Uniqueness of Norm Properties of Calkin Algebras

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A thesis submitted for the degree of
Doctor of Philosophy
of the

Australian National University

Mathematical Sciences Institute

January, 2014

Statement of Originality

The research presented in this thesis is, to the best of my knowledge and belief, original and my own work, except as acknowledged herein.

Griffith Kuskie Ware

For Dad and my Great Uncle Griff

Acknowledgements

A PhD thesis is not written in a vacuum, and I have been supported in my endeavours by a wide variety of colleagues, friends and family. First and foremost, I am extremely grateful to my long time mentor and supervisor, Dr Rick Loy, whose unwavering faith and kindness was necessary for me to make it to this stage. Rick has always had time for me, and he has gone well beyond the call of duty in continuing to supervise me during his retirement. My eternal thanks, Rick.

I am also grateful for helpful correspondence I have had with Prof H. Garth Dales, Dr Matthew Daws, and Prof Hans-Olav Tylli. In particular, Prof Tylli had a major impact on my choice of subject matter when he kindly sent me copies of various papers, which became the background material for my eventual thesis topic. Many others also made me feel very welcome at the recent Banach Algebras and Applications conference in Göteborg, for which I am most thankful.

I am indebted to the administrative team at the MSI. In particular, my thanks go to the amazing Kelly Wicks who has been a constant help throughout the course of my candidature. Recently, Michelle Cowie has also done a stellar job stepping into Kelly's very large shoes. Kelly, I hope you had a great holiday.

I very much appreciate the support of my extended family and all my friends, who have stuck with me through thick and thin. In particular, thank you to Rosemary, Dad, Adrian, Christy, and Daniel, for your valuable contributions in the past few weeks, and to Izaak, for being there at the end. Also, Alex, Chenni, and Mat did a marvellous job of lifting my spirits at critical points. Additionally, I would like to make special mention of Dr Pierre Portal, whose advice, encouragement, and friendship I highly value.

Many thanks to those who have given feedback on various drafts of chapters: Alex, Alison, Izaak, John, Mark, Nic, and Seb, you were a great help. Mark, I hope to have followed in your footsteps in making less mistakes. Thanks to Tristan for chats about set theory. Of course, any errors that remain are my own.

I have reserved my most significant acknowledgement until last. My darling Alison, without your love and continued support, I would be lost. You are the most important part of my life, and your kind heart has meant everything to me.

Abstract

A classical result due to M. Eidelheit and B. Yood states that the standard algebra norm on the algebra of bounded linear operators on a Banach space is minimal, in the sense that the norm must be less than a multiple of any other submultiplicative norm on the same algebra. This definition does not assume that the arbitrary algebra norm is complete. In cases when the standard algebra norm is, in addition, maximal, it is therefore unique up to equivalence. More recently, M. Meyer showed that the Calkin algebras of a very restricted class of Banach spaces also have unique algebra norms.

We generalise the Eidelheit-Yood method of proof, to show that the conventional quotient norm on a larger class of Calkin algebras is minimal. Since maximality of the norm is a presumed property for the class, the norm is also unique. We thus extend the result of Meyer. In particular, we establish that the Calkin algebras of canonical Banach spaces such as James' space and Tsirelson's space have unique algebra norms, without assuming completeness. We also prove uniqueness of norm for quotients of the algebras of operators on classical non-separable spaces, the closed ideals of which were previously studied by M. Daws.

One aspect of the Eidelheit-Yood method is a dependence on the uniform boundedness principle. As a component of our generalisation, we prove an analogue of that principle which applies to Calkin algebra elements rather than bounded linear operators. In order to translate the uniform boundedness principle into this new setting, we take the perspective that non-compact operators map certain well-separated sequences to other well-separated sequences. We analyse the limiting separation of such sequences, using these values to measure the non-compactness of operators and define the requisite notion of a bounded set of non-compact operators. In the cases when the underlying Banach space has a Schauder basis, we are able to restrict attention to seminormalised block basic sequences. As a consequence, our main uniqueness of norm result for Calkin algebras relies on the existence of bounded mappings between, and projections onto, the spans of block basic sequences in the relevant Banach spaces.

Contents

Terminology

It is assumed that the reader is familiar with elementary results and definitions relating to normed spaces, Banach spaces, algebras, and Banach algebras, as appear in texts such as [Meg], [Pal], and [All]. The notions of quotient spaces, closed subspaces, complements, projections, compact operators, and general bounded linear operators between Banach spaces, will be particularly central. Further background material, terminology, and notation, shall be introduced when needed. However, we make the following initial definitions clear:

By a *unique algebra norm* or a *unique Banach algebra norm* on an algebra $\mathfrak{A}$, we mean a norm $\|\cdot\|$ that is unique up to equivalence within its respective class of norms on $\mathfrak{A}$. That is, given another such norm $|||\cdot|||$, there exists a constant $C \geq 1$ such that $C^{-1}\|a\| \leq |||a||| \leq C\|a\|$ for all $a \in \mathfrak{A}$. If two norms $\|\cdot\|$ and $|||\cdot|||$ are thereby *equivalent*, we write $\|\cdot\| \sim |||\cdot|||$. As shall be observed at the start of Chapter 1, on a normed space the notions of a unique norm or a unique complete norm are essentially trivial, and therefore we will often abuse this terminology when discussing (Banach) algebras and use *unique norm* to mean a unique algebra norm and *unique complete norm* to mean a unique Banach algebra norm.

On occasion it will be convenient to state several results or definitions in parallel, with multiple components substituted in a respective fashion. In such cases we shall write 'p ⟨resp. q⟩' to mean 'p, respectively q'.

The natural numbers $\{1, 2, \ldots\}$ are represented by $\mathbb{N}$, while $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. When $\mathbb{N}_0$ is given its natural ordering, the first infinite ordinal ω and the first infinite cardinal $\aleph_0$ are identically considered to be $\omega = \aleph_0 = \mathbb{N}_0$.

Infinite sequences will be represented by (x_n) or similar. When this notation is used, the variable n is always assumed to range over the whole of $\mathbb{N}$. In places where a different indexing sequence or set is utilised, it will always be denoted explicitly.

The only vector spaces we shall consider will be over either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. For this reason, we adopt the convention that $\mathbb{F}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$, and that a reference to a vector space implies that it is over $\mathbb{F}$ even if this is not explicitly stated. This includes references to normed spaces, Banach spaces, algebras, and Banach algebras. When several such spaces are discussed in the same context, unless otherwise indicated they are assumed to be over the same field $\mathbb{F}$, be it $\mathbb{R}$ or $\mathbb{C}$. As is standard, $|\cdot|$ denotes the absolute value function on $\mathbb{F}$. The set $\{x \in \mathbb{R} : x > 0\}$ is denoted $\mathbb{R}_+$.

The closed linear span of a set A in a normed space is represented by $[A]$. In particular, $[(x_n)]$ is the closed linear span of the sequence (x_n).

The default notation used for a norm is $\|\cdot\|$; in cases where the meaning is unambiguous, $\|\cdot\|$ may be used simultaneously for the (different) norms on two or more normed spaces. The closed ball with centre x and radius r in a normed space X is denoted $B_X(x,r) = \{y \in X : \|y-x\| \leq r\}$, and we write B_X for $B_X(0,1)$, the *unit ball*. Note, therefore, that $rB_X = B_X(0,r)$. The boundary of B_X is the *unit sphere*: $S_X = \{x \in X : \|x\| = 1\}$. The standard operator norm for a linear operator $T : X \to Y$ is $\|T\| = \sup\{\|Tx\| : x \in B_X\}$. The identity operator on a normed space X is denoted I. As with norms, I may be used simultaneously for the identity operators on multiple normed spaces.

In general, Banach spaces and their subspaces will be represented by the symbols X, Y or Z, with additional subscripts or superscripts in some cases. If X and Y are isomorphic $\langle$resp. isometric$\rangle$, we write $X \approx Y$ $\langle$resp. $X = Y\rangle$. The space of all bounded $\langle$resp. compact; finite rank$\rangle$ linear operators from X to Y is denoted $\mathscr{B}(X,Y)$ $\langle$resp. $\mathscr{K}(X,Y)$; $\mathscr{F}(X,Y)\rangle$, and we set

$$\mathscr{B}(X) = \mathscr{B}(X,X), \quad \mathscr{K}(X) = \mathscr{K}(X,X), \quad \text{and} \quad \mathscr{F}(X) = \mathscr{F}(X,X).$$

Other spaces of operators will be introduced at various points, and we shall also use ‘script’ letters to represent them in a similar fashion.

Finally, we highlight an important point about the notation used for the quotient norm on the Calkin algebra $\mathscr{C}(X) = \mathscr{B}(X)/\mathscr{K}(X)$ of a Banach space X. We call that quotient norm the *essential norm* and denote it $\|\cdot\|_e$. The essential norm induces a semi-norm on $\mathscr{B}(X)$ with kernel $\mathscr{K}(X)$. We also denote that semi-norm $\|\cdot\|_e$, and refer to the *essential norm* $\|T\|_e$ of an operator $T \in \mathscr{B}(X)$. This abuse of notation, in using $\|\cdot\|_e$ as a function on both $\mathscr{B}(X)$ and $\mathscr{C}(X)$, will occur frequently and within the same context. The same will be true of other norms $\langle$resp. semi-norms$\rangle$ defined on $\mathscr{C}(X)$ $\langle$resp. $\mathscr{B}(X)\rangle$.

Chapter 1

Introduction

Wenn jede beschränkte Folge von Elementen einer linearen Mannigfaltigkeit kompakt ist, so ist die Mannigfaltigkeit von endlicher Dimensionszahl.

— Friedrich Riesz

A fundamental result in the theory of Banach spaces is that any two normed spaces X and Y of the same finite dimension are isomorphic. That is, the natural linear map between X and Y induced by identifying any choices of respective bases is automatically continuous. Thus the norm on a finite dimensional space is unique.

The distinctive nature of this uniqueness of norm property was first exhibited by Riesz in 1916, when he characterised finite dimensional spaces by showing that a given normed space X is finite dimensional if and only if B_X is compact in the norm topology [Rie, Hilfssätze 4–5]. Conversely, when X is infinite dimensional, the presence of non-compact, linearly independent sequences in the unit ball allows the construction of unbounded linear operators and inequivalent norms on X. If, in addition, X is a Banach space, then an inequivalent complete norm on X can be constructed by the same process:

Theorem. *Let $(X, \|\cdot\|)$ be an infinite dimensional normed $\langle$resp. Banach$\rangle$ space. Then X admits an inequivalent norm $\langle$resp. an inequivalent complete norm$\rangle$.*

***Proof*.** Let $\mathfrak{B}$ be a normalised Hamel basis for X, and select $(b_n) \subset \mathfrak{B}$ such that b_n is distinct for each $n \in \mathbb{N}$. Define an unbounded linear bijection T by setting $T(b_n) = nb_n$ for $n \in \mathbb{N}$, $T(b) = b$ for $b \in \mathfrak{B} \setminus (b_n)$, and extending linearly. Set $|||x||| = \|Tx\|$ for $x \in X$. Then $|||\cdot|||$ is a norm $\langle$resp. a complete norm$\rangle$ on X, however for all $n \in \mathbb{N}$ we have $|||b_n||| = n$, while $\|b_n\| = 1$, so that $|||\cdot||| \nsim \|\cdot\|$. ■

Hence, since the inception of Banach space theory, it was known that a normed space ⟨resp. a Banach space⟩ has a unique norm ⟨resp. a unique complete norm⟩ if and only if it is finite dimensional.

While these results for normed spaces constitute a rudimentary component of any introductory course in functional analysis, it is only at a particularly specialised graduate level that one normally studies uniqueness properties of norms on *algebras*.[1] In stark contrast to the straightforward results for normed spaces, a rich variety of uniqueness and related properties can arise regarding potential algebra norms on an algebra $\mathfrak{A}$, due to the added constraint of submultiplicativity requiring that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathfrak{A}$. Indeed, there exist infinite dimensional algebras which have a unique Banach algebra norm. Furthermore, there are infinite dimensional Banach algebras which have a unique algebra norm, even without the assumption of completeness. This is surprising when one considers that the basic axioms for an unnormed algebra have no apparent topological properties, yet in the cases in question the algebraic structure can completely determine the topology no matter how an algebra norm is imposed[3] — and despite the ease with which we can define an inequivalent norm on the underlying vector space! A detailed treatment of many such results and the related theory can be found in [Dal1], but for the sake of context their history is summarised in §1.1 below, along with an exposition of the more pertinent examples. Note, however, that some additional structure is always assumed: to completely catalogue those algebras which admit a unique algebra or Banach algebra norm is likely as impossible a task as a complete general classification of Banach spaces.

In this work we shall primarily restrict the question of uniqueness of norm to the Calkin algebras $\mathscr{C}(X) = \mathscr{B}(X)/\mathscr{K}(X)$ of various Banach spaces X.[4] Of course, the quotient norm on $\mathscr{C}(X)$ is defined relative to the norm on $\mathscr{B}(X)$. Thus, a more fundamental question is whether $\mathscr{B}(X)$ admits an algebra norm that is not equivalent

[1] This is not to suggest that Banach algebra theory has in any way lagged behind the study of Banach spaces. Algebras, as they are defined today, were first considered by Peirce in 1870 [Pei] under the guise of *linear associative algebras*.[2] Complete normed algebras were implicitly considered by Bennett as early as 1916 [Ben], albeit in a very convoluted fashion as one example of a broader axiomatic framework: Bennett's paper is cited and discussed in detail in [Pie2, §1.6]. Banach algebras were first explicitly defined (as *linearen, metrischen Ringen*) by Nagumo in 1936 [Nag, §1].[2]

[2] See [Pal, §1.1] for more details of these and several other interesting historical observations.

[3] There are also algebras which admit no algebra norm at all, although this is perhaps less remarkable. For example, the algebra $\mathscr{L}(X)$, of all (possibly unbounded) linear operators from an infinite dimensional vector space X to itself, cannot be given an algebra norm (see [Dal1, Prop. 2.1.14(ii)]).

[4] Calkin algebras are named after J. W. Calkin. His 1941 study of the Calkin algebra of a separable Hilbert space, which is $\mathscr{C}(\ell_2)$ up to isomorphism, was the first to consider such quotient algebras: Calkin established certain results regarding the representation theory of $\mathscr{C}(\ell_2)$, while also proving that $\mathscr{K}(\ell_2)$ is the only non-trivial closed ideal in $\mathscr{B}(\ell_2)$ [Cal].

to the standard supremum norm of an operator. If not, then the supremum norm is the only way to measure the 'size' of an operator on X in a composition-compatible way. This happens to be the situation for a large family of cases, as shown by the classical findings of Johnson and their precursors. These results are detailed in §1.1 below.

Subsequent sections of this chapter provide further foundational material, upon which the various results discussed in this thesis rely. §1.2 catalogues some definitions and properties that concern ideals and quotients in the algebra of bounded linear operators on a Banach space. §1.3 covers the necessary background about basic sequences in Banach spaces, which will be central to our investigation of uniqueness of norm properties for Calkin algebras. Finally, §1.4 gives a general overview of the structure and results of this thesis.

1.1 Classical uniqueness of norm results

To determine whether a norm ⟨resp. a complete norm⟩ $\|\cdot\|$ is unique, it is natural to consider the issue in two parts. Given an arbitrary norm ⟨resp. complete norm⟩ $|||\cdot|||$, on the same algebra as $\|\cdot\|$, we can investigate if there is a constant C such that $\|\cdot\| \leq C|||\cdot|||$ separately to whether there is a C such that $|||\cdot||| \leq C\|\cdot\|$. In the light of this, we adopt the following terminology.

1.1.1 Definition. Let $(\mathfrak{A}, \|\cdot\|)$ be a Banach algebra.

(i) $\|\cdot\|$ is *minimal* if for any (not necessarily complete) algebra norm $|||\cdot|||$ on $\mathfrak{A}$, there exists a constant C (dependent on $|||\cdot|||$) such that $\|\cdot\| \leq C|||\cdot|||$. That is, the identity map $I : (\mathfrak{A}, |||\cdot|||) \to (\mathfrak{A}, \|\cdot\|)$ is bounded.

(ii) $\|\cdot\|$ is *maximal* if for any (not necessarily complete) algebra norm $|||\cdot|||$ on $\mathfrak{A}$, there exists a constant C (dependent on $|||\cdot|||$) such that $|||\cdot||| \leq C\|\cdot\|$. That is, the identity map $I : (\mathfrak{A}, \|\cdot\|) \to (\mathfrak{A}, |||\cdot|||)$ is bounded.

Thus a norm is unique if and only if it is both minimal and maximal. In general, it is possible that a norm might have one of these properties without having the other. Illustrating this potential separation, the first result relating to the question of norm uniqueness (for $\mathscr{B}(X)$, and indeed any Banach algebra) was established by Eidelheit in 1940 [Eid],[5] and its full significance demonstrated in an independent proof by Yood in 1958 [Yoo]. Eidelheit's result lays the conceptual foundation for

[5]The claim that this was the earliest such result appears in both [Pal, p12] and [Dal1, p608].

a significant portion of this thesis. Hence, we state it in broadest generality and provide a detailed proof, which will be referred to in subsequent discussion.

1.1.2 Theorem ([Eid, Lem. 1], [Yoo, Thm. 3], [Pal, §1.7.15], [Dal1, Thm. 5.1.14]). *Let X be a Banach space and let $\mathfrak{A}$ be $\mathscr{B}(X)$, or any subalgebra of $\mathscr{B}(X)$ that contains $\mathscr{F}(X)$. Then the standard operator norm $\|\cdot\|$ is minimal.*

***Proof*.** Let $|||\cdot|||$ be any algebra norm on $\mathfrak{A}$. Suppose, contrary to minimality, that there is no C such that $\|\cdot\| \le C|||\cdot|||$. Then we can find $(T_n) \subset \mathfrak{A}$ such that $\|T_n\| > n\,|||T_n|||$ for each $n \in \mathbb{N}$. By scaling T_n as necessary, without loss of generality we may assume $|||T_n||| = 1$ for all $n \in \mathbb{N}$. Thus $(\|T_n\|)$ is unbounded, so by the uniform boundedness principle there exists $x_0 \in X$ for which $(\|T_n x_0\|)$ is unbounded. Employing the identification of $(T_n x_0)$ with its canonical image in X^{**}, a further application of the uniform boundedness principle implies that there is $x_0^* \in X^*$ such that $(|x_0^*(T_n x_0)|)$ is unbounded.

Now define $S \in \mathscr{B}(X)$ by

$$Sx = x_0^*(x)x_0$$

for $x \in X$. Note that $S \neq 0$ and S has rank 1, so $|||S||| > 0$ and $S \in \mathscr{F}(X) \subset \mathfrak{A}$. Therefore $ST_nS \in \mathfrak{A}$ and

$$|||ST_nS||| \le |||S|||\cdot|||T_n|||\cdot|||S||| = |||S|||^2$$

for all $n \in \mathbb{N}$. However, for all $x \in X$ we have

$$ST_nSx = ST_n(x_0^*(x)x_0) = x_0^*(x)x_0^*(T_nx_0)x_0 = x_0^*(T_nx_0)Sx\,.$$

Thus $ST_nS = x_0^*(T_nx_0)S$, and so

$$|x_0^*(T_nx_0)| = \frac{|||ST_nS|||}{|||S|||} \le |||S|||\,, \tag{1.1.2a}$$

in contradiction to the previous conclusion that $(|x_0^*(T_nx_0)|)$ is unbounded. ■

1.1.3 Remark. An important element of all published proofs of Theorem 1.1.2, which we shall also refer to as the Eidelheit-Yood theorem, is the reliance on the uniform boundedness principle (UBP). As it is normally stated, and as we used it above, the UBP says that, if $\{T_\lambda\}_{\lambda\in\Lambda}$ is a collection of operators in $\mathscr{B}(X)$, then $\{\|T_\lambda\| : \lambda \in \Lambda\}$ is bounded if and only if $\{\|T_\lambda x\| : \lambda \in \Lambda\}$ is bounded for each $x \in X$. Thus if $\{T_\lambda\}_{\lambda\in\Lambda}$ is unbounded, then there is an $x \in X$ such that $\{\|T_\lambda x\|\}_{\lambda\in\Lambda}$ is unbounded, and therefore we can extract a sequence $(T_n) \subset \{T_\lambda\}_{\lambda\in\Lambda}$ such that $\|T_n x\| \uparrow \infty$. This standard argument tells us nothing about the rate of growth of $\|T_n x\|$, and most published proofs of the UBP do not concern themselves with this detail, whether they be based on the Baire category theorem or the so-called 'gliding hump' argument. However, careful analysis shows that a quantitative estimate relating the growth of $\|T_n x\|$ to the growth of $\|T_n\|$ is possible, and we shall use this stronger version at one stage in Chapter 6. The following statement, extracted from a recent interesting note in *The American Mathematical Monthly*, will suffice for our purposes.

1.1.4 Theorem (The (quantitative) uniform boundedness principle; e.g. see [Sok])**.** *Let X be a Banach space and let $(T_n) \subset \mathscr{B}(X)$ be such that, for all $n \in \mathbb{N}$, $\|T_n\| \geq 4^n$. Then there exists $x \in S_X$ such that, for all $n \in \mathbb{N}$, $\|T_n x\| \geq {}^{4^n}/{}_{3^{n+1}}$.*

Applying the two-step process from the Eidelheit-Yood proof gives:

1.1.5 Corollary. *Let X be a Banach space and let $(T_n) \subset \mathscr{B}(X)$ be such that, for all $n \in \mathbb{N}$, $\|T_n\| \geq 10^n$. Then there exist $x \in S_X$ and $x^* \in S_{X^*}$ such that, for all $n \in \mathbb{N}$, $|x^*(T_n x)| \geq {}^{10^n}/{}_{9^{n+1}}$.*

Proof. By Theorem 1.1.4, there exists $x \in S_X$ such that, for all $n \in \mathbb{N}$, $\left\|\frac{3^{n+1}\cdot 4^n}{10^n} T_n x\right\| \geq 4^n$. Identifying $\left(\frac{3^{n+1}\cdot 4^n}{10^n} T_n x\right)$ with its canonical image in X^{**}, another application of Theorem 1.1.4 gives an $x^* \in X^*$ such that, for all $n \in \mathbb{N}$, $\left|x^*\left(\frac{3^{n+1}\cdot 4^n}{10^n} T_n x\right)\right| \geq \frac{4^n}{3^{n+1}}$. The required result follows. ∎

1.1.6 Remark. While the uniform boundedness principle is an important tool in the proof of Theorem 1.1.2, its use is not the most significant aspect. Rather, the key idea in the Eidelheit-Yood method is the construction of the operator S, and this construction underlies all of the various published accounts. The purpose of S is threefold. First, on the right it maps onto a restricted subspace (the span of x_0), on which we know the operators T_k are unbounded thanks to the UBP. Second, after the T_k have been applied, on the left it brings the various images of that subspace back together, preserving the unbounded nature of (T_k) as it does so. Third, this 'shift' back into a common space factorises S itself, which has whatever constant norm $|||S|||$ it has, in such a way that, as a result of the submultiplicative property of $|||\cdot|||$, the unboundedness of $\|T_k\|$ is transferred to $|||T_k|||$. The fact that these three purposes can be achieved with the same operator, which in no way varies with k, yields an extremely elegant argument. However, the contradiction of 1.1.2a would remain even if S varied with k, so long as $|||S|||$ remained nonzero and bounded (which may not be easy to control, given that $|||\cdot|||$ is arbitrary). Furthermore, there appears to be no reason why a similar shift/factorisation effect could not be achieved with different operators on the left and right. Exploiting the flexibility offered by these possibilities will be central to our proof that various Calkin algebras have a unique algebra norm. Further discussion of this approach appears in §3.4. Note that the Eidelheit-Yood proof does not immediately generalise to show minimality of the essential norm on $\mathscr{C}(X)$, because the operator S is finite rank. This means that its norm collapses to zero under any arbitrary algebra norm on $\mathscr{C}(X)$. Hence, the contradiction of 1.1.2a does not exist in the Calkin algebra setting, if we simply follow the proof given for Theorem 1.1.2.

Remarks 1.1.3 and 1.1.6 discussed the essential similarities possessed by the versions of the Eidelheit-Yood theorem (1.1.2) that can be found in the literature. However, those previous presentations have also been somewhat varied. Among the references cited prior to the statement of the theorem, the proof as given above is closest to the one in [Dal1]. The original proofs by Eidelheit and Yood were used for purposes beyond the content of Theorem 1.1.2, and that theorem is not directly contained, as expressed here, in either [Eid] or [Yoo]. In fact, Eidelheit relied on a similar argument to establish only the weaker result that $\mathscr{B}(X)$ has a unique complete norm. This follows due to an application of the open mapping theorem:

1.1.7 Theorem. *Suppose that* $(\mathfrak{A}, \|\cdot\|)$ *is a Banach algebra, and that either* $\|\cdot\|$ *is minimal or that* $\|\cdot\|$ *is maximal. Then* $\mathfrak{A}$ *has a unique Banach algebra norm.*

Proof. Suppose that $|||\cdot|||$ is an arbitrary complete algebra norm on $\mathfrak{A}$. If $\|\cdot\|$ is minimal, then $I : (\mathfrak{A}, |||\cdot|||) \to (\mathfrak{A}, \|\cdot\|)$ is bounded and therefore continuous, so by the open mapping theorem I has a continuous inverse (e.g., see [All, Coroll. 3.41]). Similarly, if $\|\cdot\|$ is maximal, then $I : (\mathfrak{A}, \|\cdot\|) \to (\mathfrak{A}, |||\cdot|||)$ is continuous, so by the open mapping theorem has a continuous inverse. Thus, in either case, we can conclude that $|||\cdot||| \sim \|\cdot\|$. ■

1.1.8 Corollary. *Let* X *be a Banach space. Then* $\mathscr{B}(X)$*, and all its norm-closed subalgebras that contain* $\mathscr{F}(X)$*, have unique Banach algebra norms.*

Proof. This is an immediate consequence of Theorems 1.1.2 and 1.1.7. ■

Hence, if we wish to show that an algebra has a unique complete norm, it suffices to address just one of the two separate questions of whether $\|\cdot\|$ is minimal or maximal.[6] We have seen that, in the case of $\mathscr{B}(X)$, the Eidelheit-Yood theorem answers the first of these questions no matter the choice of the Banach space X. As alluded to at the end of Remark 1.1.6, much of this thesis will be concerned with how to apply a similar method to establish the minimality of the quotient norm for the Calkin algebras of a restricted class of Banach spaces. In Chapter 3, we will present results of prior authors which show that, in contrast to the case of $\mathscr{B}(X)$, there are Banach spaces X such that the quotient norm on $\mathscr{C}(X)$ is not minimal. Therefore, some restrictions on X are necessary for the quotient norm on $\mathscr{C}(X)$ to be minimal, and the Eidelheit-Yood theorem can not be completely generalised in this context.

The reason we will be less focussed on the question of the maximality of the norm on $\mathscr{C}(X)$ is due to the next definition and sequence of results.

1.1.9 Definition. Let X be a Banach space. A *continued bisection of the identity* on X is a pair $\{(Y_n), (Z_n)\}$ of sequences of closed subspaces of X such that $X = Y_1 \oplus Z_1$ and such that, for each $n \in \mathbb{N}$, we have

$$Y_n = Y_{n+1} \oplus Z_{n+1} \quad \text{and} \quad Y_n \approx Z_n .$$

Definition 1.1.9 is a cross between the original [Joh1, Defn. 3.1] and the now standard definition of a continued bisection of the identity for a general unital algebra (see, e.g., [Dal2, Defn. 5.2.3]). The correspondence between those definitions comes from associating the subspaces Y_n and Z_n with their complementing projections on X. We have given the version of the definition best suited for our purposes, which will involve applications of the following classical automatic continuity results.

1.1.10 Theorem ([Joh1, Thm. 3.3])**.** *Let X be a Banach space with a continued bisection of the identity. Then all homomorphisms from $\mathscr{B}(X)$ into a Banach algebra are continuous.*

1.1.11 Proposition (see, e.g., [Pal, Prop. 6.1.6])**.** *Let $\mathfrak{A}$ be a Banach algebra such that all homomorphisms from $\mathfrak{A}$ into a Banach algebra are continuous. If $\mathscr{I}$ is a closed ideal of $\mathfrak{A}$, then all homomorphisms from $\mathfrak{A}/\mathscr{I}$ into a Banach algebra are continuous.*

***Proof*.** Note that this is a direct result of the open mapping theorem: the quotient

[6] However, note that for a norm to be a unique complete norm, it is not a necessary condition that it is either minimal or maximal.

map is open, thus composing it with a hypothetical discontinuous homomorphism from $\mathfrak{A}/\mathscr{I}$ would yield a discontinuous homomorphism from $\mathfrak{A}$. ∎

1.1.12 Corollary. *Let X be a Banach space with a continued bisection of the identity. Then the standard norm on $\mathscr{B}(X)$, and the quotient norms on all quotients of $\mathscr{B}(X)$ by its closed ideals, are maximal. In particular, $\mathscr{B}(X)$ has a unique algebra norm.*

Proof. Let $(\mathfrak{B}, \|\cdot\|)$ be either the Banach algebra $\mathscr{B}(X)$, or the Banach algebra $\mathscr{B}(X)/\mathscr{I}(X)$ for a closed ideal $\mathscr{I}(X) \subset \mathscr{B}(X)$. Suppose that $|||\cdot|||$ is an arbitrary algebra norm on $\mathfrak{B}$. By Theorem 1.1.10 and Proposition 1.1.11, the homomorphism

$$I : (\mathfrak{B}, \|\cdot\|) \to \overline{(\mathfrak{B}, |||\cdot|||)}$$

must be continuous, and is therefore bounded. Hence $\|\cdot\|$ is maximal. If $\mathscr{B} = \mathscr{B}(X)$, then $\|\cdot\|$ is also minimal by Theorem 1.1.2, and thus is unique. ∎

The above proof contains the result that the automatic continuity of homomorphisms from a Banach algebra $(\mathfrak{A}, \|\cdot\|)$ implies that $\|\cdot\|$ is maximal. It is also the case that, if there is a discontinuous homomorphism θ from $\mathfrak{A}$ into a Banach algebra, then we may use it to define an algebra norm $|||\cdot|||$ on $\mathfrak{A}$ such that there is no C for which $|||\cdot||| \le C\|\cdot\|$. For example, set $|||a||| = \max\{\|a\|, \|\theta(a)\|\}$ for all $a \in \mathfrak{A}$. That is, we have the following equivalence.

1.1.13 Proposition (see, e.g., [Dal1, Prop. 2.1.7]). *Let $(\mathfrak{A}, \|\cdot\|)$ be a Banach algebra. Then $\|\cdot\|$ is maximal if and only if all homomorphisms from $\mathfrak{A}$ into a Banach algebra are continuous.*

A particular case of the above Theorem 1.1.10 and Corollary 1.1.12 is:

1.1.14 Corollary. *Let X be a Banach space such that $X \approx X \oplus X$. Then X has a continued bisection of the identity, all homomorphisms from $\mathscr{B}(X)$ into a Banach algebra are continuous, and the standard norms on all quotients of $\mathscr{B}(X)$ by its closed ideals are maximal.*

Proof. Since $X \approx X \oplus X$, for each $n \in \mathbb{N}$ we may choose subspaces $Y_n \approx Z_n \approx X$ inductively, such that $Y_n = Y_{n+1} \oplus Z_{n+1}$. Therefore X has a continued bisection of the identity, and the remaining conclusions follow from Theorem 1.1.10 and Corollary 1.1.12. ∎

In his original paper on the topic, Johnson claims that the various spaces which have a continued bisection of the identity form "quite a large class of Banach spaces" ([Joh1, p1]). While the later construction of various pathological counterexamples[7] might be considered to refute this statement, it is nonetheless the case that many of the Banach spaces which enjoy mainstream attention have a continued bisection of the identity. For example, the function spaces $L_p[0,1]$, $1 \leq p \leq \infty$, are such that

$$L_p[0,1] \;\approx\; L_p[0,{}^1\!/\!_2] \oplus L_p[{}^1\!/\!_2,1] \;\approx\; L_p[0,1] \oplus L_p[0,1]\,,$$

and numerous other classical spaces X also have the property that $X \oplus X \approx X$. So we have that the standard norm on $\mathscr{B}(X)$, and therefore also $\mathscr{C}(X)$, is maximal in a "large" number of interesting cases.

These cases additionally include some spaces X for which the norm on $\mathscr{B}(X)$ is maximal, yet X does not have a continued bisection of the identity. One such example is James' quasi-reflexive space J, for which the norms on $\mathscr{B}(J)$ and $\mathscr{C}(J)$ are maximal as a result of [Wil2, Prop. 8] and Propositions 1.1.11 and 1.1.13. We shall further study James' space in §5.5.

Because of the various results related above, the question of whether the norm on $\mathscr{C}(X)$ is maximal often reduces to the same question on $\mathscr{B}(X)$, particularly in many of the cases of interest. It is for this reason that, subsequently, we shall be more concerned with the minimality of the norm on a given Calkin algebra, when investigating the question of whether that norm is unique.

However, before we conclude our discussion of situations in which the maximality property holds, there is one further aspect to be considered. Note that the Eidelheit-Yood theorem (1.1.2) not only established minimality of the operator norm for $\mathscr{B}(X)$, but also for closed ideals in $\mathscr{B}(X)$. So it is natural to question whether, on the spaces X for which Corollary 1.1.12 guarantees the operator norm is maximal, that result also passes to closed ideals. That is, when X has a continued bisection of the identity, do the closed ideals of $\mathscr{B}(X)$ have a maximal, and therefore unique, norm? The answer is that we can still can say something under an additional assumption:

1.1.15 Definition (see, e.g., [Dal1, Defn. 2.9.1])**.** Let $\mathfrak{A}$ be a normed algebra. A *left approximate identity* for $\mathfrak{A}$ is a net $(e_\beta)_{\beta\in B} \subset \mathfrak{A}$ such that $\lim_{\beta\in B} e_\beta a = a$ for each $a \in A$. Given $m > 0$, $(e_\beta)_{B\in B}$ is *bounded by* m if $\sup_{\beta\in B} e_\beta \leq m$, and is *bounded* if $\sup_{\beta\in B} e_\beta < \infty$.

[7] Such counterexamples include the recent plethora of indecomposable spaces that have their origin in [GoMa], and which clearly fail to have a continued bisection of the identity.

We will utilise the following test of the existence of a left approximate identity in Chapter 6.

1.1.16 Proposition ([Dal1, Prop. 2.9.14]). *Let $\mathfrak{A}$ be a normed algebra, and $c \geq 1$. Suppose that there exists a dense subset A of $\mathfrak{A}$ such that, for each $a \in A$ and $\varepsilon > 0$, there exists $u \in cB_{\mathfrak{A}}$ with $\|a - ua\| < \varepsilon$. Then $\mathfrak{A}$ has a left approximate identity of bound c.*

The relevance of left approximate identities to the theory that has been previously discussed is shown by:

1.1.17 Theorem ([Dal1, Thm. 5.4.11]). *Let $(\mathfrak{B}, \|\cdot\|)$ be a unital Banach algebra with a continued bisection of the identity, and let $\mathfrak{A}$ be a closed ideal in $\mathfrak{B}$ with a bounded left approximate identity. Then each homomorphism from $\mathfrak{A}$ into a Banach algebra is automatically continuous and $\|\cdot\|$ is maximal on $\mathfrak{A}$.*

Hence, given a Banach space X, maximality of the norm on $\mathscr{B}(X)$ does pass to closed ideals in some circumstances, although not in every case. So the situation is somewhat different compared to the analogous one for minimality. In Chapter 6, we will apply Theorem 1.1.17 in the following context.

1.1.18 Corollary. *Let X be a Banach space such that $X \approx X \oplus X$, and let $\mathscr{I}_1(X), \mathscr{I}_2(X)$ be closed ideals in $\mathscr{B}(X)$ such that $\mathscr{I}_1(X) \subset \mathscr{I}_2(X)$. Suppose that $\mathscr{I}_2(X)/\mathscr{I}_1(X)$, a closed ideal in $\mathscr{B}(X)/\mathscr{I}_1(X)$, has a bounded left approximate identity. Then the standard quotient norm on $\mathscr{I}_2(X)/\mathscr{I}_1(X)$ is maximal.*

Proof. This is a direct consequence of Corollary 1.1.14 and Theorem 1.1.17. ■

In particular, if $\mathscr{I}_1(X) = \{0\}$ in the above corollary, we have that the operator norm on $\mathscr{I}_2(X)$ is maximal and therefore unique, because it is minimal by the Eidelheit-Yood theorem (1.1.2).

From Definition 1.1.9 onwards we have discussed specific conditions which ensure the maximality of the algebra norm on $\mathscr{B}(X)$ and related algebras, when X is a Banach space. However, we have not yet given an example to show that the standard algebra norm on these algebras can fail to be maximal, to justify the need for specificity. It would not be surprising if there were no such example in the case of $\mathscr{B}(X)$: at the other end of the spectrum, the Eidelheit-Yood theorem guarantees the norm is always minimal. However, the appearance of [Rea], published in 1989, established the existence of a Banach space X such that there is a discontinuous derivation from $\mathscr{B}(X)$ to a Banach $\mathscr{B}(X)$-bimodule. A *derivation* from a Banach algebra $\mathfrak{A}$ to a Banach $\mathfrak{A}$-bimodule Y is a linear map $D : \mathfrak{A} \to Y$ such that, for all $a, b \in \mathfrak{A}$,

$$D(ab) = a \cdot Db + Da \cdot b\,.$$

Given such a derivation D that is also discontinuous, the space $\mathfrak{A} \oplus Y$ is a Banach algebra, with the product defined by $(a, x)(b, y) = (ab, a \cdot y + x \cdot b)$ for all $a, b \in \mathfrak{A}$ and $x, y \in Y$, and furthermore the map $\mathfrak{A} \to \mathfrak{A} \oplus Y$ given by $a \mapsto (a, Da)$ is a discontinuous homomorphism (see, e.g., [Dal1, Thm. 2.7.5]). Hence, the existence of a discontinuous derivation from a Banach algebra $\mathfrak{A}$ to a Banach $\mathfrak{A}$-bimodule implies that there is a discontinuous homomorphism from $\mathfrak{A}$, and thus the norm on $\mathfrak{A}$ fails to be maximal by Proposition 1.1.13.

Therefore, we have that:

1.1.19 Theorem ([Rea])**.** *There exist Banach spaces X for which the standard norm on $\mathscr{B}(X)$ fails to be maximal.*

Hence we have examples of spaces for which $\mathscr{B}(X)$ does not have a unique algebra norm. We shall discuss further details of the Read example in Chapter 3. At that point we shall also study a recent construction from [Tar2] that gives another example of a Banach space X, of a different nature to the example from [Rea], for which the standard norm on $\mathscr{B}(X)$ is not maximal. For the moment, it is sufficient to note that these spaces are very unnatural: Read's space is a particular quotient of an infinite direct sum of James-like spaces generated by an elaborate choice of members from an infinite family of incomparable symmetric norms, and Tarbard's space is an indecomposable, but not hereditarily indecomposable, space that builds on the recent Argyros-Haydon construction ([ArHa]) of a Banach space on which every operator is a scalar multiple of the identity plus a compact operator. So Banach spaces X for which $\mathscr{B}(X)$ does not have a unique algebra norm seem to be very peculiar indeed.

The *radical* of a unital Banach algebra $\mathfrak{A}$ with identity e, denoted $\operatorname{rad}\mathfrak{A}$, is the intersection of all the maximal left ideals of $\mathfrak{A}$, and is a closed (two-sided) ideal of $\mathfrak{A}$. Equivalently, the radical may be defined as the intersection of all the maximal right ideals of $\mathfrak{A}$, or by:

$$\operatorname{rad}\mathfrak{A} \;=\; \{a \in \mathfrak{A} : e - ba \in \operatorname{Inv}\mathfrak{A}\ (\forall b \in \mathfrak{A})\}\,,$$

where $\operatorname{Inv}\mathfrak{A}$ denotes the set of invertible elements in $\mathfrak{A}$. We say that $\mathfrak{A}$ is *semisimple* if $\operatorname{rad}\mathfrak{A} = \{0\}$. The relevance of semisimplicity to our discussion is that $\mathscr{B}(X)$ is always semisimple, for every Banach space X (see, e.g., [Dal2, Prop. 2.2.4]).

We have seen that the standard norm on $\mathscr{B}(X)$ is always minimal, but may fail to be maximal. In the cases when it is maximal, we have two independent ways to establish that $\mathscr{B}(X)$ has a unique complete norm, by Theorem 1.1.7. In fact, there is a yet another way:

1.1.20 Theorem ([Joh2, Thm. 2]). *A semisimple Banach algebra has a unique complete algebra norm.*

This completes our discussion of uniqueness of norm properties on $\mathscr{B}(X)$, when X is a Banach space. As with the minimality of the norm on $\mathscr{B}(X)$, the semisimplicity of $\mathscr{B}(X)$ does not pass to the Calkin algebra $\mathscr{C}(X)$. We previously noted that in the cases when the algebra norm on $\mathscr{B}(X)$ is maximal, maximality does pass to the quotient norm on $\mathscr{C}(X)$. Hence, in such cases, $\mathscr{C}(X)$ is also known to have a unique complete norm. Like maximality, we therefore will be less concerned with whether a given Calkin algebra has a unique complete algebra norm, and our focus is on the stronger property of uniqueness of the algebra norm. In this context it is worth noting the following.

1.1.21 Lemma. *Let $(\mathfrak{A}, \|\cdot\|_1)$ be a Banach algebra such that any algebra norm $\|\cdot\|_2$ on $\mathfrak{A}$ is necessarily complete. Then $\mathfrak{A}$ has a unique algebra norm.*

Proof. Let $\|\cdot\|_2$ be an arbitrary algebra norm on $\mathfrak{A}$. Then, by assumption, both $\|\cdot\|_2$ and $|||\cdot||| = \|\cdot\|_1 + \|\cdot\|_2$ are complete. $|||\cdot|||$ is an algebra norm because, for all $a, b \in \mathfrak{A}$,

$$|||ab||| \;\le\; \|a\|_1\|b\|_1 + \|a\|_2\|b\|_2 \;\le\; (\|a\|_1 + \|a\|_2)(\|b\|_1 + \|b\|_2) \;=\; |||a|||\,|||b|||\,.$$

Now, $\|\cdot\|_1 \le |||\cdot|||$ and $\|\cdot\|_2 \le |||\cdot|||$, so by the open mapping theorem we must have

$$\|\cdot\|_1 \sim |||\cdot||| \sim \|\cdot\|_2\,. \qquad \blacksquare$$

1.2 Closed ideals and quotients of $\mathscr{B}(X)$

For a given Banach space X, §1.1 detailed several results that applied not only to $\mathscr{B}(X)$, but also to its closed ideals and quotients by them. In this section, we consider some specific examples of ideals in $\mathscr{B}(X)$, to which much of that theory can be applied. While our primary focus in this thesis is on the Calkin algebra $\mathscr{C}(X)$ and by association the ideal $\mathscr{K}(X)$ of compact operators, there are also other ideals in which we shall have a passing interest.

Suppose that X and Y are Banach spaces. An operator $T \in \mathscr{B}(X,Y)$ is *weakly compact* if $T(B_X)$ is a relatively weakly compact subset of Y. We write $\mathscr{W}(X,Y)$ for the space of all weakly compact operators in $\mathscr{B}(X,Y)$. T is *approximable* if it is lies in the closure of the finite rank operators $\mathscr{F}(X,Y)$, and we write $\mathscr{A}(X,Y)$ for the space of all approximable operators in $\mathscr{B}(X,Y)$. T is *strictly singular* if it is not an isomorphism onto its range when restricted to any infinite dimensional subspace of X. We write $\mathscr{S}(X,Y)$ for the space of all strictly singular operators in $\mathscr{B}(X,Y)$. Finally, T is *inessential* if $(I - ST)$ is a Fredholm operator for every $S \in \mathscr{B}(Y,X)$, which means that $\ker(I - ST)$ and $X/(I - ST)(X)$ are both finite dimensional. We write $\mathscr{E}(X,Y)$ for the space of all inessential operators in $\mathscr{B}(X,Y)$.

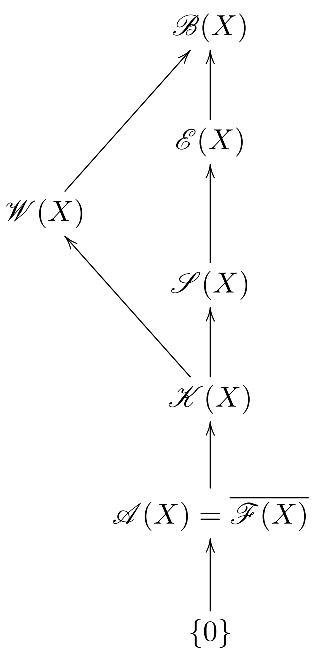

Figure 1: The hierarchy amongst certain standard closed operator ideals

When $Y = X$, we write simply $\mathscr{W}(X)$, $\mathscr{A}(X)$, $\mathscr{S}(X)$, and $\mathscr{E}(X)$ respectively, and each of these spaces of operators forms an ideal in $\mathscr{B}(X)$. $\mathscr{E}(X)$ can be characterised as the inverse image of $\operatorname{rad}\mathscr{C}(X)$ under the quotient map from $\mathscr{B}(X)$ to $\mathscr{C}(X)$. The relationship between these ideals and $\mathscr{K}(X)$ is shown in the preceding diagram. Each inclusion depicted may be strict.

While our main concern will be whether the Calkin algebra of a given Banach space X has a unique algebra norm, the same question can be asked of other quotients of $\mathscr{B}(X)$. The following proposition shows that there is a degree of connection between these questions. It does not seem to have been previously noted in the literature.

1.2.1 Proposition. *Let $\mathfrak{A}$ be a Banach algebra, and let $\mathfrak{I}_1$, $\mathfrak{I}_2$ be closed ideals in $\mathfrak{A}$ with $\mathfrak{I}_1 \subset \mathfrak{I}_2$. Suppose that both $\mathfrak{A}/\mathfrak{I}_2$ and $\mathfrak{I}_2/\mathfrak{I}_1$ have unique algebra norms. Then $\mathfrak{A}/\mathfrak{I}_1$ has a unique algebra norm.*

Proof. Let $|||\cdot|||$ be an arbitrary algebra norm on $\mathfrak{A}/\mathfrak{I}_1$. Then $|||\cdot|||_{|\mathfrak{I}_2/\mathfrak{I}_1}$ is an algebra norm on $\mathfrak{I}_2/\mathfrak{I}_1$, and is therefore equivalent to $\|\cdot\|$ on $\mathfrak{I}_2/\mathfrak{I}_1$. In particular it is complete on $\mathfrak{I}_2/\mathfrak{I}_1$, so $\mathfrak{I}_2/\mathfrak{I}_1$ is closed in $(\mathfrak{A}/\mathfrak{I}_1, |||\cdot|||)$. We may identify $\mathfrak{A}/\mathfrak{I}_2$ with $\frac{\mathfrak{A}}{\mathfrak{I}_1}/\frac{\mathfrak{I}_2}{\mathfrak{I}_1}$, and thus consider the quotient norm on $\mathfrak{A}/\mathfrak{I}_2$ induced by $|||\cdot|||$, which we will also denote $|||\cdot|||$.

$|||\cdot|||$ is an algebra norm on $\mathfrak{A}/\mathfrak{I}_2$, so it is equivalent to $\|\cdot\|$ and hence is complete. Therefore $\left(\frac{\mathfrak{A}}{\mathfrak{I}_1}/\frac{\mathfrak{I}_2}{\mathfrak{I}_1}, |||\cdot|||\right)$ and $(\mathfrak{I}_2/\mathfrak{I}_1, |||\cdot|||)$ are both Banach spaces. Since completeness is a three-space property,[8] we conclude that $|||\cdot|||$ is a complete norm on $\mathfrak{A}/\mathfrak{I}_1$. Thus any algebra norm on $\mathfrak{A}/\mathfrak{I}_1$ must be complete, and so $\mathfrak{A}/\mathfrak{I}_1$ has a unique algebra norm by Lemma 1.1.21. ∎

In the context of Proposition 1.2.1, it is natural to ask whether, for a Banach algebra $\mathfrak{A}$ and closed ideal $\mathfrak{I} \subset \mathfrak{A}$, the weaker assumption that both $\mathfrak{A}/\mathfrak{I}$ and $\mathfrak{I}$ have unique complete norms implies that $\mathfrak{A}$ has a unique complete norm. However, this is false: an example from the introductory section of [Fel] (see also [Dal1, Ex. 5.4.6]) gives a complex Banach algebra $\mathfrak{A}$, without a unique complete norm, and such that $\operatorname{rad}\mathfrak{A} \approx \mathbb{C}$. Clearly $\operatorname{rad}\mathfrak{A}$ has a unique complete norm, and $\mathfrak{A}/\operatorname{rad}\mathfrak{A}$ is semisimple and thus has unique complete norm by Theorem 1.1.20.

Another consideration is whether a Calkin algebra having a unique algebra norm might dualise in any way. In this regard, the following result is relevant.

[8] See, for example, [Meg, Defn. 1.7.8 and Thm. 1.7.9].

1.2.2 Theorem. *Let X be a reflexive Banach space. Then $\mathscr{C}(X)$ has a unique algebra norm if and only if $\mathscr{C}(X^*)$ has a unique algebra norm.*

Proof. Because X is reflexive, the isometric antihomomorphism $T \mapsto T^*$ maps $\mathscr{B}(X)$ onto $\mathscr{B}(X^*)$. By Schauder's theorem ([Scha, Satz I–II]), T is compact if and only if T^* is compact. Thus, $\mathscr{C}(X)$ and $\mathscr{C}(X^*)$ are isometric under the antihomomorphism $T + \mathscr{K}(X) \mapsto T^* + \mathscr{K}(X^*)$. Therefore, any inequivalent algebra norm on $\mathscr{C}(X)$ induces a likewise inequivalent algebra norm on $\mathscr{C}(X^*)$, and vice versa. ■

1.3 Bases in Banach spaces

Much of the technical theory we shall rely upon relates to the properties of basic sequences in specific Banach spaces. In this section, we collate the various elementary definitions and results relating to basic sequences that are needed, and introduce some useful notation. See, for example, [Meg, Chp. 4] and [AlKa, Chps. 1,9] for more details.

As is standard, we shall use the term *basis* to mean a Schauder basis. Given a Banach space X, a *Schauder basis* for X is a sequence $(x_n) \subset X$ such that every $x \in X$ has a unique representation

$$x = \sum_n c_n x_n$$

for some $(c_n) \subset \mathbb{F}$. Note that the order of summation, and therefore the order of the basis, may be important, as the sum may not converge unconditionally. If the sum converges unconditionally for all $x \in X$, we call the basis *unconditional.*

Given a basis (x_n) for a Banach space X, for each $m \in \mathbb{N}$ we can define the *basis projection* $P_m : X \to X$ by

$$P_m : \sum_n c_n x_n \mapsto \sum_{n=1}^{m} c_n x_n \, .$$

We also set $P_0 = 0$. The basis projections of (x_n) are all bounded, finite rank linear operators on X, and have the property that $\sup_n \|P_n\|$ is finite. We call the value $\sup_n \|P_n\|$ the *basis constant* of (x_n). A basis is *monotone* if it has basis constant equal to 1, meaning that $\|P_n\| = 1$ for all $n \in \mathbb{N}$. If, in addition, $\|P_m - P_n\| = 1$ for all $m, n \in \mathbb{N}$ such that $m \neq n$, then the basis is *bimonotone.*

For $E \subset \mathbb{N}$, we also use E to denote the projection $E : X \to X$ that is formally given by

$$E\left(\sum_{n=1}^{\infty} c_n x_n\right) = \sum_{n\in E} c_n x_n\,,$$

in the cases where it is well defined. When E is infinite, the sum over $n \in E$ in the above is taken in the natural order inherited from $\mathbb{N}$. Note that every $E \subset \mathbb{N}$ gives a well defined projection if and only if the underlying basis is unconditional, in which case the *suppression constant* $\sup_{E\subset\mathbb{N}} \|E\|$ of the basis is finite. A basis (x_n) is also unconditional if and only if, for all sequences of signs $(\sigma_n) \subset \{-1, 1\}$, the mapping $\sum_n c_n x_n \mapsto \sum_n \sigma_n c_n x_n$ is a bounded operator, in which case the supremum of the norms of all such operators is called the *unconditional constant* K_u of the basis. For all $K \geq K_u$, we say (x_n) is *K-unconditional*, and have that

$$\|\sum_n a_n x_n\| \;\leq\; 2K\|\sum_n b_n x_n\|$$

whenever $\sum_n b_n x_n \in X$ and $|a_n| \leq |b_n|$ for all $n \in \mathbb{N}$.

A sequence $(x_n) \subset X$ is, more generally, a *basic sequence* ⟨resp. an *unconditional basic sequence*⟩ if it is a basis ⟨resp. an unconditional basis⟩ for its closed linear span $[(x_n)]$. Two basic sequences (x_n) and (y_n), in potentially different Banach spaces, are called *equivalent*, and we write $(x_n) \sim (y_n)$, if for all choices of scalars $(c_n) \subset \mathbb{F}$ it is the case that $\sum_n c_n x_n$ converges if and only if $\sum_n c_n y_n$ converges. If (x_n) is equivalent to (y_n), then there is an isomorphism $T : [(x_n)] \to [(y_n)]$ such that, for all $n \in \mathbb{N}$, $Tx_n = y_n$. This means there exists $C \geq 1$ such that, for all $(c_n) \subset \mathbb{F}$ for which $\sum_n c_n x_n$ and $\sum_n c_n y_n$ converge,

$$C^{-1}\left\|\sum_n c_n y_n\right\| \;\leq\; \left\|\sum_n c_n x_n\right\| \;\leq\; C\left\|\sum_n c_n y_n\right\|.$$

For any such C, we say that (x_n) and (y_n) are *C-equivalent*. If, for some $C \geq 1$, a closed subspace of a Banach space X is complemented by a projection P, with $\|P\| \leq C$, we say that the subspace is *C-complemented*. A basic sequence $(x_n) \subset X$ is also termed *complemented* ⟨resp. *C-complemented*⟩ if $[(x_n)]$ is complemented ⟨resp. C-complemented⟩.

A basis or basic sequence (x_n) is *symmetric* if $(x_{\pi(n)}) \sim (x_n)$ for any permutation π of $\mathbb{N}$. Symmetric bases (x_n) are necessarily unconditional, and have the property that there is a constant $K \geq 1$ such that, for each $\sum_n c_n x_n \in [(x_n)]$,

$$\left\|\sum_n \sigma_n c_n x_{\pi(n)}\right\| \;\leq\; K\left\|\sum_n c_n x_n\right\|$$

holds for all sequences of signs $(\sigma_n) \subset \{-1, 1\}$ and all permutations π of $\mathbb{N}$. For such a K we say that (x_n) is *K-symmetric*. Symmetric bases all have the more general property that they are subsymmetric:

For a sequence (y_n) in an arbitrary set (not necessarily in a Banach space), we shall write $(y_{n_k}) \subset (y_n)$ to indicate that (y_{n_k}) is a subsequence $(y_{n_k})_{k=1}^{\infty}$ of (y_n), with (n_k) implied to be a strictly increasing sequence in $\mathbb{N}$. A basis or basic sequence (x_n) is *subsymmetric* if (x_n) is unconditional and if, for every $(x_{n_k}) \subset (x_n)$, we have $(x_{n_k}) \sim (x_n)$. Subsymmetric bases (x_n) have the property that there is a constant $C \geq 1$ such that, for each $\sum_n c_n x_n \in [(x_n)]$,

$$\left\| \sum_n \sigma_n c_n x_{n_k} \right\| \leq C \left\| \sum_n c_n x_n \right\|$$

holds for all sequences of signs $(\sigma_n) \subset \{-1, 1\}$ and all strictly increasing sequences $(n_k) \subset \mathbb{N}$. For such a C we say that (x_n) is *C-subsymmetric*.

Given a Banach space X with a basis ⟨resp. an unconditional basis; a subsymmetric basis; or a symmetric basis⟩, it is possible to give X an equivalent norm such that the basis is monotone and bimonotone ⟨resp. has suppression constant and unconditional constant equal to 1; is 1-subsymmetric; is 1-symmetric⟩. Since much of what we shall consider in this thesis holds up to isomorphism, this means that in most cases we can assume that all the relevant basis constants associated with a basis are all equal to 1, without affecting the validity of results.

1.3.1 Example. The Banach space c_0 is the sequence space of all $(\alpha_n) \subset \mathbb{F}$ such that $\alpha_n \to 0$, with the norm given by

$$\|(\alpha_n)\|_{c_0} = \sup_n |\alpha_n| .$$

The Banach space ℓ_p, when $1 \leq p < \infty$, is the sequence space of all $(\alpha_n) \subset \mathbb{F}$, such that the norm

$$\|(\alpha_n)\|_{\ell_p} = \left(\sum_n |\alpha_n|^p \right)^{1/p}$$

is finite. For each of these spaces, the *standard unit vector basis* (e_n) of a sequence space, given by

$$e_n = (\delta_{nm})_{m=1}^{\infty} ,$$

is monotone, bimonotone, has unconditional basis constant and unconditional constant equal to 1, and is both 1-subsymmetric and 1-symmetric. Here δ_{nm} is the Kronecker delta, which is 1 if $n = m$ and 0 otherwise.

Let X be a Banach space with a basis (x_n). The *support* of a vector $x = \sum_n c_n x_n \in X$ is the set

$$\operatorname{supp}(x) = \{n \in \mathbb{N} : c_n \neq 0\}.$$

For $A, B \subset \mathbb{N}_0$ we write $A < B$ when $n < m$ for all $n \in A$ and $m \in B$. We also write $n < B$ in the case that $\{n\} < B$, and $A < m$ in the case that $A < \{m\}$. For $x, y \in X$ and $n \in \mathbb{N}_0$, we write $x \prec y$ if $\operatorname{supp}(x) < \operatorname{supp}(y)$, $n \prec x$ if $n < \operatorname{supp}(x)$, and $x \prec n$ if $\operatorname{supp}(x) < n$.

A sequence of non-zero vectors $(y_n) \subset X$ for which $y_n \prec y_{n+1}$ for all $n \in \mathbb{N}$ is called a *block basic sequence of* (x_n), or just a *block basic sequence* or *block basis* (for its closed linear span) in cases where the respective basis is clear. We write $A < \infty$ if A is a finite subset of $\mathbb{N}$, and $x \prec \infty$ if $\operatorname{supp}(x) < \infty$. Note that if (y_n) is a block basic sequence then necessarily $y_n \prec \infty$ for all $n \in \mathbb{N}$; vectors x for which $x \prec \infty$ are called *blocks*. Vectors x and y for which $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$ are called *disjoint*; clearly, any two members of a block basis are disjoint.

Further to Example 1.3.1, we have the following very strong property of the standard unit vector basis in c_0 and ℓ_p, $1 \leq p < \infty$, which characterises those spaces, as the subsequent theorem shows.

1.3.2 Proposition (see, e.g., [AlKa, Lem. 2.1.1]). *Let X be c_0 or ℓ_p, $1 \leq p < \infty$, and suppose (y_n) is a normalised block basic sequence of the standard unit vector basis in X. Then (y_n) is 1-complemented and 1-equivalent to the standard unit vector basis in X.*

1.3.3 Theorem ([Zip, Thm. 3.1]). *Let (x_n) be a normalised basis for a Banach space X. Then (x_n) is equivalent to the standard unit vector basis of c_0 or ℓ_p, $1 \leq p < \infty$, if and only if every normalised block basis (y_n) of (x_n) is equivalent to (x_n).*

We now establish two preparatory results that relate the essential norm of an operator its action on blocks and block bases.

1.3.4 Lemma. *Suppose X is a Banach space with a bimonotone basis (x_n) and that $T \in \mathscr{B}(X)$. Then for all $i, j \in \mathbb{N}_0$ and $\delta > 0$ there exists $x \in S_X$ such that $i \prec x \prec \infty$ and $\|(I - P_j)Tx\| > \|T\|_e - \delta$.*

Proof. Let $i, j \in \mathbb{N}_0$ and $\delta > 0$ be given. For all $n \in \mathbb{N}_0$, $P_n \in \mathscr{F}(X)$. Thus,

$TP_i \in \mathscr{F}(X)$ and $T(I - P_i) \in T + \mathscr{F}(X) \subset T + \mathscr{K}(X)$. We can similarly conclude that $(I - P_j)T(I - P_i) \in T + \mathscr{K}(X)$ and hence $\|(I - P_j)T(I - P_i)\|_e = \|T\|_e$. Since $\|(I - P_j)T(I - P_i)\| \geq \|(I - P_j)T(I - P_i)\|_e$, this means that there exists $y \in B_X$ such that

$$\|(I - P_j)T(I - P_i)y\| > \|T\|_e - \delta.$$

Now, because (x_n) is a basis, $\lim_{n\to\infty} P_n(I - P_i)y = (I - P_i)y$. Therefore, there is $N \in \mathbb{N}$ such that $\|(I - P_j)TP_N(I - P_i)y\| > \|T\|_e - \delta$. Set $x = P_N(I - P_i)y$. Since (x_n) is bimonotone, $\|x\| \leq \|y\|$. Thus $x \in B_X$, and without loss of generality we may scale x so that $\|x\| = 1$. Furthermore, $i \prec x \prec N + 1 < \infty$, as required. ■

1.3.5 Corollary. *Suppose X is a Banach space with a bimonotone basis (x_n), $T \in \mathscr{B}(X)\backslash\mathscr{K}(X)$ and that $(\delta_n) \subset \mathbb{R}_+$. Then there exist block basic sequences (y_n) and (z_n) of (x_n), and an increasing sequence $(N_n)_{n=0}^{\infty} \subset \mathbb{N}_0$ with $N_0 = 0$, such that, for all $n \in \mathbb{N}$,*

(i) $\|y_n\| = 1$,

(ii) $z_n = (P_{N_n} - P_{N_{n-1}})Ty_n$,

(iii) $\|(I - P_{N_n})Ty_n\| < \delta_n$, *and*

(iv) $\|z_n\| > \|T\|_e - \delta_n$.

Proof. We proceed by induction by repeatedly applying Lemma 1.3.4. First choose $y_1 \in S_X$ such that $y_1 \prec \infty$ and $\|Ty_1\| > \|T\|_e - \frac{\delta_1}{2}$. Let z_1 be a large enough initial segment of Ty_1, with $z_1 = P_{N_1}Ty_1 = (P_{N_1} - P_{N_0})Ty_1$ for some $N_1 \in \mathbb{N}$, such that

$$\|z_1 - Ty_1\| = \|(I - P_{N_1})Ty_n\| < \frac{\delta_1}{2}.$$

Thus we also have

$$\|z_1\| > \|Ty_1\| - \frac{\delta_1}{2} > \|T\|_e - \delta_1 .$$

Now assume $y_1 \prec \cdots \prec y_m \prec \infty$, $z_1 \prec \cdots \prec z_m \prec \infty$, and $(N_n)_{n=0}^{m}$ have been chosen in accordance with requirements. Then, by Lemma 1.3.4, there is $y_{m+1} \in S_X$ such that $y_m \prec y_{m+1} \prec \infty$ and $\|(I - P_{N_m})Ty_{m+1}\| > \|T_e\| - \frac{\delta_{m+1}}{2}$. Let z_{m+1} be an initial segment of $(I - P_{N_m})Ty_{m+1}$, large enough that

$$\|z_{m+1} - (I - P_{N_m})Ty_{m+1}\| < \frac{\delta_{m+1}}{2} ,$$

and choose $N_{m+1} \in \mathbb{N}$ such that $z_{m+1} = P_{N_{m+1}}z_{m+1}$. Note that it follows that $\|z_{m+1}\| > \|T\|_e - \delta_{m+1}$, $z_{m+1} = (P_{N_{m+1}} - P_{N_m})Ty_{m+1}$, and clearly $z_m \prec z_{m+1}$. So by induction we can construct block bases (y_n) and (z_n) as required. ■

The final result in this section that concerns basic sequences is commonly called the *principle of small perturbations*. It was originally established in 1940 by Krein, Milman, and Rutman, however it has become ubiquitous in the literature in various forms. We will give the version from [AlKa]. For ease of future reference, we base our presentation on the following definition.

1.3.6 Definition. Let (x_n) be a basic sequence in a Banach space X, with basis constant K. If (y_n) is a sequence in X such that

$$2K\sum_n \frac{\|x_n - y_n\|}{\|x_n\|} \;\leq\; \delta \;<\; 1\,,$$

then we call (y_n) a *δ-small perturbation of* (x_n).

1.3.7 Theorem (see, e.g., [AlKa, Thm. 1.3.9])**.** *Let (x_n) be a basic sequence in a Banach space X with basis constant K. If (y_n) is a δ-small perturbation of (x_n), then (y_n) is a basic sequence with basis constant at most $(1+\delta)(1-\delta)^{-1}K$, and $(x_n) \sim (y_n)$. In particular, there is an invertible $U \in \mathscr{B}(X)$ such that $Ux_n = y_n$, $\|U\| \leq 1+\delta$, $\|U^{-1}\| \leq (1-\delta)^{-1}$, and $(I-U) \in \mathscr{K}(X)$.*

If, furthermore, $[(x_n)]$ is complemented in X by a projection P, then $[(y_n)]$ is complemented in X by a projection Q with norm at most $(1+\delta)(1-\delta)^{-1}\|P\|$.

Proof. We have added slightly to the content of [AlKa, Thm. 1.3.9]. In particular, the claim about the norm of Q and the fact that $(I-U) \in \mathscr{K}(X)$ do not appear there. To substantiate the additional details, note that U is given by, for each $x \in X$,

$$Ux \;=\; x + \sum_n x_n^*(x)(y_n - x_n)\,,$$

for certain $x_n^* \in X^*$ such that $\|x_n^*\| \leq 2K\|x_n\|^{-1}$ for all $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, define $F_m \in \mathscr{F}(X)$ by

$$F_m x \;=\; \sum_{n=1}^{m} x_n^*(x)(x_n - y_n)\,.$$

Then

$$\|(I - U - F_m)x\| \;\leq\; \sum_{n=m+1}^{\infty} \|x_n^*\|\|x\|\|y_n - x_n\|$$

for all $x \in X$, and so

$$\|I - U - F_m\| \leq \sum_{n=m+1}^{\infty} \|x_n^*\|\|y_n - x_n\| \leq 2K \sum_{n=m+1}^{\infty} \frac{\|x_n - y_n\|}{\|x_n\|}.$$

This gives $\|I - U - F_m\| \to 0$, since by assumption

$$\sum_n \frac{\|x_n - y_n\|}{\|x_n\|}$$

converges. Thus $(I - U) \in \mathscr{A}(X) \subset \mathscr{K}(X)$. Now suppose that P is a complementing projection for $[(x_n)]$. Then set $Q = UPU^{-1}$, so that

$$\|Q\| \leq \|U\|\|P\|\|U^{-1}\| \leq (1+\delta)(1-\delta)^{-1}\|P\|.$$

We have $Q(X) \subset UP(X) = U([(x_n)]) = [(y_n)]$, and $Qy_n = y_n$ for all $n \in \mathbb{N}$, so Q is a projection onto $[(y_n)]$. ■

In the above theorem, it is also possible to conclude that $(I - U^{-1}) \in \mathscr{K}(X)$ from the fact that $(I - U) \in \mathscr{K}(X)$, as the following corollary shows.[9] However, note that the U in this corollary is the inverse of the U in the above. We state the result in this form because it is the specific version of the principle of small perturbations we shall require in §3.3 and §5.1.

1.3.8 Corollary. *Let (x_n) be a basic sequence in a Banach space X, and suppose $0 < \delta < 1$. If (y_n) is a δ-small perturbation of (x_n), then (y_n) is a basic sequence, and there is $U \in \mathscr{B}(X)$ such that*

(i) $Uy_n = x_n$ for all $n \in \mathbb{N}$, and

(ii) $(I - U) \in \mathscr{K}(X)$.

Proof. Theorem 1.3.7 gives that (y_n) is a basic sequence, and that there exists an invertible $V \in \mathscr{B}(X)$ such that (i) $Vx_n = y_n$ for all $n \in \mathbb{N}$, and for which (ii) $T = (I - V) \in \mathscr{K}(X)$. Set $U = V^{-1}$, so that $Uy_n = x_n$ for all $n \in \mathbb{N}$. Also,

$$I - U = I - U(I - T + T) = I - V^{-1}V + V^{-1}T = V^{-1}T \in \mathscr{K}(X). \quad ■$$

[9] As indicated in the proof of Theorem 1.3.7, we actually have the stronger result that $(I - U)$, $(I - U^{-1}) \in \mathscr{A}(X)$. In all situations in which we shall apply the result, the whole of X has a basis, so the distinction between $\mathscr{A}(X)$ and $\mathscr{K}(X)$ is not relevant.

1.4 Overview

The main idea which drives all of the results we shall present, is that the uniform boundedness principle, and as a consequence the Eidelheit-Yood method, can be generalised to apply to Calkin algebras, if we take the perspective that non-compact operators must map some bounded well-separated sequences to other well-separated sequences.

With that concept in mind, in Chapter 2 we shall investigate, from a completely elementary standpoint, the action of non-compact operators on sequences. The primary result of Chapter 2 is to establish a measure of the non-compactness of an operator, which relates to the limiting separation of relevant image sequences. We will continue investigation of the properties of that measure in Chapter 4, which also contains the analogous version of the uniform boundedness principle for Calkin algebras that was mentioned above. In between, Chapter 3 gives further background material, relating to results of which the author was unaware while completing the work presented in Chapter 2.

In the case that the Banach space underlying a given Calkin algebra has a basis, we can, within our framework, restrict attention to the action of non-compact operators on normalised block bases, whose images are themselves small perturbations of constant-norm block bases. This is due to arguments involving the principle of small perturbations (1.3.7). We term this result a 'block-to-block' lemma.

In combination with the Calkin algebra version of the uniform boundedness principle, we are thus able to derive a 'block-to-blocks' lemma in the case that a basis exists: given a (countable) set of non-compact operators which is unbounded in the essential norm, there must be a common normalised block basis in the underlying Banach space, the images of which, under the various members of the set, are 'almost' block bases, and whose norms are uniformly unbounded. This is the final result of Chapter 4.

Previously, in [Mey], it was shown that the Calkin algebras $\mathscr{C}(c_0)$ and $\mathscr{C}(\ell_p)$, $1 \leq p < \infty$, have minimal and thus unique algebra norms (note that each of the respective Banach spaces X satisfy $X \oplus X \approx X$, and so the essential norm is maximal on these Calkin algebras). This result, among others, will be further discussed in Chapter 3. [Mey] utilises a block-to-block lemma for the spaces c_0 and ℓ_p, $1 \leq p < \infty$, and then exploits the characteristic property of block bases in these spaces that was quoted in Proposition 1.3.2.

One of our major results, given in Chapter 5, will be to show how these strong requirements of the basis for a space can be somewhat relaxed, yet still achieve the uniqueness of norm result for the Calkin algebra. We call this sufficient, but by no means necessary, condition on the underlying Banach space the 'UN property'. Given a space with UN property, our method for establishing the result is to apply an anologue of the Eidelheit-Yood method in combination with the block-to-blocks lemma from Chapter 4.

The remainder of Chapter 5 gives examples of spaces with the UN property, and thus new examples of spaces whose Calkin algebras have unique algebra norms. We thus extend the result of [Mey]. In particular, we establish that the Calkin algebras of canonical Banach spaces such as James' space and Tsirelson's space have unique algebra norms. In Chapter 6, we also prove uniqueness of norm for quotients of the algebras of operators on classical non-separable spaces, the closed ideals of which were previously studied in [Daw]. The methods used in Chapter 6 are closer to the original ones from [Mey], in contrast to those of Chapter 5. In Chapter 7, we outline some possibilities for further applications of the results and perspectives of this thesis.

Chapter 2

An Alternative Algebra Norm on $\mathscr{C}(X)$

While it is possible, in some cases, to define an algebra norm on $\mathscr{B}(X)$ that is not equivalent to the operator norm (assuming that the Banach space X is sufficiently peculiar), we have seen that such examples do not arise naturally (see §1.1). Indeed, if one starts with only classical facts, writing down an algebra norm on a general $\mathscr{B}(X)$ that might not be equivalent to $\|\cdot\|$ is a seemingly impossible task. If we were unaware of the Read and Tarbard examples, then the Eidelheit-Yood theorem (1.1.2) would significantly support this belief since it holds for all Banach spaces. However, as discussed in Remark 1.1.6, in the case of $\mathscr{C}(X)$ the collapse of $\mathscr{K}(X)$ to $\{0\}$ means that the Eidelheit-Yood proof does not easily generalise. Thus on $\mathscr{C}(X)$ there appears, at least initially, to be more scope for the existence of an inequivalent norm. So a reasonable first step in investigating uniqueness of norm properties for Calkin algebras is to attempt a definition of an alternative algebra norm; in particular we wish to make a definition that is not obviously equivalent to the essential/quotient norm. The purpose is to explore how much leeway is readily discernible.

In this chapter we will pursue this initial task along the same lines as originally taken by the author, without the knowledge of prior results that will be discussed in Chapter 3.

2.1 Another way to measure the 'compactness' of an operator

Since any norm on $\mathscr{C}(X)$ can be associated with a semi-norm on $\mathscr{B}(X)$ with kernel $\mathscr{K}(X)$, and vice versa (refer to §1.2), we seek a definition of a semi-norm on $\mathscr{B}(X)$

whose value for an operator T will be 0 if and only if T is compact. Such a semi-norm should thus involve a measure of how close an operator is to being compact, but we must use a quantity that is sufficiently different to the essential norm in order that the new definition might not be equivalent. To achieve this, we consider the formulation of compactness in terms of the presence of convergent subsequences amongst the images of bounded sequences. In the following, Definition 2.1.1, Proposition 2.1.3, and its proof, were inspired by [Fer, Defn. 2 and Lem. 2], which drew similar conclusions about an analogous norm on quotient algebras of the form $\mathscr{B}(X)/\mathscr{S}(X)$.

2.1.1 Definition. Let X and Y be Banach spaces. For $(x_n) \subset X$ and $T \in \mathscr{B}(X, Y)$, set

$$\lceil T\,, (x_n) \rceil \;=\; \lim_{N\to\infty} \sup_{n_1,n_2 \geq N} \|T(x_{n_1} - x_{n_2})\| \,.$$

For $\lambda > 0$, denote by Δ_X^λ the collection of all sequences in X of diameter at most λ, that is,

$$\Delta_X^\lambda \;=\; \{(x_n) \subset X : (\forall n_1, n_2 \in \mathbb{N})\ \|x_{n_1} - x_{n_2}\| \leq \lambda\}.$$

Now define

$$\|T\|_K \;=\; \sup_{(x_n)\subset \Delta_X^1} \; \inf_{(x_{n_k})\subset (x_n)} \lceil T\,, (x_{n_k}) \rceil \,.$$

2.1.2 Remark. Note that $\|T\|_K \leq \|T\|$ for all $T \in \mathscr{B}(X, Y)$. The quantity $\lceil T\,, (x_n) \rceil$ is in some sense a measure of the divergence of the sequence (Tx_n), and equals 0 if and only if (Tx_n) converges. As such, the following facts regarding $\lceil \cdot\,, \cdot \rceil$ are not surprising. Given $T \in \mathscr{B}(X, Y)$ and $(x_n) \subset X$, suppose $(x_{n_k}) \subset (x_n)$. Then, comparing $\lceil T\,, (x_{n_k}) \rceil$ to $\lceil T\,, (x_n) \rceil$, for a given $N \in \mathbb{N}$ the supremum in the definition for $\lceil \cdot\,, \cdot \rceil$ ranges over a restricted subset, so it follows that

$$\lceil T\,, (x_{n_k}) \rceil \;\leq\; \lceil T\,, (x_n) \rceil \,.$$

Also, note that removing a finite number of members from (x_n) will not change the value of the limit in the definition of $\lceil T\,, (x_n) \rceil$, so,

$$\lceil T\,, (x_n)_{n=M}^{\infty} \rceil \;=\; \lceil T\,, (x_n) \rceil$$

for all $M \in \mathbb{N}$.

With the idea that $\lceil T\,, (x_n) \rceil$ measures the divergence of the sequence (Tx_n)

in mind, it follows that the quantity

$$\inf_{(x_{n_k})\subset(x_n)} \lceil T\,,(x_{n_k})\rceil$$

measures how 'close' (Tx_n) is to having a convergent subsequence. Thus, the outer supremum in the definition (2.1.1) of $\|T\|_K$ detects the maximum 'distance' images of sequences in $T(\Delta_X^1)$ are from having convergent subsequences.[1] This provides the intended conceptual basis for the following elementary properties of $\|\cdot\|_K$.

2.1.3 Proposition. *Let X and Y be Banach spaces. Then the function $\|\cdot\|_K$ is a semi-norm on $\mathscr{B}(X,Y)$. Furthermore, its kernel is the space of compact operators $\mathscr{K}(X,Y)$.*

Proof. We will identify the kernel of $\|\cdot\|_K$ first. Since an operator $T\in\mathscr{B}(X,Y)$ is compact if and only if every bounded sequence (x_n) in X has a subsequence (x_{n_k}) such that the sequence $(Tx_{n_k})\subset Y$ converges, it is immediate from Definition 2.1.1 that if T is compact then $\|T\|_K=0$. Conversely, if $\|T\|_K=0$ then we have that

$$\inf_{(x_{n_k})\subset(x_n)} \lceil T\,,(x_{n_k})\rceil \;=\; 0$$

for all $(x_n)\in\Delta_X^1$. A diagonalisation argument then shows that this infimum is actually attained, so that each $(x_n)\in\Delta_X^1$ has a subsequence (x_{n_k}) such that the sequence $(Tx_{n_k})\subset Y$ converges (the argument will not be given here since, in a more general context, it forms part of the proof of Lemma 2.2.2 below). Hence $\|T\|_K=0$ only if T is compact, and so the kernel of $\|\cdot\|_K$ is $\mathscr{K}(X,Y)$.

To show that $\|\cdot\|_K$ is a semi-norm, it suffices to check the triangle inequality; clearly $\|cT\|_K=|c|\|T\|_K$ for $c\in\mathbb{F}$ and $T\in\mathscr{B}(X,Y)$. Let $(x_n)\subset X$ be an arbitrary sequence and note that $\lceil\cdot\,,(x_n)\rceil$ satisfies the triangle inequality (apply the triangle inequality for the norm $\|\cdot\|$ of Y to the respective part of Definition 2.1.1). Now let T and U be elements of $\mathscr{B}(X,Y)$ and suppose $(x_n)\in\Delta_X^1$ is such that the supremum in the definition of $\|T+U\|_K$ is attained up to a given $\varepsilon>0$. It follows that

$$\|T+U\|_K \;\le\; \lceil T+U\,,(x_{n_k})\rceil+\varepsilon \;\le\; \lceil T\,,(x_{n_k})\rceil+\lceil U\,,(x_{n_k})\rceil+\varepsilon \qquad (2.1.3a)$$

for all $(x_{n_k})\subset(x_n)$.

[1] Using Δ_X^1 as the canonical source of bounded sequences in X is more natural than using B_X, in this setting.

Now let $(x_{n_k^1}) \subset (x_n)$ be such that

$$\inf_{(x_{n_k}) \subset (x_n)} \lceil T, (x_{n_k}) \rceil$$

is attained up to ε. By Remark 2.1.2, for all $(x_{n_k^2}) \subset (x_{n_k^1})$ we have

$$\left\lceil T, (x_{n_k^2}) \right\rceil \leq \left\lceil T, (x_{n_k^1}) \right\rceil ,$$

so the above infimum will also be attained up to ε by every $(x_{n_k^2}) \subset (x_{n_k^1})$. Thus from (2.1.3a) it follows that

$$\begin{aligned} \|T+U\|_K &\leq \left\lceil T, (x_{n_k^2}) \right\rceil + \left\lceil U, (x_{n_k^2}) \right\rceil + \varepsilon \\ &\leq \inf_{(x_{n_k}) \subset (x_n)} \lceil T, (x_{n_k}) \rceil + \varepsilon + \left\lceil U, (x_{n_k^2}) \right\rceil + \varepsilon \\ &\leq \|T\|_K + \left\lceil U, (x_{n_k^2}) \right\rceil + 2\varepsilon \end{aligned}$$

for all $(x_{n_k^2}) \subset (x_{n_k^1})$. Hence

$$\begin{aligned} \|T+U\|_K &\leq \|T\|_K + \inf_{(x_{n_k^2}) \subset (x_{n_k^1})} \left\lceil U, (x_{n_k^2}) \right\rceil + 2\varepsilon \\ &\leq \|T\|_K + \|U\|_K + 2\varepsilon \,. \end{aligned}$$

This holds for all $\varepsilon > 0$, therefore

$$\|T+U\|_K \leq \|T\|_K + \|U\|_K \,. \qquad \blacksquare$$

So, if we restrict attention to operators in $\mathscr{B}(X)$ for some Banach space X and naturally identify $\|\cdot\|_K$ with a norm on $\mathscr{C}(X)$, we have that $\|\cdot\|_K$ is a potential candidate for a new algebra norm. It remains to check that $\|\cdot\|_K$ is submultiplicative. This is the conclusion of the next section, which also amasses some technical facts of later importance.

2.2 Technicalities relating to $\|\cdot\|_K$

2.2.1 Definition. Let X and Y be Banach spaces and $T \in \mathscr{B}(X,Y)$. We call a sequence $(x_n) \subset X$ T-*substable* if (x_n) is bounded and

$$\lceil T, (x_{n_k}) \rceil = \lceil T, (x_n) \rceil$$

for every $(x_{n_k}) \subset (x_n)$. Denote by Δ_T the set of all T-substable sequences in X and by Δ_T^λ the set of all T-substable sequences that are members of Δ_X^λ (see Definition 2.1.1), that is, $\Delta_T^\lambda = \Delta_T \cap \Delta_X^\lambda$.

Note that if $(x_n) \in \Delta_T$ then we trivially have

$$\inf_{(x_{n_k})\subset(x_n)} \lceil T\,,(x_{n_k})\rceil \;=\; \lceil T\,,(x_n)\rceil\;,$$

and clearly any subsequence of a T-substable sequence will itself be T-substable. Less immediate, but still elementary, is the following lemma.

2.2.2 Lemma. *Let X and Y be Banach spaces and $T \in \mathscr{B}(X,Y)$. Then every bounded sequence $(x_n) \subset X$ has a T-substable subsequence.*

Proof. Given a bounded sequence $(x_n) \subset X$, set

$$C \;=\; \lceil T\,,(x_n)\rceil$$

and

$$\delta \;=\; C \;-\; \inf_{(x_{n_k})\subset(x_n)} \lceil T\,,(x_{n_k})\rceil\;.$$

If $\delta = 0$ then $(x_n) \in \Delta_T$ and we are done. So assume $\delta > 0$ and set $\delta_1 = \delta$. Note that

$$C - \delta_1 \;=\; \inf_{(x_{n_k})\subset(x_n)} \lceil T\,,(x_{n_k})\rceil\;.$$

Thus there is $(x_n^1) \subset (x_n)$ such that

$$C - \delta_1 \;\le\; \left\lceil T\,,(x_n^1)\right\rceil \;<\; C - \delta_1 + \frac{1}{2}\delta$$

and it also follows that, for all $(x_{n_k}^1) \subset (x_n^1)$,

$$C - \delta_1 \;\le\; \left\lceil T\,,(x_{n_k}^1)\right\rceil.$$

We proceed by induction, followed by a diagonalisation argument. Assume that, for some $m \in \mathbb{N}$, we have already chosen $(x_n^m) \subset (x_n^{m-1}) \subset \cdots \subset (x_n^1) \subset (x_n)$ and $\delta_m \le \delta_{m-1} \le \cdots \le \delta_1 = \delta$ such that, for all $1 \le l \le m$,

$$C - \delta_l \;\le\; \left\lceil T\,,(x_n^l)\right\rceil \;<\; C - \delta_l + \frac{1}{2^l}\delta \tag{2.2.2a}$$

and furthermore that, for all $(x^l_{n_k}) \subset (x^l_n)$,

$$C - \delta_l \;\le\; \left\lceil T\,,(x^l_{n_k})\right\rceil\,. \tag{2.2.2b}$$

Note that our choices of (x^1_n) and δ_1 above are appropriate when $m = 1$. With this assumption, either there is $(x^{m+1}_n) \subset (x^m_n)$ such that

$$\lceil T\,,(x^{m+1}_n)\rceil \;<\; C - \delta_m + \frac{1}{2^{m+1}}\delta$$

and so, by setting $\delta_{m+1} = \delta_m$, we can conclude that (2.2.2a) and (2.2.2b) hold for $l = m+1$; otherwise, for all $(x^m_{n_k}) \subset (x^m_n)$,

$$C - \delta_m + \frac{1}{2^{m+1}}\delta \;\le\; \lceil T\,,(x^m_{n_k})\rceil$$

and so, by setting $\delta_{m+1} = \delta_m - \frac{1}{2^{m+1}}\delta$ and $(x^{m+1}_n) = (x^m_n)$, we can also conclude that (2.2.2a) and (2.2.2b) hold for $l = m+1$. Thus in either possible case there exist appropriate choices of (x^{m+1}_n) and δ_{m+1}. Hence by induction there are

$$(x_n) \supset (x^1_n) \supset (x^2_n) \supset \dots$$

and

$$\delta \;=\; \delta_1 \;\ge\; \delta_2 \;\ge\; \dots$$

such that, for all $m \in \mathbb{N}$ and all $(x^m_{n_k}) \subset (x^m_n)$,

$$C - \delta_m \;\le\; \lceil T\,,(x^m_n)\rceil \;<\; C - \delta_m + \frac{1}{2^m}\delta$$

and

$$C - \delta_m \;\le\; \lceil T\,,(x^m_{n_k})\rceil\,.$$

Let

$$D \;=\; \lim_{m\to\infty} \delta_m \qquad \text{and} \qquad (y_n) \;=\; (x^n_n)\,.$$

Now, given $(y_{n_k}) \subset (y_n)$ and $m \in \mathbb{N}$, we have that

$$(y_{n_k})_{k=m}^{\infty} \subset (x^m_n)\,,$$

so the properties noted in Remark 2.1.2 apply. Also, $D \le \delta_m$, therefore

$$C - \delta_m \;\le\; \lceil T\,,(y_{n_k})_{k=m}^{\infty}\rceil \;\le\; \lceil T\,,(x^m_n)\rceil \;<\; C - \delta_m + \frac{1}{2^m}\delta \;\le\; C - D + \frac{1}{2^m}\delta.$$

Since $\lceil T\,,(y_{n_k})\rceil = \lceil T\,,(y_{n_k})_{k=m}^{\infty}\rceil$, we can conclude that, for all m,

$$C - \delta_m \;\leq\; \lceil T\,,(y_{n_k})\rceil \;\leq\; C - D + \frac{1}{2^m}\delta\,.$$

Hence, taking the limit as $m \to \infty$,

$$C - D \;\leq\; \lceil T\,,(y_{n_k})\rceil \;\leq\; C - D\,.$$

So for all $(y_{n_k}) \subset (y_n)$,

$$\lceil T\,,(y_{n_k})\rceil \;=\; C - D\,,$$

which shows that $(y_n) \in \Delta_T$. Therefore, for all bounded sequences $(x_n) \subset X$, there is a subsequence $(y_n) \subset (x_n)$ such that $(y_n) \in \Delta_T$. ■

2.2.3 Corollary. *Let X and Y be Banach spaces. Then for each $T \in \mathscr{B}(X,Y)$ and all $\lambda > 0$,*

$$\|T\|_K \;=\; \frac{1}{\lambda} \sup_{(x_n)\in\Delta_T^{\lambda}} \lceil T\,,(x_n)\rceil\;.$$

Proof. Given $T \in \mathscr{B}(X,Y)$, note that

$$\begin{aligned}
\lambda \sup_{(x_n)\in\Delta_T^{1}} \lceil T\,,(x_n)\rceil \;&=\; \sup_{(x_n)\in\Delta_T^{1}} \lim_{N\to\infty} \sup_{n_1,n_2\geq N} \|T(\lambda x_{n_1} - \lambda x_{n_2})\| \\
&=\; \sup_{(x_n)\in\Delta_T^{\lambda}} \lim_{N\to\infty} \sup_{n_1,n_2\geq N} \|T(x_{n_1} - x_{n_2})\| \\
&=\; \sup_{(x_n)\in\Delta_T^{\lambda}} \lceil T\,,(x_n)\rceil
\end{aligned}$$

and so it suffices to check the case when $\lambda = 1$. We have

$$\begin{aligned}
\|T\|_K \;&=\; \sup_{(x_n)\subset\Delta_X^{1}} \inf_{(x_{n_k})\subset(x_n)} \lceil T\,,(x_{n_k})\rceil \\
&\geq\; \sup_{(x_n)\in\Delta_X^{1}\cap\Delta_T} \left\{ \inf_{(x_{n_k})\subset(x_n)} \lceil T\,,(x_{n_k})\rceil \right\} \\
&=\; \sup_{(x_n)\in\Delta_T^{1}} \lceil T\,,(x_n)\rceil\;.
\end{aligned}$$

Also,

$$\|T\|_K \;=\; \sup_{(x_n)\subset\Delta_X^{1}} \left\{ \inf_{(x_{n_k})\subset(x_n)} \lceil T\,,(x_{n_k})\rceil \right\}$$

$$\begin{aligned} &\leq \sup_{(x_n)\subset\Delta_X^1} \{\lceil T\,,(x_{n_k})\rceil \;:\; (x_{n_k}) \subset (x_n),\, (x_{n_k}) \in \Delta_T\} \\ &= \sup_{(x_n)\in\Delta_T^1} \lceil T\,,(x_n)\rceil\ . \end{aligned}$$

Thus,

$$\|T\|_K \;=\; \sup_{(x_n)\in\Delta_T^1} \lceil T\,,(x_n)\rceil\ . \qquad \blacksquare$$

2.2.4 Corollary. *Let X and Y be Banach spaces. Then for all $T, S \in \mathscr{B}(X,Y)$,*

$$\|T\|_K \;=\; \sup_{(x_n)\in\Delta_T\cap\Delta_S^1} \lceil T\,,(x_n)\rceil\ .$$

Proof. From Corollary 2.2.3 we know that

$$\|T\|_K \;=\; \sup_{(x_n)\in\Delta_T^1} \lceil T\,,(x_n)\rceil\ .$$

Given $(x_n) \in \Delta_T$, by Lemma 2.2.2 (x_n) has an S-substable subsequence. Since passing to this subsequence does not affect the value of $\lceil T\,,(x_n)\rceil$, we have

$$\|T\|_K \;=\; \sup_{(x_n)\in\Delta_T^1\cap\Delta_S} \lceil T\,,(x_n)\rceil\ .$$

But, from Definition 2.2.1, $\Delta_T^1 \cap \Delta_S = \Delta_T \cap \Delta_X^1 \cap \Delta_S = \Delta_T \cap \Delta_S^1$ and so

$$\|T\|_K \;=\; \sup_{(x_n)\in\Delta_T\cap\Delta_S^1} \lceil T\,,(x_n)\rceil\ . \qquad \blacksquare$$

With the above initial facts established, we are able to confirm that $\|\cdot\|_K$ behaves as expected:

2.2.5 Proposition. *For all Banach spaces X, $\|\cdot\|_K$ is an algebra norm for $\mathscr{C}(X)$ and $(\mathscr{C}(X), \|\cdot\|_K)$ is unital.*

Proof. Let S and T be members of $\mathscr{B}(X)$ and let $\varepsilon > 0$ be given. Set

$$\Delta_T^{1\,\prime} \;=\; \{(x_n) \in \Delta_T^1 : (\forall n_1, n_2 \in \mathbb{N})\ \|Tx_{n_1} - Tx_{n_2}\| \leq \|T\|_K + \varepsilon\}$$

and note that

$$\Delta_T^{1\,\prime} \;=\; \left\{(x_n) \in \Delta_T^1 : (Tx_n) \in \Delta_X^{(\|T\|_K+\varepsilon)}\right\}. \tag{2.2.5a}$$

By Corollary 2.2.3,

$$\|T\|_K = \sup_{(x_n)\in\Delta_T^1} \lceil T\,,(x_n)\rceil\;,$$

therefore each $(x_n) \in \Delta_T^1$ has the property that there is some $N \in \mathbb{N}$ for which

$$\sup_{n_1,n_2\geq N} \|Tx_{n_1} - Tx_{n_2}\| \leq \|T\|_K + \varepsilon\,.$$

Hence, each $(x_n) \in \Delta_T^1$ must have a subsequence (x_{n_k}), also in Δ_T^1, such that $(x_{n_k}) \in \Delta_T^1{}'$. If it is additionally the case that $(x_n) \in \Delta_{ST}$, then passing to a subsequence does not affect the value of $\lceil ST\,,(x_n)\rceil$, so we have that

$$\begin{aligned}\sup_{(x_n)\in\Delta_{ST}\cap\Delta_T^1} \lceil ST\,,(x_n)\rceil &= \sup_{(x_n)\in\Delta_{ST}\cap\Delta_T^1\cap\Delta_T^1{}'} \lceil ST\,,(x_n)\rceil \\ &\leq \sup_{(x_n)\in\Delta_{ST}\cap\Delta_T^1{}'} \lceil ST\,,(x_n)\rceil\;. \qquad (2.2.5\text{b})\end{aligned}$$

Thus

$$\begin{aligned}\|ST\|_K &= \sup_{(x_n)\in\Delta_{ST}\cap\Delta_T^1} \lceil ST\,,(x_n)\rceil && \text{[by Coroll. 2.2.4]}\\ &\leq \sup_{(x_n)\in\Delta_{ST}\cap\Delta_T^1{}'} \lceil ST\,,(x_n)\rceil && \text{[by 2.2.5b]}\\ &= \sup_{(x_n)\in\Delta_{ST}\cap\Delta_T^1{}'} \lim_{N\to\infty} \sup_{n_1,n_2\geq N} \|S(Tx_{n_1} - Tx_{n_2})\| \\ &\leq \sup_{(y_n)\in\Delta_S\cap\Delta_X^{(\|T\|_K+\varepsilon)}} \lim_{N\to\infty} \sup_{n_1,n_2\geq N} \|S(y_{n_1} - y_{n_2})\| && \text{[by 2.2.5a]}\\ &= \sup_{(y_n)\in\Delta_S^{(\|T\|_K+\varepsilon)}} \lceil S\,,(y_n)\rceil \\ &= (\|T\|_K + \varepsilon)\|S\|_K\,. && \text{[by Coroll. 2.2.3]}\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we have $\|ST\|_K \leq \|S\|_K\|T\|_K$. Also, $\|I\|_K \leq \|I\| = 1$, hence $(\mathscr{C}(X), \|\cdot\|_K)$ is a unital algebra. ■

2.3 Cases for which $\|\cdot\|_K \sim \|\cdot\|_e$

Having established that $\|\cdot\|_K$ is actually an algebra norm, we turn our attention to the question of its equivalence to $\|\cdot\|_e$. Chapter 3 will detail earlier results from which we can show that $\|\cdot\|_K \sim \|\cdot\|_e$ on the Calkin algebras of all Banach spaces X that have the bounded compact approximation property. Here, we will give a direct proof of this fact when X has a basis. In the case when $X = c_0$, our argument yields the stronger result of equality between the (semi-)norms. This stronger result is also true for other spaces, including $X = \ell_p$, $1 \leq p < \infty$, but the intricacies of the following only allow for a proof of equivalence for such X. To achieve the equality result more generally, we also defer to the prior publications to be discussed in the next chapter. The critical difference in approach relates to the manner in which we prove the following elementary lemma, various analogues of which are very important for the remainder of this thesis.

2.3.1 Lemma (The block-to-block lemma for c_0)**.** *Suppose that* $T \in \mathscr{B}(c_0)$, $T \notin \mathscr{K}(c_0)$ *and* $(\delta_n) \subset \mathbb{R}_+$. *Then there exist block basic sequences* (x_n) *and* (z_n) *of* (e_n) *such that* (x_n) *is normalised and, for all* $n \in \mathbb{N}$, $\|z_n - Tx_n\| < \delta_n$ *and* $\|z_n\| > \|T\|_e - \delta_n$.

Proof. By Lemma 1.3.4, we may choose $x_1 \in S_{c_0}$ such that $x_1 \prec \infty$ and $\|Tx_1\| > \|T\|_e - \frac{\delta_1}{2}$. Let z_1 be an initial segment of Tx_1, large enough that $\|z_1 - Tx_1\| < \frac{\delta_1}{2}$. Note, therefore, that $\|z_1\| > \|T\|_e - \delta_1$. We proceed by induction. Assume $x_1 \prec \cdots \prec x_m \prec \infty$ and $z_1 \prec \cdots \prec z_m \prec M+1$ have been chosen in accordance with requirements, for some $M \in \mathbb{N}$. By Corollary 1.3.5, there exist block bases (y_n) and (z'_n), and $(N_n)_{n=0}^{\infty} \subset \mathbb{N}_0$, such that, for all $n \in \mathbb{N}$, $\|y_n\| = 1$, $z'_n = (P_{N_n} - P_{N_{n-1}})Ty_n$, $\|(I - P_{N_n})Ty_n\| < \frac{\delta_{m+1}}{2}$, and $\|z'_n\| > \|T\|_e - \frac{\delta_{m+1}}{2}$. Now, $P_M \in \mathscr{K}(c_0)$, thus $P_M T \in \mathscr{K}(c_0)$, hence there is a subsequence (y_{n_k}) of (y_n) for which $P_M T y_{n_k}$ converges. Therefore, for sufficiently large $k_1, k_2 \in \mathbb{N}$, we have

$$\|P_M T(y_{n_{k_1}} - y_{n_{k_2}})\| \; < \; \frac{\delta_{m+1}}{2}.$$

Without loss of generality we may ensure $x_m \prec y_{n_{k_1}} \prec y_{n_{k_2}}$ and $M \leq n_{k_1}$.
Set

$$x_{m+1} \; = \; y_{n_{k_1}} - y_{n_{k_2}}$$

and let z_{m+1} be an initial segment of $(I - P_M)Tx_{m+1}$, say $z_{m+1} = (P_L - P_M)Tx_{m+1}$, where $L \geq N_{n_{k_2}}$, large enough that

$$\|z_{m+1} - (I - P_M)Tx_{m+1}\| \; < \; \frac{\delta_{m+1}}{2}.$$

Thus

$$\|z_{m+1} - Tx_{m+1}\| \; < \; \frac{\delta_{m+1}}{2} + \|P_M T x_{m+1}\| \; < \; \delta_{m+1}$$

and, since (e_n) is bimonotone in c_0,

$$\begin{aligned}
\|z_{m+1}\| &\geq \|(P_{N_{n_{k_2}}} - P_{N_{n_{k_2}-1}})z_{m+1}\| \\
&= \|(P_{N_{n_{k_2}}} - P_{N_{n_{k_2}-1}})(I - P_M)z_{m+1}\| \\
&= \|(P_{N_{n_{k_2}}} - P_{N_{n_{k_2}-1}})(I - P_M)Tx_{m+1}\| \\
&= \|(P_{N_{n_{k_2}}} - P_{N_{n_{k_2}-1}})Tx_{m+1}\| \\
&= \|(P_{N_{n_{k_2}}} - P_{N_{n_{k_2}-1}})T(y_{n_{k_1}} - y_{n_{k_2}})\| \\
&\geq \|(P_{N_{n_{k_2}}} - P_{N_{n_{k_2}-1}})Ty_{n_{k_2}}\| - \|(P_{N_{n_{k_2}}} - P_{N_{n_{k_2}-1}})Ty_{n_{k_1}}\| \\
&\geq \|z'_{n_{k_2}}\| - \|(I - P_{N_{n_{k_2}-1}})Ty_{n_{k_1}}\| \\
&> \|T\|_e - \frac{\delta_{m+1}}{2} - \|(I - P_{N_{n_{k_1}}})Ty_{n_{k_1}}\| \\
&> \|T\|_e - \delta_{m+1}\,,
\end{aligned}$$

as required. From the basic properties of the supremum norm we also obtain

$$\|x_{m+1}\| \;=\; \|y_{n_{k_1}} - y_{n_{k_2}}\| \;=\; \sup\{\|y_{n_{k_1}}\|, \|y_{n_{k_2}}\|\} \;=\; 1\,.$$

Thus x_{m+1} and z_{m+1} can be chosen to satisfy the requirements of the lemma, and so by induction there exist suitable block basic sequences (x_n) and (z_n). ■

Note that the above proof used only the fact that (e_n) is bimonotone in c_0, until the final calculation of the norm of $\|x_{m+1}\|$. In ℓ_p, $1 \leq p < \infty$, (e_n) is also bimonotone, however we would instead have:

$$\|x_{m+1}\| \;=\; \|y_{n_{k_1}} - y_{n_{k_2}}\| \;=\; \Big(\|y_{n_{k_1}}\|^p + \|y_{n_{k_2}}\|^p\Big)^{\frac{1}{p}} \;=\; 2^{\frac{1}{p}}\,.$$

Thus, to normalise $(x_n)_{n=2}^{\infty}$, each block must be scaled by a factor of $2^{-\frac{1}{p}}$ and this will similarly scale $(z_n)_{n=2}^{\infty}$, which justifies the following.

2.3.2 Lemma (The block-to-block lemma for ℓ_p — weak version)**.** *Suppose that $T \in \mathscr{B}(\ell_p)$, $1 \leq p < \infty$, $T \notin \mathscr{K}(\ell_p)$ and $(\delta_n) \subset \mathbb{R}_+$. Then there exist block basic sequences (x_n) and (z_n) of (e_n) such that (x_n) is normalised and, for all $n \in \mathbb{N}$, $\|z_n - Tx_n\| < \delta_n$ and $\|z_n\| > 2^{-\frac{1}{p}}\|T\|_e - \delta_n$.*

Indeed, as x_{m+1} is defined to be the difference of two normalised blocks, it will always be the case that $\|x_{m+1}\| \leq 2$, no matter the ambient space. Thus we obtain:

2.3.3 Lemma (The block-to-block lemma)**.** *Suppose that X is a Banach space with a bimonotone basis, $T \in \mathscr{B}(X)\backslash\mathscr{K}(X)$ and $(\delta_n) \subset \mathbb{R}_+$. Then there exist block basic sequences (x_n) and (z_n) in X such that (x_n) is normalised and, for all $n \in \mathbb{N}$, $\|z_n - Tx_n\| < \delta_n$ and $\|z_n\| > \frac{1}{2}\|T\|_e - \delta_n$.*

We will discuss the differences between Lemma 2.3.2 and previously published proofs of alternative versions in Chapter 3. For now, Lemma 2.3.3 allows us to establish the main result of this section.

2.3.4 Theorem. *Suppose that X is a Banach space with a basis. Then $\|\cdot\|_K \sim \|\cdot\|_e$. Furthermore, when $X = c_0$, if $T \in \mathscr{B}(c_0)$ then $\|T\|_K = \|T\|_e$.*

Proof. Without loss of generality we may give X an equivalent norm so that its basis is bimonotone, since this will only affect the constant of equivalence and not the fact that $\|\cdot\|_K \sim \|\cdot\|_e$. With the additional assumption that the basis is bimonotone, we shall show that $\frac{1}{4}\|T\|_e \leq \|T\|_K \leq \|T\|_e$ for all $T \in \mathscr{B}(X)$.

If $T \in \mathscr{K}(X)$, then trivially $\|T\|_K = \|T\|_e = 0$. So suppose that $T \notin \mathscr{K}(X)$. Writing $t = T + \mathscr{K}(X)$, we have that $\|T\|_K = \|T'\|_K \leq \|T'\|$ for all $T' \in t$ and that $\|T\|_e = \inf\{\|T'\| : T' \in t\}$. Hence $\|T\|_K \leq \|T\|_e$. To show that $\frac{1}{4}\|T\|_e \leq \|T\|_K$, we will demonstrate the existence of a sequence $(y_n) \in \Delta^1_X$ such that $\lceil T, (y_{n_k})\rceil \geq \frac{1}{4}\|T\|_e$ for all subsequences $(y_{n_k}) \subset (y_n)$. In the case when $X = c_0$, we will obtain the stronger result that $\lceil T, (y_{n_k})\rceil \geq \|T\|_e$ for such (y_{n_k}).

Fix a sequence $(\delta_n) \subset \mathbb{R}_+$ which monotonically decreases to limit 0. By Lemma 2.3.3 ⟨resp. Lemma 2.3.1⟩, there exist block bases (x_n) and (z_n) in X ⟨resp. in c_0⟩ such that (x_n) is normalised and, for all $n \in \mathbb{N}$, $\|z_n - Tx_n\| < \delta_n$ and $\|z_n\| > \frac{1}{2}\|T\|_e - \delta_n$ ⟨resp. $\|z_n\| > \|T\|_e - \delta_n$⟩. Note that, for all $n_1, n_2 \in \mathbb{N}$ for which $n_1 \neq n_2$, we have $\|x_{n_1} - x_{n_2}\| \leq \|x_{n_1}\| + \|x_{n_2}\| = 2$ ⟨resp. $\|x_{n_1} - x_{n_2}\| = \sup\{\|x_{n_1}\|, \|x_{n_2}\|\} = 1$⟩, so that $(\frac{1}{2}x_n) \in \Delta^1_X$ ⟨resp. $(x_n) \in \Delta^1_{c_0}$⟩. Furthermore, if $n_2 > n_1 \geq N \in \mathbb{N}$, then

$$\begin{aligned}
\|T(x_{n_1} - x_{n_2})\| &\geq \|z_{n_1} - z_{n_2}\| - \|(z_{n_1} - z_{n_2}) - T(x_{n_1} - x_{n_2})\| \\
&\geq \sup\{\|z_{n_1}\|, \|z_{n_2}\|\} - \|z_{n_1} - Tx_{n_1}\| - \|z_{n_2} - Tx_{n_2}\| \\
&> \frac{1}{2}\|T\|_e - \delta_{n_2} - \delta_{n_1} - \delta_{n_2} \\
&\geq \frac{1}{2}\|T\|_e - 3\delta_N \\
\langle \text{resp. } \|T(x_{n_1} - x_{n_2})\| &> \|T\|_e - 3\delta_N\rangle\,.^2 \qquad (2.3.4a)
\end{aligned}$$

[2] See Remark 2.3.5 on the next page.

Thus, for any subsequence $(x_{n_k}) \subset (x_n)$,

$$\lceil T\,,(x_{n_k})\rceil \;=\; \lim_{N\to\infty}\sup_{k_1,k_2\geq N}\|T(x_{n_{k_1}}-x_{n_{k_2}})\| \;\geq\; \lim_{N\to\infty}(\frac{1}{2}\|T\|_e - 3\delta_N) \;=\; \frac{1}{2}\|T\|_e$$

$$\langle\text{resp. } \lceil T\,,(x_{n_k})\rceil \;\geq\; \|T\|_e\rangle\,.$$

Setting $(y_n) = (\frac{1}{2}x_n)$ $\langle$resp. $(y_n) = (x_n)\rangle$, we have $(y_n) \in \Delta^1_X$, so

$$\|T\|_K \;\geq\; \inf_{(y_{n_k})\subset(y_n)}\lceil T\,,(y_{n_k})\rceil \;=\; \inf_{(x_{n_k})\subset(x_n)}\frac{1}{2}\lceil T\,,(x_{n_k})\rceil \;\geq\; \frac{1}{4}\|T\|_e$$

or

$$\|T\|_K \;\geq\; \inf_{(x_{n_k})\subset(x_n)}\lceil T\,,(x_{n_k})\rceil \;\geq\; \|T\|_e$$

in the respective case when $X = c_0$. ■

2.3.5 Remark. For future reference in Chapter 4, note that from equation 2.3.4a in the preceding proof we can conclude that, in the case of c_0, the sequence (x_n) is T-substable, and additionally has the property that

$$\lim_{N\to\infty}\inf_{n_1,n_2\geq N}\|T(x_{n_1}-x_{n_2})\| \;\geq\; \|T\|_e \;\geq\; \|T\|_K \;\geq\; \lim_{N\to\infty}\sup_{n_1,n_2\geq N}\|T(x_{n_1}-x_{n_2})\|\,,$$

hence

$$\lim_{N\to\infty}\inf_{n_1,n_2\geq N}\|T(x_{n_1}-x_{n_2})\| \;=\; \lim_{N\to\infty}\sup_{n_1,n_2\geq N}\|T(x_{n_1}-x_{n_2})\|\,.$$

In fact, the use of the block-to-block lemma (2.3.3) in the above proof introduced an extra factor of $\frac{1}{2}$, which can be seen to be unnecessary when compared to a similar argument direct from Corollary 1.3.5 (on which Lemma 2.3.3 is based). So the same method of proof can be used to show $\frac{1}{2}\|T\|_e \leq \|T\|_K \leq \|T\|_e$ for all $T \in \mathscr{B}(X)$, when X has a bimonotone basis. However, we chose to emphasise the utility of the block-to-block lemma for consistency with future chapters, in which reliance on certain properties of block basic sequences in specific Banach spaces will be vital.

Chapter 3

Previous Uniqueness of Norm Results for $\mathscr{C}(X)$

After the research detailed in the previous chapter was completed, Prof. Hans-Olav Tylli drew attention to prior results concerning the question of uniqueness of norm on Calkin algebras. These results are summarised in [Tyl2], a survey paper published in 2004, which appears in the proceedings of a conference held in 2001 at the University of Oulu in Finland. This paper brings together multiple strands that were formerly "scattered in the literature, and no [other] expositions seem to exist" [Tyl2, p212]. Until now, there have been no apparent additional publications on the subject.

[Tyl2] details the state of knowledge concerning "positive" examples of Calkin algebras which have a unique algebra norm, as well as providing several "negative" examples of inequivalent algebra norms on various quotient algebras of $\mathscr{B}(X)$, for certain Banach spaces X that fail particular approximation properties. A summary of relevant material from [Tyl2], and from the previous works to which it refers, is provided in §§3.2–3.3 below. §3.2 covers the negative results, and also includes two examples of Calkin algebras that lack a unique algebra norm which are not included in [Tyl2]. Then §3.3 outlines the examples of Calkin algebras with a unique algebra norm that are known in the literature. Included are some improvements to the previous proof that $\mathscr{C}(X)$ has a unique algebra norm when X is c_0 or ℓ_p, $1 \leq p < \infty$. In §3.4 we shall discuss a novel approach to that proof, which provides the conceptual basis for results in later chapters. First, however, in §3.1 we will examine another definition made by earlier authors to alternatively measure the 'compactness' of an operator, and how it relates to the seminorm $\|\cdot\|_K$. As indicated in Chapter 2, the definition (2.1.1) we gave for $\|\cdot\|_K$ was developed independently.

3.1 Other measures of non-compactness

3.1.1 Definition. Let X and Y be Banach spaces. For $T \in \mathscr{B}(X, Y)$, define the *measure of non-compactness of* T, denoted $\gamma(T)$, to be the infimum of all $\varepsilon > 0$ such that TB_X can be covered by a finite number of balls of radius ε. Equivalently,

$$\gamma(T) = \inf\{\varepsilon > 0 : (\exists \text{ compact } K \subset Y)\ TB_X \subset K + \varepsilon B_Y\}.$$

This measure of non-compactness was first systematically analysed in [LeSc, §3]; the introductory section of that paper gives several additional references, which detail the evolution of the concept. It is also noted that:

3.1.2 Proposition ([LeSc, p7])**.** *Let X and Y be Banach spaces. Then*

(i) $\gamma(\cdot)$ is a seminorm on $\mathscr{B}(X, Y)$ with $\ker \gamma(\cdot) = \mathscr{K}(X, Y)$,

(ii) $\gamma(\cdot) \leq \|\cdot\|_e$, and

(iii) when $X = Y$, $\gamma(\cdot)$ induces an algebra norm on $\mathscr{C}(X)$.

Recall that a Banach space X has the *bounded compact approximation property* (abbreviated BCAP) if there is a constant $C \geq 1$ such that, for all compact subsets $K \subset X$ and $\varepsilon > 0$, there exists $T \in \mathscr{K}(X)$ such that $\|T\| \leq C$ and

$$\sup\{\|x - Tx\| : x \in K\} < \varepsilon.$$

The BCAP has implications in determining whether $\gamma(\cdot)$ is equivalent to $\|\cdot\|_e$:

3.1.3 Theorem ([LeSc, Thm. 3.6])**.** *Let X and Y be Banach spaces. If Y has the bounded compact approximation property, then there exists $C \geq 1$ such that, for all $T \in \mathscr{B}(X, Y)$,*

$$\gamma(T) \leq \|T\|_e \leq C\gamma(T).$$

3.1.4 Corollary. *Let X be a Banach space with the bounded compact approximation property, and let the seminorms $\gamma(\cdot)$ and $\|\cdot\|_e$ on $\mathscr{B}(X)$ be identified with norms on $\mathscr{C}(X)$. Then $\gamma(\cdot) \sim \|\cdot\|_e$.*

The measure of non-compactness $\gamma(\cdot)$ is central to the example from [Tyl2], originally provided in [AsTy], of a Calkin algebra for which $\|\cdot\|_e$ is not minimal (see §3.2 below). It happens that the seminorm $\|\cdot\|_K$ defined in Chapter 2 is closely related to $\gamma(\cdot)$:

3.1.5 Theorem. *$\|\cdot\|_K \sim \gamma(\cdot)$ on $\mathscr{B}(X)$ for all Banach spaces X.*

***Proof*.** Let $T \in \mathscr{B}(X)$ be given. If $T \in \mathscr{K}(X)$ then $\gamma(T) = \|T\|_K = 0$, hence we may assume T is non-compact, so that $\gamma(T) > 0$. Suppose that $\varepsilon > \gamma(T)$ and $(x_n) \in \Delta_T^1$. Without loss of generality, we may translate (x_n) by $-x_1$ and thus assume that $x_1 = 0$ and $(x_n) \subset B_X$. Choose a compact set $K_\varepsilon \subset X$ such that $TB_X \subset K_\varepsilon + \varepsilon B_X$. For each $n \in \mathbb{N}$, let $y_n \in K_\varepsilon$ and $z_n \in \varepsilon B_X$ be such that $Tx_n = y_n + z_n$. Then, since $(y_n) \subset K_\varepsilon$, there is $(y_{n_k}) \subset (y_n)$ such that (y_{n_k}) converges and hence such that

$$\lim_{N\to\infty} \sup_{k_1,k_2\geq N} \|y_{n_{k_1}} - y_{n_{k_2}}\| = 0\,.$$

Thus

$$\begin{aligned}
\lceil T\,,(x_n)\rceil &= \left\lceil T\,,(x_{n_k})\right\rceil && \text{[because (x_n) is T-substable]}\\
&= \lim_{N\to\infty} \sup_{k_1,k_2\geq N} \|Tx_{n_{k_1}} - Tx_{n_{k_2}}\| \\
&= \lim_{N\to\infty} \sup_{k_1,k_2\geq N} \|y_{n_{k_1}} - y_{n_{k_2}} + z_{n_{k_1}} - z_{n_{k_2}}\| \\
&\leq \lim_{N\to\infty} \left(\sup_{k_1,k_2\geq N} \|y_{n_{k_1}} - y_{n_{k_2}}\| + \sup_{k_1,k_2\geq N} \|z_{n_{k_1}} - z_{n_{k_2}}\| \right) \\
&\leq \sup_{n_1,n_2\in\mathbb{N}} (\|z_{n_1}\| + \|z_{n_2}\|) \\
&\leq 2\varepsilon\,.
\end{aligned}$$

This holds for all $(x_n) \in \Delta_T^1$, therefore by Corollary 2.2.3 we have $\|T\|_K \leq 2\varepsilon$ for all $\varepsilon > \gamma(T)$, and so $\|T\|_K \leq 2\gamma(T)$.

Now suppose $0 < \varepsilon < \gamma(T)$. Choose $x_1 \in B_X$. We will inductively construct a sequence (x_n). Assume $x_1, \ldots, x_k \in B_X$ have been chosen. Since $\varepsilon < \gamma(T)$, we have that

$$TB_X \not\subset \bigcup_{n=1}^{k} B_X(Tx_n, \varepsilon)\,.$$

Hence we may choose $x_{k+1} \in B_X$ such that

$$Tx_{k+1} \notin \bigcup_{n=1}^{k} B_X(Tx_n, \varepsilon)\,.$$

Thus, by induction there exists a sequence $(x_n) \subset B_X$ such that, for all $n_1, n_2 \in \mathbb{N}$,

$\|Tx_{n_1} - Tx_{n_2}\| > \varepsilon$ and therefore $\|T(\frac{1}{2}x_{n_1} - \frac{1}{2}x_{n_2})\| > \frac{\varepsilon}{2}$. This implies

$$\inf_{(x_{n_k})\subset(\frac{1}{2}x_n)} \lim_{N\to\infty} \sup_{k_1,k_2\geq N} \|T(x_{n_{k_1}} - x_{n_{k_2}})\| \geq \frac{\varepsilon}{2}$$

and hence $\frac{\varepsilon}{2} \leq \|T\|_K$, because $(\frac{1}{2}x_n) \in \Delta_X^1$ whenever $(x_n) \subset B_X$. This holds for all $\varepsilon < \gamma(T)$, therefore $\frac{1}{2}\gamma(T) \leq \|T\|_K$ and we have

$$\frac{1}{2}\gamma(T) \leq \|T\|_K \leq 2\gamma(T)$$

for all $T \in \mathscr{B}(X)$. ∎

Thus, the theory regarding $\gamma(\cdot)$ that has been established in previous studies is applicable to $\|\cdot\|_K$. In particular, we have the following result:

3.1.6 Corollary. *Let X be a Banach space with the bounded compact approximation property. Then $\|\cdot\|_K \sim \|\cdot\|_e$ on $\mathscr{C}(X)$.*

Proof. This follows immediately from Corollary 3.1.3 and Theorem 3.1.5. ∎

There are several other 'measures' of the non-compactness of an operator that have been considered by various authors; for further details, see [AsTy, Rem. 2.7]. In many cases, these measures can also be shown to be equivalent to $\gamma(\cdot)$. One such example is given by [LeSc, Thm. 3.1]: for Banach spaces X and Y, and $T \in \mathscr{B}(X,Y)$, if we define

$$\|T\|_m = \inf\{\|T_{|M}\| : M \text{ is a finite codimensional subspace of } X\},$$

then

$$\gamma(\cdot)/2 \leq \|\cdot\|_m \leq \gamma(\cdot).$$

This example is a little surprising, since intuitively one might assume that $\|\cdot\|_m$ has $\mathscr{A}(X,Y)$ as its kernel. That the kernel is actually $\mathscr{K}(X,Y)$ can be seen as one of the quirks of spaces which fail the approximation property. Such quirks will be important in the next section.

3.2 Previous negative results

An example of a Calkin algebra for which $\gamma(\cdot) \nsim \|\cdot\|_e$, and hence which does not have a unique algebra norm, was first given in [AsTy]. We follow the presentation of [Tyl2, §1], which summarises [Tyl1], where the construction of a counterexample was extended to more general quotients of $\mathscr{B}(X)$. Let $\mathscr{I}$ be a closed surjective operator ideal (in the sense of [Pie1]), and let X and Y be Banach spaces. Then, as originally conceived in [Ast], we define the *outer $\mathscr{I}$-variation* $\gamma_{\mathscr{I}}(T)$ of an operator $T \in \mathscr{B}(X,Y)$ by

$$\gamma_{\mathscr{I}}(T) \;=\; \inf\{\varepsilon > 0 \;:\; TB_X \subset UB_Z + \varepsilon B_Y \;,\; U \in \mathscr{I}(Z,Y)\}\,,$$

where the infimum is taken over all Banach spaces Z. If $\mathscr{I}$ is a closed injective operator ideal (in the sense of [Pie1]), then we similarly define the *inner $\mathscr{I}$-variation* $\beta_{\mathscr{I}}(T)$ of an operator $T \in \mathscr{B}(X,Y)$ by

$$\beta_{\mathscr{I}}(T) \;=\; \inf\{\varepsilon > 0 \;:\; \|Tx\| \;\le\; \|Ux\| + \varepsilon\|x\| \;(\forall x \in X)\,,\; U \in \mathscr{I}(X,Z)\}\,,$$

where the infimum is again taken over all Banach spaces Z.

3.2.1 Proposition ([Tyl2, Prop. 1.1])**.** *Given a closed surjective ⟨resp. injective⟩ operator ideal $\mathscr{I}$ and Banach spaces X, Y:*

(i) $\gamma_{\mathscr{I}}(\cdot)$ *⟨resp.* $\beta_{\mathscr{I}}(\cdot)$*⟩ is a seminorm on* $\mathscr{B}(X,Y)$ *with kernel* $\mathscr{I}(X,Y)$ *,*

(ii) $\gamma_{\mathscr{I}}(\cdot) \le \|\cdot\|_{\mathscr{I}}$ *⟨resp.* $\beta_{\mathscr{I}}(\cdot) \le \|\cdot\|_{\mathscr{I}}$*⟩ , where* $\|\cdot\|_{\mathscr{I}}$ *is the standard quotient norm on* $\mathscr{B}(X,Y)/\mathscr{I}(X,Y)$ *, and*

(iii) when $X = Y$*,* $\gamma_{\mathscr{I}}(\cdot)$ *⟨resp.* $\beta_{\mathscr{I}}(\cdot)$*⟩ induces an algebra norm on* $\mathscr{B}(X)/\mathscr{I}(X)$ *.*

Note that $\mathscr{K}$ is both an injective and surjective operator ideal, and that $\gamma_{\mathscr{K}}(\cdot) = \gamma(\cdot)$. Since $\gamma(\cdot) \sim \|\cdot\|_e$ on the Calkin algebras of all Banach spaces that have the BCAP (Theorem 3.1.3), it is no surprise that the failure of certain approximation properties is required to produce examples of quotient algebras for which $\gamma_{\mathscr{I}}(\cdot) \nsim \|\cdot\|_{\mathscr{I}}$ or $\beta_{\mathscr{I}}(\cdot) \nsim \|\cdot\|_{\mathscr{I}}$, and thus which fail to have a unique algebra norm.

The relevant approximation properties are as follows. We say that a Banach space X has the *outer $\mathscr{I}$-approximation property* (outer $\mathscr{I}$-AP) if there is a constant $C < \infty$ such that, for all Banach spaces Z and operators $U \in \mathscr{I}(Z,X)$,

$$\inf\{\|U - VU\| \;:\; V \in \mathscr{I}(X),\, \|I - V\| \;\le\; C\} \;=\; 0\,.$$

Similarly, we say that X has the *inner $\mathscr{I}$-approximation property* (inner $\mathscr{I}$-AP) if there is a constant $C < \infty$ such that, for all Banach spaces Z and operators

$U \in \mathscr{I}(X, Z)$,

$$\inf\{\|U - UV\| : V \in \mathscr{I}(X),\, \|I - V\| \leq C\} = 0\,.$$

For the question of uniqueness of norm on Calkin algebras, the following equivalences in the case that $\mathscr{I} = \mathscr{K}$ are of interest:

3.2.2 Proposition ([Tyl2, Ex. 1.7]). *Let X be a Banach space.*

(i) The following conditions are equivalent: (a) X has the outer $\mathscr{K}$-AP; (b) X has the BCAP; (c) $\mathscr{K}(X)$ has a bounded left approximate identity.

(ii) X has the inner $\mathscr{K}$-AP if and only if $\mathscr{K}(X)$ has a bounded right approximate identity.

The main result of [Tyl1] is:

3.2.3 Theorem ([Tyl1, Thms. 1.2–1.3]). *Suppose that $\mathscr{I}$ is a closed surjective $\langle$resp. injective$\rangle$ operator ideal. Then a Banach space X has the outer $\mathscr{I}$-AP $\langle$resp. the inner $\mathscr{I}$-AP$\rangle$ if and only if $\gamma_{\mathscr{I}}(\cdot) \sim \|\cdot\|_{\mathscr{I}}$ $\langle$resp. $\beta_{\mathscr{I}}(\cdot) \sim \|\cdot\|_{\mathscr{I}}\rangle$ on $\mathscr{B}(Z, X)$ $\langle$resp. on $\mathscr{B}(X, Z)\rangle$ for all Banach spaces Z.*

As a direct consequence of Theorems 3.1.5 and 3.2.3, we have the following extension of Theorem 2.3.4:

3.2.4 Corollary. *Let X be a Banach space with the BCAP. Then $\|\cdot\|_K \sim \|\cdot\|_e$ on $\mathscr{C}(X)$.*

As a corollary to Theorem 3.2.3, in [Tyl2] Tyilli deduces:

3.2.5 Theorem ([Tyl2, Coroll. 1.4]). *Let X be a Banach space, and suppose that $\mathscr{I}$ is a surjective $\langle$resp. an injective$\rangle$ closed operator ideal such that X fails to have the outer $\mathscr{I}$-AP $\langle$resp. the inner $\mathscr{I}$-AP$\rangle$. Then there is a Banach space Y such that the quotient norm on $\mathscr{B}(X \oplus Y)/\mathscr{I}(X \oplus Y)$ is not minimal.*

The space Y in Theorem 3.2.5 is constructed as an infinite direct sum, whose summands are copies of X with equivalent, yet increasingly distorted, norms. These norms are chosen based on the failure of the outer $\mathscr{I}$-AP $\langle$resp. the inner $\mathscr{I}$-AP$\rangle$, such that the identity maps between X and the summands have standard quotient norm 1, but whose outer $\langle$resp. inner$\rangle$ $\mathscr{I}$-variations tend to 0. Extending these maps to the whole of $X \oplus Y$ shows that there is no $c \in \mathbb{R}$ such that $\| \cdot \|_{\mathscr{I}} \leq c\gamma_{\mathscr{I}}(\cdot)$ $\langle$resp. $\| \cdot \|_{\mathscr{I}} \leq c\beta_{\mathscr{I}}(\cdot)\rangle$ on $\mathscr{B}(X \oplus Y)$.

3.2.6 Corollary. *There exist Calkin algebras on which $\| \cdot \|_e$ is not minimal; in particular, there exist Calkin algebras on which $\gamma(\cdot) \nsim \| \cdot \|_e$ and $\| \cdot \|_K \nsim \| \cdot \|_e$.*

Proof. [Sza, §1] provides examples of Banach spaces that do not have the BCAP. By Proposition 3.2.2, such spaces do not have the outer $\mathscr{K}$-AP. Therefore, by Theorem 3.2.5, there are Calkin algebras for which $\| \cdot \|_e$ is not minimal, such that $\gamma(\cdot) \nsim \|\cdot\|_e$. Theorem 3.1.5 shows that $\|\cdot\|_K \sim \gamma(\cdot)$, so we also have $\|\cdot\|_K \nsim \|\cdot\|_e$. ∎

Furthermore, [Tyl1, Ex. 2.5] also provides an example to show that the seminorm given by the map $T \mapsto \|T^*\|_e$, for bounded linear operators T, fails to be equivalent to $\| \cdot \|_e$ on certain Calkin algebras. This relies on a similar construction to the one needed to establish Theorems 3.2.3 and 3.2.5, as described above. However, the situation is more delicate: the 'dual' norm is shown to be equivalent to $\beta_{\mathscr{K}}(\cdot)$, which in turn fails to be equivalent to $\| \cdot \|_e$. To achieve this, [Tyl1] exploits certain properties of spaces which have the BCAP, yet fail to have the approximation property. That such spaces exist was first established in [Wil1].

A remark on [Tyl2, p220] notes that there is an example of a Banach space X, such that $\gamma_{\mathscr{W}}(T) = \beta_{\mathscr{W}}(T) = \|T\|_{\mathscr{W}}$ for all $T \in \mathscr{B}(X)$, yet X has neither the outer $\mathscr{W}$-AP nor the inner $\mathscr{W}$-AP. Here $\mathscr{W}$ is the closed injective and surjective operator ideal of weakly compact operators. Thus Theorem 3.2.3 does not hold if we restrict attention to the case $Z = X$.

However, when $\mathscr{I} = \mathscr{K}$, there is no known example of a Banach space X_γ without the BCAP and such that $\gamma(\cdot) \sim \|T\|_e$ on $\mathscr{C}(X_\gamma)$. Thus, although Corollary 3.2.4 stipulates that $\| \cdot \|_K$, as defined in Chapter 2, must be equivalent to $\| \cdot \|_e$ whenever X has the BCAP, whether the converse is true is unknown. Note that the method of constructing the space Y in Theorem 3.2.3, as described above, is similar in nature to an argument that can be used to show that the existence of a Banach space without the approximation property implies that there is a Banach space X for which $\mathscr{A}(X) \neq \mathscr{K}(X)$ (see, e.g., [Ale]). Therefore, it seems possible that a line of reasoning which demonstrated whether or not X_γ can exist, might be sufficient to also resolve the question of whether $\mathscr{A}(X) = \mathscr{K}(X)$ only when X has the approximation property. Unfortunately, to date this question remains intractable.

Negative examples of Banach spaces X for which $\mathscr{C}(X)$ fails to have a unique algebra norm shall not concern us much further, however it is worthwhile to note two particular cases that were not reported in [Tyl2]. Indeed, all such examples of Calkin algebras that have been previously recorded were counterexamples to the minimality of $\|\cdot\|_e$, whereas the cases we examine below show that $\|\cdot\|_e$ can fail to be maximal.

The example due to Read, as discussed in §1.1 (see Theorem 1.1.19 and [Rea]), of a Banach space X_{R} such that the standard operator norm on $\mathscr{B}(X_{\mathrm{R}})$ is not maximal, is presented in [Tyl2, Ex. 2.6]. In a subsequent remark in [Tyl2], it is questioned whether there exists an analogous example of a Banach space X such that the quotient norm on $\mathscr{C}(X)$ is not maximal. In fact, X_{R} itself has this property. The main result of [Rea] is that the Banach algebra $\mathfrak{A} = \mathscr{B}(X_{\mathrm{R}})$ satisfies the requirements of the following theorem:

3.2.7 Theorem ([Rea, Thm. 1]). *If a unital Banach algebra $\mathfrak{A}$ contains a closed (two-sided) ideal $\mathscr{I}$ such that $\mathscr{I}$ has codimension 1 in $\mathfrak{A}$ but $\mathscr{I}^2$ is of infinite codimension in $\mathfrak{A}$, then there is a discontinuous derivation $D : \mathfrak{A} \to M$ for some Banach $\mathfrak{A}$-bimodule M.*

In particular, by [Rea, Coroll. 4.2] we have

(i) $\mathscr{B}(X_{\mathrm{R}})$ has a closed ideal $\mathscr{I}$ of codimension 1 in $\mathscr{B}(X_{\mathrm{R}})$,

(ii) $\mathscr{B}(X_{\mathrm{R}})/\mathscr{W}(X_{\mathrm{R}})$ is infinite dimensional, and

(iii) $\mathscr{I}^2 \subset \mathscr{W}(X_{\mathrm{R}})$.

Since $\mathscr{K}(X_{\mathrm{R}}) \subset \mathscr{W}(X_{\mathrm{R}})$, we therefore have that

(i) $\mathscr{C}(X_{\mathrm{R}})$ has a closed ideal $\mathscr{I}/\mathscr{K}(X_{\mathrm{R}})$ of codimension 1 in $\mathscr{C}(X_{\mathrm{R}})$,

(ii) $\mathscr{C}(X_{\mathrm{R}})/\frac{\mathscr{W}(X_{\mathrm{R}})}{\mathscr{K}(X_{\mathrm{R}})}$ is infinite dimensional, and

(iii) $(\mathscr{I}/\mathscr{K}(X_{\mathrm{R}}))^2 \subset \mathscr{W}(X_{\mathrm{R}})/\mathscr{K}(X_{\mathrm{R}})$.

Thus Theorem 3.2.7 applies to $\mathscr{C}(X_{\mathrm{R}})$. Our discussion prior to the statement of Theorem 1.1.19 showed that the existence of a discontinuous derivation from a Banach algebra $\mathfrak{A}$ to a Banach $\mathfrak{A}$-bimodule implies that the norm on $\mathfrak{A}$ fails to be maximal, so we have:

3.2.8 Example. $\|\cdot\|_e$ is not maximal on $\mathscr{C}(X_{\mathrm{R}})$, and therefore $\mathscr{C}(X_{\mathrm{R}})$ does not have a unique algebra norm.

Our other example is very new, and has not yet been published except in the recent PhD thesis [Tar2]. One of the major results given there is:

3.2.9 Theorem ([Tar2, Thm. 4.1.1]). *There exists a Banach space X_{Ta} over $\mathbb{R}$, which has a basis and the following properties*

(i) $X_{\mathrm{Ta}}^* = \ell_1$.

(ii) There exists a non-compact operator S on X_{Ta} which is not a scalar multiple of the identity. The sequence $(S^j + \mathscr{K}(X_{\mathrm{Ta}}))_{j=0}^{\infty} \subset \mathscr{C}(X_{\mathrm{Ta}})$ is a basic sequence 1-equivalent to the canonical basis of ℓ_1.

(iii) If $T \in \mathscr{B}(X_{\mathrm{Ta}})$ then there are unique scalars $(c_i)_{i=0}^{\infty} \subset \mathbb{R}$ and an operator $K \in \mathscr{K}(X_{\mathrm{Ta}})$ with $\sum_{i=0}^{\infty} |c_i| < \infty$ and

$$T = \sum_{i=0}^{\infty} c_i S^i + K .$$

The operator S appearing in the above sum is the same operator as described in Property (ii).

(iv) $\mathscr{C}(X_{\mathrm{Ta}})$ is isometrically isomorphic as a Banach algebra to the convolution algebra $\ell_1(\mathbb{N}_0)$.

Let $S + \mathscr{K}(X_{\mathrm{Ta}})$ be denoted by s. Property (ii) then implies that $(s^j)_{j=0}^{\infty}$ spans $\mathscr{C}(X_{\mathrm{Ta}})$ and is such that

$$\left\| \sum_{i=0}^{\infty} c_i s^i \right\|_e = \sum_{i=0}^{\infty} |c_i| .$$

In particular, $\mathscr{C}(X_{\mathrm{Ta}})$ is isometrically isomorphic to a Banach algebra of power series in the sense of [Dal1, Defn. 4.6.4]. Hence, there exists a discontinuous derivation from $\mathscr{C}(X_{\mathrm{Ta}})$ into a Banach $\mathscr{C}(X_{\mathrm{Ta}})$-module, by [Dal1, Thm. 5.6.79]. As in the case of the Read example, we therefore have that $\|\cdot\|_e$ is not maximal on $\mathscr{C}(X_{\mathrm{Ta}})$.

Additionally, in recent correspondence Prof. H. Garth Dales pointed out that the norm $\|\cdot\|_{\ell_1}$ on the convolution algebra $\ell_1(\mathbb{N}_0)$ is not minimal. We translate his illustration of a relevant inequivalent norm on $\ell_1(\mathbb{N}_0)$ into the above context of $\mathscr{C}(X_{\mathrm{Ta}})$. For each $i \in \mathbb{N}_0$, set

$$w_i = \exp(-i^2) .$$

Many other weights will also lead to inequivalent norms: for example, take any other radical weight, in the sense of [Dal1, Ex. 2.1.13(v)]. Define $\|\cdot\|_w$ on $\mathscr{C}(X_{\mathrm{Ta}})$ by

$$\left\|\sum_{i=0}^{\infty} c_i s^i\right\|_w = \sum_{i=0}^{\infty} |c_i| w_i \,.$$

Then $\|\cdot\|_w$ is an algebra norm, but $\|s^i\|_e = 1$ for all $i \in \mathbb{N}_0$, while $\|s^i\|_w = w_i \to 0$. Therefore $\|\cdot\|_e$ is not minimal on $\mathscr{C}(X_{\mathrm{Ta}})$.

The fact the algebra norm on $\mathscr{C}(X_{\mathrm{Ta}})$ is neither minimal nor maximal leaves open the possibility that $\mathscr{C}(X_{\mathrm{Ta}})$ might have an inequivalent complete algebra norm. However, the convolution algebra $\ell_1(\mathbb{N}_0)$ is semisimple by [Dal1, Coroll. 4.5.5]. Therefore $\mathscr{C}(X_{\mathrm{Ta}})$ has a unique complete algebra norm by Theorem 1.1.20. Also implied is that $\mathscr{K}(X_{\mathrm{Ta}}) = \mathscr{E}(X_{\mathrm{Ta}})$.

It remains unknown whether there exists a Banach space X such that $\|\cdot\|_e$ is not the unique complete algebra norm on $\mathscr{C}(X)$. For such an X, it would be necessary that $\mathscr{K}(X) \neq \mathscr{E}(X)$, so that $\mathscr{C}(X)$ was not semisimple. Furthermore, we are not aware of an example, other that $\mathscr{C}(X_{\mathrm{Ta}})$, such that $\|\cdot\|_e$ is neither maximal nor minimal.

3.2.10 Remark. X_{Ta} has a basis. Hence, it provides an example of a space that has the BCAP, while its Calkin algebra fails to have $\|\cdot\|_e$ as a minimal algebra norm. In particular, $\gamma(\cdot) \sim \|\cdot\|_e$ on $\mathscr{C}(X_{\mathrm{Ta}})$, by Proposition 3.2.2 and Theorem 3.2.3. Thus it is also the case that $\gamma(\cdot)$ is not minimal on $\mathscr{C}(X_{\mathrm{Ta}})$.

3.3 Previous positive results

We have seen that it is possible to define an inequivalent algebra norm on $\mathscr{C}(X)$, in some limited cases. However, for each known example, the underlying Banach space X has rather extreme properties, either failing the BCAP or being an otherwise particularly pathological counterexample. By Corollary 1.1.12, we also know that $\mathscr{B}(X)$ has a unique algebra norm for a wide variety of more classical Banach spaces. In light of these points, it might be expected that $\mathscr{C}(X)$ has a unique algebra norm in many cases, and that a severe pathology of some description is a necessary condition for uniqueness to fail. It is thus surprising that the class of Banach spaces known in the literature whose Calkin algebras have unique algebra norms is very limited. Even more surprising is that this class is split into two radically different halves. We shall discuss the more recent subclass of known positive examples first.

3.3.1 Example. Let Z be:

(i) the Banach space constructed in [ArHa], such that every $T \in \mathscr{B}(Z)$ is the sum of a compact operator and a scalar multiple of the identity,

(ii) one of the Banach spaces $\mathscr{X}_k$ constructed in [Tar1], such that $\mathscr{C}(\mathscr{X}_k)$ is k-dimensional, or

(iii) any other Banach space such that $\mathscr{C}(Z)$ is finite dimensional.

Then $\mathscr{C}(Z)$ has a unique norm as a normed space, because it is finite dimensional (see p1). In particular, $\mathscr{C}(Z)$ has a unique algebra norm. Note that each of the spaces in (i) or (ii) above is *hereditarily indecomposable* (HI), which means that Z is not the direct sum of any two infinite dimensional subspaces and that this is also true for every subspace of Z. Hence Z does not have a continued bisection of the identity, and so gives an example of a space Z for which the conditions of Corollary 1.1.12 do not hold, yet the norm on $\mathscr{C}(Z)$ is still maximal.

HI spaces were first shown to exist in [GoMa] and have some very unusual properties. If X is HI, then $\mathscr{B}(Y, X)/\mathscr{S}(Y, X)$ is finite dimensional for all closed subspaces Y of X (see [Fer]). Thus the quotient algebra $\mathscr{B}(X)/\mathscr{S}(X)$ is finite dimensional and has a unique algebra norm, for all HI Banach spaces X.

Example 3.3.1 contrasts markedly with the other previously known class of Banach spaces X such that $\mathscr{C}(X)$ has a unique algebra norm:

3.3.2 Theorem ([Mey, Thm. 2.2])**.** *Let X be c_0 or ℓ_p, $1 \leq p < \infty$. Then $\|\cdot\|_e$ is the unique algebra norm on $\mathscr{C}(X)$.*

So, other than 'trivial' examples, previous studies have only revealed that $\mathscr{C}(X)$ has a unique algebra norm when X is one of the 'nicest' of all possible Banach spaces. Aspects of the proof of Theorem 3.3.2 will be our focus us for the remainder of this section. The original result from [Mey] is discussed in [Tyl2, §2], where the presentation is a bit different to [Mey], although the primary features are the same.

[Mey] contains a proof of a statement that is similar to the following improvement to Lemma 2.3.2.

3.3.3 Lemma (The block-to-block lemma for ℓ_p — strong version [Mey, Eqn. 0])**.** *Suppose that $T \in \mathscr{B}(\ell_p)$, $1 \leq p < \infty$, $T \notin \mathscr{K}(\ell_p)$ and $(\delta_n) \subset \mathbb{R}_+$. Then there exist block basic sequences (x_n) and (z_n) of (e_n) such that (x_n) is normalised and, for all $n \in \mathbb{N}$, $\|z_n - Tx_n\| < \delta_n$ and $\|z_n\| > \|T\|_e - \delta_n$.*

The sketched proof of [Tyl2, Lem 2.1] also includes a method that can be used to establish the above lemma. Both sources rely on some version of Corollary 1.3.5, and an inductive process similar to the one we used to prove the weaker Lemma 2.3.2. However, the important difference in our proof is as follows. At each stage in the process, one must choose a new pair of blocks x_{m+1} and z_{m+1}, supported after some $M \in \mathbb{N}$ that bounds the maximum of the supports of the previous vectors. To do this, we relied on the compactness of the projections P_n. That is, given that Corollary 1.3.5 guarantees that there is a block basis (y_n) such that the images Ty_n are 'large' on disjointly supported sets, we chose x_{m+1} as the difference of two blocks $y_{n_{k_1}}$ and $y_{n_{k_2}}$, from a subsequence such that $(P_M T y_{n_k})$ converges. By ensuring these two blocks were late enough in the subsequence, this meant that Tx_{m+1} was not only 'large' after M but also 'small' before it, and the required block-to-block behaviour could be inductively produced as a result. However, after normalising x_{m+1}, the final size of $\|z_{m+1}\|$ may be reduced by a factor that can be as bad as 2^{-1}, because we defined x_{m+1} as a combination of two of the y_n.

The method given in [Mey] and [Tyl2] varies from the above in the manner by which x_{m+1} is produced from (y_n). Excluding the case when $p = 1$, ℓ_p is reflexive. Thus (y_n) and (Ty_n) are weakly null. Therefore, $e_m^*(y_n) \to 0$ and $e_m^*(Ty_n) \to 0$ for each functional e_m^* in the sequence biorthogonal to (e_m). This shows that, in fact, one of the y_n themselves is such that both y_n and Ty_n are 'small' before M. In the case of ℓ_1, a similar argument based on the weak* nullity of the relevant sequences can be used. Because x_{m+1} is simply one of the y_n, the factor of 2^{-1} is not introduced.

In fact, by utilising Rosenthal's ℓ_1 theorem (see, e.g., [AlKa, Thm. 10.2.1]) and the Bessaga-Pełzyński selection principle (see, e.g., [AlKa, Prop. 1.3.10]), it would be possible to extend this method to more general circumstances. In contrast, our proof of the block-to-block lemma (2.3.3) for all Banach spaces with a basis was entirely elementary, and this 'weaker' strategy suffices for our purposes in future chapters (however, see Remark 3.3.4 and Lemma 3.3.6 below). At the end of Chapter 4 we shall again use the trick of selecting successive terms x_{m+1} as the difference of two blocks: it shall appear in our proof of a similar 'block-to-blocks' lemma, as an application of our generalisation of the uniform boundedness principle.

3.3.4 Remark. The stronger version (3.3.3) of the block-to-block lemma for ℓ_p (and our own version (2.3.1) for c_0), can be used to show that $\|\cdot\|_e$ has a very strong isometric property on the Calkin algebras in question:

([Mey, Thm. 2.1]). *Let X be c_0 or ℓ_p, $1 \le p < \infty$, and let $|||\cdot|||$ be any* *algebra norm on* $\mathscr{C}(X)$*. Then* $|||\cdot||| \leq \|\cdot\|_e$ *implies that* $|||\cdot||| = \|\cdot\|_e$.

This result extends [Bon, Thm. 8], which shows that a similar result is true for $\mathscr{B}(X)$, where X can be any Banach space. The result for $\mathscr{B}(X)$ can be seen as an isometric version of the Eidelheit-Yood theorem (1.1.2). In parallel, the above theorem is an isometric version of Theorem 3.3.2.

We shall now substantiate Theorem 3.3.2. Our proof is modified from that of [Tyl2, §2], but we shall make clear some pertinent differences between our version and the ones in [Tyl2] and [Mey]. Firstly, note that $c_0 \approx c_0 \oplus c_0$ and $\ell_p \approx \ell_p \oplus \ell_p$, for all $1 \leq p < \infty$. Thus, by Corollary 1.1.14, $\|\cdot\|_e$ is maximal on the Calkin algebras of these spaces, and it remains to show that $\|\cdot\|_e$ is also minimal.

[Tyl2] includes the following factorisation result, based on a 'strong' version of the block-to-block lemma for c_0 and ℓ_p (slightly different to the one we gave in Lemmas 2.3.1 and 3.3.3), and the characteristic properties of block bases in those spaces (see Proposition 1.3.2 and Theorem 1.3.3).

3.3.5 Lemma ([Tyl2, Lem. 2.1])**.** *Let* X *be* c_0 *or* ℓ_p, $1 \leq p < \infty$*. Then for each* $T \in \mathscr{B}(X)\backslash\mathscr{K}(X)$ *and* $\varepsilon > 0$ *there are operators* $A_\varepsilon, B_\varepsilon \in \mathscr{B}(X)$ *such that*

$$I \;=\; B_\varepsilon T A_\varepsilon \qquad \text{and} \qquad \|A_\varepsilon\|\|B_\varepsilon\| \;\leq\; (1+\varepsilon)\|T\|_e^{-1}.$$

From the above, we have that

$$\|A_\varepsilon\|_e\|B_\varepsilon\|_e \;\leq\; \|A_\varepsilon\|\|B_\varepsilon\| \;\leq\; (1+\varepsilon)\|T\|_e^{-1},$$

which is how Lemma 3.3.5 is used in [Tyl2, §2]. However, our next result shows that we can improve this estimate, if we apply the 'strongest' version of the block-to-block lemma for c_0 and ℓ_p instead.

3.3.6 Lemma. *Let* X *be* c_0 *or* ℓ_p, $1 \leq p < \infty$*. Then for each* $T \in \mathscr{B}(X)\backslash\mathscr{K}(X)$ *there are operators* $A, B \in \mathscr{B}(X)$ *such that*

$$I \;=\; BTA \qquad \text{and} \qquad \|A\|_e\|B\|_e \;=\; \|T\|_e^{-1}\,.$$

Proof. By Lemma 2.3.1 or 3.3.3, there exist block bases $(x_n), (z_n) \subset X$ such that (x_n) is normalised and, for all $n \in \mathbb{N}$,

$$\|z_n - Tx_n\| \;<\; 2^{-(n+2)}\|T\|_e \qquad \text{and} \qquad \|z_n\| \;>\; (1 - 2^{-(n+2)})\|T\|_e\,.$$

In fact, it can be shown that $\|z_n\| \to \|T\|_e$. One way to do so is to use the theory developed in Chapter 2: if instead $\|z_n\| \nrightarrow \|T\|_e$, then because we also have $\liminf \|z_n\| \geq \|T\|_e$ there must be $\varepsilon > 0$ and $(z_{n_k}) \subset (z_n)$ such that $\|z_{n_k}\| \geq \|T\|_e + \varepsilon$ for all $k \in \mathbb{N}$. Set $\alpha = \|e_1 + e_2\|^{-1}$. The properties of the norm on X and the fact that (αx_{n_k}) is a block basis make it easy to check that $(\alpha x_{n_k}) \in \Delta_X^1$ and $\lceil T, (\alpha x_{n_k}) \rceil \geq \|T\|_e + \varepsilon$, in contradiction to Corollary 2.2.3 and the fact that $\|T\|_K \leq \|T\|_e$ (see Remark 2.1.2).

Because $\|z_n\| \to \|T\|_e$, we may pass to subsequences if necessary to ensure that, for all $n \in \mathbb{N}$,

$$(1 - 2^{-(n+2)})\|T\|_e \;<\; \|z_n\| \;<\; (1 + 2^{-(n+2)})\|T\|_e\,,$$

and recall that we already had

$$\|z_n - Tx_n\| \;<\; 2^{-(n+2)}\|T\|_e\,,$$

which is unaffected by passing to subsequences, so long as we choose the respective subsequence of (x_n) that corresponds to the one chosen from (z_n).

For each $n \in \mathbb{N}$, set $y_n = \|z_n\|^{-1}\|T\|_e z_n$, so that (y_n) is the block basis (z_n), rescaled to have constant norm $\|T\|_e$. Then

$$\|y_n - z_n\| \;\leq\; 2^{-(n+2)}\|T\|_e\,,$$

and thus

$$\|y_n - Tx_n\| \;\leq\; 2^{-(n+1)}\|y_n\|\,.$$

It follows from Definition 1.3.6 that (Tx_n) is a $\frac{1}{2}$-small perturbation of (y_n). Therefore, by Corollary 1.3.8, there is $U \in \mathscr{B}(X)$ such that $(I - U) \in \mathscr{K}(X)$ and $UTx_n = y_n$, for all $n \in \mathbb{N}$.

Now, by Proposition 1.3.2, both (x_n) and $(\|T\|_e^{-1}y_n)$ are 1-complemented and 1-equivalent to the standard unit vector basis in X. Thus there is $A \in \mathscr{B}(X)$ such that $\|A\| = 1$ and $Ae_n = x_n$, for all $n \in \mathbb{N}$. Furthermore, there is a projection $P \in \mathscr{B}(X)$ onto $[(y_n)]$ with $\|P\| = 1$, and an operator $S \in \mathscr{B}([(y_n)], X)$ such that $\|S\| = \|T\|_e^{-1}$ and $Sy_n = e_n$, for all $n \in \mathbb{N}$. Set $B = SPU$. Then, for all $n \in \mathbb{N}$,

$$BTAe_n \;=\; SPUTx_n \;=\; SPy_n \;=\; Sy_n \;=\; e_n\,.$$

Thus $I = BTA$. Also, it is that case that $\|A\|_e \leq \|A\| = 1$ and

$$\|B\|_e \;\leq\; \|S\|_e\|P\|_e\|U\|_e \;=\; \|S\|_e\|P\|_e\|I\|_e \;\leq\; \|S\|\|P\| = \|T\|_e^{-1}\,.$$

In fact, we must have $\|B\|_e = \|T\|_e^{-1}$ and $\|A\|_e = 1$, as otherwise

$$1 = \|I\|_e < \|T\|_e^{-1} \cdot \|T\|_e \cdot 1 = 1,$$

a contradiction. ■

At this stage in their proofs of Theorem 3.3.2, both [Tyl2] and [Mey] rely on the following lemma, and give an elegant proof due to Meyer.

3.3.7 Lemma ([Mey, Lem. 4]; [Tyl2, Lem. 2.3])**.** *Let* $(\mathfrak{A}, \|\cdot\|)$ *be a simple unital Banach algebra, and* $|||\cdot|||$ *be any algebra norm on* $\mathfrak{A}$*. Then there is an algebra norm* $|||\cdot|||_0$ *on* $\mathfrak{A}$ *and a constant* $C < \infty$ *such that* $|||\cdot|||_0 \leq |||\cdot|||$ *and* $|||\cdot|||_0 \leq C\|\cdot\|$.

Recall that a Banach algebra $\mathfrak{A}$ is *simple* if the only closed two-sided ideals in $\mathfrak{A}$ are the trivial ones: $\mathfrak{A}$ and $\{0\}$. [GMF, Thm. 5.1] showed that $\mathscr{K}(X)$ is the only non-trivial closed ideal in $\mathscr{B}(X)$, when X is c_0 or ℓ_p, $1 \leq p < \infty$.[1] Thus the Calkin algebra $\mathscr{C}(X)$ is simple, and the above lemma applies. However, relying on the simplicity of $\mathscr{C}(X)$ in those cases is unnecessary. Instead, we may use the fact that (by Corollary 1.1.14) $\|\cdot\|_e$ is maximal on $\mathscr{C}(X)$: for any algebra norm $|||\cdot|||$ on $\mathscr{C}(X)$ there exists C such that $|||\cdot||| \leq C\|\cdot\|_e$. Hence, in the cases of $\mathscr{C}(c_0)$ and $\mathscr{C}(\ell_p)$, $1 \leq p < \infty$, in Lemma 3.3.7 it suffices to take $|||\cdot|||_0 = |||\cdot|||$.

Indeed, Lemma 3.3.6 and the maximality of $\|\cdot\|_e$ are all that is required to establish the minimality of $\|\cdot\|_e$ in these cases, as the following shows:

Proof of Theorem 3.3.2. Let $|||\cdot|||$ be an arbitrary algebra norm on $\mathscr{C}(X)$. Since $\|\cdot\|_e$ is maximal on $\mathscr{C}(X)$, there is $C > 0$ such that $|||\cdot||| \leq C\|\cdot\|_e$. Identify $|||\cdot|||$ with a seminorm on $\mathscr{B}(X)$, such that $\ker|||\cdot||| = \mathscr{K}(X)$. Suppose $T \in \mathscr{B}(X)$. If $T \in \mathscr{K}(X)$, then $|||T||| = \|T\|_e = 0$, so we may assume $T \notin \mathscr{K}(X)$. By Lemma 3.3.6, there are operators $A, B \in \mathscr{B}(X)$ such that

$$I = BTA \quad \text{and} \quad \|A\|_e\|B\|_e = \|T\|_e^{-1}.$$

Hence

$$\begin{aligned} 1 \leq |||I||| &\leq |||B|||\cdot|||T|||\cdot|||A||| \\ &\leq C^2|||T|||\cdot\|A\|_e\cdot\|B\|_e \qquad (3.3.2a) \\ &= C^2\|T\|_e^{-1}|||T|||. \end{aligned}$$

[1][GMF, Thm. 5.1] is actually an immediate consequence of Lemma 3.3.5 or Lemma 3.3.6: the identity on c_0 or ℓ_p, $1 \leq p < \infty$, must be in any ideal which contains a non-compact operator.

Therefore $C^{-2}\|T\|_e \leq |||T|||$, and so

$$C^{-2}\|\cdot\|_e \leq |||\cdot||| \leq C\|\cdot\|_e\,. \tag{3.3.2b}$$

That is, $|||\cdot||| \sim \|\cdot\|_e$. ■

3.3.8 Remark. The way in which the maximality of $\|\cdot\|_e$ was required in the above proof is important to note. Firstly, it was necessary for $\|\cdot\|_e$ to be maximal in order to show that $\|\cdot\|_e$ is unique (see 3.3.2b). However, the maximality of $\|\cdot\|_e$ was also relied upon to show that $\|\cdot\|_e$ is minimal (see 3.3.2a). That is, without knowing that $\|\cdot\|_e$ is maximal, we would not have been able to use the above method of proof to show that $\|\cdot\|_e$ is minimal.

Remark 3.3.8 was not observed in previous accounts of Theorem 3.3.2. This is because they relied instead on the simplicity of the Calkin algebras in question, and the results of Lemmas 3.3.5 and 3.3.7, to give a similar proof of the theorem. However, this meant the simplicity of $\mathscr{C}(c_0)$ and $\mathscr{C}(\ell_p)$, for $1 \leq p < \infty$, was seen as a crucial element of the approach. Indeed, [Tyl2, p220] questions which other Calkin algebras have $\|\cdot\|_e$ as a minimal norm, and gives $\mathscr{C}(\ell_p \oplus \ell_q)$ as a test case (for $1 \leq p < q < \infty$), noting that it is not simple. As we have seen, instead it is the maximality of $\|\cdot\|_e$ on $\mathscr{C}(X)$ which may be important, rather than the very restrictive criteria of simplicity. We have $\ell_p \oplus \ell_q \oplus \ell_p \oplus \ell_q \approx \ell_p \oplus \ell_q$, so $\|\cdot\|_e$ is maximal on $\mathscr{C}(\ell_p \oplus \ell_q)$ by Corollary 1.1.14.

That said, another obstruction to the generalisation of Theorem 3.3.2, raised by [Tyl2, p220] in the context of $\mathscr{C}(\ell_p \oplus \ell_q)$, is of greater significance: given a more general Banach space X, we need a factorisation result similar to Lemmas 3.3.5 and 3.3.6, in order to apply the method of proof used above. It is the development of replacements for this factorisation lemma that will primarily concern us for the remainder of this thesis. The proof we gave for Lemma 3.3.6, and the ones given by previous authors for Lemma 3.3.5, all rely on the strong properties of block bases in the spaces c_0 and ℓ_p, $1 \leq p < \infty$, which are unique to those spaces (see Proposition 1.3.2 and Theorem 1.3.3) and were used in [GMF] to show that $\mathscr{C}(c_0)$ and $\mathscr{C}(\ell_p)$ are simple (see Footnote 1 on p53). Thus, the simplicity of the Calkin algebras still seems to be important: if we wish arbitrary non-compact operators in $\mathscr{B}(X)$ to be factors of the identity, then we necessarily require that there are no non-trivial ideals in $\mathscr{C}(X)$.

Therefore, the question of whether it must be the identity operator which is factored arises. A different, possibly variable, operator could be factored instead, so long as its arbitrary algebra (semi-)norm was known to be bounded below, away from 0. This lower bound is needed in order that the inequalities 3.3.2a and 3.3.2b remain meaningful: we knew that $1 \leq |||I|||$, but with an arbitrary substitute a value of 0 on the left hand side may result. Because the aim is to establish minimality, controlling the norm of a replacement for I, away from 0, is not always an easy task: we are trying to prove that a situation in which $\|T_k\|_e = 1$ while $|||T_k||| \downarrow 0$ is impossible, so we certainly can not assume to have a lower bound to begin with. Two possible approaches are:

(i) Non-zero idempotent elements must have algebra norms greater than 1 (this is what guarantees that $1 \leq |||I|||$). Hence, if a factorisation lemma could be established, in the style of Lemmas 3.3.5 and 3.3.6, by replacing the identity with a non-compact projection (which could vary with T), then the method used to prove Theorem 3.3.2 would generalise. We shall use this approach in Chapter 6, in order to prove that non-separable analogues of c_0 and ℓ_p, $1 \leq p < \infty$, have Calkin algebras with unique algebra norms. However, in those cases strong properties of transfinite analogues of block-basic sequences are still required.

(ii) Rather than attempt to prove a factorisation lemma for all non-compact $T \in \mathscr{B}(X)$ simultaneously, we shall see that it suffices to be able to show that (a subsequence of) the members of a given sequence $(T_k) \subset \mathscr{B}(X)$ each factor a common non-compact operator. This perspective parallels the discussion in Remark 1.1.6 of the proof of the Eidelheit-Yood theorem (1.1.2), and will be the focus of Chapters 4 and 5. First, in the next section, we shall examine the concept behind this approach in more detail.

3.4 A connection to the Eidelheit-Yood theorem

Recall that the proof of the Eidelheit-Yood theorem (1.1.2), for an arbitrary Banach space X, relied on the construction of an operator $S \in \mathscr{B}(X)$ defined by

$$Sx = x_0^*(x)x_0$$

and such that

$$ST_kS = x_0^*(T_kx_0)S \qquad \text{and} \qquad |x_0^*(T_kx_0)| \uparrow \infty$$

for (a subsequence of) a given sequence $(T_k) \subset \mathscr{B}(X)$ for which $\|T_k\| \uparrow \infty$.

It was noted in Remark 1.1.6 that the same construction does not produce a contradiction if instead we assume $|||T_k||| = 1$ and $\|T_k\|_e \uparrow \infty$ for an arbitrary algebra norm $|||\cdot|||$ on $\mathscr{C}(X)$, as opposed to $\mathscr{B}(X)$. This is because S has rank 1, giving $|||S||| = 0$ and

$$|||ST_kS||| \;=\; |||x_0^*(T_kx_0)S||| \;=\; 0\,.$$

However, we have not yet considered how such a construction might be modified, to give a similar factorisation for an operator S that is not compact. If that were possible for a particular Banach space X, then the Eidelheit-Yood method of proof would generalise to $\mathscr{C}(X)$.

We shall sketch a 'best-case' scenario. Suppose we could choose normalised basic sequences $(x_n) \subset X$ and $(x_n^*) \subset X^*$, for which

$$x_m^*(T_kx_n) \;=\; \delta_{nm}\alpha\|T_k\|_e\,,$$

for all $n, m, k \in \mathbb{N}$ and some $\alpha \in \mathbb{F}\setminus\{0\}$, where δ_{nm} is the Kronecker delta. In particular, we assume that $(T_kx_n)_{n=1}^{\infty}$ is a seminormalised basic sequence for each $k \in \mathbb{N}$. Furthermore, suppose that the mapping S given by

$$Sx \;=\; \sum_n x_n^*(x)x_n \tag{3.4a}$$

is a well defined bounded linear operator. If it is, then it can not be compact, because of its action on $[(T_kx_n)_{n=1}^{\infty}]$ for any $k \in \mathbb{N}$. So $|||S||| \neq 0$ for an arbitrary algebra norm $|||\cdot|||$ on $\mathscr{C}(X)$. We would also have

$$\begin{aligned} ST_kSx \;&=\; S\left(\sum_n x_n^*(x)T_kx_n\right) \\ &=\; \sum_m\left(\sum_n x_n^*(x)x_m^*(T_kx_n)x_m\right) \\ &=\; \sum_m\left(\sum_n x_n^*(x)\delta_{nm}\alpha\|T_k\|_e x_m\right) \\ &=\; \alpha\|T_k\|_e\sum_m x_m^*(x)x_m \\ &=\; \alpha\|T_k\|_e Sx\,. \end{aligned}$$

This would guarantee that $|||T_k||| \uparrow \infty$ when $\|T_k\|_e \uparrow \infty$, because we can derive

$$|||T_k||| \;\geq\; \frac{|\alpha|\cdot\|T_k\|_e}{|||S|||}\,. \tag{3.4b}$$

If we allow compact perturbations of the various operators involved, this idealised set-up is actually attainable on c_0 and ℓ_p, $1 \leq p < \infty$. Thus, we could use this approach to give an alternative proof of Theorem 3.3.2. The 'quantitative' version of the uniform boundedness principle, which we gave as Corollary 1.1.5, is required.

Such a proof has one clear benefit: it does not require using the maximality of $\|\cdot\|_e$ to establish its minimality, in contrast to the original proof (as discussed in Remark 3.3.8). The difference arises due to the following. Without the maximality of $\|\cdot\|_e$, in 3.3.2a we would have had no control over how large $|||A||| \cdot |||B|||$ could be. The same is true in 3.4b, in the sense that we have no control over how large $|||S|||$ is. However, because we have constructed S to 'transfer' the unboundedness of $\|T_k\|_e$, in 3.4b we know that $\|T_k\|_e$ will outgrow $|||S|||$ eventually, whatever value it might have.

Despite this benefit of using a 'best-case' Eidelheit-Yood proof for $\mathscr{C}(c_0)$ and $\mathscr{C}(\ell_p)$, we shall not go through the details here. The reasons we choose not to are:

(i) We will show how an analogous construction can be made precise at the end of Chapter 6, to give an alternative proof of the minimality of the algebra norm on certain quotients of $\mathscr{B}(X)$, when X is a non-separable analogue of c_0 or ℓ_p, $1 < p < \infty$. In the context of those quotients, the 'up to compact perturbation' part of the argument becomes unnecessary and we can employ a completely pure version of the Eidelheit-Yood method. This will demonstrate the possibility of proving minimality without relying on maximality. In comparison to the proof required in the separable case, fewer technicalities obscure the nature of the factorisation that can be achieved.

(ii) To ensure that the operator S will always be well-defined (as given by 3.4a) requires a large amount of control over the possible sequences $(T_k x_n)_{n=1}^{\infty}$. While we can employ small perturbation arguments to restrict attention to block bases, the prescribed behaviour of (x_n^*) seems to require that all block bases are equivalent to one another, if we wish S to be properly defined in all circumstances. Hence, even allowing for compact perturbation adjustments, because of Theorem 1.3.3 it does not seem to be possible to employ the 'best-case' single-operator-S version of the Eidelheit-Yood method, other than for c_0 and ℓ_p, $1 \leq p < \infty$. Modifications to the method, whereby the different purposes of S are achieved with different operators, are therefore of more concern. For the remainder of this section we shall discuss the possibilities and obstacles that arise in making such modifications.

Given a Banach space X, an arbitrary algebra norm $|||\cdot|||$ on $\mathscr{C}(X)$, and $(T_k) \subset \mathscr{B}(X)$

with $\|T_k\|_e \uparrow \infty$, we now consider possible replacements for the factorisation

$$\alpha\|T_k\|_e S \;=\; ST_kS\,,$$

with whatever substitutes we find for S required to be non-compact, and whatever scalars $(\alpha_k) \subset \mathbb{F}$ we substitute for $\alpha\|T_k\|_e$ required to have the property that $|\alpha_k| \uparrow \infty$. The broadest possibility is that we can find sequences of operators $(A_k), (B_k), (R_k) \subset \mathscr{B}(X)\backslash\mathscr{K}(X)$ such that

$$\alpha_k R_k \;=\; B_k T_k A_k\,.$$

Seen from this perspective, the connection of the Eidelheit-Yood method with the factorisation lemma technique (Lemma 3.3.5 or 3.3.6) is clear. We could then derive

$$|||T_k||| \;\geq\; \frac{|\alpha_k|\cdot|||R_k|||}{|||A_k|||\cdot|||B_k|||}\,.$$

To guarantee that $|\alpha_k| \uparrow \infty$ implied $|||T_k||| \uparrow \infty$, we would need to be able to bound $|||A_k|||\cdot|||B_k|||$ from above and $|||R_k|||$ from below. As discussed in point (i) at the end of §3.3 (see p55), bounding $|||R_k|||$ from below is possible in the case that the R_k are known to be projections. However, such a strong factorisation result appears to be impossible in the desired general setting. The other option is to ensure that the R_k are 'essentially' all the same operator R. That is, we require $R_k \in R + \mathscr{K}(X)$ for all $k \in \mathbb{N}$. Then we would have $|||R_k||| = |||R|||$ for all $k \in \mathbb{N}$, and

$$|||T_k||| \;\geq\; \frac{|\alpha_k|\cdot|||R|||}{|||A_k|||\cdot|||B_k|||}\,,$$

so that $|\alpha_k|$ would eventually overpower $|||R|||$, no matter how small a (non-zero) value $|||R|||$ had. This feature is the major benefit that this version of the Eidelheit-Yood method has in comparison to a 'pure' factorisation technique, with which the above equation is derived for only a single k.

Assuming an essentially constant operator R, we have the situation described in Remark 1.1.6. The operators A_k have the first task of mapping onto a restricted subspace, on which we know the operators T_k are unbounded. Because we need the A_k to be non-compact, we analogously require a 'Calkin algebra version' of the uniform boundedness principle to guarantee the existence of a (normalised) sequence $(x_n) \subset X$, such that $\|T_k x_n\| \geq |\alpha_k|$ for all $n, k \in \mathbb{N}$. This might then allow the A_k to be chosen as mappings onto subspaces of $[(x_n)]$. Then, after the T_k have been applied, on the left the B_k 'shift' the various images of $[(x_n)]$ together, producing a factorisation of $\alpha_k R$. In the case that X has a basis, we may as well assume that the sequences (x_n) and $(T_k x_n)$ are block bases, as a result of small perturbation arguments. That is, if X has a basis, then our version of the uniform boundedness principle can take the form of a block-to-block lemma, by which each block x_n is known to map to small perturbations of the blocks $T_k x_n$, $k \in \mathbb{N}$. We shall term this result the 'block-to-blocks' lemma, and it will have a role analogous to the one the block-to-block lemmas for c_0 and ℓ_p played in establishing Theorem 3.3.2.

However, we still require $|||A_k||| \cdot |||B_k|||$ to be bounded from above. In this regard, the fact we have actually mixed the 'pure' Eidelheit-Yood method with the factorisation technique causes issues: we can no longer dispense with the requirement that $\|\cdot\|_e$ is maximal on $\mathscr{C}(X)$. We can choose the projections (A_k) and shifts (B_k) such that $\|A_k\|_e \|B_k\|_e$ is bounded, but to control $|||A_k||| \cdot |||B_k|||$ we need to be able to relate the essential norm to the arbitrary norm using maximality. Thus we abandon the elegance of the 'pure' Eidelheit-Yood method, which proves minimality of the canonical algebra norm without requiring that norm to be maximal (whenever it can actually be applied).

In summary, we have seen that the Eidelheit-Yood method has connections with the factorisation technique we used to prove that $\mathscr{C}(c_0)$ and $\mathscr{C}(\ell_p)$, $1 \leq p < \infty$, have unique algebra norms. In order to use a version of that method to extend this result to the Calkin algebras of other Banach spaces X that have a basis, we will need the following components:

(i) a Calkin algebra version of the uniform boundedness principle, with which to prove a 'block-to-blocks' lemma, and

(ii) weakened factorisation properties that guarantee the existence of 'shift' operators on X, which map certain block basic sequences to a common subspace.

In addition, we must restrict attention to spaces for which $\|\cdot\|_e$ is maximal on $\mathscr{C}(X)$. Fortunately, this is known to hold for a wide variety of spaces, as discussed in §1.1.

Chapter 4

The $\|\cdot\|_K$-UBP and the Block-to-Blocks Lemma

This chapter provides the technical base that we shall rely upon to generalise the uniqueness of norm property of $\mathscr{C}(X)$, beyond the case when X is c_0 or ℓ_p, $1 \leq p < \infty$. As previously indicated, such a generalisation appears to require two main features: first, an appropriate version of the block-to-block lemma for the spaces in question, and second, an analogue of the factorisation lemma to 'shift' between images of non-compact operators.

It is the first feature that will concern us for the time being; the second will be discussed in Chapter 5. Taking the perspective presented in §3.4, we intend to emulate the Eidelheit-Yood method of proof. So, we require a substitute for the uniform boundedness principle that will be applicable to collections of elements of Calkin algebras, rather than collections of bounded linear operators. We shall see that, in establishing such a result, the appropriate notion of the 'size' of $T + \mathscr{K}(X) \in \mathscr{C}(X)$ is the semi-norm $\|T\|_K$ introduced in Chapter 2, or equivalently $\gamma(T)$, instead of the more customary $\|T\|_e$. Of course, for spaces over which $\|\cdot\|_K \sim \|\cdot\|_e$, in particular those that have a basis, this distinction is unnecessary. However, our focus on $\|\cdot\|_K$ means that our uniform boundedness result applies to all Banach spaces. This contrasts with subsequent conclusions, which progressively require more structure.

Since we again concern ourselves with $\|\cdot\|_K$, we shall require further refinements of the concepts from, and technical lemmas proven in, §2.2: these refinements are presented in §4.1. In §4.2 we use the results from §4.1 to prove the $\|\cdot\|_K$ version of the uniform boundedness principle, from which follows the required version of the block-to-block lemma (for spaces which have a basis) in §4.3.

4.1 T-stability

4.1.1 Definition. Let X and Y be Banach spaces and $T \in \mathscr{B}(X,Y)$. We call a sequence $(x_n) \subset X$ *T-stable* if (x_n) is bounded and

$$\lim_{N\to\infty} \inf_{n_1>n_2\geq N} \|T(x_{n_1}-x_{n_2})\| = \lim_{N\to\infty} \sup_{n_1,n_2\geq N} \|T(x_{n_1}-x_{n_2})\| .$$

Denote by ∇_T the set of all T-stable sequences in X, and by ∇_T^λ the set of all T-stable sequences that are members of Δ_X^λ (see Definition 2.1.1), that is, $\nabla_T^\lambda = \nabla_T \cap \Delta_X^\lambda$, for $\lambda > 0$. Also, as an analogue of our usage of $\lceil\cdot\,,\cdot\rceil$, we set

$$\lfloor T\,,(x_n)\rfloor = \lim_{N\to\infty} \inf_{n_1>n_2\geq N} \|T(x_{n_1}-x_{n_2})\| ,$$

so that the T-stability condition for (x_n) can be written as

$$\lfloor T\,,(x_n)\rfloor = \lceil T\,,(x_n)\rceil .$$

4.1.2 Remark. As previously noted in Remark 2.1.2, if $T \in \mathscr{B}(X,Y)$, $(x_n) \subset X$, and $(x_{n_k}) \subset (x_n)$, then

$$\lceil T\,,(x_{n_k})\rceil \leq \lceil T\,,(x_n)\rceil .$$

Similarly, we have

$$\lfloor T\,,(x_{n_k})\rfloor \geq \lfloor T\,,(x_n)\rfloor ,$$

and it is also immediate that, in general,

$$\lfloor T\,,(x_n)\rfloor \leq \lceil T\,,(x_n)\rceil .$$

Conceptually, a sequence is T-stable if the images of its elements under T are eventually almost equally-separated from all subsequent images: in the limit those images form an infinite dimensional 'simplex' whose vertices approximate those of a regular 'hyper-pyramid'. Passing to a subsequence removes some of those vertices, but an infinite 'hyper-pyramid' remains, so the following immediate consequences of the inequalities established in Remark 4.1.2 are not surprising.

4.1.3 Proposition. *Let X and Y be Banach spaces and $T \in \mathscr{B}(X,Y)$. Then*

(i) any T-stable sequence is T-substable, that is, $\nabla_T \subset \Delta_T$, and

(ii) any subsequence of a T-stable sequence is itself T-stable.

Proof. Let $(x_n) \subset X$ be a T-stable sequence. Suppose (x_{n_k}) is a subsequence of

(x_n). Then

$$\lceil T\,,(x_n)\rceil \;=\; \lfloor T\,,(x_n)\rfloor \;\le\; \lfloor T\,,(x_{n_k})\rfloor \;\le\; \lceil T\,,(x_{n_k})\rceil \;\le\; \lceil T\,,(x_n)\rceil\;.$$

Thus all the values appearing in the above are equal, in particular

$$\lceil T\,,(x_{n_k})\rceil \;=\; \lceil T\,,(x_n)\rceil \quad \text{and} \quad \lfloor T\,,(x_{n_k})\rfloor \;=\; \lceil T\,,(x_{n_k})\rceil\;.$$

Hence (x_n) is T-substable by Definition 2.2.1, and (x_{n_k}) is T-stable by Definition 4.1.1. ■

In Remark 2.3.5, it was noted that in the proof of Theorem 2.3.4 the sequence $(x_n) \subset c_0$, which was constructed to be T-substable, was additionally T-stable. That it was possible to choose it as such was no coincidence:

4.1.4 Lemma. *Let X and Y be Banach spaces and $T \in \mathscr{B}(X,Y)$. Then every T-substable sequence has a T-stable subsequence.*

We shall prove Lemma 4.1.4 by relying on the following initial step.

4.1.5 Lemma. *Let X and Y be Banach spaces, $T \in \mathscr{B}(X,Y)$, and suppose $\varepsilon > 0$. If $(x_n) \subset X$ is T-substable, then there is an $m \in \mathbb{N}$ such that, for infinitely many $l \in \mathbb{N}$,*

$$\|T(x_m - x_l)\| \;>\; \lceil T\,,(x_n)\rceil - \varepsilon\,.$$

***Proof*.** We will prove the contrapositive statement. Assume that there is no such m. Set $n_1 = 1$. We proceed by induction. Assume that, for some $k \in \mathbb{N}$, we have chosen $n_1 < \cdots < n_k \in \mathbb{N}$ such that, whenever $1 \le j < k$, $l \ge n_{j+1}$, with $j, l \in \mathbb{N}$, we have

$$\|T(x_{n_j} - x_l)\| \le \lceil T\,,(x_n)\rceil - \varepsilon,$$

and note that this is vacuously true for $k = 1$. Then because there is no $m \in \mathbb{N}$ such that, for infinitely many $l \in \mathbb{N}$,

$$\|T(x_m - x_l)\| > \lceil T\,,(x_n)\rceil - \varepsilon,$$

we can find $n_{k+1} \in \mathbb{N}$ for which $n_{k+1} > n_k$ and such that, for all $l \ge n_{k+1}$, $l \in \mathbb{N}$, we have

$$\|T(x_{n_{k+1}} - x_l)\| \le \lceil T\,,(x_n)\rceil - \varepsilon\,.$$

Hence by induction there is $(x_{n_k}) \subset (x_n)$ such that, for all $k, l \in \mathbb{N}$,

$$\|T(x_{n_k} - x_{n_l})\| \;\leq\; \lceil T\,,(x_n)\rceil - \varepsilon\,.$$

In particular this means that

$$\lceil T\,,(x_{n_k})\rceil \;<\; \lceil T\,,(x_n)\rceil\,,$$

which shows that (x_n) is not T-substable. ■

Proof of Lemma 4.1.4. Let $(x_n) \subset X$ be a T-substable sequence. We shall inductively construct a T-stable subsequence $(x_{n_k}) \subset (x_n)$ by repeatedly applying Lemma 4.1.5. First, use it to choose n_1 and an infinite subsequence $(x_{n_l})_{l=2}^{\infty}$ such that, for all $l > 1$,

$$\|T(x_{n_1} - x_{n_l})\| \;>\; \lceil T\,,(x_n)\rceil - \frac{1}{2}\,.$$

Now assume that, for some $k \in \mathbb{N}$, we have chosen $n_1, \ldots, n_k$ and have a subsequence $(x_{n_l})_{l=k+1}^{\infty} \subset (x_n)$, such that if $j \leq k$ and $l > j$, with $j, l \in \mathbb{N}$, it is the case that

$$\|T(x_{n_j} - x_{n_l})\| \;>\; \lceil T\,,(x_n)\rceil - \frac{1}{2^j}\,.$$

Since (x_n) is T-substable, so too is $(x_{n_l})_{l=k+1}^{\infty} \subset (x_n)$. Therefore, by Lemma 4.1.5, we may pass to a further subsequence and then fix a new n_{k+1}, such that if $l > k+1$ we have

$$\|T(x_{n_{k+1}} - x_{n_l})\| \;>\; \lceil T\,,(x_n)\rceil - \frac{1}{2^{k+1}}\,.$$

Hence by induction there is $(x_{n_k}) \subset (x_n)$ such that, if $N, k, l \in \mathbb{N}$ with $k, l \geq N$ and $k \neq l$,

$$\|T(x_{n_k} - x_{n_l})\| \;>\; \lceil T\,,(x_n)\rceil - \frac{1}{2^N}\,.$$

Thus

$$\lfloor T\,,(x_{n_k})\rfloor \;=\; \lim_{N\to\infty}\inf_{k>l\geq N}\|T(x_{n_k} - x_{n_l})\| \;\geq\; \lceil T\,,(x_n)\rceil - \lim_{N\to\infty}\frac{1}{2^N} \;=\; \lceil T\,,(x_n)\rceil\,,$$

which together with

$$\lceil T\,,(x_n)\rceil \;=\; \lceil T\,,(x_{n_k})\rceil \;\geq\; \lfloor T\,,(x_{n_k})\rfloor$$

gives

$$\lfloor T\,,(x_{n_k})\rfloor \;=\; \lceil T\,,(x_{n_k})\rceil\,.$$ ■

Lemma 4.1.4 spawns further technical properties, which are analogous to Lemma 2.2.2, Corollary 2.2.3, and Corollary 2.2.4:

4.1.6 Corollary. *Let X and Y be Banach spaces and $T \in \mathscr{B}(X,Y)$. Then every bounded sequence $(x_n) \subset X$ has a T-stable subsequence.*

Proof. By Lemma 2.2.2, (x_n) has a T-substable subsequence, which in turn has a T-stable subsequence by Lemma 4.1.4. ∎

4.1.7 Corollary. *Let X and Y be Banach spaces. Then, for all $T \in \mathscr{B}(X,Y)$ and all $\lambda > 0$,*

$$\|T\|_K \ = \ \frac{1}{\lambda} \sup_{(x_n)\in\nabla_T^\lambda} \lceil T\,,(x_n)\rceil \ = \ \frac{1}{\lambda} \sup_{(x_n)\in\nabla_T^\lambda} \lfloor T\,,(x_n)\rfloor \ .$$

Proof. By Corollary 2.2.3 we have

$$\|T\|_K \ = \ \frac{1}{\lambda} \sup_{(x_n)\in\Delta_T^\lambda} \lceil T\,,(x_n)\rceil \ .$$

If $(x_n) \in \Delta_T^\lambda$, then passing to a subsequence does not affect the value of $\lceil T\,,(x_n)\rceil$, and, by Lemma 4.1.4, each $(x_n) \in \Delta_T^\lambda$ has a subsequence $(x_{n_k}) \in \nabla_T$. Thus

$$\sup_{(x_n)\in\Delta_T^\lambda} \lceil T\,,(x_n)\rceil \ = \ \sup_{(x_n)\in\Delta_T^\lambda\cap\nabla_T} \lceil T\,,(x_n)\rceil \ = \ \sup_{(x_n)\in\nabla_T^\lambda} \lceil T\,,(x_n)\rceil \ ,$$

where we know that $\Delta_T^\lambda \cap \nabla_T \ = \ \nabla_T^\lambda$ as a consequence of Proposition 4.1.3. The required result follows. ∎

4.1.8 Corollary. *Let X, Y and Z be Banach spaces. Then for all $T \in \mathscr{B}(X,Y)$ and $S \in \mathscr{B}(X,Z)$,*

$$\|T\|_K \ = \ \sup_{(x_n)\in\nabla_T\cap\nabla_S^1} \lceil T\,,(x_n)\rceil \ .$$

Proof. From Corollary 4.1.7 we know that

$$\|T\|_K \ = \ \sup_{(x_n)\in\nabla_T^1} \lceil T\,,(x_n)\rceil \ .$$

Given $(x_n) \in \nabla_T$, by Corollary 4.1.6 (x_n) has an S-stable subsequence. Since passing to this subsequence does not affect the value of $\lceil T\,,(x_n)\rceil$, we have

$$\|T\|_K \ = \ \sup_{(x_n)\in\nabla_T^1\cap\nabla_S} \lceil T\,,(x_n)\rceil \ .$$

From Definition 4.1.1, $\nabla_T^1 \cap \nabla_S = \nabla_T \cap \Delta_X^1 \cap \nabla_S = \nabla_T \cap \nabla_S^1$, as required. ■

In Chapter 2, we did not comment on the results of Corollaries 2.2.3 and 2.2.4. However, at this point it is worthwhile to consider the conceptual meaning of the analogous Corollaries 4.1.7 and 4.1.8. Suppose that $T \in \mathscr{B}(X,Y)$. In essence, Corollary 4.1.7 tells us that the value of $\|T\|_K$ can be calculated by restricting attention to T-stable sequences, and looking at the values of the 'limiting separations' of their images under T. Recall that $\|\cdot\|_K \sim \gamma(\cdot)$, so that this also gives us a way to estimate $\gamma(\cdot)$.

Corollary 4.1.8 tells us more: in calculating the value of $\|T\|_K$, we can further restrict attention to those T-stable sequences (x_n) which are *also* S-stable for $S \in \mathscr{B}(X,Z)$, despite the fact that the choice of S is completely independent of T. At first glance, this is somewhat surprising, until it is noted that the value of $\lceil S\,,(x_n)\rceil$ for the sequences in question need not be anywhere near to $\|S\|_K$. That is, although we can approximate $\|T\|_K$ with a T-stable sequence in Δ_X^1 that is furthermore S-stable, and we can also approximate $\|S\|_K$ with an S-stable sequence in Δ_X^1 that is furthermore T-stable, we cannot necessarily simultaneously well-approximate $\|T\|_K$ and $\|S\|_K$ with the same sequence.

4.1.9 Example. To show that such a simultaneous approximation might not be possible, we will exploit the properties of the ℓ_1 norm. Let X be ℓ_1, and let O and E be the sets of odd and even natural numbers respectively. Recall that we identify O $\langle$resp. $E\rangle$ with a projection onto $[(e_n)_{n\in O}]$ $\langle$resp. $[(e_n)_{n\in E}]\rangle$. The sequence

$$(x_n) \;=\; \left(\frac{e_{2n-1}}{2}\right) \;\in\; \Delta_X^1$$

is O-stable, and $\lceil O\,,(x_n)\rceil = 1$, so $1 \le \|O\|_K \le \|O\|_e \le \|O\| = 1$. Hence $\|O\|_K = 1$. Similarly, the sequence

$$(y_n) \;=\; \left(\frac{e_{2n}}{2}\right) \;\in\; \Delta_X^1$$

is E-stable, and $\lceil E\,,(y_n)\rceil = 1$, so $\|E\|_K = 1$. Note that, for all $n \in \mathbb{N}$, $Ex_n = 0$ and $Oy_n = 0$, thus $\lceil O\,,(y_n)\rceil = \lceil E\,,(x_n)\rceil = 0$, so (x_n) is trivially E-stable and (y_n) is trivially O-stable. Therefore, we have that both (x_n) and (y_n) are elements of $\nabla_O \cap \nabla_E \cap \Delta_X^1$, and $\lceil O\,,(x_n)\rceil$ and $\lceil E\,,(y_n)\rceil$ are good approximations for $\|O\|_K$ and $\|E\|_K$ respectively, while $\lceil O\,,(y_n)\rceil$ and $\lceil E\,,(x_n)\rceil$ are very bad approximations respectively. To approximate both simulaneously, we might try

$$(z_n) \;=\; \left(\frac{1}{4}(e_{2n-1}+e_{2n})\right) \;\in\; \Delta_X^1\,,$$

which is both O-stable and E-stable, and for which $\lceil O\,,(z_n)\rceil = \lceil E\,,(z_n)\rceil = \frac{1}{2}$. However, a better simultaneous approximation is impossible: given $(w_n) \in \Delta_X^1$, suppose (w_n) is both O-stable and E-stable. Then we can find, for all $\varepsilon > 0$, an $N \in \mathbb{N}$ such that, for all $n_1 > n_2 \geq N$, $n_1, n_2 \in \mathbb{N}$,

$$\lceil O\,,(w_n)\rceil - \varepsilon \;\leq\; \|O(w_{n_1} - w_{n_2})\|_{\ell_1}$$

and

$$\lceil E\,,(w_n)\rceil - \varepsilon \;\leq\; \|E(w_{n_1} - w_{n_2})\|_{\ell_1}\,,$$

which gives

$$\begin{aligned} \lceil O\,,(w_n)\rceil + \lceil E\,,(w_n)\rceil \;&\leq\; \|O(w_{n_1} - w_{n_2})\|_{\ell_1} + \|E(w_{n_1} - w_{n_2})\|_{\ell_1} + 2\varepsilon \\ &=\; \|(w_{n_1} - w_{n_2})\|_{\ell_1} + 2\varepsilon \\ &\leq\; 1 + 2\varepsilon\,. \end{aligned}$$

Hence, for all sequences $(w_n) \in \Delta_X^1$ which are both O-stable and E-stable, it must be that $\lceil O\,,(w_n)\rceil + \lceil E\,,(w_n)\rceil \leq 1$.

Despite the above example and preceding discussion, in the specific case when $S = I \in \mathscr{B}(X)$ we have the following specialised version of Corollary 4.1.8.

4.1.10 Corollary. *Let X and Y be Banach spaces, $T \in \mathscr{B}(X, Y)$, and $\varepsilon > 0$. Then there exists $(x_n) \in \nabla_T^1$ such that*

$$\lceil T\,,(x_n)\rceil \;\geq\; \|T\|_K - \varepsilon$$

and

$$1 \;\geq\; \lim_{N\to\infty} \sup_{n_1,n_2\geq N} \|x_{n_1} - x_{n_2}\| \;=\; \lim_{N\to\infty} \inf_{n_1>n_2\geq N} \|x_{n_1} - x_{n_2}\| \;\geq\; 1 - \varepsilon\,.$$

Proof. If $\|T\|_K = 0$, then $\lceil T\,,(x_n)\rceil \geq \|T\|_K - \varepsilon$ for all $(x_n) \in \nabla_T^1$, and by Corollary 4.1.8 we have

$$1 \;=\; \|I\|_K \;=\; \sup_{(x_n)\in\nabla_I\cap\nabla_T^1} \lceil I\,,(x_n)\rceil\,,$$

so that there must be $(x_n) \in \nabla_I \cap \nabla_T^1$ for which

$$\lim_{N\to\infty} \sup_{n_1,n_2\geq N} \|x_{n_1} - x_{n_2}\| \;=\; \lim_{N\to\infty} \inf_{n_1>n_2\geq N} \|x_{n_1} - x_{n_2}\| \;\geq\; 1 - \varepsilon\,.$$

Therefore we may assume $\|T\|_K > 0$. Fix a sequence $(\varepsilon_k) \subset \mathbb{R}_+$ with $\varepsilon_k \to 0$. By Corollary 4.1.8, we know that

$$\|T\|_K \;=\; \sup_{(x_n)\in\nabla_T\cap\nabla_I^1} \lceil T\,,(x_n)\rceil\ .$$

Hence, for all $k \in \mathbb{N}$, there is $(x_n^k) \in \Delta_X^1$ such that $(x_n^k) \in \nabla_T$,

$$\left\lceil T\,,(x_n^k)\right\rceil \;\geq\; \|T\|_K - \varepsilon_k\,,$$

and

$$\lim_{N\to\infty} \inf_{n_1>n_2\geq N} \|x_{n_1}^k - x_{n_2}^k\| \;=\; \lim_{N\to\infty} \sup_{n_1,n_2\geq N} \|x_{n_1}^k - x_{n_2}^k\| \;\leq\; 1\,.$$

Choose $k \in \mathbb{N}$ such that $\varepsilon_k < \min\{\varepsilon, \varepsilon\|T\|_K\}$. Then

$$\left\lceil T\,,(x_n^k)\right\rceil \;\geq\; \|T\|_K - \varepsilon\,.$$

The result will follow if we can also show that

$$\lim_{N\to\infty} \sup_{n_1,n_2\geq N} \|x_{n_1}^k - x_{n_2}^k\| \;\geq\; 1-\varepsilon\,.$$

Suppose, in order to gain a contradiction, that this is not true. Then we must have

$$\lim_{N\to\infty} \sup_{n_1,n_2\geq N} \left\|\frac{x_{n_1}^k}{1-\varepsilon} - \frac{x_{n_2}^k}{1-\varepsilon}\right\| \;<\; 1\,.$$

Hence there exists an $N \in \mathbb{N}$ such that

$$\left(\frac{x_n^k}{1-\varepsilon}\right)_{n=N}^{\infty} \in\; \Delta_X^1\,.$$

Since $(x_n^k) \in \nabla_T$, we also have $\left(\frac{x_n^k}{1-\varepsilon}\right)_{n=N}^{\infty} \in \nabla_T \cap \Delta_X^1 = \nabla_T^1$. So, by Corollary 4.1.7,

$$\|T\|_K \;\geq\; \left\lceil T\,,\left(\frac{x_n^k}{1-\varepsilon}\right)_{n=N}^{\infty}\right\rceil \;=\; \frac{\lceil T\,,(x_n^k)\rceil}{1-\varepsilon} \;\geq\; \frac{\|T\|_K - \varepsilon_k}{1-\varepsilon} \;>\; \frac{\|T\|_K - \varepsilon\|T\|_K}{1-\varepsilon}\,,$$

which gives $\|T\|_K > \|T\|_K$, a contradiction. ■

Thus it *is* possible to simultaneously approximate $\|T\|_K$ and $\|I\|_K$ using a sequence that is both T-stable and I-stable. In particular, in calculating $\|T\|_K$ we can further restrict attention within ∇_T^1 to sequences which, prior to T being applied, are themselves asymptotically equally separated (by a distance as close to 1 as we like; however, see Remark 4.1.13 below). This version of Corollary 4.1.7 can be restated by utilising the following notation.

4.1.11 Definition. Let X be a Banach space. Denote by ∇_X^λ the set of all sequences $(x_n) \in \Delta_X^\lambda$ such that

$$\lim_{N\to\infty} \inf_{n_1>n_2\geq N} \|x_{n_1} - x_{n_2}\| = \lim_{N\to\infty} \sup_{n_1,n_2\geq N} \|x_{n_1} - x_{n_2}\| .$$

Note that, for $(x_n) \in \nabla_X^\lambda$, the common value of the above two limits must be less than or equal to λ.

4.1.12 Corollary. *Let X and Y be Banach spaces, $T \in \mathscr{B}(X,Y)$, and $\lambda > 0$. Then*

$$\|T\|_K = \frac{1}{\lambda} \sup_{(x_n)\in\nabla_X^\lambda\cap\nabla_T} \lceil T\,,(x_n)\rceil \ .$$

***Proof*.** The statement follows as an immediate consequence of Definition 4.1.11 and Corollaries 4.1.7 and 4.1.10. ∎

4.1.13 Remark. Conceptually, Definition 4.1.11 is not the most natural. We would intuitively prefer to insist that the common value of the limits in that definition is *equal* to λ, and drop the requirement that $(x_n) \in \Delta_X^\lambda$. This would allow us to prove that if $(x_n) \in \nabla_X^1$, then $(Tx_n) \in \nabla_X^{\lceil T\,,(x_n)\rceil}$, which would clearly be a useful feature. The problem with the alternative definition is that we would have sequences in ∇_X^1 with elements whose differences can have norm greater than 1, even though the limiting separation must be 1. In contrast, the given definition ensures that all differences of pairs of elements are actually in the unit ball, which is consistent with our previous framework and also technically preferable. Note that, by Corollary 4.1.10, we know that the expression for $\|T\|_K$, as given in Corollary 4.1.12, would still hold if we required that sequences in ∇_X^λ had limiting separations as close to λ as we liked. However, it is unclear whether Corollary 4.1.12 would still be true if we were to redefine ∇_X^λ to consist of those sequences that are both in Δ_X^λ *and* have limiting separations that actually equal λ. In particular, for an arbitrary T, it is not obvious whether such a sequence exists that is also T-stable.

We now come to the main result of this section: an extension of Corollaries 4.1.8 and 4.1.12 to countable collections of operators. We will once more employ some new notation.

4.1.14 Definition. Let X and Y be Banach spaces and $(T_k) \subset \mathscr{B}(X,Y)$. Denote by $\nabla^\lambda_{(T_k)}$ the set of all sequences in ∇^λ_X that are T_k-stable for all $k \in \mathbb{N}$. That is,

$$\nabla^\lambda_{(T_k)} = \nabla^\lambda_X \cap \bigcap_{k=1}^{\infty} \nabla_{T_k} .$$

4.1.15 Theorem. *Let X and Y be Banach spaces, $(T_k) \subset \mathscr{B}(X,Y)$, and $\lambda > 0$. Then, for all $k \in \mathbb{N}$,*

$$\|T_k\|_K \;=\; \frac{1}{\lambda} \sup_{(x_n)\,\in\,\nabla^\lambda_{(T_k)}} \lceil T_k\,, (x_n) \rceil \;.$$

The proof relies on the following analogue of Corollary 4.1.6.

4.1.16 Lemma. *Let X and Y be Banach spaces and $(T_k) \subset \mathscr{B}(X,Y)$. Then every bounded sequence $(x_n) \subset X$ has a subsequence which is T_k-stable for all $k \in \mathbb{N}$.*

Proof. As on previous occasions, we shall construct the required subsequence by induction, followed by a diagonalisation argument. Firstly, by Corollary 4.1.6, there is $(x^1_n) \subset (x_n)$ such that (x^1_n) is T_1-stable. Assume that, for some $j \in \mathbb{N}$, we have already chosen $(x^j_n) \subset (x^{j-1}_n) \subset \cdots \subset (x^1_n) \subset (x_n)$ such that, if $1 \le i \le j$, $i \in \mathbb{N}$, then (x^i_n) is T_i-stable. Corollary 4.1.6 gives a further subsequence $(x^{j+1}_n) \subset (x^j_n)$ such that (x^{j+1}_n) is T_{j+1}-stable. Hence, by induction, there are

$$(x_n) \supset (x^1_n) \supset (x^2_n) \supset \dots$$

such that, for all $j \in \mathbb{N}$, (x^j_n) is T_j-stable. We also have that $(x^n_n)^\infty_{n=j} \subset (x^j_n)$, hence

$$\lfloor T_j\,, (x^n_n) \rfloor \;=\; \lfloor T_j\,, (x^j_n) \rfloor \;=\; \lceil T_j\,, (x^j_n) \rceil \;=\; \lceil T_j\,, (x^n_n) \rceil \;.$$

Thus (x^n_n) is T_j-stable for all $j \in \mathbb{N}$, and clearly $(x^n_n) \subset (x_n)$. ∎

Proof of Theorem 4.1.15. Given $k \in \mathbb{N}$, by Corollary 4.1.12 we know that

$$\|T_k\|_K \;=\; \frac{1}{\lambda} \sup_{(x_n)\,\in\,\nabla^\lambda_X \cap \nabla_{T_k}} \lceil T_k\,, (x_n) \rceil \;.$$

Suppose $(x_n) \in \nabla^\lambda_X \cap \nabla_{T_k}$. Passing to a subsequence will not affect the value of $\lceil T_k\,, (x_n) \rceil$, and Lemma 4.1.16 gives $(y_n) \subset (x_n)$ such that $(y_n) \in \bigcap_{j=1}^\infty \nabla_{T_j}$. Hence the values of $\lceil T_k\,, (x_n) \rceil$ attained when (x_n) ranges over $\nabla^\lambda_X \cap \nabla_{T_k}$ are the same as when (x_n) is restricted to $\nabla^\lambda_{(T_k)}$. The required result follows. ∎

4.2 The $\|\cdot\|_K$-UBP

There is an important perspective to be gained from the technical theory that was accumulated in the previous section: a non-compact operator can be viewed as a process that takes asymptotically equally-separated sequences and maps them to other asymptotically equally-separated sequences. In passing from a sequence in the 'domain' to its image in the 'range', the maximum factor by which the equal-separation changes is measured by $\|\cdot\|_K$. Additionally, if given a sequence of non-compact operators, it is possible to restrict attention to some sort of common domain of relevant sequences. In this context, the inclination to prove a uniform boundedness principle is very natural. To do so, it is convenient to use a version of the classical gliding hump argument; perhaps the result could also be established by utilising the Baire category theorem in some form. The presentation of the gliding hump method we give below owes much to the proof of the standard uniform boundedness principle to be found in [Pie2, §2.4.3].

4.2.1 Theorem (The uniform $\|\cdot\|_K$-boundedness principle)**.** *Let X and Y be Banach spaces, and $(T_k) \subset \mathscr{B}(X,Y)$ be a sequence of operators such that*

$$\lceil T_k\,,(x_n)\rceil \;\leq\; c_{(x_n)} \tag{4.2.1a}$$

for all $k \in \mathbb{N}$ and all $(x_n) \in \nabla^1_{(T_k)}$, where the constants $c_{(x_n)} > 0$ depend only on (x_n). Then there is a uniform constant $C > 0$ such that

$$\|T_k\|_K \;\leq\; C$$

for all $k \in \mathbb{N}$.

Proof. Suppose, in order to gain a contradiction, that $(\|T_k\|_K)$ is unbounded. By Theorem 4.1.15 we know that, for all $j \in \mathbb{N}$ and each $m \in \mathbb{N}$, there is $(x_n) \in \nabla^{1/3^j}_{(T_k)}$ such that $\lceil T_m\,,(x_n)\rceil$ is as close to $\frac{1}{3^j}\|T_m\|_K$ as we like. Hence, taking $j \in \mathbb{N}$ in their natural order, we may choose, in alternating fashion, $k_j \in \mathbb{N}$ and then $(x^j_n) \in \nabla^{1/3^j}_{(T_k)}$ such that

$$\frac{1}{4\cdot 3^j}\|T_{k_j}\|_K \;\geq\; j + \sum_{i<j,\, i\in\mathbb{N}} c_{(x^i_n)} \tag{4.2.1b}$$

and

$$\lfloor T_{k_j}\,,(x^j_n)\rfloor \;=\; \lceil T_{k_j}\,,(x^j_n)\rceil \;\geq\; \frac{3}{4\cdot 3^j}\|T_{k_j}\|_K\,. \tag{4.2.1c}$$

Without loss of generality, for each $j \in \mathbb{N}$ we can shift (x^j_n) by $-x^j_1$, so that x^j_1

becomes 0. Since $(x_n^j) \in \nabla_{(T_k)}^{1/3^j} \subset \nabla_X^{1/3^j} \subset \Delta_X^{1/3^j}$, this means we can assume that, for all $n \in \mathbb{N}$, $\|x_n^j\| = \|x_n^j - x_1^j\| \leq \frac{1}{3^j}$. Thus the following definition is valid, because the infinite sum converges: let

$$y_n = \sum_{j=1}^{\infty} x_n^j$$

for all $n \in \mathbb{N}$. Note that $\|y_n\| \leq \sum_{j=1}^{\infty} \frac{1}{3^j} = \frac{1}{2}$, thus $(y_n) \in \Delta_X^1$. Applying Corollary 4.1.6 to $I \in \mathscr{B}(X)$ and (y_n) gives a subsequence of (y_n) in ∇_X^1, and then Lemma 4.1.16 guarantees a further subsequence $(y_{n_l}) \in \nabla_{(T_k)}^1$.

Now, for all $j \in \mathbb{N}$ we have

$$\begin{aligned}
\lceil T_{k_j}, (y_{n_l}) \rceil &= \lfloor T_{k_j}, (y_{n_l}) \rfloor \geq \lfloor T_{k_j}, (y_n) \rfloor = \left\lfloor T_{k_j}, \left(\sum_{j=1}^{\infty} x_n^j \right) \right\rfloor \\
&\geq -\sum_{i<j,\, i\in\mathbb{N}} \lceil T_{k_j}, (x_n^i) \rceil + \lfloor T_{k_j}, (x_n^j) \rfloor - \sum_{i>j,\, i\in\mathbb{N}} \lceil T_{k_j}, (x_n^i) \rceil \\
&\geq -\sum_{i<j,\, i\in\mathbb{N}} c_{(x_n^i)} + \frac{3}{4\cdot 3^j} \|T_{k_j}\|_K - \sum_{i>j,\, i\in\mathbb{N}} \frac{1}{3^i} \|T_{k_j}\|_K \\
&\qquad\qquad \text{[by 4.2.1a, 4.2.1c, and Thm. 4.1.15]} \\
&= \frac{1}{4\cdot 3^j} \|T_{k_j}\|_K - \sum_{i<j,\, i\in\mathbb{N}} c_{(x_n^i)} \\
&\geq j, \qquad\qquad \text{[by 4.2.1b]}
\end{aligned}$$

which contradicts 4.2.1a. ■

We have stated the uniform $\|\cdot\|_K$-boundedness principle akin to the usual statement of the standard uniform boundedness principle. However, it is the following extension of the contrapositive statement that we shall actually use.

4.2.2 Corollary. *Let X be a Banach space, and suppose $\Lambda \subset \mathscr{B}(X)$ is a collection of operators with $\sup_{T\in\Lambda} \|T\|_K = \infty$. Then there is $(T_k) \subset \Lambda$ and $(x_n) \in \nabla_{(T_k)}^1$ such that $(\lceil T_k, (x_n) \rceil)$ is unbounded.*

Proof. Choose $(T_k) \subset \Lambda$ such that $\|T_k\|_K \uparrow \infty$. There must be $(x_n) \in \nabla_{(T_k)}^1$ for which $(\lceil T_k, (x_n) \rceil)$ is unbounded, as else the result of Theorem 4.2.1 would give a contradictory bound for $\|T_k\|_K$. ■

Recall that, by Example 4.1.9, we can not always find a commonly-stable sequence $(x_n) \in \nabla_X^1 \subset \Delta_X^1$ for which $\lceil \cdot, (x_n) \rceil$ simultaneously well-approximates the $\|\cdot\|_K$ semi-norm of multiple operators. The uniform $\|\cdot\|_K$-boundedness principle tells us that, despite this, we can find a commonly-stable (x_n) for which $\lceil\cdot,(x_n)\rceil$ simultaneously approximates the $\|\cdot\|_K$ semi-norm of a $\|\cdot\|_K$-unbounded sequence of operators, *well enough* that the unboundedness is preserved. This $\lceil\cdot,(x_n)\rceil$-unboundedness is the technical leverage we shall exploit to produce the version of the block-to-block lemma (2.3.3) that was called for in §3.4.

4.3 The block-to-blocks lemma

Our final result for this chapter generalises the block-to-block lemma (2.3.3) to $\|\cdot\|_e$-unbounded collections of operators, in a similar fashion to the way Lemma 4.1.16 and Theorem 4.1.15 generalised Corollaries 4.1.6 and 4.1.12 respectively. Because we are now concerned with mapping block bases to block bases, we restrict our attention to Banach spaces X which have a basis. Since our result holds for all such spaces, we see that the technical framework for the envisioned Eidelheit-Yood style argument can be achieved in general. That is, on spaces which have a basis, proving the uniqueness of the Calkin algebra norm via the strategy outlined in §3.4 is reduced to establishing the existence of certain 'shift' operators.

In order to concisely state the result, we need the notion of δ-small perturbations (see Definition 1.3.6), as well as the following concept.

4.3.1 Definition. Let X be a Banach space with a basis. We say that a set $\{(x_n^\alpha) : \alpha \in A\}$ of block bases in X is *aligned* if there exist $(p_n)_{n=0}^\infty \subset \mathbb{N}_0$ with $0 = p_0 < p_1 < p_2 < \dots$, such that

$$p_{n-1} \prec x_n^\alpha \prec p_n + 1$$

for all $n \in \mathbb{N}$ and $\alpha \in A$.

4.3.2 Lemma (The block-to-blocks lemma)**.** *Let X be a Banach space with a basis. Suppose $\Lambda \subset \mathscr{B}(X)$ is a collection of operators with* $\sup_{T\in\Lambda} \|T\|_e = \infty$*, and* $0 < \delta < 1$*. Then there are $(T_k) \subset \Lambda$, $(C_k) \subset \mathbb{R}_+$, a normalised block basis $(y_n) \subset X$, and, for all $k \in \mathbb{N}$, normalised block bases $(z_n^k)_{n=k}^\infty \subset X$, such that:*

(i) for each $k \in \mathbb{N}$, $(T_k y_n)_{n=k}^\infty$ is a δ-small perturbation of $(C_k z_n^k)_{n=k}^\infty$,

(ii) $C_k \uparrow \infty$, and

(iii) for all $k \in \mathbb{N}$, $\{(y_n)_{n=k}^\infty , (z_n^1)_{n=k}^\infty , (z_n^2)_{n=k}^\infty , \dots , (z_n^k)_{n=k}^\infty\}$ are aligned.

Proof. Since X has a basis, $\|\cdot\|_e \sim \gamma(\cdot) \sim \|\cdot\|_K$ by Corollary 3.2.4. Therefore $\sup_{\lambda\in\Lambda}\|T_\lambda\|_K = \infty$. So, by Corollary 4.2.2, there are $(T_k) \subset \Lambda$ and $(x_n) \in \nabla^1_{(T_k)}$ such that $(\lceil T_k\,, (x_n)\rceil)$ is unbounded. Since $(x_n) \in \nabla^1_X$, we have

$$C'_0 \;=\; \lim_{N\to\infty}\inf_{n_1>n_2\geq N}\|x_{n_1}-x_{n_2}\| \;=\; \lim_{N\to\infty}\sup_{n_1,n_2\geq N}\|x_{n_1}-x_{n_2}\| \;\leq\; 1\,.$$

Pass to a subsequence of (T_k) such that $\lceil T_k\,, (x_n)\rceil \geq 1$ and $\lceil T_k\,, (x_n)\rceil \uparrow_k \infty$, and set $C'_k = \lceil T_k\,, (x_n)\rceil$ and $C_k = {C'_k}/{C'_0} \geq 1$ for all $k \in \mathbb{N}$. Let K be the basis constant of the given basis for X.

Claim. *Suppose that $l \in \mathbb{N}$, $M \in \mathbb{N}_0$, and $\varepsilon > 0$. Then there is a normalised block $y \in X$ with $M \prec y$, and for each $k \leq l$, $k \in \mathbb{N}$, there is a normalised block $z^k \in X$ with $M \prec z^k$, such that, for all $k \leq l$, $k \in \mathbb{N}$,*

$$2K\frac{\|C_k z^k - T_k y\|}{C_k} \;\leq\; \varepsilon\,.$$

We shall prove the claim later. First, we show that it implies the result of the lemma. Set $p_0 = 0$. Using the claim, choose normalised blocks y_1 and z^1_1, with $p_0 \prec y_1$ and $p_0 \prec z^1_1$, such that

$$2K\frac{\|C_1 z^1_1 - T_1 y_1\|}{C_1} \;\leq\; \frac{\delta}{2}\,.$$

We proceed by induction. Assume that, for some $l \in \mathbb{N}$, we have chosen $(p_n)_{n=0}^{l-1} \subset \mathbb{N}_0$, normalised blocks $y_1 \prec \cdots \prec y_l$, and normalised blocks

$$\begin{matrix} z^1_1\,, & z^1_2\,, & \ldots\,, & z^1_l\,, \\ & z^2_2\,, & \ldots\,, & z^2_l\,, \\ & & \cdots & , \\ & & & z^l_l\,, \end{matrix}$$

such that, for all $k \in \mathbb{N}$, $k \leq l$, and $n \in \mathbb{N}$, $k \leq n \leq l$, we have

$$2K\frac{\|C_k z^k_n - T_k y_n\|}{C_k} \;\leq\; \frac{\delta}{2^n}\,,$$

and that, for all $k \in \mathbb{N}$, $k \leq l$, the finite sequences $(y_n)_{n=k}^l\,, (z^1_n)_{n=k}^l\,, \ldots\,, (z^k_n)_{n=k}^l$ are fragments of aligned sequences with corresponding alignment numbers $(p_n)_{n=k-1}^{l-1} \subset \mathbb{N}_0$. Choose $p_l \in \mathbb{N}$ such that $y_l \prec p_l + 1$ and $z^k_l \prec p_l + 1$ for all $k \in \mathbb{N}$, $k \leq l$. Then, setting $\varepsilon = {\delta}/{2^{l+1}}$, the claim allows us to choose blocks y_{l+1} and

$(z^n_{l+1})^{l+1}_{n=1}$, such that all of these blocks are supported after p_l (thereby maintaining the alignment property), and such that, for all $k \in \mathbb{N}$, $k \leq l+1$,

$$2K \frac{\|C_k z^k_{l+1} - T_k y_{l+1}\|}{C_k} \leq \frac{\delta}{2^{l+1}} .$$

Therefore, our assumption remains true when l is replaced by $l+1$, and so by induction we can find a normalised block basis $(y_n) \subset X$, and, for all $k \in \mathbb{N}$, normalised block bases $(z^k_n)^\infty_{n=k} \subset X$, such that, for each $k \in \mathbb{N}$, the block bases $(y_n)^\infty_{n=k}\,, (z^1_n)^\infty_{n=k}\,, \ldots\,, (z^k_n)^\infty_{n=k}$ are aligned, and that

$$2K \sum_{n=k}^{\infty} \frac{\|C_k z^k_n - T_k y_n\|}{C_k} \leq \sum_{n=k}^{\infty} \frac{\delta}{2^n} \leq \delta\,,$$

which shows that $(T_k y_n)^\infty_{n=k}$ is a δ-small perturbation of $(c_k z^k_n)^\infty_{n=k}$. Thus the conclusions of the lemma have all been established. All that remains to show is the following:

Proof of claim. Define $T_0 = I$. The established properties of (x_n) imply that (x_n) is T_0-stable and $\lceil T_0\,, (x_n) \rceil = C'_0$. Set

$$L = \max_{k \leq l,\, k \in \mathbb{N}_0} \|T_k\| \geq 1$$

and

$$\varepsilon' = \min\{\frac{C'_0}{6}, \frac{C'_0 \varepsilon}{40KL}\} .$$

Temporarily fix a $k \leq l$, $k \in \mathbb{N}_0$, and let (x'_n) be any subsequence of (x_n). Since (x_n) is T_k-stable, so too is (x'_n), by Proposition 4.1.3. Thus, $\lfloor T_k\,, (x'_n) \rfloor = \lceil T_k\,, (x'_n) \rceil$, and hence there is $N \in \mathbb{N}$ such that, for all $n_1 > n_2 \geq N$, $n_1, n_2 \in \mathbb{N}$, we have

$$C'_k - \varepsilon' < \|T_k x'_{n_1} - T_k x'_{n_2}\| < C'_k + \varepsilon' .$$

Further, since P_M is compact and (x'_n) is bounded (by, for instance, $\|x'_1\| + 1$), there is a subsequence $(x''_n) \subset (x'_n)$ such that $(P_M T_k x''_n)$ converges.[1] Thus we can find $N' \geq N$, $N' \in \mathbb{N}$, such that $n_1 > n_2 \geq N'$, $n_1, n_2 \in \mathbb{N}$, implies both

$$C'_k - \varepsilon' < \|T_k(x''_{n_1} - x''_{n_2})\| < C'_k + \varepsilon' \qquad (4.3.2a)$$

[1] Note that here we use the same trick as we did in the proof of Lemmas 2.3.1–2.3.3; c.f. discussion following Lemma 3.3.3.

and

$$\|P_M T_k(x''_{n_1} - x''_{n_2})\| < \varepsilon' . \tag{4.3.2b}$$

Restricting the sequence (x''_n) to $n \geq N$ establishes that, for all $k \leq l$, $k \in \mathbb{N}_0$, and for any subsequence $(x'_n) \subset (x_n)$, there is a further subsequence $(x''_n) \subset (x'_n)$ such that both 4.3.2a and 4.3.2b hold for all $n_1, n_2 \in \mathbb{N}$. By repeatedly applying this result, for each $0 \leq k \leq l$, $k \in \mathbb{N}_0$, we can choose subsequences (x^k_n) of (x_n) such that

$$(x^0_n) \supset \ldots \supset (x^l_n),$$

and such that, for all $n_1, n_2 \in \mathbb{N}$ and $0 \leq k \leq l$, $k \in \mathbb{N}_0$, both 4.3.2a and 4.3.2b hold when x''_{n_1} and x''_{n_2} are replaced by $x^k_{n_1}$ and $x^k_{n_2}$, respectively. But then $(x^l_n) \subset (x^k_n)$ for all $0 \leq k \leq l$, $k \in \mathbb{N}_0$, so we have that both

$$C'_k - \varepsilon' < \|T_k(x^l_{n_1} - x^l_{n_2})\| < C'_k + \varepsilon'$$

and

$$\|P_M T_k(x^l_{n_1} - x^l_{n_2})\| < \varepsilon'$$

hold for all $n_1, n_2 \in \mathbb{N}$ and $0 \leq k \leq l$, $k \in \mathbb{N}_0$. Choose $M' > M$, $M' \in \mathbb{N}$, large enough that

$$\|(I - P_{M'})T_k(x^l_1 - x^l_2)\| < \varepsilon'$$

for all $0 \leq k \leq l$, $k \in \mathbb{N}_0$. This is possible because $\|(I - P_n)x\| \to 0$ for all $x \in X$, and we only have to control the tails of $l + 1$ different vectors (i.e., finitely many). Set

$$y' = x^l_1 - x^l_2 .$$

For each $0 \leq k \leq l$, $k \in \mathbb{N}_0$, note that

$$\begin{aligned} C'_k - 3\varepsilon' &\leq \|T_k y'\| - \|P_M T_k y'\| - \|(I - P_{M'})T_k y'\| \\ &\leq \|(P_{M'} - P_M)T_k y'\| \end{aligned}$$

and

$$\begin{aligned} C'_k + 3\varepsilon' &\geq \|T_k y'\| + \|P_M T_k y'\| + \|(I - P_{M'})T_k y'\| \\ &\geq \|(P_{M'} - P_M)T_k y'\| . \end{aligned}$$

Setting, for each $0 \leq k \leq l$, $k \in \mathbb{N}_0$,

$$w_k = (P_{M'} - P_M)T_k y' - T_k(P_{M'} - P_M)y',$$

we also have

$$\begin{aligned}\|w_k\| &= \|T_k(P_{M'}-P_M)y' - (P_{M'}-P_M)T_k y'\| \\ &= \|T_k(P_{M'}-P_M)T_0 y' - T_k T_0 y' + T_k y' - (P_{M'}-P_M)T_k y'\| \\ &= \|T_k(P_{M'}-I-P_M)T_0 y' + (I-P_{M'}+P_M)T_k y'\| \\ &\le \|T_k\|\|(I-P_{M'})T_0 y'\| + \|T_k\|\|P_M T_0 y'\| + \|(I-P_{M'})T_k y'\| + \|P_M T_k y'\| \\ &< 2L\varepsilon' + 2\varepsilon' \\ &\le 4L\varepsilon'.\end{aligned}$$

Now define

$$y = \frac{(P_{M'}-P_M)y'}{\|(P_{M'}-P_M)y'\|} = \frac{(P_{M'}-P_M)T_0 y'}{\|(P_{M'}-P_M)T_0 y'\|},$$

and

$$z^k = \frac{(P_{M'}-P_M)T_k y'}{\|(P_{M'}-P_M)T_k y'\|},$$

for all $k \le l$, $k \in \mathbb{N}$. Then y and $z^1, \ldots, z^l$ are normalised, and are all supported after M. Furthermore, for a given $k \le l$, $k \in \mathbb{N}$,

$$\begin{aligned}C_k z^k - T_k y &= \frac{C_k(P_{M'}-P_M)T_k y'}{\|(P_{M'}-P_M)T_k y'\|} - \frac{T_k(P_{M'}-P_M)y'}{\|(P_{M'}-P_M)T_0 y'\|} \\ &= \frac{C_k(P_{M'}-P_M)T_k y'}{\|(P_{M'}-P_M)T_k y'\|} - \frac{(P_{M'}-P_M)T_k y'}{\|(P_{M'}-P_M)T_0 y'\|} + \frac{w_k}{\|(P_{M'}-P_M)T_0 y'\|},\end{aligned}$$

which gives

$$\begin{aligned}\|C_k z^k - T_k y\| \le\ & (C'_k + 3\varepsilon')\max\left\{\left|\frac{C_k}{C'_k - 3\varepsilon'} - \frac{1}{C'_0 + 3\varepsilon'}\right|, \left|\frac{1}{C'_0 - 3\varepsilon'} - \frac{C_k}{C'_k + 3\varepsilon'}\right|\right\} \\ & + \frac{4L\varepsilon'}{C'_0 - 3\varepsilon'}.\end{aligned}$$

Because $C'_k = C_k C'_0$, we also have

$$\frac{C_k}{C'_k - 3\varepsilon'} - \frac{1}{C'_0 + 3\varepsilon'} = \frac{C'_k + 3\varepsilon' C_k - C'_k + 3\varepsilon'}{(C'_k - 3\varepsilon')(C'_0 + 3\varepsilon')} = \frac{3\varepsilon'(C_k+1)}{(C'_k - 3\varepsilon')(C'_0 + 3\varepsilon')}$$

and

$$\frac{1}{C'_0 - 3\varepsilon'} - \frac{C_k}{C'_k + 3\varepsilon'} = \frac{C'_k + 3\varepsilon' - C'_k + 3\varepsilon' C_k}{(C'_k + 3\varepsilon')(C'_0 - 3\varepsilon')} = \frac{3\varepsilon'(C_k+1)}{(C'_k + 3\varepsilon')(C'_0 - 3\varepsilon')}.$$

Since $C'_k \geq 1 \geq C'_0 > 3\varepsilon'$, we know $3\varepsilon'(C'_k - C'_0) > 3\varepsilon'(C'_0 - C'_k)$, and hence that

$$(C'_k - 3\varepsilon')(C'_0 + 3\varepsilon') \; > \; (C'_k + 3\varepsilon')(C'_0 - 3\varepsilon') \; > \; 0\,.$$

So the second of the two options in the above maximum is the larger. Therefore,

$$\begin{aligned}
2K\frac{\|C_k z^k - T_k y\|}{C_k} &\leq \frac{2K}{C_k}\left((C'_k + 3\varepsilon')\cdot\frac{3\varepsilon'(C_k+1)}{(C'_k+3\varepsilon')(C'_0-3\varepsilon')} + \frac{4L\varepsilon'}{C'_0 - 3\varepsilon'}\right)\\
&= \frac{K\varepsilon'\big(6(C_k+1)+8L\big)}{C_k(C'_0-3\varepsilon')}\\
&\leq \frac{K\varepsilon'(12C_k+8L)}{C_k(C'_0-\frac{C'_0}{2})}\\
&= \frac{24KC_k\varepsilon'}{C_kC'_0} + \frac{16KL\varepsilon'}{C_kC'_0}\\
&\leq \frac{24KL\varepsilon'}{C'_0} + \frac{16KL\varepsilon'}{C'_0}\\
&\leq \frac{40KL\varepsilon'}{C'_0}\\
&\leq \varepsilon\,.
\end{aligned}$$

■

Note that there are two aspects of the block-to-blocks lemma which do not precisely follow the construction suggested in §3.4. Firstly, we have introduced a criterion requiring that the sequence of images of a given block be aligned. This could have been left out of the conclusions of the lemma, but is included because it will be of value when resolving the uniqueness of norm question for the Calkin algebras of certain spaces in which aligned block bases have special properties.

Secondly, and seemingly more significantly, we have only controlled the images under T_k of the blocks y_n for finitely many k: in particular, z_n^k is only defined for $1 \leq k \leq n$. We still preserve the unboundedness of $(\|T_k\|_e)$ overall, but on a specific block y_n, that information is lost. So our construction has a triangular form, rather than being a full 'matrix' of blocks. However, for each k, the sequence $(z_n)_{n=k}^{\infty}$ corresponding to $(T_k y_n)$ is only 'missing' a finite number of terms, which means it can be viewed as a compact perturbation. Therefore, in the context where we are interested in the behaviour of the images of (T_k) in the Calkin algebra, these missing terms are actually not significant.

4.3.3 Remark. It would be extremely difficult, in general, to extend the construction in Lemma 4.3.2 to a full 'matrix'. If we could, then for a given non-compact operator T on ℓ_2 (for example), we could set $T_k = T^k$ and, in a similar fashion to the above, find blocks y_n such that $(T^k y_n)$ were 'almost' aligned block sequences. In particular, this would mean that the sets $\text{supp}(T^k y_1)$ would be 'essentially' bounded above by a uniform constant $M \in \mathbb{N}$. But this is very near to saying that we can find a non-trivial invariant subspace $[(T^k y_1)_{k=1}^{\infty}]$ for T. As exciting as this result would be, it is not within the scope of this thesis.

Having proven a 'Calkin algebra' version of the uniform boundedness principle, we are now half way to finding new examples of Calkin algebras which have a unique algebra norm, as per the plan outlined in §3.4. In this regard, the next chapter demonstrates how the block-to-blocks lemma can be applied.

Chapter 5

New Examples of the Uniqueness of $\|\cdot\|_e$

Given a Banach space X with a basis, and a sequence $(T_k) \subset \mathscr{B}(X)$ such that $\|T_k\|_e \uparrow \infty$, the block-to-blocks lemma (4.3.2) gives a normalised block basis (x_n) such that the operators T_k are 'non-compactly' unbounded on $[(x_n)]$. Furthermore, the sequences $(T_k x_n)_{n=k}^{\infty}$ can also be regarded as block bases. This generalises the conclusion of the uniform boundedness principle, which was the first step in the proof of the Eidelheit-Yood theorem (1.1.2).

Taking the perspective of §3.4, we thus have the first component required to generalise the Eidelheit-Yood method to Calkin algebras. The objective of this chapter is to identify spaces for which the second feature, the existence of certain 'shift' operators, can be guaranteed. That is, we wish to find operators $A_k, B_k, R \in \mathscr{B}(X)$, such that

$$C_k R \in B_k T_k A_k + \mathscr{K}(X)$$

for all $k \in \mathbb{N}$ (where (C_k) is the sequence of constants provided by the block-to-blocks lemma), and such that we have certain controls on the norms of those operators. The operators A_k should map onto a subspace of $[(x_n)]$. A natural way to ensure that such A_k exist is to insist that a subsequence of (x_n) is complemented. Similarly, each B_k must map a subspace of $[(T_k x_n)]$ onto a common image, and we can most easily achieve this if subsequences of the 'almost' block bases $\{(T_k x_n)_{n=1}^{\infty}\}_{k=1}^{\infty}$ are all equivalent to each other and complemented.

Making these rough requirements precise is the purpose of §5.1, in which we define the *UN property* to identify a class of spaces on which we can guarantee that the required shift operators exist. We also provide some more streamlined descriptions of certain subclasses of spaces with the UN property. §5.1 concludes with a proof that, if X has the UN property and $\|\cdot\|_e$ is maximal on $\mathscr{C}(X)$, then $\mathscr{C}(X)$ does indeed have a unique algebra norm, as was anticipated by the idea sketched in §3.4.

In §5.2, we show that this result extends to finite direct sums of spaces with the UN property. Then, the remainder of the chapter presents three examples of spaces which have the UN property and whose Calkin algebras have unique algebra norms: §5.3 studies the spaces $c_0(\ell_r^n)$ and $\ell_p(\ell_r^n)$, where $1 \le p < \infty$ and $1 \le r \le \infty$; §5.4 discusses Tsirelson's space; and §5.5 examines James' space.

5.1 The UN Property

5.1.1 Definition. Let X be a Banach space. We say that X has the *UN property* if X has a basis such that, for each countable set $\{(u_n^k) : k \in \mathbb{N}\}$ of normalised aligned block bases, there exists a constant $C_{\mathrm{U}} \ge 1$ and a strictly increasing sequence $(n_m) \subset \mathbb{N}$ for which

(i) for all $k \in \mathbb{N}$, $(u_{n_m}^k)_{m=k}^\infty$ is C_{U}-complemented,

(ii) there are only finitely many different equivalence classes of bases represented in $\{(u_{n_m}^k) : k \in \mathbb{N}\}$, and

(iii) within each such equivalence class, any two representatives $(u_{n_m}^{k_1})$ and $(u_{n_m}^{k_2})$, with $k_1 < k_2$, are such that $(u_{n_m}^{k_1})_{m=k_2}^\infty$ and $(u_{n_m}^{k_2})_{m=k_2}^\infty$ are C_{U}-equivalent.

This definition leads to a major result of this thesis:

5.1.2 Theorem. *Suppose that X is a Banach space with the UN property, and that $\|\cdot\|_e$ is a maximal algebra norm on $\mathscr{C}(X)$. Then $\mathscr{C}(X)$ has a unique algebra norm.*

Because it is not immediately clear which Banach spaces have the UN property, we shall consider the requirements of the property carefully, and establish results to show it is satisfied by certain more readily discernible classes of Banach spaces, before giving the proof of Theorem 5.1.2 at the end of this section.

5.1.3 Remark. The specific components of Definition 5.1.1 are constructed such that, in a Banach space with the UN property, the various block bases given by the block-to-blocks lemma (4.3.2) can be mapped easily to one another. The complementation and equivalence requirements are made in line with the general discussion at the beginning of this chapter. We restrict attention to aligned block bases because (a) the conclusions of the block-to-blocks lemma allow it, and (b) if instead we insisted that wider classes of block bases were homogeneous up to equivalence, then we would be near to excluding all spaces other than c_0 and ℓ_p, $1 \leq p < \infty$, in virtue of Theorem 1.3.3. Regarding conditions (i) and (iii), it might seem that stating their requirements only for the 'tails' $(u^k_{n_m})^\infty_{m=k}$ of the subsequences in question is unnecessarily cumbersome: we could instead have required the only slightly stronger conditions that each $(u^k_{n_m})^\infty_{m=1}$ is C_{U}-complemented, and that any two equivalence class representatives $(u^{k_1}_{n_m})$ and $(u^{k_2}_{n_m})$ are C_{U}-equivalent. However, the criteria as actually given ensure that various spaces, whose block bases have certain subsymmetric properties, satisfy the UN property (see Proposition 5.1.7 below); these spaces would otherwise have been technically excluded. Also, the weaker conditions more closely match the 'triangular' output of the block-to-blocks lemma. Finally, the relevance of condition (ii) is that we can immediately move to a more convenient situation: if there are only finitely many different equivalence classes of block bases in $\{(u^k_{n_m}) : k \in \mathbb{N}\}$, then there must be an infinite subset of block bases which are mutually equivalent. That is, without loss of generality we can pass from finitely many equivalence classes to just one.

Clearly c_0 and ℓ_p, $1 \leq p < \infty$, have the UN property: conditions (i), (ii), and (iii) are automatically satisfied because all normalised block bases are 1-complemented and 1-equivalent to one another, by Proposition 1.3.2. Therefore, Theorem 5.1.2 implies Theorem 3.3.2.

However, equivalence between certain aligned block bases is all that is actually required. Hence, there are other spaces which also have the UN property. For instance, we can relax the characteristic property of block bases in c_0 and ℓ_p, $1 \leq p < \infty$, by assuming only that any two normalised aligned block bases are C-equivalent and C-complemented, for some $C \geq 1$. A surprising result is that the complementation part of this assumption is unnecessary. That is, if a Banach space with a basis is such that any two aligned block bases are uniformly equivalent, then all block bases are uniformly complemented. This result is Proposition 5.1.4 below. It is stated as an exercise in [AlKa], without proof or reference, so it is difficult to attribute it correctly. As we shall see in §5.4, one space which satisfies the conditions of this theorem is Tsirelson's space. Both the assumed and derived properties are important to the study of Tsirelson's space, and so are cited widely for that space in the literature. For example, a proof of the conclusion of Proposition 5.1.4, for the specific case of Tsirelson's space only, is presented as a "major result" of [CaSh, Chp. II]. Extending the method from [CaSh] to the general scenario requires minor alterations, and it is a consolidated version of that proof which we give here.

5.1.4 Proposition (see [AlKa, Prob. 9.1] for the statement, or [CaSh, Prop. II.6] for a similar proof)**.** *Let X be a Banach space with a basis (x_n). Suppose there is a constant $C \geq 1$ such that any two normalised aligned block basic sequences of (x_n) are C-equivalent. Then any block basic sequence of (x_n) is $2C^2$-complemented.*

Proof. Let

$$(y_n) = \left(\sum_{i=p_{n-1}+1}^{p_n} c_i x_i \right)_{n=1}^{\infty}$$

be a block basic sequence of (x_n), for some $(c_i) \subset \mathbb{F}$ and $0 = p_0 < p_1 < p_2 < \dots$, with $(p_n)_{n=0}^{\infty} \subset \mathbb{N}_0$. For each $n \in \mathbb{N}$, set

$$E_n = \{p_{n-1} + 1 \,,\, p_{n-1} + 2 \,,\, \dots \,,\, p_n\} \,.$$

Without loss of generality, we may assume that both (x_n) and (y_n) are normalised, because the span of a basic sequence is unaffected by normalisation. Note that $\{(y_n), (\sigma_n y_n)\}$ are normalised aligned block basic sequences for any sequence of signs $(\sigma_n) \subset \{-1, 1\}$. Thus, by assumption, $(\sigma_n y_n)$ is C-equivalent to (y_n) for any such (σ_n), which shows that (y_n) is a C-unconditional basic sequence. (5.1.4a)

Now, by the Hahn-Banach extension theorem, we may choose $(y_n^*) \subset X^*$ such that $y_n^*(y_n) = 1$ and $\|y_n^*\| = 1$ for all $n \in \mathbb{N}$ (see, e.g., [Meg, Coroll. 1.9.8]). Recall that we treat each E_n as a projection, as well as a subset of $\mathbb{N}$ (see p15). Suppose $x \in X$, and define, for each $n \in \mathbb{N}$,

$$z_n = \begin{cases} \|E_n x\|^{-1} E_n x \,, & \text{if } E_n x \neq 0 \,, \text{ or} \\ y_n \,, & \text{if } E_n x = 0 \,. \end{cases}$$

Then $\{(y_n), (z_n)\}$ are normalised aligned block basic sequences. So, by assumption, (z_n) is C-equivalent to (y_n). (5.1.4b)

We also have that $x = \sum_n E_n x$, so

$$\begin{aligned} 2C^2 \|x\| &= 2C^2 \left\| \sum_n E_n x \right\| \\ &= 2C^2 \left\| \sum_n \|E_n x\| z_n \right\| \\ &\geq 2C \left\| \sum_n \|E_n x\| y_n \right\| \qquad \text{[by 5.1.4b]} \end{aligned}$$

$$\geq \left\| \sum_n y_n^*(E_n x) y_n \right\| . \qquad \text{[by 5.1.4a]}$$

In particular, $\sum_n y_n^*(E_n x) y_n$ converges. So we may define a bounded operator $P : X \to [(y_n)]$ by

$$Px = \sum_n y_n^*(E_n x) y_n \,,$$

for each $x \in X$, with $\|P\| \leq 2C^2$. For all $n, m \in \mathbb{N}$, we have $E_n y_m = \delta_{nm} y_n$, where δ_{nm} is the Kronecker delta. Therefore, for all $m \in \mathbb{N}$,

$$Py_m = \sum_n y_n^*(E_n y_m) y_n = y_m^*(y_m) y_m = y_m \,.$$

Hence P is a projection onto $[(y_n)]$, and we have that (y_n) is $2C^2$-complemented. ∎

Proposition 5.1.4 allows us to give a cleaner description of a subclass of Banach spaces that have the UN property, and which also automatically have a continued bisection of the identity (see Definition 1.1.9).

5.1.5 Theorem. *Let X be a Banach space with a basis (x_n). Suppose there is a constant $C \geq 1$ such that any two normalised aligned block basic sequences of (x_n) are C-equivalent. Then X has both the UN property and a continued bisection of the identity, and $\mathscr{C}(X)$ has a unique algebra norm.*

Proof. Set $C_{\mathrm{U}} = 2C^2$, and assume that $\{(u_n^k) : k \in \mathbb{N}\}$ is a set of normalised aligned block bases of (x_n). By assumption, $(u_n^{k_1})$ is C-equivalent to $(u_n^{k_2})$ for all $k_1, k_2 \in \mathbb{N}$, and the same is true of their tails $(u_n^{k_1})_{n=k_2}^{\infty}$ and $(u_n^{k_2})_{n=k_2}^{\infty}$. By Proposition 5.1.4, any block basis in X is C_{U}-complemented, so in particular $(u_n^k)_{n=k}^{\infty}$ is C_{U}-complemented for all $k \in \mathbb{N}$. Hence, setting $n_m = m$ for all $m \in \mathbb{N}$, we have shown in order that conditions (ii), (iii), and (i) in Definition 5.1.1 are satisfied, therefore X has the UN property.

Now, as noted in the proof of Proposition 5.1.4, the assumption that any two normalised aligned block bases of (x_n) are C-equivalent ensures that (x_n) is unconditional (see 5.1.4a). Hence, for every $E \subset \mathbb{N}$ there is a bounded projection onto the subspace $[\{x_n : n \in E\}]$. For each $m \in \mathbb{N}$, set

$$Y_m = \Big[\big(x_{2^m n}\big)\Big] \qquad \text{and} \qquad Z_m = \Big[\big(x_{2^m n - 2^{m-1}}\big)\Big].$$

By setting $p_0 = 0$ and $p_n = 2^m n$ for all $n \in \mathbb{N}$, we see that

$$\left\{ \left(x_{2^m n}\right) , \left(x_{2^m n - 2^{m-1}}\right) \right\}$$

are aligned block bases, by Definition 4.3.1. By assumption, they are therefore equivalent, and so $Y_m \approx X_m$ for all $m \in \mathbb{N}$. Also, for each $m \in \mathbb{N}$ we have

$$Y_{m+1} \oplus Z_{m+1} = \left[\left(x_{2^{m+1}n}\right)\right] \oplus \left[\left(x_{2^{m+1}n-2^m}\right)\right] = \left[\left(x_{2^m n}\right)\right] = Y_m ,$$

and $X = [(x_{2n})] \oplus [(x_{2n-1})] = Y_1 \oplus Z_1$. Thus $\{(Y_n), (Z_n)\}$ is a continued bisection of the identity on X, by Definition 1.1.9. Hence $\|\cdot\|_e$ is a maximal algebra norm on $\mathscr{C}(X)$, by Corollary 1.1.12. Therefore, we conclude that $\mathscr{C}(X)$ has a unique algebra norm, by Theorem 5.1.2. ■

5.1.6 Remark. We have not shown that a Banach space X which satisfies the assumptions of Proposition 5.1.5 is necessarily such that $X \approx X \oplus X$. As noted in [LoWi, §1], the space constructed in [Fig], which was the first example of a reflexive Banach space not isomorphic to its square, gives an example of a space with a continued bisection of the identity, yet $X \not\approx X \oplus X$.

Our next result shows that, as anticipated by Remark 5.1.3, a degree of subsymmetry allows us to manufacture the necessary block basis alignments without them being assumed.

5.1.7 Proposition. *Let X be a Banach space with a basis, and suppose that there exists a constant $C > 0$ and a finite set of subsymmetric basic sequences $\{(v_n^m)\}_{m=1}^M$, such that every normalised block basis in X has a subsequence which is C-complemented and C-equivalent to some sequence in $\{(v_n^m)\}_{m=1}^M$. Then X has the UN property.*

Proof. Define K to be the maximum value of the subsymmetric constants of the sequences in $\{(v_n^m)\}_{m=1}^M$. Let $\{(u_n^k) : k \in \mathbb{N}\}$ be a set of of normalised aligned block bases in X. We aim to show that there is a strictly increasing sequence $(n_m) \subset \mathbb{N}$ for which properties (i), (ii), and (iii) in Definition 5.1.1 hold. To do so, we employ a diagonalisation argument familiar from previous chapters.

Set $n_m^0 = m$ for all $m \in \mathbb{N}$. By assumption, we can choose a subsequence

$$\left(u^1_{n^1_m}\right) \subset \left(u^1_{n^0_m}\right)_{m=2}^{\infty} = (u_n)_{n=2}^{\infty}$$

that is C-complemented and C-equivalent to some sequence in $\{(v_n^m)\}_{m=1}^M$.

We proceed by induction. Suppose we have chosen, for some $l \in \mathbb{N}$,

$$(n_m^l) \subset (n_m^{l-1}) \subset \ldots \subset (n_m^1)$$

such that, for each $k \leq l$, $k \in \mathbb{N}$, the subsequence

$$\left(u_{n_m^k}^k\right) \subset \left(u_{n_m^{k-1}}^k\right)_{m=2}^{\infty}$$

is C-complemented and C-equivalent to some sequence in $\{(v_n^m)\}_{m=1}^M$. (5.1.7a)

Then, by the assumption of the proposition, we may choose a subsequence

$$\left(u_{n_m^{k+1}}^{k+1}\right) \subset \left(u_{n_m^k}^{k+1}\right)_{m=2}^{\infty}$$

which is C-complemented and C-equivalent to some sequence in $\{(v_n^m)\}_{m=1}^M$.

Therefore, we can inductively select

$$(n_m^1) \supset (n_m^2) \supset \ldots$$

such that 5.1.7a holds for all $k \in \mathbb{N}$. Note that this implies $n_k^k < n_{k+1}^{k+1}$. Now set

$$n_m = n_m^m$$

for all $m \in \mathbb{N}$. Then for each $k \in \mathbb{N}$ we have that $(u_{n_m}^k)_{m=k}^{\infty} \subset (u_{n_m^k}^k)$. Hence we can deduce the following properties.

(i) $(u_{n_m^k}^k)$ is C-complemented and C-equivalent to a K-subsymmetric sequence in $\{(v_n^m)\}_{m=1}^M$, so $(u_{n_m}^k)_{m=k}^{\infty}$ is C^2K-complemented and CK-equivalent to that sequence.

(ii) Therefore, $(u_{n_m}^k)$ is also equivalent to that same sequence, and so there are at most M different equivalence classes of bases represented in $\{(u_{n_m}^k) : k \in \mathbb{N}\}$.

(iii) Finally, note that any two sequences that are CK-equivalent to a common sequence must be C^2K^2-equivalent to each other.

Setting $C_U = C^2K^2$, and recalling that $\{(u_n^k) : k \in \mathbb{N}\}$ was an arbitrary set of normalised aligned block bases, we thus conclude that X has the UN property. ■

Theorem 5.1.5 and Proposition 5.1.7 provide criteria for spaces to have the UN property that are more tractable than our original Definition 5.1.1. Now that we have a better idea of the sorts of spaces to which that definition will apply, we shall prove the main result of this chapter.

Proof of Theorem 5.1.2. Let $|||\cdot|||$ be an arbitrary algebra norm on $\mathscr{C}(X)$. Because $\|\cdot\|_e$ is maximal on $\mathscr{C}(X)$, there is $C > 0$ such that $|||\cdot||| \leq C\|\cdot\|_e$. Identify $|||\cdot|||$ with a seminorm on $\mathscr{B}(X)$, such that $\ker|||\cdot||| = \mathscr{K}(X)$. In order to gain a contradiction, suppose that there exists $(T_k) \subset \mathscr{B}(X)$ such that $\|T_k\|_e \uparrow \infty$ and $|||T_k||| = 1$, for all $k \in \mathbb{N}$. Choose $0 < \delta < 1$. By the block-to-blocks lemma (4.3.2), we may pass to a subsequence of (T_k) for which there exist $(C_k) \subset \mathbb{R}_+$, a normalised block basis $(x_n) \subset X$, and, for all $k \in \mathbb{N}$, normalised block bases $(z_n^k)_{n=k}^\infty \subset X$, such that:

(i) for each $k \in \mathbb{N}$, $(T_k x_n)_{n=k}^\infty$ is a δ-small perturbation of $(C_k z_n^k)_{n=k}^\infty$,

(ii) $C_k \uparrow \infty$, and

(iii) for all $k \in \mathbb{N}$, the block bases $(x_n)_{n=k}^\infty, (z_n^1)_{n=k}^\infty, \dots, (z_n^k)_{n=k}^\infty$ are aligned.

For the sake of notational convenience, we shall fill in the 'missing' blocks $\{z_n^k : k > n\}$ with 'dummy' blocks: for all $n, k \in \mathbb{N}$ such that $k > n$, set $z_n^k = z_n^1$. Then $\{(x_n)\} \cup \{(z_n^k) : k \in \mathbb{N}\}$ is a countable set of normalised aligned block bases. Because X has the UN property, there is a constant $C_{\mathrm{U}} \geq 1$ and a strictly increasing sequence $(n_m) \subset \mathbb{N}$ for which

(iv) (x_{n_m}) is C_{U}-complemented and, for all $k \in \mathbb{N}$, $(z_{n_m}^k)_{m=k}^\infty$ is C_{U}-complemented,[1]

(v) there are only finitely many different equivalence classes of bases represented in $\{(z_{n_m}^k) : k \in \mathbb{N}\}$, and

(vi) within each such equivalence class, any two representatives $(z_{n_m}^{k_1})$ and $(z_{n_m}^{k_2})$, with $k_1 < k_2$, are such that $(z_{n_m}^{k_1})_{m=k_2}^\infty$ and $(z_{n_m}^{k_2})_{m=k_2}^\infty$ are C_{U}-equivalent.

Because of property (v), there must be a single equivalence class of bases which has infinitely many representatives in $\{(z_{n_m}^k) : k \in \mathbb{N}\}$. Thus we may find a strictly increasing sequence $(k_l) \in \mathbb{N}$ such that all the block bases in $\{(z_{n_m}^{k_l}) : l \in \mathbb{N}\}$ are equivalent, and such that their 'tails', in the sense of property (vi), are C_{U}-equivalent.

To avoid the proliferation of subscripts and superscripts, we shall now carry out some relabelling. First we define $N_l = k_l$, for all $l \in \mathbb{N}$, in order to explicitly

[1] To be precise, in (iv) the UN property actually guaranteed that $(z_{n_m}^k)_{m=k+1}^\infty$ were C_{U}-complemented, due to the addition of (x_n) to the 'front' of our set of block bases. However, without loss of generality, we may take (iv) to be true as written: simply discard the first term of (n_m), relabel indices, and note that C_{U} may need to be increased by a factor of $1 + K'$, where K' is the basis constant of the original (x_{n_m}).

preserve the current values of k_l. Redefining terms via the mappings

$$T_{k_l} \mapsto T_l\,, \qquad x_{n_m} \mapsto x_m\,, \qquad z^{k_l}_{n_m} \mapsto z^l_m\,, \qquad \text{and} \qquad C_{k_l} \mapsto C_l\,,$$

shows that, without loss of generality, we may pass to subsequences and relabel indices, to find a sequence of operators $(T_k) \subset \mathscr{B}(X)$, with $\|T_k\|_e \uparrow \infty$ and $|||T_k||| = 1$, for which there are $(C_k) \subset \mathbb{R}_+$, a normalised block basis $(x_n) \subset X$, and, for all $k \in \mathbb{N}$, normalised block bases $(z^k_n) \subset X$, such that:

(a) for each $k \in \mathbb{N}$, $(T_k x_n)^\infty_{n=N_k}$ is a δ-small perturbation of $(C_k z^k_n)^\infty_{n=N_k}$,

(b) (x_n) is C_{U}-complemented and, for all $k \in \mathbb{N}$, $(z^k_n)^\infty_{n=N_k}$ is C_{U}-complemented,

(c) for all $k \in \mathbb{N}$, $(z^k_n)^\infty_{n=N_k}$ is C_{U}-equivalent to $(z^1_n)^\infty_{n=N_k}$, and

(d) $(C_k) \uparrow \infty$.

Properties (a)–(c) allow us to define various projections and 'shift' operators:

(a′) By (a) and Corollary 1.3.8, for each $k \in \mathbb{N}$ there is an operator $U_k \in \mathscr{B}(X)$ such that $(I - U_k) \in \mathscr{K}(X)$ and $U_k T_k x_n = C_k z^k_n$, for all $n \geq N_k$, $n \in \mathbb{N}$.

(b′) By (b), for each $k \in \mathbb{N}$ we can find a projection $Q_k \in \mathscr{B}(X)$ onto $[(z^k_n)^\infty_{n=N_k}]$, such that $\|Q_k\| \leq C_{\mathrm{U}}$. Therefore, we additionally have $\|Q_k\|_e \leq C_{\mathrm{U}}$. Also, (x_n) is C_{U}-complemented, so $(x_n)^\infty_{n=N_k}$ is $(1+K)C_{\mathrm{U}}$-complemented, where K is the basis constant of (x_n). In particular, there is a projection A onto $[(x_n)]$ with $A \leq C_{\mathrm{U}}$, and for each $k \in \mathbb{N}$ we can find a projection $A_k \in \mathscr{B}(X)$ onto $[(x^k_n)^\infty_{n=N_k}]$, such that $A_k = A_k A$. Then $A - A_k$ is a projection onto $[(x_n)^{N_k-1}_{n=1}]$ and hence is compact. Thus $\|A_k\|_e = \|A\|_e \leq C_{\mathrm{U}}$.

(c′) By (c), for each $k \in \mathbb{N}$ there is $S'_k \in \mathscr{B}([(z^k_n)^\infty_{n=N_k}], X)$ such that $\|S'_k\| \leq C_{\mathrm{U}}$ and $S'_k z^k_n = z^1_n$ for all $n \geq N_k$, $n \in \mathbb{N}$. Now define $S_k = C_k^{-1} S'_k$, so that $\|S_k\| \leq C_k^{-1} C_{\mathrm{U}}$ and $S_k : C_k z^k_n \mapsto z^1_n$, for such n.

Finally, for each $k \in \mathbb{N}$ we define $B_k = S_k Q_k U_k \in \mathscr{B}(X)$. Note that

$$\|B_k\|_e \,\leq\, \|S_k\|_e \|Q_k\|_e \|U_k\|_e \,\leq\, \|S_k\| \cdot C_{\mathrm{U}} \cdot \|I\|_e \,\leq\, C_k^{-1} C_{\mathrm{U}}^2\,,$$

and thus

$$\|A_k\|_e \|B_k\|_e \,=\, \|A\|_e \|B_k\|_e \,\leq\, C_k^{-1} C_{\mathrm{U}}^3\,. \tag{5.1.2a}$$

For each $k \in \mathbb{N}$ and every $n \geq N_k$, $n \in \mathbb{N}$, we have

$$B_k T_k A_k x_n \,=\, S_k Q_k U_k T_k x_n \,=\, S_k Q_k (C_k z^k_n) \,=\, S_k (C_k z^k_n) \,=\, z^1_n \,=\, C_1^{-1} U_1 T_1 x_n\,.$$

Because A_k is a projection onto $[(x_n)_{n=N_k}^{\infty}]$, this gives

$$C_1^{-1}U_1T_1A_k \;=\; B_kT_kA_k\,. \tag{5.1.2b}$$

Since $(I-U_1)\in\mathscr{K}(X)$ and $(A-A_k)\in\mathscr{K}(X)$, we also have

$$\begin{aligned} U_1T_1A_k+\mathscr{K}(X) \;&=\; U_1T_1A_k+(I-U_1)T_1A_k+T_1(A-A_k)+\mathscr{K}(X)\\ &=\; T_1A+\mathscr{K}(X)\,. \end{aligned}$$

Thus $|||T_1A|||\;=\;|||U_1T_1A_k|||$. Therefore,

$$\begin{aligned} |||T_1A||| \;&\le\; C_1|||B_k|||\cdot|||T_k|||\cdot|||A_k||| && \text{[by 5.1.2b]}\\ &=\; C_1|||B_k|||\cdot|||A_k||| \\ &\le\; C_1C^2\|A_k\|_e\|B_k\|_e && [\text{since } |||\cdot|||\le C\|\cdot\|_e]\\ &\le\; C_1C^2C_k^{-1}C_{\mathrm{U}}^3\,. && \text{[by 5.1.2a]} \end{aligned}$$

The operator T_1A is clearly not compact, since $(T_1A(C_1^{-1}x_n))_{n=N_1}^{\infty} = (C_1^{-1}T_1x_n)_{n=N_1}^{\infty}$ is a δ-small perturbation of the normalised block basis $(z_n^1)_{n=N_1}^{\infty}$. Thus $|||T_1A|||>0$, and so we have

$$C_k \;\le\; \frac{C_1C^2C_{\mathrm{U}}^3}{|||T_1A|||}\,. \tag{5.1.2c}$$

This bound for (C_k) contradicts property (d) above. Hence, there is no sequence $(T_k)\subset\mathscr{B}(X)$ such that $\|T_k\|_e\uparrow\infty$ and $|||T_k|||=1$.

Because $|||\cdot|||$ was an arbitrary algebra norm on $\mathscr{C}(X)$, we conclude that $\|\cdot\|_e$ is minimal on $\mathscr{C}(X)$, and thus unique. ■

Equations 5.1.2a and 5.1.2b are analogous to the factorisation of the identity operator that was given by Lemma 3.3.5, in the cases of c_0 and ℓ_p, $1\le p<\infty$. Thus, we have achieved the appropriate generalisation of the factorisation lemma, as called for by [Tyl2, p220].

5.2 Finite Direct Sums

In §3.3 (see p54), the question of whether $\mathscr{C}(\ell_p\oplus\ell_q)$, $1\le p<q<\infty$, has $\|\cdot\|_e$ as a minimal (and hence unique) algebra norm, was raised as a problem from [Tyl2, p220]. Here we give a positive answer to that question, by showing that Theorem 5.1.2 generalises to finite direct sums. We suppress some of the details of the proof, which is essentially identical to that of Theorem 5.1.2, with a minor initial step added.

5.2.1 Theorem. *Let* $X_1, X_2, \ldots, X_n$ *be Banach spaces with the UN property. Suppose that* $\|\cdot\|_e$ *is a maximal algebra norm on* $\mathscr{C}(\bigoplus_{i=1}^n X_i)$*. Then* $\mathscr{C}(\bigoplus_{i=1}^n X_i)$ *has a unique algebra norm.*

Proof. Let $|||\cdot|||$ be an arbitrary algebra norm on $\mathscr{C}(\bigoplus_{i=1}^n X_i)$. Because $\|\cdot\|_e$ is maximal on $\mathscr{C}(\bigoplus_{i=1}^n X_i)$, there is $C > 0$ such that $|||\cdot||| \leq C\|\cdot\|_e$. Identify $|||\cdot|||$ with a seminorm on $\mathscr{B}(X)$, such that $\ker |||\cdot||| = \mathscr{K}(X)$. In order to gain a contradiction, suppose that there exists $(T_k) \subset \mathscr{B}(\bigoplus_{i=1}^n X_i)$ such that $\|T_k\|_e \uparrow \infty$ and $|||T_k||| = 1$, for all $k \in \mathbb{N}$. For each $j \leq n$, $j \in \mathbb{N}$, let $Q_j \in \mathscr{B}(\bigoplus_{i=1}^n X_i)$ be the projection onto the embedded copy of X_j, given by

$$Q_j(x_1, \ldots, x_n) \;=\; (0, \ldots, 0, x_j, 0, \ldots, 0)\,,$$

for all $(x_1, \ldots, x_n) \in (\bigoplus_{i=1}^n X_i)$. Then, for all $k \in \mathbb{N}$,

$$\|T_k\|_e \;=\; \left\| \sum_{i=1}^n \sum_{j=1}^n Q_j T_k Q_i \right\|_e \;\leq\; \sum_{i=1}^n \sum_{j=1}^n \|Q_j T_k Q_i\|_e\,.$$

Hence, there must be $i, j \leq n$, $i, j \in \mathbb{N}$, such that $(\|Q_j T_k Q_i\|_e)_{k=1}^\infty$ is unbounded, else $\|T_k\|_e$ would be bounded. Fix such i, j. Since X_i and X_j have the UN property, we may use the same method of proof we used to establish Theorem 5.1.2, passing to a subsequence of (T_k) if necessary, to show that there are

(i) $(C_k) \subset \mathbb{R}_+$ with $C_k \uparrow \infty$, and

(ii) non-compact operators $A, A_k, B_k \in \mathscr{B}(\bigoplus_{i=1}^n X_i)$ for which $A_k = Q_i A_k Q_i$ and $B_k = Q_j B_k Q_j$ for all $k \in \mathbb{N}$,

such that $\|A_k\|_e \|B_k\|_e \leq C_k^{-1}$ and

$$|||Q_j T_1 Q_i A||| \;=\; C_1 |||B_k Q_j T_k Q_i A_k||| \;=\; C_1 |||B_k T_k A_k||| \;\leq\; C_1 C^2 C_k^{-1}$$

for all $k \in \mathbb{N}$, yet $Q_j T_1 Q_i A \notin \mathscr{K}(\bigoplus_{i=1}^n X_i)$, a contradiction. ∎

5.2.2 Corollary. *Let* $X_1, X_2, \ldots, X_n$ *be Banach spaces with the UN property, and such that* $X_i \oplus X_i \approx X_i$ *for all* $i \leq n$, $i \in \mathbb{N}$*. Then* $\mathscr{C}(\bigoplus_{i=1}^n X_i)$ *has a unique algebra norm.*

Proof. We have

$$\left(\bigoplus_{i=1}^{n} X_i\right) \oplus \left(\bigoplus_{i=1}^{n} X_i\right) \;\approx\; \bigoplus_{i=1}^{n} (X_i \oplus X_i) \;\approx\; \left(\bigoplus_{i=1}^{n} X_i\right),$$

thus $\|\cdot\|_e$ is a maximal algebra norm on $\mathscr{C}(\bigoplus_{i=1}^{n} X_i)$ by Corollary 1.1.14. The required result then follows from Theorem 5.2.1. ■

5.2.3 Corollary. *Let X be c_0 or ℓ_p, $1 \le p < \infty$, and Y be c_0 or ℓ_p, $1 \le p < \infty$. Then $\mathscr{C}(X \oplus Y)$ has a unique algebra norm.*

Proof. By Proposition 1.3.2, X and Y have the UN property. Thus Corollary 5.2.2 applies, since $X \oplus X \approx X$ and $Y \oplus Y \approx Y$. ■

Corollary 5.2.3 answers the question raised by [Tyl2, p220], which we discussed in §3.3 (see p54), of whether $\mathscr{C}(\ell_p \oplus \ell_q)$ has $\|\cdot\|_e$ as a minimal algebra norm. Of note is that we have resolved this issue without a complete knowledge of the lattice of closed ideals in $\mathscr{B}(\ell_p \oplus \ell_q)$. This contrasts with the indication provided by [Mey] and [Tyl2], which suggested that such knowledge might be a necessary component. Known partial results concerning the structure of the lattice of closed ideals in $\mathscr{B}(\ell_p \oplus \ell_q)$ have recently been summarised and extended in [Schl].

With all the general theorems we need in place, we now progress to some more varied examples of Calkin algebras with unique algebra norms.

5.3 The spaces $c_0(\ell_r^n)$ and $\ell_p(\ell_r^n)$

For each $1 \le p < \infty$ and $1 \le r \le \infty$, let the standard unit vector basis of $\ell_p(\ell_r^n)$ be identified with (e_k), such that

$$\|(\alpha_k)\| \;=\; \left\|\sum_k \alpha_k e_k\right\| \;=\; \left(\sum_n \left(\sum_{k=t_{n-1}+1}^{t_n} |\alpha_k|^r\right)^{p/r}\right)^{1/p}, \tag{5.3a}$$

where $(t_n)_{n=0}^{\infty}$ is the sequence of triangular numbers:

$$t_0 = 0,\; t_1 = 1,\; t_2 = 3,\; t_3 = 6,\; \ldots$$

For $n \in \mathbb{N}$, define $T_n = \{t_{n-1}+1\,,\; t_{n-1}+2\,,\; \ldots\,,\; t_n\}$, and recall that we identify T_n with the operator

$$T_n : (\alpha_k) \mapsto \sum_{k \in T_n} \alpha_k e_k\,.$$

Thus T_n is the natural projection onto the ℓ_r^n summand in $\ell_p(\ell_r^n)$.

On $c_0(\ell_r^n)$, we also identify the standard unit vector basis in the same way, so that

$$\|(\alpha_k)\| = \sup_n \left(\sum_{k=t_{n-1}+1}^{t_n} |\alpha_k|^r \right)^{1/r} .$$

These spaces give us the first specific new examples to which we can apply the theory we built up in §5.1:

5.3.1 Theorem. *Let X be $c_0(\ell_r^n)$ or $\ell_p(\ell_r^n)$, for some $1 \leq p < \infty$ and $1 \leq r \leq \infty$. Then $\mathscr{C}(X)$ has a unique algebra norm.*

Proof. Apply Theorem 5.1.5, in light of Lemma 5.3.2 below. ■

5.3.2 Lemma. *Let X be as in the statement of Theorem 5.3.1, and (u_m) and (v_m) be normalised aligned block bases of $(e_k) \subset X$. Identify $\frac{1}{\infty} = 0$, and set $p = \infty$ when X is $c_0(\ell_r^n)$. Then (u_m) and (v_m) are $3^{\frac{1}{p}+\frac{1}{r}}$-equivalent.*

Proof.[2] We shall present the proof for $\ell_p(\ell_r^n)$ when $r < \infty$. The cases when $X = c_0(\ell_r^n)$ $\langle$resp. $r = \infty\rangle$ are similar, with p-norm $\langle$resp. r-norm$\rangle$ computations replaced by suprema over n.

The technical calculation in this proof relies on the following two simple arithmetic facts: if $\beta, x, y, z \geq 0$, then

$$(x+y+z)^\beta \leq (3\sup\{x,y,z\})^\beta \leq 3^\beta(x^\beta + y^\beta + z^\beta) \tag{5.3.2a}$$

and

$$(x^\beta + y^\beta + z^\beta) \leq 3\sup\{x^\beta, y^\beta, z^\beta\}) \leq 3(x+y+z)^\beta . \tag{5.3.2b}$$

Because (u_m) and (v_m) are aligned, by Definition 4.3.1 there are $(p_m)_{m=0}^\infty \subset \mathbb{N}_0$ with

[2] In combination with the solution to [AlKa, Problem 9.1] (discussed prior to the proof of Proposition 5.1.4), Lemma 5.3.2 can be used to solve [AlKa, Problem 9.2], which is an exercise given there without substantiation or reference. The progression of the exercises implicitly suggests that Lemma 5.3.2 holds and therefore that it is a known result. Indeed, more general c_0- and ℓ_p- sums of finite dimensional spaces have been extensively studied, including a classification of their subspaces and quotients in [JoZi1] and [JoZi2]. However, we are not aware of any published proof of Lemma 5.3.2.

$0 = p_0 < p_1 < p_2 < \dots$, such that, for all $m \in \mathbb{N}$,

$$p_{m-1} \prec u_m \prec p_m + 1 \qquad \text{and} \qquad p_{m-1} \prec v_m \prec p_m + 1 \, .$$

For each $n \in \mathbb{N}$, define

$$A_n = \{ m \in \mathbb{N} \, : \, p_{m-1} + 1 \in T_n \quad \text{and} \quad p_m \in T_n \} \, ,$$

and

$$B_n = \left\{ m \in \mathbb{N} \, : \, \begin{array}{lll} (\, p_{m-1} + 1 < T_n & \text{and} & T_n < p_m \,) \quad \text{or} \\ (\, p_{m-1} + 1 < T_n & \text{and} & p_m \in T_n \,) \quad \text{or} \\ (\, p_{m-1} + 1 \in T_n & \text{and} & T_n < p_m \,) \end{array} \right\} .$$

Clearly, A_n and B_n are disjoint, and B_n can have at most 2 elements because $(p_m)_{m=0}^{\infty}$ is strictly increasing. Note that, for all $n \in \mathbb{N}$ and $m \in A_n$,

$$T_n u_m = u_m \qquad \text{and} \qquad T_n v_m = v_m \, . \tag{5.3.2c}$$

Conceptually, A_n consists of m for which both $\operatorname{supp}(u_m)$ and $\operatorname{supp}(v_m)$ are fully contained in T_n, and B_n consists of those m such that $\operatorname{supp}(u_m)$ and $\operatorname{supp}(v_m)$ may cross 'into' or 'out of' T_n (or both).

We also define, for all $m \in \mathbb{N}$,

$$\begin{aligned} \Psi_m &= \{ n \in \mathbb{N} : m \in A_n \cup B_n \} \, , \ \text{and} \\ \Omega_m &= \{ n \in \mathbb{N} : m \in B_n \} \, . \end{aligned} \tag{5.3.2d}$$

For a given n, if $m \notin A_n \cup B_n$ then $T_n u_m = T_n v_m = 0$, so for all $m \in \mathbb{N}$ we have

$$u_m = \sum_{n \in \Psi_m} T_n u_m \qquad \text{and} \qquad v_m = \sum_{n \in \Psi_m} T_n v_m \, . \tag{5.3.2e}$$

Since A_n and B_n are disjoint, $\Omega_m \neq \emptyset$ implies

$$u_m = \sum_{n \in \Omega_m} T_n u_m \qquad \text{and} \qquad v_m = \sum_{n \in \Omega_m} T_n v_m \, ,$$

which gives

$$\sum_{n \in \Omega_m} \|T_n u_m\|^p = \|u_m\| = 1 = \|v_m\| = \sum_{n \in \Omega_m} \|T_n v_m\|^p \, . \tag{5.3.2f}$$

Therefore, if $\sum_m c_m v_m$ is a finite sum in $\mathrm{Span}\{v_m : m \in \mathbb{N}\}$, we have

$$\begin{aligned}
\left\|\sum_m c_m u_m\right\|^p &= \left\|\sum_m \left(\sum_{n\in\Psi_m} c_m T_n u_m\right)\right\|^p && \text{[by 5.3.2e]}\\
&= \left\|\sum_n \left(\sum_{m\in A_n\cup B_n} c_m T_n u_m\right)\right\|^p && \text{[by 5.3.2d]}\\
&= \sum_n \left\|T_n \left(\sum_{m\in A_n\cup B_n} c_m u_m\right)\right\|^p && \text{[by 5.3a]}\\
&= \sum_n \left\|\sum_{m\in A_n} c_m u_m + \left(\sum_{m\in B_n} c_m T_n u_m\right)\right\|^p && \text{[by 5.3.2c]}\\
&= \sum_n \left(\sum_{m\in A_n} |c_m|^r + \sum_{m\in B_n} |c_m|^r \|T_n u_m\|^r\right)^{p/r} && \text{[by 5.3a]}\\
&\le 3^{\frac{p}{r}} \left(\sum_n \left(\sum_{m\in A_n} |c_m|^r\right)^{p/r} + \sum_n \sum_{m\in B_n} |c_m|^p \|T_n u_m\|^p\right)\\
& && \text{[by 5.3.2a, since } |B_n| \le 2]\\
&= 3^{\frac{p}{r}} \left(\sum_n \left(\sum_{m\in A_n} |c_m|^r\right)^{p/r} + \sum_{m\in \bigcup_n B_n} |c_m|^p \sum_{n\in\Omega_m} \|T_n u_m\|^p\right)\\
&= 3^{\frac{p}{r}} \left(\sum_n \left(\sum_{m\in A_n} |c_m|^r\right)^{p/r} + \sum_{m\in \bigcup_n B_n} |c_m|^p \sum_{n\in\Omega_m} \|T_n v_m\|^p\right)\\
& && \text{[by 5.3.2f]}\\
&= 3^{\frac{p}{r}} \sum_n \left(\left(\sum_{m\in A_n} |c_m|^r\right)^{p/r} + \sum_{m\in B_n} |c_m|^p \|T_n v_m\|^p\right)\\
&\le 3^{1+\frac{p}{r}} \sum_n \left(\sum_{m\in A_n} |c_m|^r + \sum_{m\in B_n} |c_m|^r \|T_n v_m\|^r\right)^{p/r}\\
& && \text{[by 5.3.2b, since } |B_n| \le 2]\\
&= 3^{1+\frac{p}{r}} \left\|\sum_m c_m v_m\right\|^p,
\end{aligned}$$

where the final equality is achieved by reversing the first five lines of the calculation, with 'v_m' replacing 'u_m'. Since $\mathrm{Span}\{v_m : m \in \mathbb{N}\}$ is dense in $[(v_m)]$, by continuity the above calculation holds for all $\sum_m c_m v_m \in [(v_m)]$. Therefore,

$$\left\| \sum_m c_m u_m \right\| \leq 3^{\frac{1}{p}+\frac{1}{r}} \left\| \sum_m c_m v_m \right\| ,$$

for all $\sum_m c_m v_m \in [(v_m)]$. Furthermore, when $\sum_m c_m u_m \in [(u_m)]$, a symmetric argument tells us that

$$\left\| \sum_m c_m v_m \right\| \leq 3^{\frac{1}{p}+\frac{1}{r}} \left\| \sum_m c_m u_m \right\| . \qquad \blacksquare$$

5.4 Tsirelson's space

A question that received considerable attention in the early development of Banach space theory was whether every Banach space must contain a copy of c_0 or ℓ_p, $1 \leq p < \infty$. An example showing that this need not be the case was first provided in [Tsi]. Subsequently, an alternative description of the dual of the original example was given in [FiJo], and it is that space which has become known as *Tsirelson's space*. We shall denote it X_{T}.

5.4.1 Theorem ([FiJo, §2])**.** *There exists a reflexive Banach space X_{T}, with the standard unit vectors (e_n) as a basis, which contains no copy of c_0 or ℓ_p, $1 \leq p < \infty$.*

Tsireslon's space has been extensively studied. For details of its construction, see [CaSh]. Because of its unusual properties, the discovery of Tsirelson's space opened up new areas of research that led to the discovery of many new counterexamples. Since it is a markedly different space to the others we consider in this thesis, which are all related in some fashion to c_0 or ℓ_p, $1 \leq p < \infty$, it is of interest that we can show that the Calkin algebra of Tsirelson's space also has $\|\cdot\|_e$ as its unique algebra norm. To establish this result using the theory developed in §5.1, we need only the following property of X_{T}.

5.4.2 Theorem ([CJT, Prop. 6]; [CaSh, Prop. II.4])**.** *Let*

$$(u_n) = \left(\sum_{i=p_{n-1}+1}^{p_n} c_i e_i \right)_{n=1}^{\infty}$$

be a normalised block basic sequence of (e_n) *in* X_{T}*. Then, if* $\sum_n c_n u_n \in [(u_n)]$,

$$\frac{1}{3}\left\|\sum_n c_n e_{p_n}\right\| \leq \left\|\sum_n c_n u_n\right\| \leq 18\left\|\sum_n c_n e_{p_n}\right\|.$$

5.4.3 Corollary. *In* X_{T}*, any two normalised aligned block basic sequences are* 54*-equivalent.*

***Proof*.** Suppose (u_n) and (v_n) are normalised aligned block basic sequences in X_{T}. By Theorem 5.4.2, if $\sum_n c_n u_n \in [(u_n)]$,

$$\frac{1}{54}\left\|\sum_n c_n u_n\right\| \leq \frac{1}{3}\left\|\sum_n c_n e_{p_n}\right\| \leq \left\|\sum_n c_n v_n\right\| \leq 18\left\|\sum_n c_n e_{p_n}\right\| \leq 54\left\|\sum_n c_n u_n\right\|.$$

■

5.4.4 Theorem. *The algebras* $\mathscr{C}(X_{\mathrm{T}})$ *and* $\mathscr{C}(X_{\mathrm{T}}^*)$ *have unique algebra norms.*

***Proof*.** By Theorem 5.1.5 and Corollary 5.4.3, $\mathscr{C}(X_{\mathrm{T}})$ has a unique algebra norm. Because X_{T} is reflexive, it follows that $\mathscr{C}(X_{\mathrm{T}}^*)$ has a unique algebra norm by Theorem 1.2.2. ■

5.4.5 Remark. We also could have shown that $\mathscr{C}(X_{\mathrm{T}}^*)$ has a unique algebra norm directly, because block bases in X_{T}^* have similar properties as those in X_{T}. Indeed, the uniqueness of norm result will likely also hold for other Tsirelson-like spaces. For instance, Theorem 5.4.4 can be proved in an identical fashion if X_{T} is replaced with one of the continuum of analogues described in [CaSh, §X.A].

5.5 James' space

Another famous counterexample in the history of Banach space theory is due to R. C. James. His space provided a negative solution to the question, which originated in Banach's book, of whether the reflexivity of a Banach space was a necessary condition for that space to be isomorphic to its bidual.

5.5.1 Theorem ([Jam])**.** *There exists a non-reflexive Banach space* X_{J} *with a basis, such that* $X_{\mathrm{J}}^{**} \approx X_{\mathrm{J}}$.

James' space is unusual in comparison to spaces we have previously discussed for two pertinent reasons. The first is that it does not have an unconditional basis.

The second is that $X_J \not\approx X_J \oplus X_J$, and X_J does not have a continued bisection of the identity. Despite this, as discussed in §1.1, $\mathscr{C}(X_J)$ still has $\|\cdot\|_e$ as a maximal algebra norm:

5.5.2 Theorem ([Wil2, Prop. 8]). *Every homomorphism from $\mathscr{B}(X_J)$ into a Banach algebra is continuous.*

5.5.3 Corollary. *$\|\cdot\|_e$ is a maximal algebra norm on $\mathscr{C}(X_J)$.*

***Proof*.** By Theorem 5.5.2 and Proposition 1.1.11, every homomorphism from $\mathscr{C}(X_J)$ into a Banach algebra is continuous. Therefore, $\|\cdot\|_e$ is maximal on $\mathscr{C}(X_J)$, by Proposition 1.1.13. ■

Hence, if we can show that James' space has the UN property, we can conclude that $\mathscr{C}(X_J)$ has a unique algebra norm by Theorem 5.1.2. To establish the UN property for X_J, we make use of the following concept.

5.5.4 Definition. Let X be a Banach space with a basis, and $(u_n) \subset X$ a block basic sequence. Then (u_n) is a *skipped block basic sequence* or *skipped block basis* if there exist strictly increasing sequences $(p_n), (q_n) \subset \mathbb{N}$ such that $p_{n+1} - q_n > 1$ and

$$p_n - 1 \prec u_n \prec q_n + 1$$

for all $n \in \mathbb{N}$.

5.5.5 Proposition ([CLL]; [HeWh]). *There exists a constant $C \geq 1$ such that every normalised skipped block basis in X_J is C-complemented and C-equivalent to the standard unit vector basis of ℓ_2.*

***Proof*.** It is difficult to be precise about the best value of the constant C, because X_J has been given different, but equivalent, norms by various authors over many years of study. However, the proof of [HeWh, Lem. 1] shows that every normalised skipped block basis (u_n) in the original X_J, from [Jam], satisfies

$$\left\|\sum_n |c_n|^2\right\|^{1/2} \leq \left\|\sum_n c_n u_n\right\| \leq \sqrt{5}\left\|\sum_n |c_n|^2\right\|^{1/2},$$

whenever $\sum_n c_n u_n \in [(u_n)]$. Thus, every such (u_n) is $\sqrt{5}$-equivalent to the standard unit vector basis of ℓ_2, and it is also the case that (u_n) is $\sqrt{5}$-equivalent to any other normalised skipped block basis in X_J.

By [CLL, Thm. 10], we also have that every normalised skipped block basis in X_{J} is complemented; this result is not affected by the equivalent norm used in [CLL]. The proof relies in part on [CLL, Thm. 5], and an examination of the proofs of [CLL, Thm. 5] and [CLL, Thm. 10] reveals that the constant of complementation is at most $2\sqrt{2}$ in all cases, under the equivalent norm. ■

In combination with the theory developed in §5.1, the above property of normalised skipped block bases in X_{J} is all we require to prove the main result of this section.

5.5.6 Theorem. *The algebra $\mathscr{C}(X_{\mathrm{J}})$ has a unique algebra norm.*

Proof. Let C be the constant from the statement of Proposition 5.5.5. Suppose that (u_n) is a normalised block basis in X_{J}. Then (u_{2n}) is a skipped block basis, and hence is C-complemented and C-equivalent to the standard unit vector basis of ℓ_2, by Proposition 5.5.5. The standard unit vector basis of ℓ_2 is 1-subsymmetric, as discussed in Example 1.3.1. Thus every normalised block basis in X_{J} has a subsequence which is C-complemented and C-equivalent to a common subsymmetric basic sequence. Therefore, X_{J} has the UN property, by Proposition 5.1.7. By Corollary 5.5.3, we also have that $\|\cdot\|_e$ is a maximal algebra norm on $\mathscr{C}(X_{\mathrm{J}})$. Consequently, $\mathscr{C}(X_{\mathrm{J}})$ has a unique algebra norm, by Theorem 5.1.2. ■

5.5.7 Remark. As with Tsirelson's space (see Remark 5.4.5), there are modified versions of James' space for which Theorem 5.5.6 is also likely to hold. In particular, the Calkin algebras of the natural generalisations of James' space that are saturated with copies of ℓ_p, for some $1 < p < \infty$, rather than ℓ_2 (see, e.g., [LoWi, §4] or [Lau, Defn. 4.1] for their definition), each have unique algebra norms: the proof follows identical lines to the one we have given for X_{J}, except that a different constant is needed in the analogue of Proposition 5.5.5 (which is provided by [Lau, Lem. 4.6]).

While the above demonstrates a use for Proposition 5.1.7, employing it to show that X_{J} has the UN property was somewhat unnatural, because the subsymmetry of the standard unit vector basis of ℓ_2 was explicitly required. In contrast, an argument direct from the properties of (aligned) skipped block bases in X_{J} would only implicitly rely on subsymmetry. For instance, the ploy of passing from an arbitrary block basis (u_n) to the skipped block subsequence (u_{2n}) can also be used to show the following.

5.5.8 Proposition. *There exists a constant $C \geq 1$ such that, for every pair*

$\{(u_n),(v_n)\}$ of aligned block bases in X_{J}, (u_{2n}) and (v_{2n}) are C-equivalent and C-complemented.

Proof. For such $\{(u_n),(v_n)\}$, (u_{2n}) and (v_{2n}) are each skipped block subsequences, so the result follows from Proposition 5.5.5. ■

It is the fact that X_{J} satisfies Proposition 5.5.8, rather than Proposition 5.1.7, which is more central to the uniqueness of $\|\cdot\|_e$ on $\mathscr{C}(X_{\mathrm{J}})$. This is because, using Proposition 5.5.8, it is immediate that X_{J} has the UN property (choose $(n_m) = (2m)$ in Definition 5.1.1). Indeed, an almost identical method of proof as was used to substantiate Proposition 5.1.4 and Theorem 5.1.5 gives the following. We omit the details.

5.5.9 Theorem. *Let X be a Banach space with a basis. Suppose that there is a constant $C \geq 1$ such that every pair of skipped block subsequences in X are C-equivalent. Then X has the UN property.*

Unlike in the case of Theorem 5.1.5, X does not automatically have a continued bisection of the identity under the assumptions of Theorem 5.5.9. So it is necessary to separately state the following.

5.5.10 Corollary. *Suppose that X is a Banach space with a basis, and that $\|\cdot\|_e$ is a maximal algebra norm on $\mathscr{C}(X)$. If there is a constant $C \geq 1$ such that every pair of skipped block subsequences in X are C-equivalent, then $\mathscr{C}(X)$ has a unique algebra norm.*

It is more natural to apply Corollary 5.5.10 to demonstrate that $\mathscr{C}(X_{\mathrm{J}})$ has a unique algebra norm, as opposed to our use of Proposition 5.1.7 in the proof of Theorem 5.5.6. However, note that the assumptions on X in Theorem 5.5.9 imply that each skipped block subsequence in X is subsymmetric. Despite this, we are currently left without an example that properly justifies the statement of Proposition 5.1.7. Spaces for which it seems Proposition 5.1.7 has greater relevance will be discussed in Chapter 7.

This brings to an end our brief survey of examples of Calkin algebras which the results of §5.1 show have unique algebra norms. It is very likely that many other examples exist: specimens such as Tsirelson's space and James' space indicate the broad extent to which we have generalised the results of [Mey]. In the next chapter, we shall continue to extend those results in a somewhat different direction.

Chapter 6

Uniqueness of Norm for κ-Calkin Algebras

The preceding chapters have demonstrated how to alter the proof of the Eidelheit-Yood theorem (1.1.2), so that it is applicable to certain Calkin algebras. In particular, we have been able to establish uniqueness of norm for the Calkin algebras of spaces of a more general nature than c_0 and ℓ_p, $1 \leq p < \infty$, and therefore extend the previous result of [Mey]. We replaced the rank 1 operator S, that appears in the Eidelheit-Yood proof, by a variable shift-like operator. This provided a sufficiently weaker alternative to the strong factorisation properties of non-compact operators, inherently relied upon by [Mey], which hold in the cases of c_0 and ℓ_p, $1 \leq p < \infty$.

However, even though the essential idea was the same, we have not followed the 'pure' Eidelheit-Yood method, because the replacement for S was not kept fixed. Hence, the 'best case' scenario outlined at the start of §3.4 has gone unfulfilled so far; for instance, we did not explicitly define a sequence of functionals to independently witness the unbounded essential norms of a given set of operators.

In this chapter, we return to the unmodified versions of both the factorisation technique, and the Eidelheit-Yood method. We use each to independently prove the same uniqueness of norm results. Our objects of interest are more general quotients of $\mathscr{B}(X)$, in the case when X is a non-separable analogue of c_0 or ℓ_p, $1 \leq p < \infty$.[1] Our main result is that all such quotients have unique algebra norms.

In §6.1, we define the non-separable spaces in question and the notion of κ-compact operators for a cardinal number κ, as previously studied in [Gra], [Luf], and [Daw]. We also quote some results from [Daw] and discuss their ramifications. Along with the underlying elementary set theory, material from [Daw] is essentially

[1] In some vague but natural sense, here we shall generalise 'upwards' within the class of all Banach spaces, rather than the 'sideways' generalisation presented in previous chapters.

all the background required; in particular we do not need to rely on results from Chapters 4 or 5. The main uniqueness of norm theorem is stated in §6.2, along with its proof in one specific instance, which reduces to the separable case of §3.3.

In §6.3 we give our first proof of the main result, using the technique of [Mey] as presented in [Tyl2, §2] (see §3.3). The factorisation lemma that we rely upon is very similar to the one which holds in the separable case (Lemma 3.3.5 or 3.3.6).

The proof via the Eidelheit-Yood method is given in §6.4. With this approach we are able to establish the minimality of the quotient norm without relying on its maximality to do so. However, we can not apply the method in general: this second proof of the main result excludes the ℓ_1 case and places a restriction on the cardinality of κ.

6.1 'Classical' non-separable spaces

The set theory we adopt is a standard version of ZFC and adheres to the development in [Jec]. In particular, we assign to each set A its cardinal number $|A|$, being the least ordinal α with cardinality the same as A.[2] The infinite ordinal numbers that are cardinals are termed, and denoted by, *alephs*. For every ordinal α there is a least cardinal α^+ strictly greater than α, and if W is a set of cardinals, then $\sup W$ is a cardinal (see [Jec, Lem. 3.4]). Hence the class of all alephs can be increasingly enumerated by the ordinal numbers, as per [Jec, p30]:

$$\aleph_0 = \omega\,, \quad \aleph_{\alpha+1} = \aleph_\alpha^+\,, \quad \text{and} \quad \aleph_\alpha = \sup\{\aleph_\beta : \beta < \alpha\}\,, \text{ if } \alpha \text{ is a limit ordinal.}$$

Because we assume the Axiom of Choice, cardinal arithmetic is trivial (see [Jec, Chp. 5]): if κ and λ are alephs, then $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$, and, if $\{\kappa_i\}_{i\in I}$ is a set of cardinal numbers, we define

$$\sum_{i\in I} \kappa_i = \left|\bigcup_{i\in I} A_i\right|,$$

where $\{A_i\}_{i\in I}$ is a disjoint family of sets such that $|A_i| = \kappa_i$ for each $i \in I$. The *cofinality* $\mathrm{cf}(\kappa)$ of an aleph κ (see [Jec, pp31&53]) is itself an aleph which can be defined by

$$\mathrm{cf}(\kappa) = \inf\left\{\lambda \;:\; \kappa = \sum_{i<\lambda} \kappa_i\,,\ (\forall i)\, \kappa_i < \kappa\right\}.$$

[2] Note that we also continue to use $|\cdot|$ for the absolute value function on $\mathbb{F}$; the intended meaning will be obvious from the context.

That is, $\mathrm{cf}(\kappa)$ is the smallest possible cardinality of a collection of strictly smaller cardinals such that their sum is κ. An aleph κ is *regular* if $\mathrm{cf}(\kappa) = \kappa$; otherwise, $\mathrm{cf}(\kappa) < \kappa$ and κ is *singular*. Note in particular that, by [Jec, Coroll. 5.3], $\aleph_1$ is a regular cardinal.

To define the aforementioned non-separable analogues of c_0 and ℓ_p, $1 \leq p < \infty$, we largely follow the notation of [Daw]; that paper is also the authority for the elementary properties we shall now list. For the rest of this chapter only, I will denote an arbitrary infinite set, rather than the identity operator. Given an infinite set I, we consider families of scalars $(x_i)_{i\in I} \subset \mathbb{F}$ to be generalised sequences, and we treat them as vectors in a similar fashion to elements of a sequence space. Write $I^{<\infty} = \{A \subset I : |A| < \infty\}$. Then we define

$$c_0(I) \;=\; \left\{(x_i)_{i\in I} \subset \mathbb{F} \;:\; (\forall \varepsilon > 0)\,\{i \in I : |x_i| \geq \varepsilon\} \in I^{<\infty}\right\},$$

which is a Banach space under the supremum norm. Similarly, for $1 \leq p < \infty$, we define

$$\ell_p(I) \;=\; \left\{(x_i)_{i\in I} \subset \mathbb{F} \;:\; \|(x_i)_{i\in I}\|_{\ell_p} := \left(\sum_{i\in I} |x_i|^p\right)^{\frac{1}{p}} < \infty\right\}.$$

As sets, $\ell_p(I) \subset c_0(I)$ for all $1 \leq p < \infty$. Let X be one of the spaces $c_0(I)$ or $\ell_p(I)$, $1 \leq p < \infty$. Then X is separable if and only if I is countable, in which case X will be isometric to c_0 or ℓ_p, $1 \leq p < \infty$, respectively. When I is uncountable, X is similarly isometrically determined by $|I|$. In particular, $X \oplus X \approx X$.

For each $i \in I$, we set $e_i = (\delta_{ij})_{j\in I}$, where δ_{ij} denotes the Kronecker delta. As in the separable case, we can define, for each $j \in I$, biorthogonal functionals $e_j^* \in X^*$ such that $e_j^*(e_i) = \delta_{ij}$ for all $i \in I$, and extend by linearity and continuity to the whole of X. Hence

$$x \;=\; \sum_{i\in I} e_i^*(x)e_i\,,$$

for all $x \in X$. The *support* $\mathrm{supp}(x) \subset I$ of a vector $x = (x_i)_{i\in I} \in X$ is defined in the usual way:

$$\mathrm{supp}(x) \;=\; \{i \in I : x_i \neq 0\} \;=\; \{i \in I : e_i^*(x) \neq 0\}\,,$$

and we can also define the *support* $\mathrm{supp}(x^*) \subset I$ of a functional $x^* \in X^*$ by

$$\mathrm{supp}(x^*) \;=\; \{i \in I : x^*(e_i) \neq 0\}\,.$$

It is immediate from the definition of X that $\mathrm{supp}(x)$ is countable for each $x \in X$, however $\mathrm{supp}(x^*)$ can be uncountable when $x^* \in \ell_1(I)^*$. Finally, for all $A \subset I$, we can define norm 1 projections $P_A \in \mathscr{B}(X)$ by

$$P_A(x) = \sum_{\alpha \in A} e_\alpha^*(x) e_\alpha .$$

We now formally introduce the notion of κ-compact operators. The definition we give is from [Daw], however equivalent definitions were originally given and independently studied in [Gra] and [Luf].

6.1.1 Definition. Let Y and Z be Banach spaces, and κ be an infinite cardinal. We say an operator $T \in \mathscr{B}(Y, Z)$ is *κ-compact* if, for each $\varepsilon > 0$, we can find $B \subset B_Y$ with $|B| < \kappa$, and such that, for all $x \in B_X$,

$$\inf_{y \in B} \|T(x-y)\| \leq \varepsilon .$$

We denote by $\mathscr{K}_\kappa(Y, Z)$ the set of all κ-compact operators $Y \to Z$, and write $\mathscr{K}_\kappa(Y)$ for $\mathscr{K}_\kappa(Y, Y)$. $\mathscr{K}_\kappa(Y)$ forms a closed ideal in $\mathscr{B}(Y)$, and $\mathscr{K}_{\aleph_0}(Y) = \mathscr{K}(Y)$.

In particular, we have the following:

6.1.2 Lemma ([Daw, Lem. 3.6])**.** *Let I be an infinite set, and let X be $c_0(I)$ or $\ell_p(I)$, $1 \leq p < \infty$. If $A \subset I$ with $|A| = \kappa$, then $P_A \in \mathscr{K}_{\kappa^+}(X) \backslash \mathscr{K}_\kappa(X)$.*

The main result of [Daw] is:

6.1.3 Theorem ([Daw, Thm. 7.4])**.** *Let I be an infinite set, and let X be $c_0(I)$ or $\ell_p(I)$, $1 \leq p < \infty$. Then every non-trivial closed ideal in $\mathscr{B}(X)$ is of the form $\mathscr{K}_\kappa(X)$ for some aleph κ, and the closed ideals in $\mathscr{B}(X)$ form an ordered chain*

$$\{0\} \subsetneq \mathscr{K}(X) = \mathscr{K}_{\aleph_0}(X) \subsetneq \mathscr{K}_{\aleph_1}(X) \subsetneq \ldots \subsetneq \mathscr{K}_{|I|^+}(X) = \mathscr{B}(X) .$$

Theorem 6.1.3 generalised the identical, yet independently derived, [Gra, Thm 3.3] and [Luf, Coroll. 6.2], which gave the above result in the specific case when X is an infinite dimensional Hilbert space and therefore isometric to $\ell_2(I)$ for some infinite set I. Theorem 6.1.3 can also be viewed as a generalisation of [GMF, Thm. 5.1], which, as discussed in §3.3, established that $\mathscr{K}(X)$ is the only non-trivial closed ideal in $\mathscr{B}(X)$ when X is c_0 or ℓ_p, $1 \leq p < \infty$.

In this context, we are led inexorably to:

6.1.4 Definition. Let X be a Banach space, and let κ be an aleph. Then the *κ-Calkin algebra* of X is the (Banach) quotient algebra $\mathscr{B}(X)/\mathscr{K}_\kappa(X)$.

We will show that the κ-Calkin algebras of $c_0(I)$ and $\ell_p(I)$, $1 \le p < \infty$, have unique algebra norms. For the spaces X in question, the next proposition from [Daw] is critical to the analysis of closed ideals in, and quotients of, $\mathscr{B}(X)$.

6.1.5 Proposition ([Daw, Prop. 5.1])**.** *Let I be an infinite set, and X be $c_0(I)$ or $\ell_p(I)$, $1 \le p < \infty$. Let λ be an aleph and $T \in \mathscr{B}(X)$. Then we have*

$$\|T + \mathscr{K}_\lambda(X)\| = \inf\{\|P_{I\setminus A}T\| \,:\, A \subset I,\, |A| < \lambda\} \tag{6.1.5a}$$
$$= \inf\{\|P_{I\setminus A}TP_{I\setminus B}\| \,:\, A, B \subset I,\, |A| < \lambda,\, |B| < \lambda\}\,. \tag{6.1.5b}$$

Now suppose that $T \in \mathscr{K}_\lambda(X)$. Then we have:

(i) if λ is a cardinal with $\mathrm{cf}(\lambda) > \aleph_0$, then there exists $A \subset I$ with $|A| < \lambda$ and $T = P_A T$;

(ii) if $\mathrm{cf}(\lambda) = \aleph_0$, then, for each $\varepsilon > 0$, there exists $A \subset I$ with $|A| < \lambda$ and $\|T - P_A T\| < \varepsilon$.

In conjunction with some fundamental results that were presented in §1.1, Proposition 6.1.5 allows us to establish an automatic continuity property:

6.1.6 Theorem. *Let I be an infinite set, and let X be $c_0(I)$ or $\ell_p(I)$, $1 \le p < \infty$. Suppose κ, λ are alephs such that $\kappa < \lambda \le |I|^+$. Then each homomorphism from $(\mathfrak{A}, \|\cdot\|) = \mathscr{K}_\lambda(X)/\mathscr{K}_\kappa(X)$ into a Banach algebra is automatically continuous. In particular, the quotient operator norm $\|\cdot\|$ is maximal.*

Proof. In the second part of Proposition 6.1.5, when $T \in \mathscr{K}_\lambda(X)$, we can clearly ensure that $|A| \ge \kappa$ for any given $\kappa < \lambda$, by taking a superset if necessary. This is true both in case (i) and in case (ii). Thus, with reference also to Lemma 6.1.2, we have that if $t = T + \mathscr{K}_\kappa(X) \in \mathfrak{A}$, then for each $\varepsilon > 0$ there is $A \in I$ with $\kappa \le |A| < \lambda$, such that if we set $u = P_A + \mathscr{K}_\kappa(X)$ then $u \in \mathfrak{A}$ and $\|t - ut\| \le \|T - P_A T\| < \varepsilon$. Also, $\|u\| \le \|P_A\| = 1$.

Therefore the conditions of Proposition 1.1.16 are satisfied for the algebra $\mathfrak{A}$, and so $\mathfrak{A}$ has a bounded left approximate identity. Since $X \approx X \oplus X$, Corollary 1.1.18 guarantees that each homomorphism from $\mathfrak{A}$ must be continuous. ■

Theorem 6.1.6 positively answers the maximality half of the uniqueness of norm question, not just for the κ-Calkin algebras of $c_0(I)$ and $\ell_p(I)$, $1 \leq p < \infty$, but for all the closed ideals and quotients of those algebras as well. We will show that the other half of the question also has a positive answer for those spaces, and thus prove the uniqueness of norm result in the fullest generality possible within this context.

6.2 Quotients of $\mathscr{B}(c_0(I))$ and $\mathscr{B}(\ell_p(I))$

Our aim is to establish the following principal result, as an analogue of Theorem 3.3.2, and also of Theorem 1.1.2 and Corollary 1.1.14.

6.2.1 Theorem. *Let I be an infinite set, and let X be $c_0(I)$ or $\ell_p(I)$, $1 \leq p < \infty$. Then for all alephs κ, λ such that $\kappa < \lambda \leq |I|^+$, the Banach algebra $\mathscr{K}_\lambda(X)/\mathscr{K}_\kappa(X)$ has a unique algebra norm. In particular, all the κ-Calkin algebras of X have unique algebra norms.*

In the specific case when $\kappa = \aleph_0$ and $\lambda = \aleph_1$, we will now explain how to reduce the proof of our main theorem to the previous uniqueness of norm result for $\mathscr{C}(c_0)$ and $\mathscr{C}(\ell_p)$, $1 \leq p < \infty$.

Proof of Theorem 6.2.1 when $\kappa = \aleph_0$**,** $\lambda = \aleph_1$.

In this case, $\mathscr{K}_\kappa(X) = \mathscr{K}(X)$. So, we are required to show that $\mathscr{K}_{\aleph_1}/\mathscr{K}(X)$ has the essential norm $\|\cdot\|_e$, familiar from previous chapters, as its unique algebra norm. Suppose $|||\cdot|||$ is another algebra norm on $\mathscr{K}_{\aleph_1}(X)/\mathscr{K}(X)$, and simultaneously identify $|||\cdot|||$ with a semi-norm on $\mathscr{K}_{\aleph_1}(X)$, as we do with $\|\cdot\|_e$. Theorem 6.1.6 guarantees the existence of a $C > 0$ such that $|||\cdot||| \leq C\|\cdot\|_e$, so it remains to show that there can not exist $(T_k) \subset \mathscr{K}_{\aleph_1}(X)\setminus\mathscr{K}(X)$ with $\|T_k\|_e \uparrow \infty$ while $|||T_k||| = 1$. Assume, in order to gain a contradiction, that such a sequence (T_k) exists. We will argue that there is $A' \subset I$ with $|A'| = \aleph_0$ and such that $\|P_{A'}T_kP_{A'}\|_e \uparrow \infty$. Fix $\varepsilon > 0$. Since each T_k is $\aleph_1$-compact, by Definition 6.1.1 there exists, for each $k \in \mathbb{N}$, a subset $\Upsilon_k \subset B_X$ with $|\Upsilon_k| \leq \aleph_0$, such that, for all $x \in B_X$,

$$\inf_{y \in \Upsilon_k} \|T_k(x-y)\| \leq \varepsilon\,.$$

Define

$$\Upsilon = \bigcup_k \Upsilon_k \subset B_X \quad \text{and} \quad A = \bigcup_{x\in\Upsilon} \operatorname{supp}(x) \subset I\,.$$

Note that, since $|\mathrm{supp}(x)| \leq \aleph_0$ for all $x \in X$, and $\mathrm{cf}(\aleph_1) = \aleph_1 > \aleph_0$,

$$|\Upsilon| \leq \sum_{k \in \aleph_0} \aleph_0 < \aleph_1 \quad \text{and} \quad |A| \leq \sum_{x \in \Upsilon} \aleph_0 < \aleph_1 ,$$

so $|A| \leq \aleph_0$. Now let $B \subset I$ be such that $|B| < \aleph_0$. For all $k \in \mathbb{N}$ and all $x \in B_X$, we have

$$\begin{aligned}
\|P_{I\setminus B}T_k x\| - \|P_{I\setminus B}T_k P_A\| &= \|P_{I\setminus B}T_k x\| - \sup_{y \in B_X} \|P_{I\setminus B}T_k P_A y\| \\
&= \inf_{y \in B_X} \big(\|P_{I\setminus B}T_k x\| - \|P_{I\setminus B}T_k P_A y\|\big) \\
&\leq \inf_{y \in B_X} \|P_{I\setminus B}(T_k x - T_k P_A y)\| \\
&\leq \inf_{y \in B_X} \|T_k x - T_k P_A y\| \qquad \text{[since } \|P_{I\setminus B}\| = 1] \\
&= \inf_{y \in P_A(B_X)} \|T_k(x-y)\| \\
&\leq \inf_{y \in \Upsilon} \|T_k(x-y)\| \\
&\leq \inf_{y \in \Upsilon_k} \|T_k(x-y)\| \\
&\leq \varepsilon .
\end{aligned}$$

Thus, for each $k \in \mathbb{N}$,

$$\|P_{I\setminus B}T_k P_A\| \geq \|P_{I\setminus B}T_k\| - \varepsilon \geq \|T_k\|_e - \varepsilon , \qquad \text{[by 6.1.5a]}$$

which gives, again by Proposition 6.1.5,

$$\|T_k P_A\|_e \geq \|T_k\|_e - \varepsilon \uparrow \infty .$$

Now, $T_k P_A$ is $\aleph_1$-compact for each $k \in \mathbb{N}$, so by Proposition 6.1.5(i) there is $A_k \subset I$ with $|A_k| \leq \aleph_0$ and $P_{A_k}T_k P_A = T_k P_A$. Define

$$A' = A \cup \bigcup_k A_k .$$

Then

$$|A'| \leq \aleph_0 + \sum_k \aleph_0 = \aleph_0 ,$$

and, for all $k \in \mathbb{N}$,

$$\|P_{A'}T_k P_{A'}\|_e \geq \|P_{A_k}T_k P_A\|_e = \|T_k P_A\|_e \uparrow \infty .$$

Note that we must have $|A'| = \aleph_0$, as else $P_{A'}$ would be compact.

Now, we can isometrically identify $c_0(A')$ and $\ell_p(A')$, $1 \le p < \infty$, with c_0 and ℓ_p, $1 \le p < \infty$, respectively. Let Y be the subspace of X that is naturally isometric to the appropriate respective space. Then $P_{A'}$ is the natural projection from X onto Y. Set $\iota : Y \to X$ to be the natural injection. We may transfer the algebra semi-norm $|||\cdot|||$ to $\mathscr{B}(Y)$ by making the definition

$$|||T|||_Y = |||\iota T P_{A'}|||$$

for each $T \in \mathscr{B}(Y)$. Since $\ker |||\cdot||| = \mathscr{K}(X)$, clearly $|||\cdot|||_Y$ has kernel $\mathscr{K}(Y)$ and can be treated as an algebra norm on $\mathscr{C}(Y)$. Further,

$$|||P_{A'}T_kP_{A'}\iota|||_Y = |||\iota P_{A'}T_kP_{A'}P_{A'}||| = |||P_{A'}T_kP_{A'}||| \le |||P_{A'}|||^2|||T_k||| \le C^2 .$$

However, under the isometry ι, the essential norm $\|\cdot\|_e$ does not change value, so on $\mathscr{B}(Y)$ we also have $\|P_{A'}T_kP_{A'}\iota\|_e \uparrow \infty$, in contradiction with Theorem 3.3.2. ■

Hence we see that, in some sense, $\mathscr{K}_{\aleph_1}(X)/\mathscr{K}_{\aleph_0}(X)$ 'contains' a Calkin algebra. The proof of Theorem 6.2.1 when $\kappa \ge \aleph_1$ will be given in the next section. In combination with the above, that result will also yield:

Proof of Theorem 6.2.1 when $\kappa = \aleph_0$***,*** $\lambda > \aleph_1$.

The previous proof showed that $\mathscr{K}_{\aleph_1}(X)/\mathscr{K}_{\aleph_0}(X)$ has a unique algebra norm. Without relying on the current result, in §6.3 we shall show that $\mathscr{K}_\lambda(X)/\mathscr{K}_{\aleph_1}(X)$ has a unique algebra norm. Therefore, by Proposition 1.2.1, $\mathscr{K}_\lambda(X)/\mathscr{K}_{\aleph_0}(X)$ has a unique algebra norm. ■

6.3 Uniqueness of norm via factorisation

The partial proofs of Theorem 6.2.1 given in §6.2 dealt with the case when $\kappa = \aleph_0$. We shall now substantiate the remaining cases, using the direct factorisation method from [Mey] and [Tyl2, §2], which was presented in §3.3. We need the following two results, which can be extracted from proofs given in [Daw]. As explained there, "unlike the separable case, there seems to be a difference between the ℓ_1 case and the other cases."

6.3.1 Proposition (see proof of [Daw, Thm. 6.2]). *Let I be an infinite set, let X be $c_0(I)$ or $\ell_p(I)$, $1 < p < \infty$, let $\kappa > \aleph_0$ be an aleph, and let $T \in \mathscr{B}(X)\backslash\mathscr{K}_\kappa(X)$. Then there exist operators $S, U \in \mathscr{K}_{\kappa^+}\backslash\mathscr{K}_\kappa(X)$ and a set $A \subset I$ with $|A| = \kappa$, such* *that*

$$P_A = STU$$

and

$$\|S + \mathscr{K}_\kappa(X)\| \cdot \|U + \mathscr{K}_\kappa(X)\| \leq 4\|T + \mathscr{K}_\kappa(X)\|^{-1}.$$

6.3.2 Proposition (see proof of [Daw, Thm. 7.3])**.** *Let I be an infinite set, let X be $\ell_1(I)$, let $\kappa > \aleph_0$ be an aleph, and let $T \in \mathscr{B}(X) \backslash \mathscr{K}_\kappa(X)$. Then for each $\varepsilon > 0$ there exists an operator $S \in \mathscr{K}_{\kappa^+} \backslash \mathscr{K}_\kappa(X)$ and a set $A \subset I$ with $|A| = \kappa$, such that*

$$P_A = STP_A$$

and

$$\|S + \mathscr{K}_\kappa(X)\| \leq (1 + \varepsilon)\|T + \mathscr{K}_\kappa(X)\|^{-1}.$$

6.3.3 Corollary. *The conclusion of Proposition 6.3.1 also holds if X is $\ell_1(I)$.*

Proof. Simply choose $0 < \varepsilon \leq 3$ and apply Proposition 6.3.2, setting $U = P_A$ and noting that, by Lemma 6.1.2, we have $P_A \in \mathscr{K}_{\kappa^+} \backslash \mathscr{K}_\kappa(X)$. ■

Clearly, the above propositions are close analogues of Lemma 3.3.5. Because they guarantee that every non-κ-compact operator factors an appropriate projection, we have the situation envisioned in point (i) at the end of §3.3 (see p55). Thus, we can prove uniqueness of norm for quotients of $\mathscr{B}(c_0(I))$ and $\mathscr{B}(\ell_p(I))$, $1 \leq p < \infty$, by an identical method as was used for $\mathscr{C}(c_0)$ and $\mathscr{C}(\ell_p)$, $1 \leq p < \infty$, in the proof of Theorem 3.3.2.

Proof of Theorem 6.2.1 when $\kappa > \aleph_0$.

Suppose $|||\cdot|||$ is another algebra norm on $\mathscr{K}_\lambda(X)/\mathscr{K}_\kappa(X)$. Theorem 6.1.6 guarantees the existence of a $C > 0$ such that $|||T + \mathscr{K}_\kappa(X)||| \leq C\|T + \mathscr{K}_\kappa(X)\|$ for all $T \in \mathscr{K}_\lambda(X)$, so it remains to show that $\|\cdot\|$ is minimal.

Suppose $T \in \mathscr{K}_\lambda(X) \backslash \mathscr{K}_\kappa(X)$. By Proposition 6.3.1 or Corollary 6.3.3, there are operators $S, U \in \mathscr{K}_\lambda(X) \backslash \mathscr{K}_\kappa(X)$ and a set $A \subset I$ with $|A| = \kappa$, such that

$$P_A = STU \quad \text{and} \quad \|S + \mathscr{K}_\kappa(X)\| \cdot \|U + \mathscr{K}_\kappa(X)\| \leq 4\|T + \mathscr{K}_\kappa(X)\|^{-1}.$$

We have

$$(P_A + \mathscr{K}_\kappa(X))^2 = P_A^2 + \mathscr{K}_\kappa(X) = P_A + \mathscr{K}_\kappa(X),$$

thus

$$|||P_A + \mathscr{K}_\kappa(X)||| \;=\; |||(P_A + \mathscr{K}_\kappa(X))^2||| \;\le\; |||P_A + \mathscr{K}_\kappa(X)|||^2 \,.$$

Also, $P_A \notin \mathscr{K}_\kappa(X)$, by Lemma 6.1.2. Hence,

$$\begin{aligned} 1 \;\le\; |||P_A + \mathscr{K}_\kappa(X)||| &\le\; |||S + \mathscr{K}_\kappa(X)|||\cdot|||T + \mathscr{K}_\kappa(X)|||\cdot|||U + \mathscr{K}_\kappa(X)||| \\ &\le\; C^2|||T + \mathscr{K}_\kappa(X)|||\cdot\|S + \mathscr{K}_\kappa(X)\|\cdot\|U + \mathscr{K}_\kappa(X)\| \\ &\le\; 4C^2\|T + \mathscr{K}_\kappa(X)\|^{-1}|||T + \mathscr{K}_\kappa(X)|||\,. \end{aligned}$$

Therefore $\frac{1}{4}C^{-2}\|T + \mathscr{K}_\kappa(X)\| \le |||T + \mathscr{K}_\kappa(X)|||$, and so

$$\frac{1}{4}C^{-2}\|\cdot\| \;\le\; |||\cdot||| \;\le\; C\|\cdot\|\,.$$

That is, $|||\cdot||| \sim \|\cdot\|$ on $\mathscr{K}_\lambda(X)/\mathscr{K}_\kappa(X)$. ■

The above completes the proof of Theorem 6.2.1, and thus shows that all the κ-Calkin algebras of the non-separable analogues of c_0 and ℓ_p, $1 \le p < \infty$, have unique algebra norms. So we have again generalised the results of [Mey], but in a somewhat different direction to the one explored in Chapter 5.

However, as in the proof we gave for Theorem 5.1.2, and Theorem 3.3.2 before it, we relied on the maximality of the quotient norm to establish its minimality. In the next section, we shall show that this reliance is unnecessary, if we exclude the ℓ_1 case and put a particular restriction on the cofinality of κ.

6.4 Eidelheit-Yood revisited

At the beginning of §3.4, we indicated how a 'pure' version of the Eidelheit-Yood method might be used to show the minimality of the essential norm on $\mathscr{C}(c_0)$ and $\mathscr{C}(\ell_p)$, $1 \le p < \infty$. In this section, we provide the details of an analogous argument for κ-Calkin algebras of $c_0(I)$ and $\ell_p(I)$, $1 < p < \infty$, where I is a suitable infinite set. This reproves certain cases of Theorem 6.2.1 by a notably different method. The argument is cleaner than the one we could provide for the separable case, since we do not need to be concerned with compact perturbations of relevant operators. Note that we exclude the case $p = 1$. This is because, as mentioned at the start of §6.3, the technicalities relating to the structure of $\ell_1(I)$ are somewhat different. We also restrict attention to alephs κ for which $\mathrm{cf}\kappa > \mathfrak{c}$. Here, $\mathfrak{c}$ denotes the *cardinality of the continuum*, $\mathfrak{c} = 2^{\aleph_0}$. This restriction is necessary due to our reliance on a particular set theoretic argument, given in Lemma 6.4.5 below.

We begin with a trivial observation:

6.4.1 Proposition. *Let $\{A_j\}_{j\in J}$ be a disjoint partition of a set $I = \bigcup_{j\in J} A_j$, and X be $\ell_p(I)$ for some $1 \le p < \infty$. Then, for all $x \in X$,*

$$\|x\| \;=\; \left\|(\|P_{A_j}x\|)_{j\in J}\right\|_{\ell_p} .$$

Similarly, for all $x \in c_0(I)$,

$$\|x\| \;=\; \sup_{j\in J} \|P_{A_j}x\| .$$

Proof. Given $(x_i)_{i\in I} \in X$,

$$\left(\sum_{i\in I} |x_i|^p\right)^{1/p} \;=\; \left(\sum_{j\in J}\sum_{i\in A_j} |x_i|^p\right)^{1/p} \;=\; \left(\sum_{j\in J} \|P_{A_j}x\|^p\right)^{1/p} ,$$

and a similar equality holds in the case of $(x_i)_{i\in I} \in c_0(I)$. ∎

This property of the respective norms is important due to the following.

6.4.2 Lemma. *Let A, I be infinite sets with $|A| \le |I|$, and let X be $c_0(I)$ or $\ell_p(I)$, $1 \le p < \infty$. Suppose that $\{x_\alpha\}_{\alpha\in A} \subset X$ is a set of normalised vectors with disjoint supports $B_\alpha = \operatorname{supp}(x_\alpha)$, and that $\{x^*_\alpha\}_{\alpha\in A} \subset X^*$ is a set of functionals with disjoint supports $B'_\alpha = \operatorname{supp}(x^*_\alpha)$, such that $\sup_{\alpha\in A} \|x^*_\alpha\| = M < \infty$. Then the map*

$$x \;\mapsto\; S(x) \;=\; \sum_{\alpha\in A} x^*_\alpha(x)x_\alpha$$

defines an operator $S \in \mathscr{K}_{|A|^+}(X)$ for which $\|S\| \le M$. If, further, we assume that $\{x^*_\alpha\}_{\alpha\in A}$ is normalised, then $S \notin \mathscr{K}_{|A|}(X)$.

Proof. We will establish the result in the case $X = \ell_p(I)$, $1 \le p < \infty$. The proof for $c_0(I)$ is essentially the same, with sums of powers of p replaced by suprema in the following calculation. For all $x \in X$, we have

$$\begin{aligned}
\left\|\sum_{\alpha\in A} x^*_\alpha(x)x_\alpha\right\|^p &= \sum_{\alpha\in A}\left\|P_{B_\alpha}\left(\sum_{\alpha\in A} x^*_\alpha(P_{B'_\alpha}x)x_\alpha\right)\right\|^p && \text{[by Prop 6.4.1]}\\
&= \sum_{\alpha\in A} |x^*_\alpha(P_{B'_\alpha}x)|^p\,\|x_\alpha\|^p\\
&\le M^p\sum_{\alpha\in A} \|P_{B'_\alpha}x\|^p && \text{[since } \|x_\alpha\| = 1 \text{ and } \|x^*_\alpha\| \le M\text{]}
\end{aligned}$$

$$\leq M^p\|x\|^p\,, \qquad \text{[by Prop 6.4.1]}$$

hence $\|S\| \leq M$, and so $S \in \mathscr{B}(X)$.

Define $B = \bigcup_{\alpha\in A} B_\alpha$. Then we have that $P_B(x_\alpha) = x_\alpha$ for all $\alpha \in A$, hence $P_B S = S$. Also, by Lemma 6.1.2, P_B is a member of the ideal $\mathscr{K}_{|B|^+}(X)$. Thus $S \in \mathscr{K}_{|B|^+}(X)$, and

$$|A| \leq |B| = \sum_{\alpha\in A} |B_\alpha| \leq \sum_{\alpha\in A} \aleph_0 = |A|\cdot\aleph_0 = |A|\,,$$

from which we deduce that $S \in \mathscr{K}_{|A|^+}(X)$.

Now assume that $\{x_\alpha^*\}_{\alpha\in A}$ is normalised. Then for each $\alpha \in A$ and $\varepsilon > 0$, there exists $y_\alpha \in P_{B'_\alpha}X$ such that $\|y_\alpha\| = 1$ and $x_\alpha^*(y_\alpha) > 1-\varepsilon$, which gives $S(y_\alpha) = x_\alpha^*(y_\alpha)x_\alpha$. Because the supports of the x_α are disjoint, we can choose $\varepsilon > 0$ sufficiently small so that the balls $B_X(S(y_\alpha), \varepsilon)$ are disjoint. Hence if B is a set such that

$$\inf_{y\in B} \|S(y_\alpha - y)\| \leq \varepsilon$$

for all $\alpha \in A$, it is necessary that $|B| \geq |A|$. Therefore, from Definition 6.1.1, $S \notin \mathscr{K}_{|A|}(X)$. ■

In the cases in question, the following 'block-to-block' result is needed as a foundation for the Eidelheit-Yood method to function.

6.4.3 Proposition ([Daw, Prop. 6.1])**.** *Let I be an infinite set, let X be $c_0(I)$ or $\ell_p(I)$, $1 < p < \infty$, let $\kappa > \aleph_0$ be an aleph, and let $T \in \mathscr{B}(X)\backslash\mathscr{K}_\kappa(X)$. Then we can find a family $(x_i)_{i\in\kappa}$ of vectors in X such that*

(i) for $i \in \kappa$, we have $\|x_i\| = 1$ and $\|T(x_i)\| \geq \frac{1}{2}\|T + \mathscr{K}_\kappa(X)\|$; and

(ii) for each $i, j \in \kappa$ with $i \neq j$, we have

$$\mathrm{supp}(Tx_i) \cap \mathrm{supp}(Tx_j) = \mathrm{supp}(x_i) \cap \mathrm{supp}(x_j) = \emptyset\,.$$

6.4.4 Corollary. *Let I be an infinite set, let X be $c_0(I)$ or $\ell_p(I)$, $1 < p < \infty$, let $\kappa > \aleph_0$ be an aleph, and let $T \in \mathscr{B}(X)\backslash\mathscr{K}_\kappa(X)$. If $A \subset I$ is such that $|A| < \kappa$, then*

$$\left\| (TP_{I\backslash A})\big|_{\ker P_A T} \right\| \geq \frac{1}{2}\|T + \mathscr{K}_\kappa(X)\|\,.$$

Proof. Choose a family $(x_i)_{i\in\kappa}$ of vectors in X satisfying the conclusions of Proposition 6.4.3. Then, since $\operatorname{supp}(x_i)$ is disjoint from $\operatorname{supp}(x_j)$ for all $i, j \in \kappa$ with $i \neq j$, the set

$$B_1 = \{i \in \kappa : \operatorname{supp}(x_i) \cap A \neq \emptyset\}$$

has cardinality at most $|A|$. Similarly, the set

$$B_2 = \{i \in \kappa : \operatorname{supp}(Tx_i) \cap A \neq \emptyset\}$$

has cardinality at most $|A|$. Since $|A| < \kappa$, we have

$$|\kappa \setminus (B_1 \cup B_2)| = \kappa \neq 0 .$$

Therefore, there exists $i \in \kappa$ such that $i \notin B_1 \cup B_2$. For such an i, we have

$$(\operatorname{supp}(x_i) \cup \operatorname{supp}(Tx_i)) \cap A = \emptyset ,$$

hence $P_{I\setminus A}x_i = x_i$ and $P_A T x_i = 0$, so that $x_i \in \ker P_A T$. From Proposition 6.4.3 we know $\|x_i\| = 1$ and $\|T(x_i)\| \geq \frac{1}{2}\|T + \mathscr{K}_\kappa(X)\|$, thus

$$\left\| (TP_{I\setminus A})\big|_{\ker P_A T} \right\| \geq \|TP_{I\setminus A}x_i\| = \|T(x_i)\| \geq \frac{1}{2}\|T + \mathscr{K}_\kappa(X)\| . \qquad \blacksquare$$

One more lemma is required before we can present the main argument. Its conditions are the cause of our restriction on the cardinality of κ.

6.4.5 Lemma. *Let κ be an aleph for which $\operatorname{cf}\kappa > \mathfrak{c}$, and let J be a set for which $|J| = \kappa$. Suppose $\{a_{n,j} : n \in \mathbb{N}, j \in J\}$ is a set of scalars in $\mathbb{F}$. Then there exists $J' \subset J$ with $|J'| = \kappa$, and $(\alpha_n) \subset \mathbb{F}$, such that, for all $n \in \mathbb{N}$ and $j \in J'$,*

$$a_{n,j} = \alpha_n .$$

Proof. Suppose, to the contrary, that for every $(\alpha_n) \in \mathbb{F}^{\mathbb{N}}$ the set

$$J_{(\alpha_n)} = \{j : a_{n,j} = \alpha_n \text{ for all } n \in \mathbb{N}\}$$

has cardinality less than κ. Then we have

$$|J| = \Big| \bigcup_{(\alpha_n)\in\mathbb{F}^{\mathbb{N}}} J_{(\alpha_n)} \Big| = \sum_{(\alpha_n)\in\mathbb{F}^{\mathbb{N}}} \left| J_{(\alpha_n)} \right| ,$$

which gives $|J| < \kappa$, since $\operatorname{cf}\kappa > \mathfrak{c} = \left|\mathbb{F}^{\mathbb{N}}\right|$. This contradicts $|J| = \kappa$, so there must be $(\alpha_n) \subset \mathbb{F}$ such that $\left|J_{(\alpha_n)}\right| = \kappa$. ■

Alternative proof of Theorem 6.2.1 when $\mathrm{cf}\kappa > \mathfrak{c}$ ***and*** $p \neq 1$.

Suppose $|||\cdot|||$ is another algebra norm on $\mathscr{K}_\lambda(X)/\mathscr{K}_\kappa(X)$. Theorem 6.1.6 guarantees the existence of a $C > 0$ such that $|||T + \mathscr{K}_\kappa(X)||| \leq C\|T + \mathscr{K}_\kappa(X)\|$ for all $T \in \mathscr{K}_\lambda(X)$, so it remains to show that there cannot exist $(T_k) \subset \mathscr{K}_\lambda(X)$ with $\|T_k + \mathscr{K}_\kappa(X)\| \uparrow \infty$, while $|||T_k + \mathscr{K}_\kappa(X)||| = 1$ for all $k \in \mathbb{N}$. Assume, in order to gain a contradiction, that such a sequence (T_k) exists, and thin (T_k) such that, for all $k \in \mathbb{N}$, $\|T_k + \mathscr{K}_\kappa(X)\| \geq 2 \cdot 10^k$, which implies that $\|T_k\| \geq 2 \cdot 10^k$. Set $C_k = {}^{10^k}/_{9^{k+1}}$. By Corollary 1.1.5, there are $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ such that, for all $k \in \mathbb{N}$,

$$|x_0^*(T_k x_0)| \geq C_k .$$

Set $A_0 = \mathrm{supp}(x_0) \cup \bigcup_k \mathrm{supp}(T_k x_0)$, and note that $|A_0| \leq \aleph_0$. Without loss of generality, we may assume $x_0^* = x_0^* P_{A_0}$, so that $\mathrm{supp}(x_0^*) \subset A_0$.

We now treat κ as an ordinal and, for each ordinal $\alpha < \kappa$, we will choose $x_\alpha \in S_X$, $x_\alpha^* \in S_{X^*}$, and $A_\alpha \subset I$, by transfinite induction over κ. Suppose $\beta < \kappa$ is an ordinal for which we have already chosen $(x_\alpha)_{\alpha<\beta}$, $(x_\alpha^*)_{\alpha<\beta}$, and $(A_\alpha)_{\alpha<\beta}$, such that

(i) for all $k \in \mathbb{N}$ and $\alpha < \beta$,
$$|x_\alpha^*(T_k x_\alpha)| \geq C_k ,$$

(ii) for all $\alpha < \beta$,
$$\mathrm{supp}(x_\alpha^*) \subset A_\alpha = \mathrm{supp}(x_\alpha) \cup \bigcup_k \mathrm{supp}(T_k x_\alpha) ,$$

(iii) $\{\mathrm{supp}(x_\alpha)\}_{\alpha<\beta}$ are disjoint subsets of I, and

(iv) $\{A_\alpha\}_{\alpha<\beta}$, and thus $\{\mathrm{supp}(x_\alpha^*)\}_{\alpha<\beta}$, are disjoint subsets of I.

Set $A = \bigcup_{\alpha<\beta} A_\alpha$. Since $|A_\alpha| \leq \aleph_0$ for each $\alpha < \beta$, we have $|A| \leq |\beta| \cdot \aleph_0 < \kappa$. Therefore, by Corollary 6.4.4,

$$\left\|(T_k P_{I\setminus A})\big|_{\ker P_A T_k}\right\| \geq \frac{1}{2}\|T_k + \mathscr{K}_\kappa(X)\| \geq 10^k$$

for all $k \in \mathbb{N}$. Hence, by Corollary 1.1.5, there are $x_\beta \in S_X$ and $x_\beta^* \in S_{X^*}$ such that, for all $k \in \mathbb{N}$,

$$\left|x_\beta^*\Big((T_k P_{I\setminus A})\big|_{\ker P_A T_k} x_\beta\Big)\right| \geq C_k .$$

We may assume that $x_\beta = P_{I\backslash A}x_\beta$, so that $\{\text{supp}(x_\alpha)\}_{\alpha<\beta+1}$ are disjoint subsets of I. Set $A_\beta = \text{supp}(x_\beta) \cup \bigcup_k \text{supp}(T_k x_\beta)$. Without loss of generality, we may assume $x_\beta^* = x_\beta^* P_{A_\beta}$, so that $\text{supp}(x_\beta^*) \subset A_\beta$. Since $x_\beta \in \ker P_A T$, we have $A_\beta \cap A = \emptyset$. So $\{A_\alpha\}_{\alpha<\beta+1}$, and thus $\{\text{supp}(x_\alpha^*)\}_{\alpha<\beta+1}$, are disjoint subsets of I. Also, this implies $x_\beta^* = x_\beta^* P_{I\backslash A}$, hence

$$|x_\beta^*(T_k x_\beta)| \;=\; |x_\beta^* P_{I\backslash A}(T_k P_{I\backslash A} x_\beta)| \;\geq\; C_k$$

for all $k \in \mathbb{N}$. Thus, adding our choices of x_β and x_β^* to the end of their respective transfinite sequences, we have established that (i), (ii), (iii), and (iv) hold when β is replaced by $\beta + 1$.

Therefore, by transfinite induction over κ, there exist normalised sets $(x_\alpha)_{\alpha<\kappa} \subset X$ and $(x_\alpha^*)_{\alpha<\kappa} \subset X^*$ such that (i), (ii), (iii), and (iv) hold when β is replaced by $J = \kappa$. Set $a_{k,\alpha} = x_\alpha^*(T_k x_\alpha)$, for each $k \in \mathbb{N}$ and $\alpha < \kappa$. Then by Lemma 6.4.5 we can pass to a subset J' (not an ordinal) of J, for which we also have $|J'| = \kappa$, and find $(\alpha_k) \subset \mathbb{F}$ such that, for all $k \in \mathbb{N}$ and $j \in J'$,

$$x_j^*(T_k x_j) = \alpha_k\,.$$

Passing to the subset J' does not affect the facts (by (iii), (iv), and (i) above, respectively) that $\{x_j\}_{j\in J'}$ is a set of normalised vectors with disjoint supports, $\{x_j^*\}_{j\in J'}$ is a set of normalised functionals with disjoint supports, and

$$|\alpha_k| \;=\; |x_j^*(T_k x_j)| \;\geq\; C_k \uparrow \infty\,,$$

for all $k \in \mathbb{N}$ and $j \in J'$. Since $|J'| = \kappa$, all assumptions of Lemma 6.4.2 hold, and thus we can define $S \in \mathscr{K}_{\kappa^+}(X)\backslash\mathscr{K}_\kappa(X)$ by

$$Sx \;=\; \sum_{j\in J'} x_j^*(x)x_j\,.$$

Now, for each $k \in \mathbb{N}$ and $x \in X$,

$$\begin{aligned} ST_kSx \;&=\; S\left(\sum_{j\in J'} x_j^*(x)T_k x_j\right) \\ &=\; \sum_{j\in J'} x_j^*(x) \sum_{i\in J'} x_i^*(T_k x_j)x_i \\ &=\; \sum_{j\in J'} x_j^*(x)x_j^*(T_k x_j)x_j \qquad \text{[by (ii) and (iv)]} \end{aligned}$$

$$= \sum_{j \in J'} x_j^*(x)\alpha_k x_j$$
$$= \alpha_k Sx \,.$$

Thus $ST_kS = \alpha_k S$, and we have precisely the same situation as in the proof of the Eidelheit-Yood theorem (1.1.2). In particular,

$$\begin{aligned} |\alpha_k| \cdot |||S + \mathscr{K}_\kappa(X)||| &= |||ST_kS + \mathscr{K}_\kappa(X)||| \\ &\leq |||S + \mathscr{K}_\kappa(X)||| \cdot |||T_k + \mathscr{K}_\kappa(X)||| \cdot |||S + \mathscr{K}_\kappa(X)||| \\ &= |||S + \mathscr{K}_\kappa(X)|||^2 \,. \end{aligned}$$

Because $S \notin \mathscr{K}_\kappa(X)$, we have $|||S + \mathscr{K}_\kappa(X)||| \neq 0$. Therefore,

$$|\alpha_k| \;\leq\; |||S + \mathscr{K}_\kappa(X)||| \,,$$

in contradiction to the fact that $|\alpha_k| \uparrow \infty$. ■

Thus we see that, in comparison to the previous proofs given for Theorems 3.3.2, 5.1.2, and 6.2.1, maximality is not always required to show the minimality of a quotient norm. The stipulation that $\mathrm{cf}\kappa > \mathfrak{c}$ in the above argument seems unlikely to be a necessary feature. Although we shall not give the details here, if instead $\mathrm{cf}\kappa = \aleph_0$ then we can still show that the quotient norm is minimal on the κ-Calkin algebra of $c_0(I)$ of $\ell_p(I)$, $1 < p < \infty$, without requiring maximality, by allowing κ-compact perturbations within the Eidelheit-Yood method. This is in line with the claim we made in §3.4 (see p57). However, if $\aleph_0 < \mathrm{cf}\kappa \leq \mathfrak{c}$, then the situation regarding reliance on maximality remains unclear.

Chapter 7

Concluding Remarks

In this final chapter, we briefly consider two additional points. Firstly, in §§7.1–7.2, we give further examples of Banach spaces that seem very likely to have the UN property, or to have a similar structure that would also allow us to show uniqueness of the Calkin algebra norm. However, we are currently unable prove that this is the case for the examples considered. Then, in §7.3, we discuss a criterion for a Banach space X which guarantees that $\mathscr{C}(X)$ is semisimple, and hence shows that $\mathscr{E}(X) = \mathscr{K}(X)$. This gives another application of the perspective taken in Chapters 2 and 4, that non-compact operators map certain well-separated sequences to other well-separated sequences.

7.1 The L_p spaces for $1 < p < \infty$

We will use the abbreviation L_p for the Banach space $L_p[0,1]$, where in general $1 \leq p \leq \infty$. Since $L_p \approx L_p \oplus L_p$, $\|\cdot\|_e$ is a maximal norm on $\mathscr{C}(L_p)$ by Corollary 1.1.14. Because the general structural theory of L_p spaces has wide ranging applications, it would be interesting to know whether $\|\cdot\|_e$ is also minimal on $\mathscr{C}(L_p)$, and hence unique. If we restrict attention to values of p in the range $1 < p < \infty$, then L_p is reflexive and $L_p^* = L_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. Note also that L_2 is isometric to ℓ_2, so we already know that $\mathscr{C}(L_2)$ has a unique algebra norm. Thus, if we are to show that the Calkin algebras $\mathscr{C}(L_p)$, $1 < p < \infty$, have unique algebra norms, it suffices to consider the cases when $2 < p < \infty$, by Theorem 1.2.2.

Due to Theorem 5.1.2, it would be sufficient to show that such L_p have the UN property. As was noted at the end of §5.5 (see p100), we are yet to provide an example that satisfactorily uses Proposition 5.1.7 to establish the UN property of a Banach space. To use Proposition 5.1.7 requires the knowledge that there is a uniform constant $C > 0$, such that every normalised block basis has a C-complemented subsequence, C-equivalent to one of a finite number of 'canonical' subsymmetric basic sequences. The statement of this proposition was made with the example of L_p, $2 < p < \infty$, in mind, in light of the following.

7.1.1 Theorem ([KaPe, Thm. 2; Coroll. 6]). *Suppose $2 < p < \infty$ and $\varepsilon > 0$. For every normalised basic sequence (y_n) in L_p, either there is a subsequence of (y_n) which is $(1+\varepsilon)$-complemented in L_p and $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p, or else there is a subsequence of (y_n) which is complemented and equivalent to the unit vector basis of ℓ_2.*

The above dichotomy result of Kadec and Pełczyński appeared in 1962, and triggered a broad analysis of the L_p spaces using basic sequence techniques. Note how close it comes to establishing, via Proposition 5.1.7, that L_p has the UN property when $2 < p < \infty$. The only missing component is a uniform bound on the equivalence and complementation of subsequences that are equivalent to the unit vector basis of ℓ_2. Unfortunately, so far this obstacle has been impervious to further scrutiny. If, in the proof of Theorem 5.1.2, we do not have a uniform bound on the norms of the shift operators B_k, then the contradiction derived in the inequality 5.1.2c does not eventuate, and the minimality of the essential norm no longer follows. However, additional hope is provided by the following recent development.

7.1.2 Theorem ([Als, Thm. 1.3]). *Suppose that X is a subspace of L_p, where $2 < p < \infty$, which is isomorphic to ℓ_2. Then, for every $\varepsilon > 0$, there is a sequence (z_n) in X that is $(1+\varepsilon)$-equivalent to the standard the unit vector basis of ℓ_2, and $(1+\varepsilon)\gamma_p$-complemented in L_p, where γ_p is the norm of a symmetric Gaussian random variable.*

Set $2 < p < \infty$. A careful reading of the proof of the above result in [Als] reveals that, given a basic sequence (x_n) in L_p that is equivalent to the unit vector basis of ℓ_2, there is a block basic sequence (y_n) of (x_n) which is $(1+\varepsilon)$-equivalent to the standard the unit vector basis of ℓ_2, and $(1+\varepsilon)\gamma_p$-complemented in L_p. However, the proof does not allow passing to a similar *subsequence* (y_n) of (x_n). The techniques we used in Chapter 5 relied on the ability to pass from (x_n) to a subsequence with good complementation and equivalence properties, however they do not extend to a scenario where we can only pass to a block basic sequence of (x_n). Thus, despite appearances, Theorem 7.1.2 does not resolve the issue at hand, and we are left with:

7.1.3 Conjecture. For each $1 < p < \infty$, $\mathscr{C}(L_p)$ has a unique algebra norm.

The indications for L_1 and L_∞ are not as convincing. In particular, L_1 does not have an unconditional basis, and so has 'fewer' complemented subspaces to aid the definition of internal shift operators.

7.2 Lorentz sequence spaces

7.2.1 Definition (see, e.g., [ACL]). Let $1 \leq p < \infty$. For any $a = (a_n) \in c_0 \setminus \ell_1$, $a_1 \geq a_2 \geq \cdots \geq 0$, we define the *Lorentz sequence space* $d(a,p)$ to be the set

$$\left\{ x = (\alpha_n) \in c_0 \; : \; \sup_{\sigma \in \pi} \sum_i |\alpha_{\sigma(i)}|^p a_n < \infty \right\},$$

where π is the set of all permutations of $\mathbb{N}$.

With the norm

$$\|x\| = \left(\sup_{\sigma \in \pi} \sum_i |\alpha_{\sigma(i)}|^p a_n \right)^{1/p},$$

$d(a,p)$ is a Banach space, and the sequence of unit vectors (e_n) is a 1-symmetric basis for $d(a,p)$.

Given a Lorentz sequence space $d(a,p)$, the question of whether $\mathscr{C}(d(a,p))$ has a unique algebra norm presents a similar problem to the one described in §7.1 for $\mathscr{C}(L_p)$. First note that the existence of a symmetric basis guarantees that $d(a,p) \approx d(a,p) \oplus d(a,p)$, and hence that $\| \cdot \|_e$ is maximal on $\mathscr{C}(d(a,p))$ by Corollary 1.1.14. The similarity with the case of $\mathscr{C}(L_p)$ is then revealed by the following.

7.2.2 Theorem ([ACL, Coroll. 2]; [CaLi, Lem. 15]). *Let (e_n) be the unit vector basis of the Banach space $d(a,p)$, and suppose $\varepsilon > 0$. For every bounded block basic sequence (y_n) of (e_n), either there is a subsequence of (y_n) which is $(1+\varepsilon)$-complemented in $d(a,p)$ and $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p, or else (y_n) dominates (e_n).*

Here, we say that a basic sequence (y_n) *dominates* another basic sequence (x_n) if, for all $(c_n) \subset \mathbb{F}$, $\sum_n c_n y_n \in [(y_n)]$ implies that $\sum_n c_n x_n \in [(x_n)]$.

Note that the situation provided by Theorem 7.2.2 is not similar in all respects to the requirements of Proposition 5.1.7, nor those of the UN property. A more closely matching scenario would be one in which, in the case that (y_n) dominates (e_n), we could in fact conclude that $(y_n) \sim (e_n)$. However, we can relax the UN property to allow only domination in this instance, and not require equivalence. This is because the 'shift' operators S_k, in the proof of Theorem 5.1.2, need only to map from $[(z_n^k)]$ into a common subspace; we did not actually require the operators S_k to be invertible. Hence, Theorem 7.2.2 comes close to establishing that $\mathscr{C}(d(a,p))$ has a unique algebra norm via a generalisation of Proposition 5.1.7.

Complementation is still an issue. However, it is plausible that the relevant block bases could each have a complemented subsequence, possibly with additional assumptions on a and p. More critical is that we have no control over the norm of the mapping from $[(y_n)]$ to $[(e_n)]$, as guaranteed by the domination established in Theorem 7.2.2. As with the L_p case, we are hampered by the lack of a uniform bound for the relevant dominations/equivalences. Hence we are again left with:

7.2.3 Conjecture. Every Calkin algebra of a Lorentz sequence space has a unique algebra norm.

Recently, [Kam+] gave a partial description of the lattice of closed ideals in $\mathscr{B}(d(a,p))$, for a Lorentz sequence space $d(a,p)$. In addition to $\mathscr{C}(d(a,p))$, it seems likely that other quotients of $\mathscr{B}(d(a,p))$ will have various properties related to uniqueness of norm.

We could also generalise Conjecture 7.2.3 and ask whether every Calkin algebra of a Banach space with a symmetric basis, or a subsymmetric basis, or even just an unconditional basis, has a unique algebra norm. All these possibilities seem plausible, however there is no real evidence of their likelihood. In comparison, it would be very surprising if either Conjecture 7.1.3 or 7.2.3 does not hold.

7.3 Semisimple Calkin algebras

In this section we give a result of a somewhat different nature to the question of uniqueness of norm. However, our method of proof relies on a similar perspective to the one we took in order to generalise the Eidelheit-Yood theorem (1.1.2). Recall that our proof of Theorem 5.1.2 mimicked the Eidelheit-Yood method of proof, except we treated operators as acting on block basic sequences rather than individual vectors. We will make a similar substitution within the standard proof that $\mathscr{B}(X)$ is semisimple for all Banach spaces X. For reference, that proof is as follows.

7.3.1 Theorem (see, e.g., [Dal2, Prop. 2.2.4])**.** *Let X be a non-zero Banach space. Then $\mathscr{B}(X)$ is semisimple.*

Proof. Suppose $T \in \mathscr{B}(X)\setminus\{0\}$. Then there exist $x_0, y_0 \in X\setminus\{0\}$ with $Tx_0 = y_0$. Let $x_0^* \in X^*$ be such that $x_0^*(y_0) = 1$, and define $S \in \mathscr{B}(X)$ by $Sx = x_0^*(x)x_0$. Then

$$(I - ST)x_0 = x_0 - Sy_0 = 0,$$

and so $(I - ST) \notin \mathrm{Inv}\mathscr{B}(X)$. Hence $T \notin \mathrm{rad}\,\mathscr{B}(X)$, and thus $\mathrm{rad}\,\mathscr{B}(X) = \{0\}$. ■

The following strengthening of the block-to-block lemma (2.3.3) could be proved directly, but the required argument has essentially already been given as part of the proof of the block-to-blocks lemma (4.3.2), so we rely on that even stronger result instead.

7.3.2 Lemma. *Let X be a Banach space with a basis. Suppose that $0 < \delta < 1$, and that $T \in \mathscr{B}(X)\setminus\mathscr{K}(X)$. Then there exists a constant $C > 0$, and normalised aligned block bases (x_n) and (z_n) in X, such that (Tx_n) is a δ-small perturbation of (Cz_n).*

Proof. Set $T_k = kT$. Then $\|T_k\|_e \uparrow \infty$, so we can apply the block-to-blocks lemma (4.3.2). In particular, there is $C = C_1 > 0$ and normalised aligned block bases (x_n) and $(z_n) = (z_n^1)$, such that (Tx_n) is a δ-small perturbation of (Cz_n). ■

Using Lemma 7.3.2, we can prove an analogue of Theorem 7.3.1, of similar ilk to Theorem 5.1.2.

7.3.3 Theorem. *Let X be a Banach space with a basis. Suppose that, for every pair of aligned block bases $\{(x_n), (y_n)\}$ in X, there is a strictly increasing sequence $(n_m) \subset \mathbb{N}$ such that (x_{n_m}) and (y_{n_m}) are complemented, and $(x_{n_m}) \sim (y_{n_m})$. Then $\mathscr{C}(X)$ is semisimple, and hence*

$$\mathscr{A}(X) = \mathscr{K}(X) = \mathscr{S}(X) = \mathscr{E}(X).$$

Proof. Suppose $T \in \mathscr{B}(X)\setminus\mathscr{K}(X)$. By Lemma 7.3.2, there exists a constant $C > 0$, and normalised aligned block bases (x_n) and (z_n) in X, such that (Tx_n) is a δ-small perturbation of (Cz_n). By assumption, we can find subsequences (x_{n_m}) and (z_{n_m}) such that (x_{n_m}) and (z_{n_m}) are complemented, and $(x_{n_m}) \sim (z_{n_m})$. Also, (Tx_{n_m}) is a δ-small perturbation of (Cz_{n_m}). Hence, by Corollary 1.3.8, (Tx_{n_m}) is a basic sequence and there is $U \in \mathscr{B}(X)$ such that $UTx_{n_m} = Cz_{n_m}$ for all $m \in \mathbb{N}$. Let P be a projection onto $[(z_{n_m})]$, and R an isomorphism from $[(z_{n_m})]$ to $[(x_{n_m})]$ with $Rz_{n_m} = x_{n_m}$ for all $m \in \mathbb{N}$. Set

$$S = C^{-1}RPU \in \mathscr{B}(X).$$

Thus

$$(I - ST)x_{n_m} = x_{n_m} - C^{-1}RPUTx_{n_m} = x_{n_m} - C^{-1}RPCz_{n_m} = 0\,,$$

for all $m \in \mathbb{N}$. Hence

$$(I - ST)\big|_{[(x_{n_m})]} = 0\big|_{[(x_{n_m})]}\,.$$

Now suppose, in order to gain a contradiction, that $I - ST + \mathscr{K}(X) \in \operatorname{Inv}\mathscr{C}(X)$. Then there is $V \in \mathscr{B}(X)$ such that

$$V(I - ST) \in I + \mathscr{K}(X)\,.$$

Hence

$$0\big|_{[(x_{n_m})]} = V(I - ST)\big|_{[(x_{n_m})]} \in I\big|_{[(x_{n_m})]} + \mathscr{K}([(x_{n_m})], X)\,.$$

This implies that the identity on $[(x_{n_m})]$ is compact, a contradiction since $[(x_{n_m})]$ is infinite dimensional. Thus $I - ST + \mathscr{K}(X) \notin \operatorname{Inv}\mathscr{C}(X)$, which shows that $T + \mathscr{K}(X) \notin \operatorname{rad}\mathscr{C}(X)$. Therefore, $\operatorname{rad}\mathscr{C}(X) = \{\mathscr{K}(X)\}$. ■

Theorem 7.3.3 gives a result of a similar nature to [LLR, Coroll. 3.8], however the Banach spaces to which Theorem 7.3.3 applies form a distinct, although overlapping, class to those covered by [LLR, Coroll. 3.8]. This distinction comes from the different perspective, which grew out of our investigations in Chapters 2 and 4, that we have taken in comparison to previous authors.

In summary, in Chapter 2 we developed a novel, equivalent description of the measure of non-compactness of an operator. In Chapter 4 we employed that description to prove a Calkin algebra version of the uniform boundedness principle, which used the measure of non-compactness to determine the boundedness of a set. This allowed us to establish block-to-block properties which could then be used to generalise the Eidelheit-Yood method, in order to prove that the algebra norm of certain Calkin algebras was minimal and consequently unique. The content of Chapters 5 and 6 significantly extends the previous theorem of [Mey]. The examples provided in the present chapter indicate that there will be a wide variety of related results, given the weird and wonderful landscape of Banach space theory.

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Note. When warranted, details of the original source(s) of results that are quoted in the main text are given in parentheses, before the relevant statement. Such statements may, in some cases, be slightly reworded, or stated with regards to more specific assumptions than the corresponding rendition in the cited source.

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