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Uniqueness of meromorphic functions sharing two sets with least possible cardinalities

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Abstract. Let f and g be two nonconstant meromorphic functions sharing two finite sets, namely $S \subset \mathbb{C}$ and $\{\infty\}$. We prove two uniqueness theorems under weaker conditions on ramification indices, reducing the cardinality of the shared set S and weakening the nature of sharing of the set $\{\infty\}$ which improve results of Fang-Lahiri [7], Lahiri [17], Banerjee -Majumder-Mukherjee [5] and others.

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1 Introduction, Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (Counting Multiplicities) and if we do not consider the multiplicities, then f and g are said to share the value a IM (Ignoring Multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [12, 23]. Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity then we replace the above set by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer m we denote by $N(r, a; f \mid\geq m)$ ($\overline{N}(r, a; f \mid\geq m)$) the counting function (reduced counting function) of those a-points of f whose multiplicities are greater than or equal to m. We put

$$N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2);$$

$$\delta_2(a; f) = 1 - \limsup_{r\to\infty} \frac{N_2(r, a; f)}{T(r, f)}$$

and

$$\delta_{(k+1}(\infty; f) = 1 - \limsup_{r\to\infty} \frac{N(r, \infty; f \mid\geq k+1)}{T(r, f)}.$$

We agree to denote by E any subset of nonnegative reals of finite measure. For a nonconstant meromorphic function $f(z)$, we denote by $S(r, f)$ any quantity such that $S(r, f) = o(T(r, f)$, as $r \to \infty$, $r \notin E$.

In 1976, F. Gross [10] raised the following question.

Question A. *Can one find finite sets S_j, $j = 1, 2$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1$, 2 must be identical?*

As a natural outcome of the above question Lin and Yi raised the following question in [22].

Question B. *Can one find finite sets S_j, $j = 1, 2$ such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1$, 2 must be identical ?*

During the last few years a great deal of works has been directed by researchers to answer the above questions. A nice source of results on the topic is the monograph written by Yang and Yi [23] (see also [1]-[8], [10], [12], [16]-[19], [22]-[30]).

In 2003, Fang-Lahiri [7] exhibited a unique range set with smaller cardinalities than that obtained previously imposing some restrictions on the poles of f and g in the following result.

Theorem A. [7] *Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n(\geq 7)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g be two nonconstant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

In 2001, Lahiri [15, 16] introduced an idea of a gradation of sharing of values and sets known as weighted sharing as follows.

Definition 1.1. [15,16] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ of multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [15] *Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a positive integer or ∞. We denote by $E_f(S, k)$the set $\bigcup_{a \in S} E_k(a; f)$.*

With the notion of weighted sharing of sets improving Theorem A, Lahiri [17] proved the following theorem.

Theorem B. [17] *Let S be defined as in Theorem A. If f and g be two non-constant meromorphic functions such that $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 1$ then $f \equiv g$.*

Suppose that the polynomial $P(w)$ is defined by

$$P(w) = a\omega^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2 \qquad (1.1)$$

where $n \geq 3$ is an integer and a and b are two nonzero complex numbers satisfying
$ab^{n-2} \neq 2$. We also define

$$R(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \qquad (1.2)$$

where α_1, α_2 are two distinct roots of $n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0$. It can be shown that $P(w)$ has only simple roots (see [2,4]).

In 2011, Banerjee [4] improved Theorem B in the following result by showing that the condition on the ramification index ceases to exists when $n \geq 8$.

Theorem C. [4] *Let $S = \{w \mid P(w) = 0\}$, where $P(w)$ is given by (1.1) and $n(\geq 6)$. Suppose that f and g are two nonconstant meromorphic functions satisfying $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ and $\Theta_f + \Theta_g + min\{\Theta(b, f), \Theta(b, g)\} > 8 - n$, where $\Theta_f = 2\Theta(0; f) + \Theta(b; f) + \Theta(\infty; f)$ and Θ_g is defined similarly. Then $f \equiv g$.*

In a recent paper Banerjee-Majumder-Mukherjee [5] raised the following questions.

Question C. *What happens if in Theorem C, $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$ is replaced by $E_f(\{\infty\},k) = E_g(\{\infty\},k)$ where k is a non-negative integer?*

Question D. *Can the deficiency condition in Theorem C be further relaxed ?*

They provided some affirmative answers to the above questions in the following theorem.

Theorem D. [5] *Let $S = \{w \mid P(w) = 0\}$, where $P(w)$ is given by* (1.1) *and $n(\geq 6)$ is an integer. Let c, $d \in \mathbb{C}$ be such that c, $d \notin S\bigcup\{0,b\}$. Suppose that f and g are two nonconstant meromorphic functions satisfying $E_f(S,m) = E_g(S,m)$ and $E_f(\{\infty\},k) = E_g(\{\infty\},k)$ and f and g have respectively c-point and d-point of multiplicity $\geq p+1$ where p, k are non-negative integers or infinity such that $p^* + \frac{1}{k+1} \leq 1$, where $p^* = 1$, if $p = 0$ and $= \frac{2}{p+1}$, if $p \geq 1$. If either*
(i) $m \geq 2$ and

$$\Theta_f + \Theta_g + min\{\delta_f, \delta_g\} + p^* min\{\delta(c; f), \delta(d; g)\} > 7 + p^* + \frac{1}{k+1} - n$$

(ii) $m = 1$ and

$$\begin{aligned}\Theta_f + \Theta_g + \frac{1}{2} min\{&\Theta(0; f) + \Theta(b; f) + \Theta(\infty; f) + \delta_f, \Theta(0; g) + \Theta(b; g)\\ &+ \Theta(\infty; g) + \delta_g\} + min\{\delta_f, \delta_g\} + p^* min\{\delta(c; f), \delta(d; g)\}\\ &> 8 + p^* + \frac{1}{k+1} - n\end{aligned}$$

or
(iii) $m = 0$ and

$$\begin{aligned}\Theta_f + \Theta_g &+ \Theta(0; f) + \Theta(b; f) + \Theta(\infty; f) + 2\delta_f + \Theta(0; g) + \Theta(b; g) + \Theta(\infty; g)\\ &+ 2\delta_g + min\{\Theta(0; f) + \Theta(b; f) + \Theta(\infty; f), \Theta(0; g) + \Theta(b; g) + \Theta(\infty; g)\}\\ &+ p^* min\{\delta(c; f), \delta(d; g)\} > 13 + p^* + \frac{1}{k+1} - n,\end{aligned}$$

then $f \equiv g$, where $\Theta_f = 2\Theta(0; f) + 2\Theta(b; f) + \Theta(\infty; f) + \frac{1}{2(k+1)}\delta_{(k+1}(\infty; f)$ and $\delta_f = \sum_{w \in S} \delta(w, f)$, and Θ_g is defined similarly .

Theorem D leads us to the following observations.

Observation 1.1. $p = 0 \Rightarrow p^* = 1 \Rightarrow k = \infty$;

Observation 1.2. $p = 1 \Rightarrow p^* = \frac{2}{p+1} = 1 \Rightarrow k = \infty$; $\Theta(c; f) \geq \frac{p}{p+1} = \frac{1}{2}$ and $\Theta(d; g) \geq \frac{p}{p+1} = \frac{1}{2}$;

Observation 1.3. $p = 2 \Rightarrow p^* = \frac{2}{p+1} = \frac{2}{3} \Rightarrow k \geq 2$; $\Theta(c; f) \geq \frac{p}{p+1} = \frac{2}{3}$ and $\Theta(d; g) \geq \frac{p}{p+1} = \frac{2}{3}$.

Observation 1.4. $p = 3 \Rightarrow k = 1$; $\Theta(c; f) \geq \frac{p}{p+1} = \frac{3}{4}$ and $\Theta(d; g) \geq \frac{p}{p+1} = \frac{3}{4}$.

Thus we observe that the least possible finite value of k is 1. The theorem is silent when $k = 0$. Above theorem, thus leads us to the following questions.

Question 1.1. Is it possible to prove the above theorem with some finite value of k, say $k = 1$ when $p = 1$?

Question 1.2. Is it possible to prove the above theorem with $k = 0$ when $p = 1$?

Question 1.3. Is it possible to reduce the cardinality of the main shared set S from $n \geq 6$ to $n \geq 4$?

Question 1.4. Is it possible to prove Theorem D under weaker conditions on ramification indices?

We answer all the above questions in affirmative in two theorems to follow. However, we consider only the case when $m = 2$, that is when f and g share the set S with weight 2. Note that in the definition of the polynomial $P(w)$, we require $ab^{n-2} \neq 2$. For our purpose, in addition to it we assume $ab^{n-2} \neq 1, 4, \pm 2\omega$, where ω is a complex cube root of unity, by which the polynomial $P(w)$ will not lose any of its properties mentioned above. Thus from now on our set S is given by $S = \{w \mid P(w) = 0\}$ where $P(w)$ is given by (1.1) with $ab^{n-2} \neq 2, 1, 4, \pm 2\omega$.

Below we state our main results.

Theorem 1.1. *Let $S = \{w \mid P(w) = 0\}$, where $P(w)$ is given by* (1.1) *and $n(\geq 4)$ and $ab^{n-2} \neq 2, 1, 4, \pm 2\omega$. If f and g be two nonconstant meromorphic functions such that $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, k) = E_g(\{\infty\}, k)$, where $k \geq 1$ and there exist $c, d \notin S \cup \{0, b, \infty\}$, such that the zeros of $f - c$ and $g - d$ are of multiplicity $\geq p + 1$, where $p \geq 1$. Then*

$$\Theta_f^* + \Theta_g^* > 6 + \frac{2}{(n-2)k + n - 3} + \frac{4}{p+1} - n, \tag{1.3}$$

implies $f \equiv g$, where

$$\Theta_f^* = 2\Theta(0; f) + 2\Theta(b; f) + \Theta(\infty; f) + \frac{2}{p+1} min\{\delta(c; f), \delta(d; g)\} + \sum_{a \notin S \cup \{0,b,c,d,\infty\}} \delta_2(a, f)$$

and Θ_g^* *is defined similarly.*

Theorem 1.2. *Let* $S = \{w \mid P(w) = 0\}$*, where* $P(w)$ *is given by* (1.1) *and* $n \geq 5$ *and* $ab^{n-2} \neq 2, 1, 4, \pm 2\omega$*. If* f *and* g *be two nonconstant meromorphic functions such that* $E_f(S, 2) = E_g(S, 2)$ *and* $E_f(\{\infty\}, k) = E_g(\{\infty\}, k)$*, where* $k \geq 0$ *and there exist* c*,* $d \notin S \cup \{0, b, \infty\}$*, such that the zeros of* $f - c$ *and* $g - d$ *are of multiplicity* $\geq p + 1$*, where* $p \geq 1$*, then the inequality* (1.3) *implies* $f \equiv g$*.*

From the definitions of Θ_f and Θ_f^* of Theorem D and Theorem 1.1 respectively we see that

$$\Theta_f^* = \Theta_f - \frac{1}{2(k+1)}\delta_{(k+1}(\infty; f) + \frac{2}{p+1}min\{\delta(c; f), \delta(d; g)\} + \sum \delta_2(a; f).$$

Thus (1.3) reduces to

$$\begin{aligned}&\Theta_f + \Theta_g + \frac{4}{p+1}min\{\delta(c; f), \delta(d; g)\} + \sum \delta_2(a; f) + \sum \delta_2(a; g)\\ &> 6 + \frac{2}{(n-2)k + n - 3} + \frac{4}{p+1} - n + \frac{1}{2(k+1)}\{\delta_{(k+1}(\infty; f) + \delta_{(k+1}(\infty; g)\},\end{aligned}$$

i.e.

$$\begin{aligned}&\Theta_f + \Theta_g\\ &> 6 + \frac{2}{(n-2)k + n - 3} + \frac{4}{p+1} - n + \frac{1}{2(k+1)}\{\delta_{(k+1}(\infty; f) + \delta_{(k+1}(\infty; g)\}\\ &- \left[\frac{4}{p+1}min\{\delta(c; f), \delta(d; g)\} + \sum \delta_2(a; f) + \sum \delta_2(a; g)\right].\end{aligned}$$

If we call the right hand side of the above inequality as A, then inequality (1.3) takes the form

$$\Theta_f + \Theta_g > A.$$

Whereas the condition for Theorem D implies

$$\Theta_f + \Theta_g > 7 + \frac{2}{p+1} + \frac{1}{k+1} - n - [min\{\delta_f, \delta_g\} + \frac{2}{p+1}min\{\delta(c; f), \delta(d; g)\}].$$

If we denote the quantity on the righthand side of the above inequality as B, then we have the condition of Theorem D, as

$$\Theta_f + \Theta_g > B.$$

We establish our claim by showing that $B \geq A$.

We see that for $k \geq 1$ with $p \geq 2$ and noting that $min\{\delta_f, \delta_g\} \leq \sum \delta_2(a; f)$,

$$\begin{aligned} A =& 6 + \frac{2}{(n-2)k+n-3} + \frac{4}{p+1} - n + \frac{1}{2(k+1)}\{\delta_{(k+1}(\infty; f) + \delta_{(k+1}(\infty; g)\} \\ &- \left[\frac{4}{p+1} min\{\delta(c; f), \delta(d; g)\} + \sum \delta_2(a; f) + \sum \delta_2(a; g)\right] \\ \leq& 6 + \frac{2}{2n-5} + \frac{4}{p+1} - n + \frac{1}{k+1} - \frac{4}{p+1} min\{\delta(c; f), \delta(d; g)\} \\ &- \sum \delta_2(a; f) - \sum \delta_2(a; g) \\ \leq& 7 + \frac{2}{p+1} + \frac{1}{k+1} - n - \left[min\{\delta_f, \delta_g\} + \frac{2}{p+1} min\{\delta(c; f), \delta(d; g)\}\right] \\ &+ \frac{2}{2n-5} + \frac{2}{p+1} - 1 - \frac{2}{p+1} min\{\delta(c; f), \delta(d; g)\} + min\{\delta_f, \delta_g\} \\ &- \sum \delta_2(a; f) - \sum \delta_2(a; g) \\ \leq& B + \frac{2}{2n-5} + \frac{2}{p+1} - 1 - \frac{2}{p+1} min\{\delta(c; f), \delta(d; g)\} - \sum \delta_2(a; g). \end{aligned}$$

Thus for $p \geq 2$, we have from above for $n \geq 6$,

$$A \leq B + \frac{2}{2n-5} + \frac{2}{3} - 1 = B - \frac{2n-11}{3(2n-5)} < B.$$

We conclude this section with the definition of a few more notations as follows.

Definition 1.3. [4,15] *Let f and g be two nonconstant meromorphic functions such that f and g share $(a, 0)$ for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, and an a-point of g of multiplicity q. We denote by $\overline{N}_L(r, a; f)(\overline{N}_L(r, a; g))$ the reduced counting function of those a-points of f and g where $p > q(q > p)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from that of the corresponding a-points of g.*

We note from the above definition that $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$. We also denote by $N_E^{1)}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q = 1$.

2 Lemmas

In this section we present some lemmas which will be required to establish our results. In the lemmas several times we use the function H defined by $H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}$ where F and G are two nonconstant meromorphic functions.

Let f and g be two nonconstant meromorphic functions and

$$F = R(f), G = R(g), \tag{2.1}$$

where $R(w)$ is given by (1.2). From (1.2) and (2.1) it is clear that

$$T(r,f) = \frac{1}{n}T(r,F) + S(r,f), T(r,g) = \frac{1}{n}T(r,G) + S(r,g). \tag{2.2}$$

Lemma 2.1. [22]*If F, G be two nonconstant meromorphic functions such that they share $(1,0)$ and $H \not\equiv 0$ then*

$$N_E^{1)}(r,1;F \mid= 1) = N_E^{1)}(r,1;G \mid= 1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *Let F, G be given by (2.1) and $H \not\equiv 0$. If F, G share $(1,m)$ and f, g share (∞,k), then for any arbitrary set of complex numbers $\{a_j\} \subset \mathbb{C} \setminus S \cup \{0,b\}$, $j = 1,2,\ldots,l$,*

$$\begin{aligned}\overline{N}(r,H) \leq & \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,0;g) \\ & + \overline{N}(r,b;g) + \overline{N}_*(r,\infty;f,g) + \sum_{j=1}^{l} \overline{N}(r,a_j;f| \geq 2) \\ & + \sum_{j=1}^{l} \overline{N}(r,a_j;g| \geq 2) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function corresponding to the zeros of f' which are not the zeros of $f(f-b)\prod_{j=1}^{l}(f-a_j)$ and $F-1$. $\overline{N}_0(r,0;g')$ is defined similarly.

Proof. From the definitions of F and G, we have

$$F' = \frac{(n-2)af^{n-1}(f-b)^2 f'}{n(n-1)(f-\alpha_1)^2(f-\alpha_2)^2}, \quad G' = \frac{(n-2)ag^{n-1}(g-b)^2 g'}{n(n-1)(g-\alpha_1)^2(g-\alpha_2)^2}. \tag{2.3}$$

It is obvious that the simple zeros of $f-\alpha_1$ and $f-\alpha_2$ are the simple poles of F, the simple zeros of $g-\alpha_1$ and $g-\alpha_2$ are the simple poles of G. It can be easily verified that the simple zeros of $f-\alpha_1$, $f-\alpha_2$, $g-\alpha_1$ and $g-\alpha_2$

are not the poles of H. Now it is easy to note that the poles of H occur at
(i) the poles of f and g of different multiplicities;
(ii) the 1-points of F and G of different multiplicities;
(iii) zeros of $f(f-b)$ and $g(g-b)$;
(iv) multiple zeros of $f-a_j$ and $g-a_j$, $j=1,2,\ldots,l$;
(v) zeros of f' and g', which are not the zeros of $f(f-b)\prod_{j=1}^{l}(f-a_j)$, $F-1$ and $g(g-b)\prod_{j=1}^{l}(g-a_j)$, $G-1$ respectively.
Since the poles of H are all simple, the lemma follows from above observations. □

Lemma 2.3. [3]*Let f and g be two nonconstant meromorphic functions sharing $(1,m)$, where $0\le m\le\infty$. Then*

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)-N_E^{1)}(r,1;f)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\le\frac{1}{2}[N(r,1;f)+N(r,1;g)].$$

Lemma 2.4. [20] *Let f be a nonconstant meromorphic function and let $R(f)=\frac{\sum_{k=0}^{n}a_kf^k}{\sum_{j=0}^{m}b_jf^j}$ be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n\neq 0$, $b_m\neq 0$. Then $T(r,R(f))=dT(r,f)+S(r,f)$, where $d=max\{m,n\}$.*

Lemma 2.5. [2] *Let F and G be given by (2.1) and $H\not\equiv 0$. If F and G share $(1,m)$ and f, g share (∞,k), where $0\le m<\infty$, $0\le k<\infty$, then*

$$\begin{aligned}&[(n-2)k+n-3]\overline{N}(r,\infty;f\mid\ge k+1)\\&\quad=[(n-2)k+n-3]\overline{N}(r,\infty;g\mid\ge k+1)\\&\quad\le\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Lemma 2.6. *Let f and g be two nonconstant meromorphic functions such that $E_f(S,m)=E_g(S,m)$, and $E_f(\{\infty\},k)=E_g(\{\infty\},k)$, where $0\le m<\infty$ and $0\le k<\infty$ are integers. Let $\{a_j\}\subset\mathbb{C}\setminus S\cup\{0,b\}$, $j=1,2,\ldots,l$, be an arbitrary set of complex numbers. Then*

$$\begin{aligned}&\left\{\frac{n}{2}+1+l\right\}\{T(r,f)+T(r,g)\}\\&\le 2[\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\&+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+\sum_{j=1}^{l}\{N_2(r,a_j;f)+N_2(r,a_j;g)\}\\&+S(r,f)+S(r,g).\end{aligned}$$

Proof. It is obvious that F and G share $(1, m)$. Then from the second main theorem we obtain from Lemmas 2.1-2.4,

$$\begin{aligned}
& \{n+l+1\}\{T(r,f)+T(r,g)\} \\
\leq\ & \overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\sum_{j=1}^{l}\overline{N}(r,a_j;f)+\overline{N}(r,1;F)+\overline{N}(r,0;g) \\
+\ & \overline{N}(r,b;g)+\overline{N}(r,\infty;g)+\sum_{j=1}^{l}\overline{N}(r,a_j;g)+\overline{N}(r,1;G)-N_0(r,0;f')-N_0(r,0;g') \\
\leq\ & \overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\sum_{j=1}^{l}\overline{N}(r,a_j;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g) \\
+\ & \overline{N}(r,\infty;g)+\sum_{j=1}^{l}\overline{N}(r,a_j;g)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)+\frac{n}{2}\{T(r,f)+T(r,g)\} \\
+\ & \{\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g) \\
+\ & \overline{N}_*(r,\infty;f,g)+\sum_{j=1}^{l}\overline{N}(r,a_j;f|\geq 2)+\sum_{j=1}^{l}\overline{N}(r,a_j;g|\geq 2) \\
+\ & \overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')\}-N_0(r,0;f')-N_0(r,0;g') \\
\leq\ & 2\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;f)+\overline{N}(r,b;f)\}+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g) \\
+\ & \overline{N}_*(r,\infty;f,g)+\sum_{i=1}^{l}\{N_2(r,a_j;f)+N_2(r,a_j;g)\}+\frac{n}{2}\{T(r,f)+T(r,g)\} \\
-\ & \left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g),
\end{aligned}$$

the Lemma follows from above. □

Lemma 2.7. [4] *Let f, g be two nonconstant meromorphic functions sharing $(\infty, 0)$ and suppose that α_1 and α_2 are two distinct roots of the equation $n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0$. Then*

$$\frac{f^n}{(f-\alpha_1)(f-\alpha_2)}.\frac{g^n}{(g-\alpha_1)(g-\alpha_2)} \not\equiv \frac{n^2(n-1)^2}{a^2},$$

where $n \geq 3$ is an integer.

Lemma 2.8. [9] *Let $Q(w) = (n-1)^2(w^n-1)(w^{n-2}-1)-n(n-2)(w^{n-1}-1)^2$, then $Q(w) = (w-1)^4(w-\beta_1)(w-\beta_2)..(w-\beta_{2n-6})$ where $\beta_j \in \mathbb{C}\setminus\{0,1\}$,$(j = 1, 2, \ldots, 2n-6)$ which are pairwise distinct.*

Lemma 2.9. *Let F, G be given by* (2.1)*, where $n \geq 4$ is an integer. If f, g share $(\infty, 0)$ then $F \equiv G \Rightarrow f \equiv g$.*

Proof. From the definitions of F, G we observe that

$$F \equiv G \Rightarrow \frac{f^n}{(f-\alpha_1)(f-\alpha_2)} \equiv \frac{g^n}{(g-\alpha_1)(g-\alpha_2)}.$$

Therefore f, g share $(0, \infty)$ and (∞, ∞). Then from above and in view of the definitions of $R(w)$ we obtain

$$\begin{aligned} n(n-1)f^2g^2(f^{n-2}-g^{n-2}) - 2n(n-2)bfg(f^{n-1}-g^{n-1}) \\ +(n-1)(n-2)b^2(f^n-g^n) = 0. \end{aligned} \tag{2.4}$$

Let $h = \frac{f}{g}$ that is $f = gh$ which on substitution in (2.4) yields

$$n(n-1)h^2g^2(h^{n-2}-1) - 2n(n-2)bhg(h^{n-1}-1) + (n-1)(n-2)b^2(h^n-1) = 0. \tag{2.5}$$

Note that since f and g share $(0, \infty)$ and (∞, ∞), 0, ∞ are the exceptional values of Picard of h. If h is nonconstant then from Lemma 2.8 and (2.5) we have

$$\{n(n-1)h(h^{n-2}-1)g - n(n-2)b(h^{n-1}-1)\}^2 = -n(n-2)b^2Q(h) \tag{2.6}$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)...(h-\beta_{2n-6})$,$\beta_j \in \mathbb{C} \setminus \{0, 1\}$,$j = 1, 2, \ldots, 2n-6$ which are pairwise distinct. From (2.6) we observe that each zero of $h - \beta_j$, $j = 1, 2, \ldots, 2n-6$ is of order at least two. Therefore by the second main theorem we obtain

$$\begin{aligned} (2n-6)T(r,h) &\leq \overline{N}(r, \infty; h) + \overline{N}(r, 0; h) + \sum_{j=1}^{2n-6} \overline{N}(r, \beta_j; h) + S(r,h) \\ &\leq \frac{1}{2}(2n-6)T(r,h) + S(r,h), \end{aligned}$$

which is a contradiction for $n \geq 4$.

Thus h must be a constant. From (2.5) and (2.6) it follows that $h^{n-2}-1 = 0$ and $h^{n-1}-1 = 0$ which implies that $h \equiv 1$.Therefore $f \equiv g$. This completes the proof. □

Lemma 2.10. [4] *Let F, G be given by* (2.1) *and S be defined as in Theorem 1, where $n \geq 4$. If $E_f(S, 0) = E_g(S, 0)$ then $S(r, f) = S(r, g)$.*

Lemma 2.11. *Let f and g be two nonconstant meromorphic functions such that $E_f(\{\infty\}, 0) = E_g(\{\infty\}, 0)$ and $E_f(S, 0) = E_g(S, 0)$, where S is as defined in Theorem 1.1. Let F and G be given by* (2.1)*. If F is a bilinear transformation of G, then $f \equiv g$.*

Proof. In this case we have

$$F \equiv \frac{AG+B}{CG+D}, \tag{2.7}$$

where A, B, C, D are constants such that $AD - BC \neq 0$. Also $T(r, F) = T(r, G) + O(1)$, and hence from (2.2)

$$T(r, f) = T(r, g) + O(1). \tag{2.8}$$

Since $R(w) - c = \frac{a(w-b)^3 Q_{n-3}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}$, where the restrictions on a, b as stated in the Theorem 1.1, shows that $c = \frac{ab^{n-2}}{2} \neq 1, \frac{1}{2}, 2, \pm\omega$, ω being a complex cube root of unity and $Q_{n-3}(w)$ is a polynomial in w of degree $n-3$. Then in view of the definitions of F and G we notice that

$$\begin{aligned} \overline{N}(r, c; F) &\leq \overline{N}(r, b; f) + (n-3)T(r, f) \leq (n-2)T(r, f) + S(r, f), \\ \overline{N}(r, c; G) &\leq \overline{N}(r, b; g) + (n-3)T(r, g) \leq (n-2)T(r, g) + S(r, g). \end{aligned} \tag{2.9}$$

Now we consider the following cases.
Case 1. $A \neq 0$.
Subcase 1.1. $C \neq 0$. Since f, g share the value ∞, it follows from (2.7) that ∞ is an exceptional value of Picard of f and g. Therefore from (1.2) and (2.1) it follows that

$$\begin{aligned} \overline{N}(r, \infty; F) &= \overline{N}(r, \alpha_1; f) + \overline{N}(r, \alpha_2; f), \\ \overline{N}(r, \infty; G) &= \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g). \end{aligned} \tag{2.10}$$

Subcase 1.1.1. $B \neq 0$. Then from (2.7) it follows that $\overline{N}(r, -\frac{B}{A}; G) = \overline{N}(r, 0; F)$.
Subcase 1.1.1.1. $c \neq -\frac{B}{A}$. Thus from the second main theorem we have from (2.7), (2.8), (2.9) and (2.10)

$$\begin{aligned} 2nT(r, g) &\leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, -\frac{B}{A}; G) + \overline{N}(r, c; G) + S(r, G) \\ &\leq \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) + \overline{N}(r, 0; f) \\ &\quad + (n-2)T(r, g) + S(r, g) \\ &\leq (n+2)T(r, g) + S(r, g). \end{aligned} \tag{2.11}$$

Clearly (2.11) leads to a contradiction if $n \geq 4$.
Subcase 1.1.1.2. $c = -\frac{B}{A}$. Then $B = -Ac$. Therefore, we have from (2.7),

$$F - c = \frac{(A - Cc)G - (Ac + Dc)}{CG + D}.$$

First let $A - Cc = 0$. Then from above we have,

$$F - c = -\frac{(A+D)c}{CG+D}.$$

Clearly $A + D \neq 0$, for otherwise F becomes constant. We also claim that $D \neq 0$. For if $D = 0$, then from above we have $F - c = -\frac{Ac}{CG}$. Since F and G share the value 1, we have $1 - c = -\frac{Ac}{C}$, which, when combined with our assumption $A - Cc = 0$, leads to $c^2 - c + 1 = 0$. Thus c becomes complex roots of $z^3 = -1$. Thus if we denote by ω, a complex cube root of unity, then our c becomes precisely $c = -\omega$, which is contrary to our assumption as has been observed at beginning of the proof of the lemma.

Since $A + D \neq 0$, $\frac{-Ac}{D} \neq c$. Therefore it follows from above by the use of the second main theorem,

$$\begin{aligned}
2nT(r,f) &\le \overline{N}(r,0;F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + \overline{N}(r,-\frac{Ac}{D};F) + S(r,f) \\
&\le \overline{N}(r,0;f) + \overline{N}(r,\infty;G) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,0;G) + S(r,f) \\
&\le \overline{N}(r,0;f) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) \\
&\quad + \overline{N}(r,0;g) + S(r,f) \\
&\le 6T(r,f) + S(r,f).
\end{aligned}$$

This leads to a contradiction for $n \geq 4$.
Next we consider $A - Cc \neq 0$. Using $B = -Ac$, we have from (2.7),

$$F = \frac{A(G-c)}{CG+D}.$$

Suppose that $D \neq 0$. It is obvious from above that $c \neq -\frac{D}{C}$, for otherwise F becomes constant. Therefore, by the second main theorem, we have,

$$\begin{aligned}
2nT(r,g) &\le \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,-\frac{D}{C};G) + \overline{N}(r,c;G) + S(r,g) \\
&\le \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) \\
&\quad + \overline{N}(r,0;f) + S(r,g) \\
&\le 6T(r,g) + S(r,g).
\end{aligned}$$

This is a contradiction as before for $n \geq 4$.
If $D = 0$, then we have

$$F = \frac{A(G-c)}{CG}.$$

Note that $c \neq \frac{Ac}{A-Cc}$. Therefore the second main theorem and (2.9) yields

$$\begin{aligned}2nT(r,g) &\leq \overline{N}(r,0;G) + \overline{N}(r,c;G) + \overline{N}(r,\frac{Ac}{A-Cc};G) + \overline{N}(r,\infty;G) + S(r,g)\\ &\leq \overline{N}(r,0;g) + \overline{N}(r,0;f) + \overline{N}(r,c;F) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + S(r,g)\\ &\leq 4T(r,g) + (n-2)T(r,f) + S(r,g)\\ &\leq (n+2)T(r,g) + S(r,g).\end{aligned}$$

This, as before, yields a contradiction for $n \geq 4$.

Subcase 1.1.2. $B = 0$. Then $F \equiv \frac{\frac{A}{C}.G}{G+\frac{D}{C}}$ and $\overline{N}(r,\frac{-D}{C};G) = \overline{N}(r,\infty;F)$. We also note that $c = \frac{ab^{n-2}}{2} \neq 0$.

If possible suppose $c = \frac{-D}{C}$. Since F, G share 1-points, we have $A = C + D = C - cC$ and hence $F = \frac{(C-cC)G}{CG-cC} = \frac{(1-c)G}{G-c}$. Then since $c \neq \frac{1}{2}$,

$$\overline{N}(r,c;F) = \overline{N}(r,\frac{c^2}{2c-1};G).$$

Thus by the second main theorem and (2.9) and (2.10) we have,

$$\begin{aligned}2nT(r,g) &\leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,c;G) + \overline{N}(r,\frac{c^2}{2c-1};G) + S(r,g)\\ &\leq \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f)\\ &\quad + (n-2)T(r,f) + S(r,g)\\ &\leq (5+n-2)T(r,g) + S(r,g),\end{aligned}$$

which leads to a contradiction for $n \geq 4$.

Next let $c \neq \frac{-D}{C}$. Hence as before by the second main theorem

$$\begin{aligned}2nT(r,g) &\leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,\frac{-D}{C};G) + \overline{N}(r,c;G) + S(r,G)\\ &\leq \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f)\\ &\quad + (n-2)T(r,g) + S(r,g)\\ &\leq (5+n-2)T(r,g) + S(r,g),\end{aligned}$$

which leads to a contradiction for $n \geq 4$.

Subcase 1.2. $C = 0$. Therefore $F \equiv \frac{A}{D}G + \frac{B}{D}$. If $B = 0$, then since F and G share the value 1, it follows that $\frac{A}{D} = 1$, and therefore $F \equiv G$. Thus by Lemma 2.9, we have $f = g$.

If $B \neq 0$, then since F and G share the value 1, it follows that $F = \eta G + (1-\eta)$, where $\eta = \frac{A}{D}$ and $1 - \eta = \frac{B}{D}$. If $c \neq 1 - \eta$, then from above we obtain by the second main theorem,

$$\begin{aligned}
2nT(r,f) &\le \overline{N}(r,0;F)+\overline{N}(r,1-\eta;F)+\overline{N}(r,c;F)+\overline{N}(r,\infty;F)+S(r,f)\\
&\le \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\alpha_1;f)+\overline{N}(r,\alpha_2;f)\\
&\quad +(n-2)T(r,f)+S(r,f)\\
&\le (n+3)T(r,f)+S(r,f),
\end{aligned}$$

which leads to a contradiction for $n \ge 4$.

If $c = 1-\eta$, then $F = (1-c)G + c$. Since by our assumption $ab^{n-2} \neq 4$, we have $c \neq 2$ and hence $c \neq \frac{c}{c-1}$. Therefore by the second main theorem we have

$$\begin{aligned}
2nT(r,g) &\le \overline{N}(r,0;G)+\overline{N}(r,c;G)+\overline{N}(r,\frac{c}{c-1};G)+\overline{N}(r,\infty;G)+S(r,g)\\
&\le \overline{N}(r,0;g)+\overline{N}(r,0;f)+\overline{N}(r,\infty;g)+\overline{N}(r,\alpha_1;g)+\overline{N}(r,\alpha_2;g)\\
&\quad +(n-2)T(r,g)+S(r,g)\\
&\le (n+3)T(r,g)+S(r,g),
\end{aligned}$$

as before this leads to a contradiction for $n \ge 4$.

Case 2. $A = 0$. Then clearly $BC \neq 0$ and $F \equiv \frac{1}{\gamma G+\delta}$ where $\gamma = \frac{C}{B}$ and $\delta = \frac{D}{B}$. Then as observed in Subcase 1.1 of Case 1, f and g will have no pole.

Since F and G have some 1-points, then $\gamma+\delta = 1$ and so $F \equiv \frac{1}{\gamma G+1-\gamma}$. If $\gamma = 1$, we arrive at a contradiction by Lemma 2.7. So let $\gamma \neq 1$. If $\frac{1}{1-\gamma} \neq c$ then by second main theorem and (2.9) and (2.10) we have,

$$\begin{aligned}
&2nT(r,f) \le \overline{N}(r,0;F)+\overline{N}(r,\frac{1}{1-\gamma};F)+\overline{N}(r,c;F)+\overline{N}(r,\infty;F)+S(r,F)\\
&\le \overline{N}(r,0;f)+(n-2)T(r,f)+\overline{N}(r,0;g)+\overline{N}(r,\alpha_1;f)+\overline{N}(r,\alpha_2;f)+S(r,f)
\end{aligned}$$

therefore

$$(n+2)T(r,f) \le \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\alpha_1;f)+\overline{N}(r,\alpha_2;f)+S(r,f),$$

which is a contradiction for $n \ge 4$.

If $c = \frac{1}{1-\gamma}$, then $F \equiv \frac{c}{(c-1)G+1}$. Note that $c \neq \frac{1}{1-c}$, for otherwise $c = \omega = \frac{ab^{n-1}}{2}$, which violates our assumption . Then by the second main theorem we obtain as before,

$$\begin{aligned}
&2nT(r,g) \le \overline{N}(r,0;G)+\overline{N}(r,c;G)+\overline{N}(r,\frac{1}{1-c};G)+\overline{N}(r,\infty;G)+S(r,g)\\
&\le \overline{N}(r,0;g)+(n-2)T(r,g)+\overline{N}(r,\infty;F)+\overline{N}(r,\alpha_1;g)+\overline{N}(r,\alpha_2;g)+S(r,g)\\
&\le \overline{N}(r,0;g)+(n-2)T(r,g)+\overline{N}(r,\alpha_1;f)+\overline{N}(r,\alpha_2;f)+\overline{N}(r,\alpha_1;g)\\
&\quad +\overline{N}(r,\alpha_2;g)+S(r,g).
\end{aligned}$$

Thus

$$\begin{aligned}(n+2)T(r,g) \le& \overline{N}(r,0;g)+\overline{N}(r,\alpha_1;f)+\overline{N}(r,\alpha_2;f)+\overline{N}(r,\alpha_1;g)\\ &+\overline{N}(r,\alpha_2;g)+S(r,g),\end{aligned}$$

which leads to a contradiction for $n \ge 4$. □

3 Proofs of theorems

Proof of Theorem 1.1. **Case 1.** $H \not\equiv 0$. Let F, G be given by (2.1). Since $E_f(S,2)=E_g(S,2)$ it follows that F, G share (1,2).

Also since $E_f(\{\infty\},k)=E_g(\{\infty\},k)$ we see that

$$\overline{N}_*(r,\infty;f,g) \le \overline{N}(r,\infty;f \mid\ge k+1).$$

Let l be any positive integer and $a_j \notin S \cup \{0,b,\infty\}$, $j=1,2,\dots,l$ be distinct complex numbers. The conditions of our theorem imply $N_2(r,c;f) \le \frac{2}{p+1}N(r,c;f)$ and $N_2(r,d;g) \le \frac{2}{p+1}N(r,d;g)$.

Thus by above and using Lemmas 2.6 and 2.5, with $m=2$, we obtain

$$\begin{aligned}&\left\{\frac{n}{2}+1+l\right\}\{T(r,f)+T(r,g)\}\\ &\le 2[\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\ &\quad+\overline{N}_*(r,\infty;f,g)-\frac{1}{2}\overline{N}_*(r,1;F,G)+\sum_{j=1}^{l}\{N_2(r,a_j;f)+N_2(r,a_j;g)\}\\ &\quad+S(r,f)+S(r,g)\\ &\le 2[\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\ &\quad+\frac{1}{(n-2)k+n-3}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,1;F,G)\}\\ &\quad+[\overline{N}(r,c;f)+N(r,c;f\mid\ge 2)]+[\overline{N}(r,d;g)+N(r,d;g\mid\ge 2)]\\ &\quad+\sum_{j=1}^{l-1}\{N_2(r,a_j;f)+N_2(r,b_j;g)\}-\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\le 2[\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\ &\quad+\frac{2}{p+1}N(r,c;f)+\frac{2}{p+1}N(r,d;g)+\frac{1}{(n-2)k+n-3}\{T(r,f)+T(r,g)\}\\ &\quad+\sum_{j=1}^{l-1}\{N_2(r,a_j;f)+N_2(r,b_j;g)\}+S(r,f)+S(r,g).\end{aligned}$$

Thus for an arbitrary $\epsilon > 0$, we have from above

$$\left\{\frac{n}{2}+1+l\right\}\{T(r,f)+T(r,g)\}$$
$$\leq \{5+l-1+\frac{1}{(n-2)k+n-3}+\frac{2}{p+1}-2\Theta(0,f)-2\Theta(b,f)-\Theta(\infty,f)$$
$$-\frac{2}{p+1}\delta(c;f)-\sum_{j=1}^{l-1}\delta_2(a_j,f)+\epsilon\}T(r,f)+\{5+l-1+\frac{1}{(n-2)k+n-3}$$
$$+\frac{2}{p+1}-2\Theta(0,g)-2\Theta(b,g)-\Theta(\infty,g)-\frac{2}{p+1}\delta(d;g)$$
$$-\sum_{j=1}^{l-1}\delta_2(b_j,g)+\epsilon\}T(r,g)+S(r),$$

i.e.,

$$\{2\Theta(0,f)+2\Theta(b,f)+\Theta(\infty,f)+\frac{2}{p+1}\delta(c,f)+\sum_{j=1}^{l-1}\delta_2(a_j,f)$$
$$-(3+\frac{1}{(n-2)k+n-3}+\frac{2}{p+1}-\frac{n}{2})-\epsilon\}T(r,f)$$
$$+\{2\Theta(0,g)+2\Theta(b,g)+\Theta(\infty,g)+\frac{2}{p+1}\delta(d,g)+\sum_{j=1}^{l-1}\delta_2(b_j,g)$$
$$-(3+\frac{1}{(n-2)k+n-3}+\frac{2}{p+1}-\frac{n}{2})-\epsilon\}T(r,g)\leq S(r).$$

Above being true for any set of complex numbers $a_j \notin S\cup\{0,b,c,\infty\}$ and $b_j \notin S\cup\{0,b,d,\infty\}$, we have

$$\left\{\Theta_f^*-\left(3+\frac{1}{(n-2)k+n-3}+\frac{2}{p+1}-\frac{n}{2}\right)-\epsilon\right\}T(r,f)$$
$$+\left\{\Theta_g^*-\left(3+\frac{1}{(n-2)k+n-3}+\frac{2}{p+1}-\frac{n}{2}\right)-\epsilon\right\}T(r,g)\leq S(r).$$

Without loss of generality we assume that $T(r,g)\leq T(r,f)$ as $r\to\infty$, $r\notin E$. Hence the above inequality reduces to

$$\left\{\Theta_f^*+\Theta_g^*-\left(6+\frac{2}{(n-2)k+n-3}+\frac{4}{p+1}-n\right)-2\epsilon\right\}T(r,g)\leq S(r),$$

which contradicts (1.3).

Case 2. $H\equiv 0$. Then $F\equiv\frac{AG+B}{CG+D}$. Hence the Theorem follows from Lemma 2.11. This completes the proof of the Theorem. □

Proof of Theorem 1.2. We omit the proof as it is the same as the above proof. □

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