}u(t) &= D_{0^+}^{\eta_i}I_{0^+}^{n-2}x(t) = I_{0^+}^{n-2-\eta_i}x(t), \quad i=1,2,\dots,m-2. \end{aligned} \tag{2.9}$$

By $u(t) = I_{0^+}^{n-2}x(t)$, $v(t) = I_{0^+}^{m-2}y(t)$, and (2.9), we have

$$D_{0^+}^{n-2}u(0) = x(0) = 0, \qquad u^{(n-2)}(1) = x(1) = \chi\int_0^\eta h(s)x(s)\,dA(s),$$
$$-D_{0^+}^{\alpha-n+2}x(t) = \lambda f\bigl(t, I_{0^+}^{n-2}x(t), I_{0^+}^{n-2-\mu_1}x(t),\dots,I_{0^+}^{n-2-\mu_{n-2}}x(t), I_{0^+}^{m-2}y(t)\bigr),$$

$$D_{0^+}^{m-2}v(0) = y(0) = 0, \qquad v^{(m-2)}(1) = y(1) = \iota \int_0^{\vartheta} a(s)y(s)\,dB(s),$$
$$-D_{0^+}^{\beta-m+2}y(t) = \mu g\big(t, I_{0^+}^{n-2}x(t), I_{0^+}^{n-2-\eta_1}x(t), \ldots, I_{0^+}^{n-2-\eta_{m-2}}x(t)\big).$$

Thus (1.1) is transformed into (2.8).

Now, assume $(x,y) \in C[0,1] \times C[0,1]$ is a positive solution for problem (2.8). Let $u(t) = I_{0^+}^{n-2}x(t)$, $v(t) = I_{0^+}^{m-2}y(t)$, by Lemma 2.2, we have

$$\begin{aligned}
-D_{0^+}^{\alpha}u(t) &= -D_{0^+}^{\alpha}I_{0^+}^{n-2}x(t) = -D_{0^+}^{\alpha-n+2}x(t)\\
&= \lambda f\big(t, I_{0^+}^{n-2}x(t), I_{0^+}^{n-2-\mu_1}x(t), \ldots, I_{0^+}^{n-2-\mu_{n-2}}x(t), I_{0^+}^{m-2}y(t)\big)\\
&= \lambda f\big(t, u(t), D_{0^+}^{\mu_1}u(t), D_{0^+}^{\mu_2}u(t), \ldots, D_{0^+}^{\mu_{n-2}}u(t), v(t)\big), \quad 0 < t < 1,\\
-D_{0^+}^{\beta}v(t) &= -D_{0^+}^{\beta}I_{0^+}^{m-2}y(t) = -D_{0^+}^{\beta-m+2}y(t)\\
&= \mu g\big(t, I_{0^+}^{n-2}x(t), I_{0^+}^{n-2-\eta_1}x(t), \ldots, I_{0^+}^{n-2-\eta_{m-2}}x(t)\big)\\
&= \mu g\big(t, u(t), D_{0^+}^{\eta_1}u(t), D_{0^+}^{\eta_2}u(t), \ldots, D_{0^+}^{\eta_{n-2}}u(t)\big), \quad 0 < t < 1.
\end{aligned}$$

On the other hand, by $u(t) = I_{0^+}^{n-2}x(t)$, $v(t) = I_{0^+}^{m-2}y(t)$, and Lemma 2.2, we also get

$$D_{0^+}^{n-2}u(t) = D_{0^+}^{n-2}I_{0^+}^{n-2}x(t) = x(t),$$
$$D_{0^+}^{m-2}v(t) = D_{0^+}^{m-2}I_{0^+}^{m-2}y(t) = y(t),$$

which implies that

$$\begin{aligned}
&u(0) = u'(0) = \cdots = u^{(n-2)}(0) = x(0) = 0,\\
&u^{(n-2)}(1) = x(1) = \chi \int_0^{\eta} h(s)x(s)\,dA(s) = \chi \int_0^{\eta} h(s)u^{(n-2)}(s)\,dA(s),\\
&v(0) = v'(0) = \cdots = v^{(m-2)}(0) = y(0) = 0,\\
&v^{(m-2)}(1) = y(1) = \iota \int_0^{\vartheta} a(s)y(s)\,dB(s) = \iota \int_0^{\vartheta} a(s)v^{(m-2)}(s)\,dB(s).
\end{aligned}$$

Clearly, $u = I_{0^+}^{n-2}x, v = I_{0^+}^{m-2}y \in C([0,1], \mathbb{R}_+^1)$. Hence, $(I_{0^+}^{n-2}x, I_{0^+}^{m-2}y)$ is a positive solution of problem (1.1). Hence, (1.1) and (2.8) are equivalent. □

The vector (x,y) is a solution of system (2.8) if and only if $(x,y) \in C[0,1] \times C[0,1]$ is a solution of the following system of nonlinear integral equations:

$$\begin{cases} x(t) = \lambda \int_0^1 G(t,s)f(s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}x(s), I_{0^+}^{m-2}y(s))\,ds, \\ y(s) = \mu \int_0^1 H(s,\tau)g(\tau, I_{0^+}^{n-2}x(\tau), I_{0^+}^{n-2-\eta_1}x(\tau), \ldots, I_{0^+}^{n-2-\eta_{m-2}}x(\tau))\,d\tau. \end{cases} \tag{2.10}$$

Obviously, system (2.10) is equivalent to the following system of integral equations:

$$\begin{aligned}
x(t) = \lambda \int_0^1 G(t,s)f\bigg(& s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}x(s),\\
& I_{0^+}^{m-2}\bigg(\mu \int_0^1 H(s,\tau)g\big(\tau, I_{0^+}^{n-2}x(\tau), I_{0^+}^{n-2-\eta_1}x(\tau), \ldots, I_{0^+}^{n-2-\eta_{m-2}}x(\tau)\big)\,d\tau\bigg)\bigg)\,ds.
\end{aligned} \tag{2.11}$$

Let P be a normal cone of a Banach space E, and $e \in P$, $e > \theta$ (θ is a zero element of E). Define a component of P by $Q_e = \{u \in P|$ there exist constants $c, C > 0$ such that $ce \leq u \leq Ce\}$. $A : Q_e \times Q_e \to P$ is said to be mixed monotone if $A(u,y)$ is nondecreasing in u and nonincreasing in y, *i.e.*, $u_1 \leq u_2$ ($u_1, u_2 \in Q_e$) implies $A(u_1,y) \leq A(u_2,y)$ for any $y \in Q_e$, and $y_1 \leq y_2$ ($y_1, y_2 \in Q_e$) implies $A(u,y_1) \geq A(u,y_2)$ for any $u \in Q_e$. The element $u^\star \in Q_e$ is called a fixed point of A if $A(u^\star, u^\star) = u^\star$.

Lemma 2.6 [31, 32] *Suppose that $A : Q_e \times Q_e \to Q_e$ is a mixed monotone operator and there exists a constant σ, $0 < \sigma < 1$, such that*

$$A\left(lx, \frac{1}{l}y\right) \geq l^\sigma A(x,y), \quad x, y \in Q_e, 0 < l < 1, \tag{2.12}$$

then A has a unique fixed point $x^\star \in Q_e$, and, for any $x_0, y_0 \in Q_e$, we have

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = x^\star,$$

where

$$x_k = A(x_{k-1}, y_{k-1}), \qquad y_k = A(y_{k-1}, x_{k-1}), \quad k = 1, 2, \ldots$$

and the convergence rate is

$$\|x_k - x^\star\| = o\big(1 - r^{\sigma^k}\big), \qquad \|y_k - x^\star\| = o\big(1 - r^{\sigma^k}\big),$$

where r is a constant, $0 < r < 1$, and dependent on x_0, y_0.

Lemma 2.7 [31, 32] *Suppose that $A : Q_e \times Q_e \to Q_e$ is a mixed monotone operator and there exists a constant $\sigma \in (0,1)$ such that* (2.12) *holds. If $x^\star_\lambda \in Q_e$ is a unique solution of equation*

$$\lambda A(x,x) = x, \quad \lambda > 0,$$

then:

(1) *For any $\lambda_0 \in (0, +\infty)$, $\|x^\star_\lambda - x^\star_{\lambda_0}\| \to 0$, $\lambda \to \lambda_0$.*
(2) *If $0 < \sigma < \frac{1}{2}$, then $0 < \lambda_1 < \lambda_2$ implies $x^\star_{\lambda_1} \leq x^\star_{\lambda_2}$, $x^\star_{\lambda_1} \neq x^\star_{\lambda_2}$.*
(3) $\lim_{\lambda\to+\infty} \|x^\star_\lambda\| = +\infty$, $\lim_{\lambda\to 0^+} \|x^\star_\lambda\| = 0$.

Let $e(t) = t^{\alpha-n+1}$, we define a normal cone of $C[0,1]$ by

$$P = \big\{x \in C[0,1] : x(t) \geq 0, 0 \leq t \leq 1\big\},$$

also define a component of P by

$$Q_e = \left\{x \in P : \text{there exists } D > 1, \frac{1}{D}e(t) \leq x(t) \leq De(t), t \in [0,1]\right\}.$$

Remark 2.2 By simple calculation, for any $t \in [0,1]$, we have

$$\begin{aligned} I_{0^+}^{n-2} e(t) &= \frac{1}{\Gamma(n-2)} \int_0^t (t-s)^{n-3} s^{\alpha-n+1}\, ds \\ &= \frac{B(\alpha-n+2, n-2)}{\Gamma(n-2)} t^{\alpha-1} = \frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha)} t^{\alpha-1}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} I_{0^+}^{m-2} t^{\beta-m+1} &= \frac{1}{\Gamma(m-2)} \int_0^t (t-s)^{m-3} s^{\beta-m+1}\, ds \\ &= \frac{B(\beta-m+2, m-2)}{\Gamma(m-2)} t^{\beta-1} = \frac{\Gamma(\beta-m+2)}{\Gamma(\beta)} t^{\beta-1}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} I_{0^+}^{n-2-\mu_\kappa} e(t) &= \frac{1}{\Gamma(n-2-\mu_\kappa)} \int_0^t (t-s)^{n-3-\mu_\kappa} s^{\alpha-n+1}\, ds \\ &= \frac{B(\alpha-n+2, n-2-\mu_\kappa)}{\Gamma(n-2-\mu_\kappa)} t^{\alpha-\mu_\kappa-1} \\ &= \frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha-\mu_\kappa)} t^{\alpha-\mu_\kappa-1}, \quad \kappa = 1,2,\ldots,n-2, \end{aligned} \tag{2.15}$$

$$\begin{aligned} I_{0^+}^{n-2-\eta_\varrho} e(t) &= \frac{1}{\Gamma(n-2-\eta_\varrho)} \int_0^t (t-s)^{n-3-\eta_\varrho} s^{\alpha-n+1}\, ds \\ &= \frac{B(\alpha-n+2, n-2-\eta_\varrho)}{\Gamma(n-2-\eta_\varrho)} t^{\alpha-\eta_\varrho-1} \\ &= \frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_\varrho)} t^{\alpha-\eta_\varrho-1}, \quad \varrho = 1,2,\ldots,m-2. \end{aligned} \tag{2.16}$$

3 Main results

Theorem 3.1 *Suppose that* (S_1)-(S_4) *hold. Then the BVP* (1.1) *has a unique positive solution* $(u_\lambda^\star, v_\lambda^\star)$, *which satisfies*

$$\begin{aligned} &\frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)} t^{\alpha-1} \le u_\lambda^\star(t) \le \frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha)} t^{\alpha-1}, \\ &\frac{\Gamma(\beta-m+2)\mu t^{\beta-1}}{\Gamma(\beta)} \left(\frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)} \right)^{\varsigma} \int_0^1 \theta_2(s) s^{\varsigma(\alpha-1)} g(s,1,1,\ldots,1)\, ds \\ &\quad \le v_\lambda^\star(t) \\ &\quad \le \frac{\Gamma(\beta-m+2) M_2 \mu t^{\beta-1}}{\Gamma(\beta)} \left(\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})} + 1 \right)^{\varsigma} \\ &\qquad \times \int_0^1 (1-s)^{\beta-m+1} g(s,1,1,\ldots,1)\, ds, \end{aligned} \tag{3.1}$$

and at the same time $u_\lambda^\star$ *satisfies*:

(1) *For* $\lambda_0 \in (0,\infty)$, $\|u_\lambda^\star - u_{\lambda_0}^\star\| \to 0$, $\lambda \to \lambda_0$.
(2) *If* $0 < \sigma < \frac{1}{2}$, *then* $0 < \lambda_1 < \lambda_2$ *implies* $u_{\lambda_1}^\star \le u_{\lambda_2}^\star$, $u_{\lambda_1}^\star \ne u_{\lambda_2}^\star$.
(3) $\lim_{\lambda\to 0} \|u_\lambda^\star\| = 0$, $\lim_{\lambda\to+\infty} \|u_\lambda^\star\| = +\infty$.

Moreover, for any u_0, *we construct a successive sequence*

$$\begin{aligned} u_{k+1}(t) = I_{0^+}^{n-2} \Big\{ \lambda \int_0^1 G(t,s) \big[\phi\big(s, u_k(s), D_{0^+}^{\mu_1} u_k(s), \\ D_{0^+}^{\mu_2} u_k(s), \ldots, D_{0^+}^{\mu_{n-2}} u_k(s), A u_k^{(n-2)}(s)\big) \end{aligned}$$

$$+\psi\left(s,u_k(s),D_{0^+}^{\mu_1}u_k(s),D_{0^+}^{\mu_2}u_k(s),\dots,D_{0^+}^{\mu_{n-2}}u_k(s),Au_k^{(n-2)}(s)\right)\big]\,ds\Big\},$$
$$k=1,2,\dots,$$

and we have $\|u_k-u_\lambda^\star\|\to 0$ *as* $k\to\infty$, *and the convergence rate*

$$\|u_k-u_\lambda^\star\| = o\left(1-r^{\sigma^k}\right),$$

where r *is a constant*, $0<r<1$, *and dependent on* u_0. *Moreover*,

$$v_\lambda^\star(t)=I_{0^+}^{m-2}\left(\mu\int_0^1 H(t,s)g\left(s,u_\lambda^\star(s),D_{0^+}^{\eta_1}u_\lambda^\star(s),\dots,D_{0^+}^{\eta_{m-2}}u_\lambda^\star(s)\right)ds\right).$$

Proof We first consider the existence of a positive solution to problem (2.10). From the discussion in Section 2, we only need to consider the existence of a positive solution to BVP (2.11). In order to realize this purpose, let

$$Ax(t)=I_{0^+}^{m-2}\left(\mu\int_0^1 H(t,s)g\left(s,I_{0^+}^{n-2}x(s),I_{0^+}^{n-2-\eta_1}x(s),\dots,I_{0^+}^{n-2-\eta_{m-2}}x(s)\right)ds\right),$$

and, for any $x,w\in Q_e$, we define the operator $T_\lambda: Q_e\times Q_e\to P$ by

$$\begin{aligned}T_\lambda(x,w)(t)={}&\lambda\int_0^1 G(t,s)\big[\phi\left(s,I_{0^+}^{n-2}x(s),I_{0^+}^{n-2-\mu_1}x(s),\dots,I_{0^+}^{n-2-\mu_{n-2}}x(s),Ax(s)\right)\\&+\psi\left(s,I_{0^+}^{n-2}w(s),I_{0^+}^{n-2-\mu_1}w(s),\dots,I_{0^+}^{n-2-\mu_{n-2}}w(s),Aw(s)\right)\big]\,ds.\end{aligned}\tag{3.2}$$

Now we prove that $T_\lambda: Q_e\times Q_e\to P$ is well defined. For any $x,w\in Q_e$, by (2.13), (2.16), we have

$$\begin{aligned}&\int_0^1 H(s,\tau)g\left(\tau,I_{0^+}^{n-2}x(\tau),I_{0^+}^{n-2-\eta_1}x(\tau),\dots,I_{0^+}^{n-2-\eta_{m-2}}x(\tau)\right)d\tau\\
&\quad\le\int_0^1 H(s,\tau)g\bigg(\tau,\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha)}\tau^{\alpha-1},\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_1)}\tau^{\alpha-\eta_1-1},\dots,\\
&\qquad\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})}\tau^{\alpha-\eta_{m-2}-1}\bigg)d\tau\\
&\quad\le\int_0^1 H(s,\tau)g\left(\tau,\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha)}+1,\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_1)}+1,\dots,\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})}+1\right)d\tau\\
&\quad\le\int_0^1 H(s,\tau)g\left(\tau,\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})}+1,\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})}+1,\dots,\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})}+1\right)d\tau\\
&\quad\le M_2\left(\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})}+1\right)^{\varsigma}s^{\beta-m+1}\int_0^1(1-\tau)^{\beta-m+1}g(\tau,1,1,\dots,1)\,d\tau,\quad s\in[0,1],\\
&\int_0^1 H(s,\tau)g\left(\tau,I_{0^+}^{n-2}x(\tau),I_{0^+}^{n-2-\eta_1}x(\tau),\dots,I_{0^+}^{n-2-\eta_{m-2}}x(\tau)\right)d\tau\\
&\quad\ge\int_0^1 H(s,\tau)g\bigg(\tau,\frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}\tau^{\alpha-1},\frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha-\eta_1)}\tau^{\alpha-\eta_1-1},\dots,\\
&\qquad\frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha-\eta_{m-2})}\tau^{\alpha-\eta_{m-2}-1}\bigg)d\tau\end{aligned}$$

$$
\begin{aligned}
&\ge \int_0^1 H(s,\tau) g\left(\tau, \frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}\tau^{\alpha-1}, \frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}\tau^{\alpha-1}, \ldots, \frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}\tau^{\alpha-1}\right) d\tau \\
&\ge \left(\frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}\right)^{\varsigma} s^{\beta-m+1} \int_0^1 \theta_2(\tau)\tau^{\varsigma(\alpha-1)} g(\tau,1,1,\ldots,1)\, d\tau, \quad s \in [0,1].
\end{aligned}
$$

Hence, by (2.14), for any $s \in [0,1]$, we have

$$
\begin{aligned}
Ax(s) &= I_{0^+}^{m-2}\left(\mu \int_0^1 H(s,\tau) g\left(\tau, I_{0^+}^{n-2}x(\tau), I_{0^+}^{n-2-\eta_1}x(\tau), \ldots, I_{0^+}^{n-2-\eta_{m-2}}x(\tau)\right) d\tau\right) \\
&\le \frac{\Gamma(\beta-m+2)M_2\mu s^{\beta-1}}{\Gamma(\beta)} \left(\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})}+1\right)^{\varsigma} \\
&\quad \times \int_0^1 (1-\tau)^{\beta-m+1} g(\tau,1,1,\ldots,1)\, d\tau, \qquad (3.3) \\
Ax(s) &= I_{0^+}^{m-2}\left(\mu \int_0^1 H(s,\tau) g\left(\tau, I_{0^+}^{n-2}x(\tau), I_{0^+}^{n-2-\eta_1}x(\tau), \ldots, I_{0^+}^{n-2-\eta_{m-2}}x(\tau)\right) d\tau\right) \\
&\ge \frac{\mu\Gamma(\beta-m+2)s^{\beta-1}}{\Gamma(\beta)} \left(\frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}\right)^{\varsigma} \int_0^1 \theta_2(\tau)\tau^{\varsigma(\alpha-1)} g(\tau,1,1,\ldots,1)\, d\tau.
\end{aligned}
$$

By (3.3), (2.13), (2.15), (S_1), and Remark 2.1, we have

$$
\begin{aligned}
&\phi\left(s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}x(s), Ax(s)\right) \\
&\quad \le \phi\left(s, \frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha)}s^{\alpha-1}+1, \frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\mu_1)}s^{\alpha-\mu_1-1}+1, \ldots,\right. \\
&\qquad \frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\mu_{n-2})}s^{\alpha-\mu_{n-2}-1}+1, \frac{\Gamma(\beta-m+2)M_2\mu}{\Gamma(\beta)}\left(\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})}+1\right)^{\varsigma} \\
&\qquad \left.\times s^{\beta-1}\int_0^1 (1-\tau)^{\beta-m+1} g(\tau,1,1,\ldots,1)\, d\tau + 1\right) \\
&\quad \le \phi(s, Db+1, Db+1, \ldots, Db+1) \\
&\quad \le (Db+1)^{\sigma}\phi(s,1,1,\ldots,1) \\
&\quad \le 2^{\sigma} b^{\sigma} D^{\sigma}\phi(s,1,1,\ldots,1), \quad s \in (0,1), \qquad (3.4)
\end{aligned}
$$

where

$$
\begin{aligned}
D > \max\Bigg\{ &\Bigg[2^{\sigma} b^{\sigma} M_1\lambda \int_0^1 (1-s)^{\alpha-n+1}\phi(s,1,1,\ldots,1)\, ds \\
&+ M_1 c^{-\sigma}\lambda \int_0^1 s^{-\sigma(\alpha-1)}(1-s)^{\alpha-n+1}\psi(s,1,1,\ldots,1)\, ds \Bigg]^{\frac{1}{1-\sigma}}, 1, 2c, b^{-1}, \\
&\left[c^{\sigma}\lambda \int_0^1 \theta_1(s)s^{\sigma(\alpha-1)}\phi(s,1,1,\ldots,1)\, ds + 2^{-\sigma}b^{-\sigma}\lambda \int_0^1 \theta_1(s)\psi(s,1,1,\ldots,1)\, ds \right]^{-\frac{1}{1-\sigma}} \Bigg\}, \\
b = \max\Bigg\{ &\frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha-\mu_{n-2})}, 1, \frac{\Gamma(\beta-m+2)}{\Gamma(\beta)(M_2\mu)^{-1}}\left(\frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})}+1\right)^{\varsigma} \\
&\times \int_0^1 (1-\tau)^{\beta-m+1} g(\tau,1,1,\ldots,1)\, d\tau \Bigg\}.
\end{aligned}
$$

By (3.3), (2.13), (2.15), (S_1), and (S_2), we also have

$$\begin{aligned}
&\psi\big(s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}x(s), Ax(s)\big)\\
&\quad\le \psi\bigg(s, \frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}s^{\alpha-1}, \frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha-\mu_1)}s^{\alpha-\mu_1-1}, \ldots, \frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha-\mu_{n-2})}s^{\alpha-\mu_{n-2}-1},\\
&\qquad \frac{\mu\Gamma(\beta-m+2)s^{\beta-1}}{\Gamma(\beta)}\bigg(\frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}\bigg)^{\varsigma}\int_0^1 \theta_2(\tau)\tau^{\varsigma(\alpha-1)}g(\tau,1,1,\ldots,1)\,d\tau\bigg)\\
&\quad\le \psi\bigg(s, \frac{c}{D}s^{\alpha-1}, \frac{c}{D}s^{\alpha-\mu_1-1}, \ldots, \frac{c}{D}s^{\alpha-\mu_{n-2}-1}, \frac{c}{D^{\varsigma}}s^{\beta-1}\bigg)\\
&\quad\le \psi\bigg(s, \frac{c}{D}s^{\alpha-1}, \frac{c}{D}s^{\alpha-1}, \ldots, \frac{c}{D}s^{\alpha-1}\bigg)\\
&\quad\le \bigg(\frac{c}{D}s^{\alpha-1}\bigg)^{-\sigma}\psi(s,1,1,\ldots,1)\\
&\quad= c^{-\sigma}D^{\sigma}s^{-\sigma(\alpha-1)}\psi(s,1,1,\ldots,1), \quad s\in(0,1),
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
c = \min\bigg\{&\frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha)}, 1, \frac{\Gamma(\beta-m+2)\mu}{\Gamma(\beta)}\bigg(\frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha)}\bigg)^{\varsigma}\\
&\times\int_0^1 \theta_2(\tau)\tau^{\varsigma(\alpha-1)}g(\tau,1,\ldots,1)\,d\tau\bigg\}.
\end{aligned}$$

Noting $\frac{c}{D}s^{\alpha-1} < 1$, by (3.3), (2.13), (2.15), (S_1), and (S_2), we have

$$\begin{aligned}
&\phi\big(s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}x(s), Ax(s)\big)\\
&\quad\ge \phi\bigg(s, \frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}s^{\alpha-1}, \frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha-\mu_1)}s^{\alpha-\mu_1-1}, \ldots, \frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha-\mu_{n-2})}s^{\alpha-\mu_{n-2}-1},\\
&\qquad \frac{\mu\Gamma(\beta-m+2)}{\Gamma(\beta)}\bigg(\frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}\bigg)^{\varsigma}s^{\beta-1}\int_0^1 \theta_2(\tau)\tau^{\varsigma(\alpha-1)}g(\tau,1,1,\ldots,1)\,d\tau\bigg)\\
&\quad\ge \phi\bigg(s, \frac{c}{D}s^{\alpha-1}, \frac{c}{D}s^{\alpha-\mu_1-1}, \ldots, \frac{c}{D}s^{\alpha-\mu_{n-2}-1}, \frac{c}{D}s^{\beta-1}\bigg)\\
&\quad\ge \phi\bigg(s, \frac{c}{D}s^{\alpha-1}, \frac{c}{D}s^{\alpha-1}, \ldots, \frac{c}{D}s^{\alpha-1}\bigg)\\
&\quad\ge \bigg(\frac{c}{D}s^{\alpha-1}\bigg)^{\sigma}\phi(s,1,1,\ldots,1)\\
&\quad= c^{\sigma}D^{-\sigma}s^{\sigma(\alpha-1)}\phi(s,1,1,\ldots,1), \quad s\in(0,1),
\end{aligned} \tag{3.6}$$

and by (3.3), (2.13), (2.15), (S_1), and Remark 2.1, we also get

$$\begin{aligned}
&\psi\big(s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}x(s), Ax(s)\big)\\
&\quad\ge \psi\bigg(s, \frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha)}s^{\alpha-1}, \frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\mu_1)}s^{\alpha-\mu_1-1}, \ldots, \frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\mu_{n-2})}s^{\alpha-\mu_{n-2}-1},\\
&\qquad \frac{\Gamma(\beta-m+2)M_2\mu}{\Gamma(\beta)}\bigg(\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha)}+1\bigg)^{\varsigma}s^{\beta-m+1}\int_0^1 (1-\tau)^{\beta-m+1}g(\tau,1,\ldots,1)\,d\tau\bigg)
\end{aligned}$$

$$
\begin{aligned}
&\geq \psi\left(s, Dbs^{\alpha-1}, Dbs^{\alpha-\mu_1-1}, \dots, Dbs^{\alpha-\mu_{n-2}-1}, D^{\varsigma} bs^{\beta-m+1}\right) \\
&\geq \psi(s, Db+1, Db+1, \dots, Db+1) \\
&\geq (Db+1)^{-\sigma} \psi(s,1,1,\dots,1) \\
&\geq 2^{-\sigma} b^{-\sigma} D^{-\sigma} \psi(s,1,1,\dots,1), \quad s \in (0,1).
\end{aligned} \tag{3.7}
$$

For $x, w \in Q_e$, it follows from (3.4), (3.5) that

$$
\begin{aligned}
&\lambda \int_0^1 G(t,s)\phi\left(s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \dots, I_{0^+}^{n-2-\mu_{n-2}}x(s), Ax(s)\right) ds \\
&\quad \leq M_1 t^{\alpha-n+1}\lambda \int_0^1 (1-s)^{\alpha-n+1}\phi\left(s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \dots, I_{0^+}^{n-2-\mu_{n-2}}x(s), Ax(s)\right) ds \\
&\quad \leq 2^{\sigma} b^{\sigma} D^{\sigma} M_1 \lambda t^{\alpha-n+1} \int_0^1 (1-s)^{\alpha-n+1}\phi(s,1,1,\dots,1)\, ds < +\infty, \quad t \in [0,1],
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
&\lambda \int_0^1 G(t,s)\psi\left(s, I_{0^+}^{n-2}w(s), I_{0^+}^{n-2-\mu_1}w(s), \dots, I_{0^+}^{n-2-\mu_{n-2}}w(s), Aw(s)\right) ds \\
&\quad \leq M_1 t^{\alpha-n+1}\lambda \int_0^1 \psi\left(s, I_{0^+}^{n-2}w(s), I_{0^+}^{n-2-\mu_1}w(s), \dots, I_{0^+}^{n-2-\mu_{n-2}}w(s), Aw(s)\right) ds \\
&\quad \leq c^{-\sigma} D^{\sigma} M_1 \lambda t^{\alpha-n+1} \int_0^1 (1-s)^{\alpha-n+1} s^{-\sigma(\alpha-1)}\psi(s,1,1,\dots,1)\, ds < +\infty, \quad t \in [0,1].
\end{aligned} \tag{3.9}
$$

By (H_4), (3.8), and (3.9), we see that $T_\lambda : Q_e \times Q_e \to P$ is well defined.

Next, we will prove that $T_\lambda : Q_e \times Q_e \to Q_e$. It follows from (3.8), (3.9) that

$$
T_\lambda(x,w)(t) \leq Dt^{\alpha-n+1} = De(t), \quad t \in [0,1]. \tag{3.10}
$$

At the same time, by (3.6), (3.7), we have

$$
\begin{aligned}
&\lambda \int_0^1 G(t,s)\phi\left(s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \dots, I_{0^+}^{n-2-\mu_{n-2}}x(s), Ax(s)\right) ds \\
&\quad \geq t^{\alpha-n+1}\lambda \int_0^1 \theta_1(s)\phi\left(s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \dots, I_{0^+}^{n-2-\mu_{n-2}}x(s), Ax(s)\right) ds \\
&\quad \geq c^{\sigma} D^{-\sigma} \lambda t^{\alpha-n+1} \int_0^1 \theta_1(s) s^{\sigma(\alpha-1)}\phi(s,1,1,\dots,1)\, ds, \quad t \in [0,1],
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
&\lambda \int_0^1 G(t,s)\psi\left(s, I_{0^+}^{n-2}w(s), I_{0^+}^{n-2-\mu_1}w(s), \dots, I_{0^+}^{n-2-\mu_{n-2}}w(s), Aw(s)\right) ds \\
&\quad \geq t^{\alpha-n+1}\lambda \int_0^1 \theta_1(s)\psi\left(s, I_{0^+}^{n-2}w(s), I_{0^+}^{n-2-\mu_1}w(s), \dots, I_{0^+}^{n-2-\mu_{n-2}}w(s), Aw(s)\right) ds \\
&\quad \geq 2^{-\sigma} D^{-\sigma} b^{-\sigma} \lambda t^{\alpha-n+1} \int_0^1 \theta_1(s)\psi(s,1,1,\dots,1)\, ds, \quad t \in [0,1].
\end{aligned} \tag{3.12}
$$

Equations (3.11) and (3.12) imply that

$$
T_\lambda(x,w)(t) \geq \frac{1}{D} t^{\alpha-n+1} = \frac{1}{D} e(t), \quad t \in [0,1]. \tag{3.13}
$$

Hence, $T_\lambda : Q_e \times Q_e \to Q_e$ is well defined.

Next, we shall prove that $T_\lambda : Q_e \times Q_e \to Q_e$ is a mixed monotone operator. In fact, for any $x_1, x_2 \in Q_e$ and $x_1 \le x_2$, by the monotonicity of $I_{0^+}^{n-2-\mu_i}$, A and ϕ, for any $t \in [0,1]$, we have

$$\begin{aligned}&\lambda \int_0^1 G(t,s)\phi\big(s, I_{0^+}^{n-2}x_1(s), I_{0^+}^{n-2-\mu_1}x_1(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}x_1(s), Ax_1(s)\big)\,ds \\ &\quad\le \lambda \int_0^1 G(t,s)\phi\big(s, I_{0^+}^{n-2}x_2(s), I_{0^+}^{n-2-\mu_1}x_2(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}x_2(s), Ax_2(s)\big)\,ds.\end{aligned} \tag{3.14}$$

Hence, by (3.14), we have

$$T_\lambda(x_1, w) \le T_\lambda(x_2, w), \quad w \in Q_e, \tag{3.15}$$

that is, $T_\lambda(x,w)$ is nondecreasing on x for any $w \in Q_e$. Similarly, if $w_1 \ge w_2$, $w_1, w_2 \in Q_e$, from (S_1), for any $t \in [0,1]$, we have

$$\begin{aligned}&\lambda \int_0^1 G(t,s)\psi\big(s, I_{0^+}^{n-2}w_1(s), I_{0^+}^{n-2-\mu_1}w_1(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}w_1(s), Aw_1(s)\big)\,ds \\ &\quad\le \lambda \int_0^1 G(t,s)\psi\big(s, I_{0^+}^{n-2}w_2(s), I_{0^+}^{n-2-\mu_1}w_2(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}w_2(s), Aw_2(s)\big)\,ds.\end{aligned} \tag{3.16}$$

Hence, by (3.16), we have

$$T_\lambda(x, w_1) \le T_\lambda(x, w_2), \quad x \in Q_e, \tag{3.17}$$

i.e., $T_\lambda(x,w)$ is nonincreasing on w for any $x \in Q_e$. Hence, by (3.15) and (3.17), we see that $T_\lambda : Q_e \times Q_e \to Q_e$ is a mixed monotone operator.

Finally, we show that the operator T_λ satisfies (2.12). For any $x, w \in Q_e$ and $l \in (0,1)$, $t \in [0,1]$, by (S_2) and (S_3), we have

$$\begin{aligned}&\lambda \int_0^1 G(t,s)\phi\big(s, I_{0^+}^{n-2}lx(s), I_{0^+}^{n-2-\mu_1}lx(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}lx(s), Alx(s)\big)\,ds \\ &\quad\ge \lambda \int_0^1 G(t,s)\phi\big(s, lI_{0^+}^{n-2}x(s), lI_{0^+}^{n-2-\mu_1}x(s), \ldots, lI_{0^+}^{n-2-\mu_{n-2}}x(s), Alx(s)\big)\,ds \\ &\quad\ge \lambda l^\sigma \int_0^1 G(t,s)\phi\big(s, I_{0^+}^{n-2}x(s), I_{0^+}^{n-2-\mu_1}x(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}x(s), Ax(s)\big)\,ds,\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\lambda \int_0^1 G(t,s)\psi\big(s, I_{0^+}^{n-2}l^{-1}w(s), I_{0^+}^{n-2-\mu_1}l^{-1}w(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}l^{-1}w(s), Al^{-1}w(s)\big)\,ds \\ &\quad\ge \lambda \int_0^1 G(t,s)\psi\big(s, l^{-1}I_{0^+}^{n-2}w(s), l^{-1}I_{0^+}^{n-2-\mu_1}w(s), \ldots, l^{-1}I_{0^+}^{n-2-\mu_{n-2}}w(s), l^{-1}Aw(s)\big)\,ds \\ &\quad\ge \lambda l^\sigma \int_0^1 G(t,s)\psi\big(s, I_{0^+}^{n-2}w(s), I_{0^+}^{n-2-\mu_1}w(s), \ldots, I_{0^+}^{n-2-\mu_{n-2}}w(s), Aw(s)\big)\,ds.\end{aligned} \tag{3.19}$$

Equations (3.18), (3.19) imply that

$$T_\lambda\left(lx, \frac{1}{l}w\right) \ge l^\sigma T_\lambda(x,w), \quad x, w \in Q_e. \tag{3.20}$$

Hence, as regards Lemma 2.6 assume that there exists a unique positive solution $x_\lambda^\star \in Q_e$ such that $T_\lambda(x_\lambda^\star, x_\lambda^\star) = x_\lambda^\star$. It is easy to check that $x_\lambda^\star$ is a unique positive solution of (2.8) for any given $\lambda > 0$. Moreover, by Lemma 2.7 we have the following conclusions:

(1) For any $\lambda_0 \in (0, +\infty)$, $\|x_\lambda^\star - x_{\lambda_0}^\star\| \to 0$, $\lambda \to \lambda_0$.
(2) If $0 < \sigma < \frac{1}{2}$, then $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1}^\star \leq x_{\lambda_2}^\star$, $x_{\lambda_1}^\star \neq x_{\lambda_2}^\star$.
(3) $\lim_{\lambda\to 0} \|x_\lambda^\star\| = 0$, $\lim_{\lambda\to+\infty} \|x_\lambda^\star\| = +\infty$.

By Lemma 2.5, we have

$$\begin{cases} u_\lambda^\star(t) = I_{0^+}^{n-2} x_\lambda^\star(t), \\ v_\lambda^\star(t) = I_{0^+}^{m-2} y_\lambda^\star(t), \quad t \in [0,1]. \end{cases} \tag{3.21}$$

Hence, by (3.21) and the monotonicity and continuity of $I_{0^+}^{n-2}$, we get:

(1) $\|u_\lambda^\star - u_{\lambda_0}^\star\| \to 0$, $\lambda \to \lambda_0$.
(2) If $0 < \sigma < \frac{1}{2}$, then $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^\star \leq u_{\lambda_2}^\star$, $u_{\lambda_1}^\star \neq u_{\lambda_2}^\star$.
(3) $\lim_{\lambda\to 0} \|u_\lambda^\star\| = 0$, $\lim_{\lambda\to+\infty} \|u_\lambda^\star\| = +\infty$.

Moreover, for any $u_0(t) = I_{0^+}^{n-2} x_0 \in Q_e$, by Lemma 2.6, constructing a successive sequence

$$\begin{aligned} x_{k+1}(t) = \lambda \int_0^1 G(t,s)\bigl[&\phi\bigl(s, I_{0^+}^{n-2} x_k(s), I_{0^+}^{n-2-\mu_1} x_k(s), \dots, I_{0^+}^{n-2-\mu_{n-2}} x_k(s), Ax_k(s)\bigr) \\ &+ \psi\bigl(s, I_{0^+}^{n-2} x_k(s), I_{0^+}^{n-2-\mu_1} x_k(s), \dots, I_{0^+}^{n-2-\mu_{n-2}} x_k(s), Ax_k(s)\bigr)\bigr]\, ds, \quad k = 1, 2, \dots, \end{aligned}$$

by $u_{k+1}(t) = I_{0^+}^{n-2} x_{k+1}(t)$, then

$$\begin{aligned} u_{k+1}(t) = I_{0^+}^{n-2} \biggl\{\lambda \int_0^1 G(t,s)\bigl[&\phi\bigl(s, u_k(s), D_{0^+}^{\mu_1} u_k(s), D_{0^+}^{\mu_2} u_k(s), \dots, D_{0^+}^{\mu_{n-2}} u_k(s), Au_k^{(n-2)}(s)\bigr) \\ &+ \psi\bigl(s, u_k(s), D_{0^+}^{\mu_1} u_k(s), D_{0^+}^{\mu_2} u_k(s), \dots, D_{0^+}^{\mu_{n-2}} u_k(s), Au_k^{(n-2)}(s)\bigr)\bigr]\, ds\biggr\}, \\ &k = 1, 2, \dots, \end{aligned}$$

and we have $\|u_k - u_\lambda^\star\| = \|I_{0^+}^{n-2} x_k - I_{0^+}^{n-2} x_\lambda^\star\| \to 0$ as $k \to \infty$, the convergence rate is

$$\bigl\|u_k - u_\lambda^\star\bigr\| = \bigl\|I_{0^+}^{n-2} x_k - I_{0^+}^{n-2} x_\lambda^\star\bigr\| = o\bigl(1 - r^{\sigma^k}\bigr),$$

r is a constant, $0 < r < 1$, and dependent on u_0, where $u_\lambda^\star(t) = I_{0^+}^{n-2} x_\lambda^\star(t)$, and we easily get

$$v_\lambda^\star(t) = I_{0^+}^{m-2} \biggl(\mu \int_0^1 H(t,s) g\bigl(s, I_{0^+}^{n-2} x_\lambda^\star(s), I_{0^+}^{n-2-\eta_1} x_\lambda^\star(s), \dots, I_{0^+}^{n-2-\eta_{m-2}} x_\lambda^\star(s)\bigr)\, ds\biggr),$$

so by $u_\lambda^\star(t) = I_{0^+}^{n-2} x_\lambda^\star(t)$, we have

$$v_\lambda^\star(t) = I_{0^+}^{m-2} \biggl(\mu \int_0^1 H(t,s) g\bigl(s, u_\lambda^\star(s), D_{0^+}^{\eta_1} u_\lambda^\star(s), \dots, D_{0^+}^{\eta_{m-2}} u_\lambda^\star(s)\bigr)\, ds\biggr). \tag{3.22}$$

By (3.3), (3.22) , for any $t \in [0,1]$, we easily get

$$\frac{\Gamma(\alpha - n + 2)}{D\Gamma(\alpha)} t^{\alpha-1} \leq u_\lambda^\star(t) = I_{0^+}^{n-2} x_\lambda^\star(t) \leq \frac{D\Gamma(\alpha - n + 2)}{\Gamma(\alpha)} t^{\alpha-1},$$

$$
\begin{aligned}
&\frac{\Gamma(\beta-m+2)\mu}{\Gamma(\beta)}\left(\frac{\Gamma(\alpha-n+2)}{D\Gamma(\alpha)}\right)^{\varsigma} t^{\beta-1}\int_0^1 \theta_2(s)s^{\varsigma(\alpha-1)}g(s,1,1,\dots,1)\,ds \\
&\quad\le v_\lambda^\star(t) = I_{0^+}^{m-2}y_\lambda^\star(t) \\
&\quad\le \frac{\Gamma(\beta-m+2)M_2\mu}{\Gamma(\beta)}\left(\frac{D\Gamma(\alpha-n+2)}{\Gamma(\alpha-\eta_{m-2})}+1\right)^{\varsigma} t^{\beta-1} \\
&\qquad\times\int_0^1 (1-s)^{\beta-m+1}g(s,1,1,\dots,1)\,ds.
\end{aligned}
$$

Therefore, the proof of Theorem 3.1 is completed. □

Remark 3.1 Compared with previous work [23, 25], the fractional orders are involved not only in the nonlinearity f but also in the nonlinearity g and the uniqueness positive solution of equation (1.1) is dependent on eigenvalue λ. Moreover, compared with [25], an iterative sequence and the convergence rate are also given.

4 Example

Example 4.1 Consider the following boundary value problem:

$$
\begin{cases}
-D_{0^+}^{\frac{5}{2}}u(t) = \lambda f(t,u(t),D_{0^+}^{\frac{1}{2}}u(t),v(t)), \quad 0<t<1, \\
-D_{0^+}^{\frac{3}{2}}v(t) = \mu g(t,u(t)), \quad 0<t<1, \\
u(0)=u'(0)=0, \qquad u'(1)=\int_0^{\eta} u'(s)\,dA(s), \\
v(0)=0, \qquad v(1)=\int_0^{\vartheta} v(s)\,dB(s),
\end{cases}
\tag{4.1}
$$

where $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$, $h(s) = a(s) = 1$, $\eta = \vartheta = \frac{3}{4}$, $\chi = \tau = 1$. Let $u(t) = I_{0^+}^1 x(t)$, $v(t) = v(t)$, the equation can be changed to the following:

$$
\begin{cases}
D_{0^+}^{\frac{1}{2}}x(t) + \lambda f(t,I_{0^+}^1 x(t),I_{0^+}^{\frac{1}{2}}x(t),v(t)) = 0, \quad 0<t<1, \\
D_{0^+}^{\frac{3}{2}}v(t) + \mu g(t,I_{0^+}^1 x(t)) = 0, \quad 0<t<1, \\
x(0)=0, \qquad x(1)=\int_0^{\frac{3}{4}} x(s)\,dA(s), \\
v(0)=0, \qquad v(1)=\int_0^{\frac{3}{4}} v(s)\,dB(s),
\end{cases}
\tag{4.2}
$$

and

$$
\begin{aligned}
&\phi(t,x_1,x_2,x_3) = \left(t^{-\frac{1}{2}}+\cos t\right)x_1^{\frac{1}{7}} + 2tx_2^{\frac{1}{4}} + 2x_3^{\frac{1}{8}}, \\
&\psi(t,x_1,x_2,x_3) = t^{-\frac{1}{2}}x_1^{-\frac{1}{4}} + x_2^{-\frac{1}{6}} + (2-t)x_3^{-\frac{1}{5}}, \\
&g(t,u) = \left(3t+t^2\right)u^{\frac{3}{5}} + (t\sin t + t)u^{\frac{2}{3}}, \\
&A(t) = \begin{cases} 0, & t\in[0,\frac{1}{2}), \\ 6, & t\in[\frac{1}{2},\frac{3}{4}), \\ 2, & t\in[\frac{3}{4},1], \end{cases} \\
&B(t) = \begin{cases} 0, & t\in[0,\frac{1}{2}), \\ 4, & t\in[\frac{1}{2},\frac{3}{4}), \\ 1, & t\in[\frac{3}{4},1]. \end{cases}
\end{aligned}
$$

Hence,

$$\begin{aligned}
\int_0^{\eta} \chi t^{\alpha-n+1} h(t)\, dA(t) &= \int_0^{\frac{3}{4}} t^{\frac{1}{2}}\, dA(t) \\
&= 6 \times \left(\frac{1}{2}\right)^{\frac{1}{2}} - 4\left(\frac{3}{4}\right)^{\frac{1}{2}} \\
&\approx 6 \times 0.7071 - 4 \times 0.8660 \\
&= 0.7786 \\
&< 1, \\
\int_0^{\vartheta} \iota t^{\beta-1} g(t)\, dB(t) &= \int_0^{\frac{3}{4}} t^{\frac{1}{2}}\, dB(t) \\
&= 4 \times \left(\frac{1}{2}\right)^{\frac{1}{2}} - 3\left(\frac{3}{4}\right)^{\frac{1}{2}} \\
&\approx 4 \times 0.7071 - 3 \times 0.8660 \\
&= 0.2304 \\
&< 1.
\end{aligned}$$

Moreover, for any $(t,x_1,x_2,x_3) \in (0,1) \times (0,\infty)^3$ and $0 < l < 1$, we have

$$\begin{aligned}
\phi(t,lx_1,lx_2,lx_3) &= \bigl(t^{-\frac{1}{2}} + \cos t\bigr)(lx_1)^{\frac{1}{7}} + 2t(lx_2)^{\frac{1}{4}} + 2(lx_3)^{\frac{1}{8}} \\
&\geq l^{\frac{1}{4}}\bigl(\bigl(t^{-\frac{1}{2}} + \cos t\bigr)x_1^{\frac{1}{7}} + 2tx_2^{\frac{1}{4}} + 2x_3^{\frac{1}{8}}\bigr) \\
&= l^{\frac{1}{4}}\phi(t,x_1,x_2,x_3), \\
\psi\bigl(t,l^{-1}x_1,l^{-1}x_2,l^{-1}x_3\bigr) &= t^{-\frac{1}{2}}\bigl(l^{-1}x_1\bigr)^{-\frac{1}{4}} + \bigl(l^{-1}x_2\bigr)^{-\frac{1}{6}} + (2-t)\bigl(l^{-1}x_3\bigr)^{-\frac{1}{5}} \\
&\geq l^{\frac{1}{4}}\bigl(t^{-\frac{1}{2}}x_1^{-\frac{1}{4}} + x_2^{-\frac{1}{6}} + (2-t)x_3^{-\frac{1}{5}}\bigr) \\
&= l^{\frac{1}{4}}\psi(t,x_1,x_2,x_3), \\
g(t,lu) &= \bigl(3t+t^2\bigr)(lu)^{\frac{3}{5}} + (t\sin t + t)(lu)^{\frac{2}{3}} \\
&\geq l^{\frac{2}{3}}\bigl(\bigl(3t+t^2\bigr)u^{\frac{3}{5}} + (t\sin t + t)u^{\frac{2}{3}}\bigr) \\
&= l^{\frac{2}{3}}g(t,u).
\end{aligned}$$

Noting $\sigma = \frac{1}{4}$, $\psi(s,1,1,1) = s^{-\frac{1}{2}} + 3 - s$, $\phi(s,1,1,1) = s^{-\frac{1}{2}} + \cos s + 2s + 2$, $g(s,1) = 4s + s\sin s + s^2$, we have

$$\begin{aligned}
\int_0^1 (1-s)^{\alpha-n+1} s^{-\sigma(\alpha-1)}\psi(s,1,1,1)\, ds &\leq \int_0^1 s^{-\sigma(\alpha-1)}\psi(s,1,1,1)\, ds \\
&= \int_0^1 s^{-\frac{1}{4}\times\frac{3}{2}}\bigl(s^{-\frac{1}{2}} + 3 - s\bigr)\, ds \\
&= 8 + \frac{24}{5} - \frac{8}{13} < +\infty,
\end{aligned}$$

$$\begin{aligned}
\int_0^1 (1-s)^{\alpha-n+1}\phi(s,1,1,1)\,ds &\le \int_0^1 \phi(s,1,1,1)\,ds \\
&= \int_0^1 \bigl(s^{-\frac{1}{2}} + \cos s + 2s + 2\bigr)\,ds \\
&= 5 + \sin 1 < +\infty, \\
\int_0^1 (1-s)^{\beta-m+1} g(s,1)\,ds &\le \int_0^1 (1-s)^{\frac{1}{2}}\bigl(4s + s\sin s + s^2\bigr)\,ds \\
&\le \int_0^1 (1-s)^{\frac{1}{2}}\bigl(s^2 + 4s + 1\bigr)\,ds \\
&= \frac{10}{3} < +\infty.
\end{aligned}$$

Thus, the assumptions (S_1)-(S_4) of Theorem 3.1 hold. Then Theorem 3.1 implies that problem (4.1) has a unique solution. Furthermore, when $\lambda \to \lambda_0$, $\lambda_0 \in (0,+\infty)$, we have

$$\bigl\|x_\lambda^\star - x_{\lambda_0}^\star\bigr\| \to 0,$$

and $0 < \lambda_1 < \lambda_2$ implies

$$x_{\lambda_1}^\star(t) \le x_{\lambda_2}^\star(t), \qquad x_{\lambda_1}^\star(t) \not\equiv x_{\lambda_2}^\star(t).$$

Since $\sigma = \frac{1}{4} \in (0, \frac{1}{2})$,

$$\lim_{\lambda\to 0}\bigl\|x_\lambda^\star\bigr\| = 0, \qquad \lim_{\lambda\to +\infty}\bigl\|x_\lambda^\star\bigr\| = +\infty.$$

By $u_\lambda^\star(t) = I_{0^+}^1 x_\lambda^\star(t)$, we can easily get:

(1) $\lambda_0 \in (0,+\infty)$, $\|u_\lambda^\star - u_{\lambda_0}^\star\| \to 0$, $\lambda \to \lambda_0$.

(2) If $0 < \sigma < \frac{1}{2}$, then $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^\star \le u_{\lambda_2}^\star$, $u_{\lambda_1}^\star \not\equiv u_{\lambda_2}^\star$.

(3) $\lim_{\lambda\to 0}\|u_\lambda^\star\| = 0$, $\lim_{\lambda\to+\infty}\|u_\lambda^\star\| = +\infty$.

In addition, for any initial $u_0 = I_{0^+}^1 x_0 \in Q_e$, we construct a successive sequence

$$\begin{aligned}
x_{k+1}(t) = \lambda \int_0^1 G(t,s)\bigl[&\phi\bigl(t, I_{0^+}^1 x_k(t), I_{0^+}^{\frac{1}{2}} x_k(t), Ax_k(t)\bigr) \\
&+ \psi\bigl(t, I_{0^+}^1 x_k(t), I_{0^+}^{\frac{1}{2}} x_k(t), Ax_k(t)\bigr)\bigr]\,ds, \quad k = 1,2,\ldots,
\end{aligned}$$

by $u_{k+1}(t) = I_{0^+}^1 x_{k+1}(t)$, then

$$\begin{aligned}
u_{k+1}(t) = I_{0^+}^1 \biggl\{\lambda \int_0^1 G(t,s)\bigl[&\phi\bigl(s, u_k(t), D_{0^+}^{\frac{1}{2}} u_k(t), Au_k'(t)\bigr) \\
&+ \psi\bigl(s, u_k(t), D_{0^+}^{\frac{1}{2}} u_k(t), Au_k'(t)\bigr)\bigr]\,ds\biggr\}, \quad k = 1,2,\ldots,
\end{aligned}$$

and we have $\|u_k - u_\lambda^\star\| = \|I_{0^+}^1 x_k - I_{0^+}^1 x_\lambda^\star\| \to 0$ as $k \to \infty$, and the convergence rate

$$\bigl\|u_k - u_\lambda^\star\bigr\| = \bigl\|I_{0^+}^1 x_k - I_{0^+}^1 x_\lambda^\star\bigr\| = o\bigl(1 - r^{\sigma^k}\bigr),$$

r is a constant, $0 < r < 1$, and dependent on u_0. Moreover,

$$v_\lambda^\star(t) = \mu \int_0^1 H(t,s)g\big(s, u_\lambda^\star(s)\big)\,ds.$$

5 Conclusions

In this paper, some existence results are obtained for the case where the nonlinearity is allowed to be singular in regard to not only the time variable but also the space variable and the fractional orders are involved in the nonlinearity of the boundary value problem (1.1). Moreover, our equation system contains many types of equation systems because there are many parameters in our equation system and the uniqueness of the positive solution of equation (1.1) is dependent on λ. An iterative sequence and convergence rate are given which are important for practical applications. The method which we used for the analysis in this paper is the fixed point theorem of a mixed monotone operator in cone.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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