TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 339, Number 2, October 1993

UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF $\Delta u + f(u) = 0$ IN $\mathbb{R}^n$, II

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ABSTRACT. We prove a uniqueness result for the positive solution of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$ which goes to 0 at ∞. The result applies to a wide class of nonlinear functions f, including the important model case $f(u) = -u + u^p$, $1 < p < (n+2)/(n-2)$. The result is proved by reducing to an initial-boundary problem for the ODE $u'' + (n-1)/r + f(u) = 0$ and using a shooting method.

1. INTRODUCTION

Let $n > 1$ be a real parameter, and let $f : [0, \infty) \to \mathbb{R}$ satisfy

(i) $f \in C^1([0, \infty))$, $f(0) = 0$, $f'(0) = -m < 0$, and

(ii) for some $\alpha > 0$, $f(u) < 0$ for $0 < u < \alpha$, $f(u) > 0$ for $u > \alpha$ and $f'(\alpha) > 0$.

We wish to study the question of uniqueness for solutions of the boundary value problem

$$u'' + \frac{n-1}{r}u' + f(u) = 0 \quad \text{for } r > 0, \tag{1}$$

$$u'(0) = 0, \quad u(r) \to 0 \quad \text{as } r \to \infty, \tag{2}$$

$$u(r) > 0 \quad \text{for } r \geq 0. \tag{3}$$

The problem (1)–(3) will be referred to as (GS), since a solution of (1)–(3) can be considered (at least when $n \geq 2$ is an integer) as a positive, radially symmetric solution $u(X) = u(\|X\|)$ of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, and in many physical situations such a positive solution will represent the state of lowest energy, or ground state, of the system.

Due to its large number of applications, (GS) has attracted considerable attention from mathematicians in recent years. In particular, the existence question is now well understood. We will mention here a special case of a result of Berestycki and Lions [1] which gives very simple conditions under which a solution of (GS) exists.

Theorem A. *In addition to* (i) *and* (ii), *let* f *satisfy*

(iii) *for some* $\beta > \alpha$, $F(\beta) = 0$, *where* $F(u) = \int_0^u f(s)\,ds$, *and*

Received by the editors May 17, 1990.
1991 *Mathematics Subject Classification.* Primary 35J60; Secondary 35B05.

(iv) $\lim_{u\to\infty} f(u)/u^l = 0$, *where* $l = (n+2)/(n-2)$ *if* $n > 2$, *and* l *can be any finite real number if* $1 < n \le 2$.

Then (*GS*) *has at least one solution.*

As an example, existence holds for the model case $f(u) = -u + u^p$ $(p > 1)$ when $1 < p < (n+2)/(n-2)$. (It follows from the Pohozaev identity [10] that (GS) has no solution when $n > 2$, $f(u) = -u+u^p$ and $p \ge (n+2)/(n-2)$). The proof of Theorem A in [1] uses PDE techniques; a similar theorem is proved by ODE methods in [2]. For a recent, simple existence proof which is less general but which still covers the model case, see [8].

The first uniqueness result for (GS) was proved by Coffman [3], who studied the model case with $n = p = 3$. Coffman's method was generalized in [7], where a result similar to Theorem 1 below was proved, with the restriction that $\lambda \ge n-2$ when $n > 2$. This assumption meant that uniqueness for the model case could only be proved in [7] for $1 < p \le n/(n-2)$, and not even for all of this range when $n > 4$. The next development was due to Kwong [5], who introduced two important new ideas: replacing certain complicated differential identities in [7] by arguments using Sturm oscillation, and a continuation argument in which the parameter n is varied continuously from $n \le 2$ (where the result is relatively simple) to larger values of n. Kwong applied his method only to the model case, but there it gave uniqueness for the full range $1 < p < (n+2)/(n-2)$. For a general f, Kwong's argument gives Theorem 1 with the additional requirement that f be convex. Recently, using different methods, Zhang [11] has succeeded in proving Theorem 1 assuming that f is convex for $u > \alpha$. The final proof of Theorem 1 resulted from combining a simplified version of Kwong's continuation argument with an essential idea from [11], which appears here as Lemma 10. Kwong and Zhang have also obtained further results related to Theorem 1 [6].

We now state the main result of the present paper. We first introduce a modified version of the I-function from [7]: for $\lambda > 0$, we define

$$I(u, \lambda) = \lambda u f'(u) - (\lambda+2) f(u). \tag{4}$$

Theorem 1. *Let* f *satisfy conditions* (i) *and* (ii), *and suppose that for each* $U > \alpha$ *there is a* $\lambda = \lambda(U) > 0$, *continuously depending on* U, *such that*

$$I(u, \lambda) \ge 0 \quad \textit{for } 0 < u < U \textit{ and} \tag{5}$$

$$I(u, \lambda) \le 0 \quad \textit{for } u > U. \tag{6}$$

Then (GS) *has at most one solution.*

The proof of this theorem will be given in §4. Since the main hypothesis can be hard to check directly, we will indicate in the final section of the paper how Theorem 1 can be applied to some specific functions. In the same section, we will state some additional results which follow from the proof of Theorem 1.

2. Preliminary results

It has become standard to study the uniqueness problem for (GS) by considering the initial value problem

$$u'' + \frac{n-1}{r} u' + f(u) = 0 \quad \text{for } r > 0, \tag{7}$$

$$u(0) = a > 0, \quad u'(0) = 0, \tag{8}$$

(9) u extends maximally to the right with $u \geq 0$.

The solution of this problem (which is unique [2]) will be denoted either by $u(r)$ or $u(r, a)$. We define

$$S^+ = \{a > 0 : u(r, a) \text{ remains bounded away from } 0\},$$
$$S^0 = \{a > 0 : u(r, a) \text{ solves (GS)}\},$$
$$S^- = \{a > 0 : u(R, a) = 0 \text{ for some (first) } R = R(a) > 0\}.$$

In case $a \in S^0$, we will also set $R(a) = \infty$. The variation $\delta(r) = \delta(r, a) = \partial u/\partial a$ satisfies

$$\delta'' + \frac{n-1}{r}\delta' + f'(u)\delta = 0, \tag{10}$$

$$\delta(0) = 1, \quad \delta'(0) = 0. \tag{11}$$

The following lemmas collect some well-known facts concerning the solutions of (7) and (10). For additional proofs of these results, the reader may consult [2] or [9].

Lemma 1. (a) *The sets S^+, S^- and S^0 are disjoint and cover the interval $(0, \infty)$, with $(0, \alpha] \in S^+$. In particular, if u is a solution of* (GS), *then $u(0) > \alpha$. Also, S^+ and S^- are both open in $(0, \infty)$.*

(b) *Any solution u with $u(0) \in S^0 \cup S^-$ is monotone decreasing. In particular, any solution of* (GS) *is monotone decreasing.*

(c) *If u is a solution of* (GS), *then for any $\varepsilon \in (0, m)$,*

$$\limsup_{r\to\infty} u(r)e^{r\sqrt{m-\varepsilon}} < \infty, \quad \limsup_{r\to\infty} |u'(r)|e^{r\sqrt{m-\varepsilon}} < \infty,$$

and

$$\frac{|u'(r)|}{u(r)} \to \sqrt{-m} \quad \text{as } r \to \infty.$$

(d) *If u is a solution of* (7) *with $u(0) \in S^+$ then u has an infinite number of local maxima and minima. Furthermore, if $u(r_0)$ is a local minimum of u then $u(r) > u(r_0)$ for $r > r_0$, while if $u(r_0)$ is a local maximum then $u(r) < u(r_0)$ for $r > r_0$.*

Lemma 2. (a) *If $u(0) \in S^0 \cup S^-$, then δ has only a finite number of critical points and a finite number of zeroes in $(0, R)$.*

(b) *If $u(0) \in S^0$, then as $r \to \infty$ either*

$$\delta(r) \to \pm\infty, \quad \delta'(r) \to \pm\infty \text{ with } \delta(r)\delta'(r) > 0 \text{ for large } r,$$

or

$$\delta(r) \to 0, \quad \delta'(r) \to 0 \text{ with } \delta(r)\delta'(r) < 0 \text{ for large } r.$$

In the second case, for any $\varepsilon \in (0, m)$,

$$\limsup_{r\to\infty} |\delta(r)|e^{r\sqrt{m-\varepsilon}} < \infty, \quad \limsup_{r\to\infty} |\delta'(r)|e^{r\sqrt{m-\varepsilon}} < \infty.$$

Proof. From (10), if u is small enough that $f'(u) < 0$, then any positive critical point of δ can only be a local minimum. Similarly, any negative critical value can only be a local maximum. Thus, δ can have at most one critical point after $f'(u) = 0$ for the last time. Since critical points of δ are isolated, δ can have at most a finite number of critical points, and hence a finite number of zeroes.

The proof of part (b) is standard, using the fact that $u(r)$ is small for large r, so that one has essentially $f'(u) = -m$.

Our next lemma gives a first relation between the sign of $\delta(R)$ and the local behaviour of solutions of (7)–(9).

Lemma 3. (a) *Let $a_0 \in S^-$ and suppose that $\delta(R(a_0)) < 0$. Then $R(a)$ is a decreasing function of a near a_0.*

(b) *Let $a_0 \in S^0$ and suppose that $\delta(r) \to -\infty$ as $r \to \infty$. Then for some $\varepsilon > 0$ the interval $(a_0, a_0 + \varepsilon)$ is contained in S^-, while $(a_0 - \varepsilon, a_0) \subset S^+$. Similarly, if $\delta(r) \to +\infty$, some right neighbourhood of a_0 is contained in S^+, while a left neighbourhood is contained in S^-.*

Proof. (a) From the implicit function theorem applied to $u(R(a), a) = 0$, we see that $R(a)$ is a differentiable function of a for $a \in S^-$, and that

$$u'(R(a), a)R'(a) + \delta(R(a), a) = 0.$$

Since $u'(R(a), a) < 0$, the assumption $\delta(R(a_0), a_0) < 0$ gives $R'(a_0) < 0$.

(b) Suppose $f'(u) < 0$ for $u \in [0, \gamma)$, and let R_1 be fixed so large that $u(r, a_0) \leq \frac{1}{2}\gamma$ when $r \geq R_1$. If $\delta(r) \to -\infty$ as $r \to \infty$, we see from Lemma 2(b) that for some $R_2 > R_1$ we have $\delta(R_2) < 0$ and $\delta'(R_2) < 0$. Thus, for $a > a_0$ but close to a_0, we have

$$u(R_2, a) < u(R_2, a_0), \quad u'(R_2, a) < u'(R_2, a_0). \tag{12}$$

If $a \in S^+$, there would be a subsequent point $R_3 > R_2$ at which $u(R_3, a) = u(R_3, a_0)$, while if $a \in S^0$, both $u(r, a)$ and $u(r, a_0)$ approach 0 as $r \to \infty$. In either case, by (12), the function $w(r) = u(r, a) - u(r, a_0)$ must have a negative minimum after R_2. However, w satisfies

$$w'' + \frac{n-1}{r}w' + f'(\theta(r))w = 0,$$

where $\theta(r)$ is between $u(r, a)$ and $u(r, a_0)$, so that $f'(\theta(r)) < 0$ in (R_2, R_3), and any negative critical point of w in this interval could only be a maximum. Thus, $a \notin S^+ \cup S^0$ and so $a \in S^-$.

The other cases are all similar. In the two cases in which $u(R_2, a) > u(R_2, a_0)$, a must be taken sufficiently close to a_0 so that $u(R_2, a) < \gamma$.

Lemma 3 shows that information about $u(r, a)$ can be obtained from an analysis of $\delta(r, a)$. In particular, information on the number of zeroes of δ plays a crucial role in the proof of Theorem 1. The following terminology is due to Kwong [5].

Definition. Let $a \in S^0 \cup S^-$. a is said to be *admissible* if $\delta(r, a)$ has exactly one zero in $[0, R)$. a is said to be *strictly admissible* if a is admissible and in addition $\delta(R, a) < 0$ (or $\delta(r) \to -\infty$ as $r \to \infty$, in case $a \in S^0$).

In the remaining sections of the paper, we will show that, under the hypotheses of Theorem 1, every $a \in S^0 \cup S^-$ is strictly admissible. Theorem 1 then follows easily. The heart of the argument is Lemma 8, in which we show that admissibility implies strict admissibility. The full proof of Lemma 8 uses the oscillation results of §3, but we can finish the present section by proving a lemma which implies Lemma 8 in case $\delta(r) = 0$ at some r where $u(r) \leq \alpha$.

Lemma 4. *Suppose that $a \in S^0 \cup S^-$ and that $\delta(r) \to 0$ as $r \to R$. Then if $r_1 \in (0, R)$ is such that $\delta(r_1) = 0$, we have $u(r_1) > \alpha$.*

Proof. Assume for contradiction that $\delta(r_1) = 0$ and $u(r_1) \le \alpha$. Without loss of generality, we may assume that r_1 is the last zero of δ in $(0, R)$, and that $\delta < 0$ in (r_1, R). From the equations satisfied by u and δ, we obtain

$$\begin{aligned}[(r^{n-1}u')(r^{n-1}\delta')]' &= -r^{2n-2}[f(u)\delta' + f'(u)u'\delta] \\ &= -r^{2n-2}[f(u)\delta]'.\end{aligned}$$

Integrating from r_1 to some $r_2 \in (r_1, R)$ and then integrating by parts, we obtain

$$\begin{aligned}&r_2^{2n-2}u'(r_2)\delta'(r_2) - r_1^{2n-2}u'(r_1)\delta'(r_1) \\ &\quad = -[s^{2n-2}f(u(s))\delta(s)]_{r_1}^{r_2} + (2n-2)\int_{r_1}^{r_2} s^{2n-3}f(u(s))\delta(s)\,ds.\end{aligned}$$

Now let $r_2 \to R$. If $R < \infty$ we note that $\delta'(R) > 0$ (since $\delta < 0$ in (r_1, R) and $\delta(R) = 0$), while if $R = \infty$ we apply Lemmas 1(c) and 2(b). Noting also that $f(u(r)) < 0$ in (r_1, R), we obtain

$$\begin{aligned}0 &> R^{2n-2}u'(R)\delta'(R) - r_1^{2n-2}u'(r_1)\delta'(r_1) \\ &= (2n-2)\int_{r_1}^{R} s^{2n-3}f(u(s))\delta(s)\,ds > 0.\end{aligned}$$

(If $R = \infty$, the term $R^{2n-2}u'(R)\delta'(R)$ is to be interpreted as 0.) This is a contradiction, and the lemma is proved.

3. Oscillation and disconjugacy

The oscillation result we need is the following lemma, which is nothing more than a special case of the Sturm Comparison Theorem. Part (b) is perhaps less well known than part (a), but can be proved by the same methods.

Lemma 5. *Let Y and Z be nontrivial solutions of*

$$Y'' + \frac{n-1}{r}Y' + g(r)Y = 0, \tag{13}$$

$$Z'' + \frac{n-1}{r}Z' + G(r)Z = 0 \tag{14}$$

respectively on some interval $(\mu, \nu) \subset (0, \infty)$, where g and G are continuous on (μ, ν), $G \ge g$ on (μ, ν) and $G \not\equiv g$. If either

(a) $\mu > 0$ and $Y(\mu) = Y(\nu) = 0$, or

(b) $\mu = 0$, Y and Z are continuous at μ and $Y'(\mu) = Z'(\mu) = Y(\nu) = 0$, then Z has at least one zero in (μ, ν). The same conclusions hold if $G \equiv g$ on (μ, ν), provided Y and Z are linearly independent.

Remark. In the situation of Lemma 5, we will say that Z oscillates faster than Y (or that Y oscillates slower than Z) on (μ, ν).

Definition. Suppose that (13) has at least one solution which does not vanish in some neighbourhood of ∞. Define

$$\rho = \inf\{r \in (0, \infty) : \text{ there is a solution of (13) with no zeroes in } (r, \infty)\}.$$

The interval (ρ, ∞) will be called the *disconjugacy interval* of (13).

Clearly, no solution of (13) can have two zeroes in (ρ, ∞), for if a solution vanishes at r_1 and r_2 in (ρ, ∞) then any linearly independent solution would vanish in (r_1, r_2) and the disconjugacy interval could not be any larger than (r_1, ∞). On the other hand, if $\rho > 0$ any solution of (13) with a zero in $(0, \rho)$ must have a subsequent zero, or the disconjugacy interval would be larger than (ρ, ∞). Thus the last zero of any solution of (13) must lie in $[\rho, \infty)$. In the next lemma, we distinguish those solutions whose last zero is precisely at ρ.

Lemma 6. *Let $g(r)$ be continuous on $(0, \infty)$, and suppose that $g(r) < 0$ for large r. Let the disconjugacy interval of* (13) *be (ρ, ∞) with $\rho > 0$, and suppose that* (13) *has a solution which goes to 0 as $r \to \infty$. If Y is a nontrivial solution of* (13) *such that $Y(\rho) = 0$, then Y has no subsequent zeroes and $Y(r) \to 0$ as $r \to \infty$. Conversely, if Y is a nontrivial solution of* (13) *with a zero in (ρ, ∞), then Y does not approach zero as $r \to \infty$.*

Proof. Let $R_0 > \rho$ be so large that $g(r) < 0$ in (R_0, ∞). From (13) we see that at any critical point in (R_0, ∞), a nontrivial solution of (13) must turn away from the r-axis. It follows that any nontrivial solution can have at most one critical point in (R_0, ∞), and so is ultimately monotone. In addition, the solution Y_2 of (13) with $Y_2(R_0) = 0$, $Y_2'(R_0) = 1$ has no critical point in (R_0, ∞), and so does not approach 0 as $r \to \infty$.

Now let Y_1 be a solution of (13) such that $Y_1(r) \to 0$ as $r \to \infty$. Then any other such solution must be a multiple of Y_1, and any solution not a multiple of Y_1 is of the form $c_1 Y_1 + c_2 Y_2$ with $c_2 \neq 0$, and so does not approach 0 as $r \to \infty$. Let ρ_1 be the last zero of Y_1. By the remarks preceding the lemma, $\rho_1 \geq \rho$. We claim that $\rho_1 = \rho$, which will imply the conclusion of the lemma.

Suppose for contradiction that $\rho_1 > \rho$. For small $\varepsilon > 0$ the solution Y_3 of (13) with $Y_3(\rho_1 - \varepsilon) = 0$, $Y_3'(\rho_1 - \varepsilon) = 1$ has $Y_3(\rho_1) > 0$, so Y_3 is not a multiple of Y_1 and Y_3 does not go to 0 as $r \to \infty$. The last zero of Y_2 before R_0 (if there is one) cannot be at ρ_1 (since Y_2 is not a multiple of Y_1) or in (ρ_1, R_0) (by Lemma 5). Therefore, if ε is sufficiently small, $Y_2 < 0$ on $[\rho_1 - \varepsilon, R_0)$. We now write $Y_3 = c_1 Y_1 + c_2 Y_2$. Evaluating at ρ_1 shows that $c_2 < 0$. Since $Y_1 \to 0$ while Y_2 is positive and bounded away from 0 for large r, Y_3 must eventually become negative. But this implies (for ε small) that Y_3 has two zeroes in (ρ, ∞).

Remark. If the disconjugacy interval of (13) is (ρ, ∞) and $g \leq G < 0$ on this interval, then the disconjugacy interval of (14) cannot be any larger than (ρ, ∞). Specifically, if Y and Z are solutions of (13) and (14) with $Y(\rho) = Z(\rho) = 0$, then $Y \to 0$ as $r \to \infty$, by Lemma 6, and an analysis of the Wronskian of Y and Z shows that if $Z > 0$ on (ρ, ∞) then $Z \to 0$ as $r \to \infty$. In case Y and Z both decay exponentially fast at ∞ (which is the only case we will use) the proof is similar to the standard proof of Lemma 5; if either Y or Z decays less rapidly, the analysis is slightly more delicate.

4. Proof of Theorem 1

Recall that our aim is to show that every $a \in S^0 \cup S^-$ is strictly admissible. As a first application of our oscillation results, we will show that for every such a the variation $\delta(r, a)$ has at least one zero in $(0, R)$. In this proof, we first use an auxiliary function which will prove to be very useful in the remainder of the paper. For $\lambda > 0$ we define

$$v(r) = v_\lambda(r) = v_\lambda(r, a) = ru'(r) + \lambda u(r). \tag{15}$$

A calculation using (7) shows that v_λ is a solution of

$$v'' + \frac{n-1}{r}v' + f'(u)v = I(u, \lambda), \tag{16}$$

which can be written as

$$v'' + \frac{n-1}{r}v' + \left[f'(u) - \frac{I(u, \lambda)}{v}\right] v = 0 \tag{17}$$

in any interval in which $v \neq 0$.

Lemma 7. *Under the assumptions of Theorem* 1, *for any* $a \in S^0 \cup S^-$ *the variation* $\delta(r, a)$ *has at least one zero in* $(0, R)$.

Proof. Suppose δ never vanishes in $(0, R)$. Then $\delta > 0$ in $(0, R)$, since $\delta(0) = 1$. Let $\lambda = \lambda(a)$ be the value corresponding to $U = a$ in the hypothesis of Theorem 1. Note that $a > \alpha$ by Lemma 1(a). Then $I(u, \lambda) \geq 0$ for $0 < u < a = u(0)$, so if $I(u, \lambda) \not\equiv 0$, comparison of (10) and (17) shows that v_λ oscillates more slowly than δ as long as $v_\lambda > 0$. But $v_\lambda(0) > 0$ and $v_\lambda(R) < 0$ (using Lemma 1(c) in case $R = \infty$), so v_λ must have a zero in $(0, R)$, which is a contradiction. (In case $I(u, \lambda) \equiv 0$, δ must be a positive multiple of v_λ and so again $\delta(R) < 0$.)

Lemma 8. *Suppose* f *satisfies the hypotheses of Theorem* 1 *and let* $a \in S^0 \cup S^-$ *be admissible. Then* a *is strictly admissible.*

Proof. Suppose that a is admissible but not strictly admissible. Then δ has exactly one zero in $(0, R)$ and $\delta(r) \to 0$ as $r \to R$. Let r_1 be the zero of δ, and note by Lemma 4 that $u(r_1) > \alpha$. Let $\lambda_1 = \lambda(u(r_1))$. Then $I(u, \lambda_1) \leq 0$ for $u > u(r_1)$, so comparison of (10) and (17) shows that if $v_{\lambda_1} > 0$ in $(0, r_1]$ it would oscillate faster than δ in this interval. But this contradicts Lemma 5(b), and we conclude that the first zero of v_{λ_1} occurs in $(0, r_1]$.

Now let $\lambda_2 = \lambda(u(0))$. As in the proof of Lemma 7, v_{λ_2} oscillates more slowly than δ in $(0, r_1)$ (as long as $v_{\lambda_2} > 0$) and so the first zero of v_{λ_2} occurs at or after r_1. It follows that there is some value of λ between λ_1 and λ_2 such that the first zero of v_λ occurs exactly at r_1. Since $\lambda(U)$ depends continuously on U, this value of λ can be chosen as $\lambda(u(r))$ for some $r \in [0, r_1]$, and then $I(u, \lambda)$ is nonnegative at r_1 and remains nonnegative in $[r_1, \infty)$. In some interval to the right of r_1, then, we have $I(u, \lambda) \geq 0$ and $v_\lambda < 0$, and as long as these inequalities persist, v_λ oscillates more quickly than δ.

Suppose first that v_λ has no zero beyond r_1. Then neither has δ, which oscillates more slowly than v_λ. If $R < \infty$, this shows that a is strictly admissible. If $R = \infty$ then $v_\lambda \to -\infty$ as $r \to \infty$ by Lemma 1(c) so, by Lemma 6, r_1 is an interior point of the disconjugacy interval for (17). The disconjugacy interval of the less oscillatory equation (10) cannot be any shorter than that of (17) so, by Lemma 6 again, $\delta(r) \to -\infty$ as $r \to \infty$, and again a is strictly admissible.

If on the other hand $v_\lambda(r_2) = 0$ for some $r_2 > r_1$, then v_λ oscillates more slowly than δ between r_2 and any subsequent zero of v_λ. Since δ has no subsequent zeroes neither does v_λ, so $v_\lambda(r) > 0$ for r close to R. It is clear from (15), however, that $v_\lambda < 0$ for r close to R, so this case cannot occur and the lemma is proved.

Lemma 9. *The set of* (*strictly*) *admissible* $a \in S^0 \cup S^-$ *is both open and closed in* $S^0 \cup S^-$.

Proof. By Lemma 7, the set of nonadmissible a's consists of those a for which $\delta(r, a)$ has at least two zeroes in $(0, R)$. This set is open, by the continuity of δ on a, so the set of admissible a's is closed. However the set of strictly admissible a's is open. This is clear from continuity if $a \in S^-$, while if $a \in S^0$ we can find R_0 so large that $\delta(r, a) < 0$, $\delta'(r, a) < 0$ and $f'(u(r)) < 0$ for $r \geq R_0$. By continuity, if we perturb a slightly we will still have $\delta'(R_0) < 0$, and since δ can then have no subsequent critical point the perturbed a is still strictly admissible. The result now follows from Lemma 8.

It should be clear that Lemma 9 will allow us to apply some type of continuation argument. It is also clear, however, that we must show that we can start somewhere; i.e. that the set of admissible a's is not empty. Our final lemma, due to Zhang [11], shows that the smallest a in $S^0 \cup S^-$ is admissible. (Of course, if no such a exists then $S^+ = (0, \infty)$, $S^0 = \varnothing$ and Theorem 1 is trivially true.)

Lemma 10. *The value* $a_0 = \inf(S^0 \cup S^-)$ *is admissible.*

Proof. Note that in fact $a_0 \in S^0$ (since S^- is open) and assume that $\delta(r, a_0)$ has two or more zeroes in $(0, \infty)$; let the first two zeroes of $\delta(r, a_0)$ be at r_1 and r_2. Then $\delta(r, a_0) > 0$ in $(0, r_1)$, $\delta(r, a_0) < 0$ in (r_1, r_2) and $\delta(r, a_0) > 0$ in some interval to the right of r_2. For $a < a_0$ but close enough to a_0, then, the solution $u(r, a)$ will intersect $u(r, a_0)$ at least twice. Let the first two intersections be at $y_1(a)$ and $y_2(a)$. Note that $y_2(a) < r_0(a)$, where $r_0(a)$ is the first minimum of $u(r, a)$, for if $u(r_0, a) > u(r_0, a_0)$ then by Lemma 1(b)(d) there can be no further intersection after r_0.

Now decrease a continuously to α. The intersection points y_1 and y_2 will vary continuously with a and cannot coalesce (for otherwise $u(r, a)$ and $u(r, a_0)$ would become tangent, contradicting uniqueness). Note that the function $w(r) = u(r) - \alpha$ satisfies

$$w'' + \frac{n-1}{r} w' + f'(\theta(r)) w = 0, \tag{18}$$

where $\theta(r)$ is between α and $u(r)$. For a close to α, the solutions of (18) behave very much like the solutions of

$$\phi'' + \frac{n-1}{r}\phi' + f'(\alpha)\phi = 0.$$

In particular, solutions of (18) will oscillate more quickly than solutions of

$$\psi'' + \frac{n-1}{r}\psi' + \frac{1}{2} f'(\alpha)\psi = 0$$

(recall that $f'(\alpha) > 0$). But this means that as $a \to \alpha$ the second intersection of $u(r, a)$ with the horizontal line $u = \alpha$ remains bounded, and hence so do $r_0(a)$ and $y_2(a)$. Since y_1 and y_2 neither coalesce nor become unbounded, we see that for all $a \in (\alpha, a_0)$ $u(r, a)$ intersects $u(r, a_0)$ at least twice.

However, for a close to α, $u(r, a)$ remains close to $u(r, \alpha) \equiv \alpha$ in any bounded interval, and its derivative remains close to $u'(r, \alpha) \equiv 0$. Since $u'(r, a_0)$ is bounded away from 0 in any compact interval not containing the origin, it follows that for a close to α, $u(r, a)$ can intersect $u(r, a_0)$ only once, and the lemma is proved.

The proof of Theorem 1 is now straightforward. Since a_0 is admissible, it is strictly admissible by Lemma 8. By Lemma 3(b), for some $\varepsilon > 0$ the interval $(a_0, a_0 + \varepsilon)$ is contained in S^-. Using Lemma 9, we see that every $a \in (a_0, a_0 + \varepsilon)$ is strictly admissible, so $R(a)$ is a strictly decreasing function of a on this interval, by Lemma 3(a). As we continue to raise a, Lemmas 10 and 3(b) show that a continues to be strictly admissible and $R(a)$ continues to decrease. Thus $(a_0, \infty) \subset S^-$, and every $a \in S^0 \cup S^-$ is strictly admissible. The proof of Theorem 1 is complete.

5. Examples and additional results

We will show first how the main hypothesis of Theorem 1 can be easily verified for a large class of functions. The following theorem is essentially Theorem 2 of [7].

Theorem 2. *Suppose f satisfies* (i) *and* (ii) *and that there is some $\tau \geq 1$ such that*

$$u^{-\tau} f(u) \text{ is increasing for } u > 0, \quad \text{and} \tag{19}$$

$$u(u^{-\tau} f(u))' \text{ is decreasing for } u > \alpha. \tag{20}$$

(In case $\tau = 1$, we require $u(u^{-\tau} f(u))'$ to be strictly decreasing for $u > \alpha$.) Then (GS) has at most one solution.

Proof. Define $J(u, \tau) = uf'(u) - \tau f(u)$ and observe that (19) implies

$$0 \leq u(u^{-\tau} f(u))' = u^{-\tau} J(u, \tau). \tag{21}$$

Further, if $\tau = 1$ and $uf'(u) - f(u) = 0$ for some $u = u_0 > \alpha$, then $uf'(u) - f(u) < 0$ for all $u > u_0$, contradicting (21). Thus, when $\tau = 1$,

$$J(u, 1) = uf'(u) - f(u) > 0 \quad \text{for } u > 0. \tag{22}$$

Now let $\sigma \geq \tau$. For $0 < u \leq \alpha$, $f(u) \leq 0$, and so $J(u, \sigma) \geq J(u, \tau) \geq 0$. Note also that $J(\alpha, \sigma) = \alpha f'(\alpha) > 0$. For $u > \alpha$ we have

$$u(u^{-\sigma} f(u))' = u(u^{\tau-\sigma} u^{-\tau} f(u))' = u^{\tau-\sigma}[u(u^{-\tau} f(u))' + (\tau - \sigma) u^{-\tau} f(u)].$$

Since the quantity in square brackets is decreasing for $u > \alpha$, if $(u^{-\sigma} f(u))'$ ever vanishes, it remains nonpositive from then on. The same is therefore true of $J(u, \sigma) = u^{\sigma+1}(u^{-\sigma} f(u))'$.

We show next that for each $U > \alpha$ there is a $\sigma \geq \tau$ $(\sigma > 1)$ such that $J(U, \sigma) = 0$. Thus, fix $U > \alpha$. From (21) and (22) we have $Uf'(U) - \tau f(U) \geq 0$, with strict inequality if $\tau = 1$. In particular, $f'(U) > 0$. Now, for $s \geq \tau$, consider the linear function of s

$$\ell(s) = \frac{Uf'(U) - sf(U)}{f'(U)} = \frac{J(U, s)}{f'(U)}.$$

$\ell(\tau) \geq 0$ $(> 0$ if $\tau = 1)$ and $\partial \ell / \partial s < 0$. It follows that $\ell(\sigma) = 0$ for some σ in the required range.

The proof of Theorem 2 is completed by putting $\lambda = 2/(\sigma - 1)$ where σ is the value found above, and noting that $I(u, \lambda) = \lambda J(u, \sigma)$ satisfies the requirements of Theorem 1.

Example. Let f be a general "polynomial",

$$f(u) = \sum_{k=1}^{\nu} a_k u^{p_k},$$

where $1 = p_1 < p_2 < \cdots < p_\nu = p$. It can be easily checked by differentiation that f will satisfy (19) and (20) with the choice $\tau = p$, provided $a_1 < 0$, $a_k \leq 0$ for $2 \leq k \leq \nu - 1$ and $a_\nu > 0$. In fact, $(u^{-p} f(u))' > 0$, which shows that $f'(u) > 0$ whenever $f(u) \geq 0$. Thus f also satisfies (i) and (ii), so uniqueness holds for such functions.

We next give a theorem which can be interpreted as a uniqueness result for the Dirichlet problem in a finite ball. Recall that in proving Theorem 1, we actually proved that every $a \in S^0 \cup S^-$ was strictly admissible, and that therefore S^- is an interval and $R(a)$ is a strictly decreasing function of a on S^-. It follows that for each fixed $R > 0$ there is at most one value of a for which $R(a) = R$, which proves the next theorem.

Theorem 3. *Let f satisfy the hypotheses of Theorem 1. Then for each $R > 0$ the problem*

$$u'' + \frac{n-1}{r} u' + f(u) = 0,$$
$$u > 0, \quad u'(0) = 0, \quad u(R) = 0$$

has at most one solution

Finally, as pointed out in [5 and 7], our methods may also be applied to the exterior Neumann problem

$$u'' + \frac{n-1}{r} u' + f(u) = 0,$$
$$u > 0 \text{ on } (b, R), \quad u'(b) = 0, \quad u(R) = 0,$$

where $0 < b < R \leq \infty$. Uniqueness will again hold if f satisfies the hypotheses of Theorem 1. For this problem, existence is known to hold for the model case for all $p > 1$ in all dimensions [4], and we obtain uniqueness also for all $p > 1$.

References

1. H. Berestycki and P.-L. Lions, *Non-linear scalar field equations.* I, *Existence of a ground state*; II, *Existence of finitely many solutions*, Arch. Rational Mech. Anal. **82** (1983), 313–375.
2. H. Berestycki, P.-L. Lions, and L. A. Peletier, *An ODE approach to the existence of positive solutions for semilinear problems in* $\mathbb{R}^n$, Indiana Univ. Math. J. **30** (1981), 141–167.
3. C. V. Coffman, *Uniqueness of the ground state for* $\Delta u - u + u^3 = 0$ *and a variational characterization of other solutions*, Arch. Rational Mech. Anal. **46** (1972), 12–95.
4. X. Garaizar, *Existence of positive radial solutions for semilinear elliptic equations in the annulus*, J. Differential Equations **70** (1987), 69–92.
5. M. K. Kwong, *Uniqueness of positive radial solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^n$, Arch. Rational Mech. Anal. **105** (1989), 243–266.

6. ——, Personal communication.

7. K. McLeod and J. Serrin, *Uniqueness of positive radial solutions of* $\Delta u + f(u) = 0$ *in* $\mathbb{R}^n$, Arch. Rational Mech. Anal. **99** (1987), 115–145.

8. K. McLeod, W. C. Troy, and F. B. Weissler, *Radial solutions of* $\Delta u + f(u) = 0$ *with prescribed numbers of zeroes*, J. Differential Equations **83** (1990), 368–378.

9. L. A. Peletier and J. Serrin, *Uniqueness of positive solutions of semilinear equations in* $\mathbb{R}^n$, Arch. Rational Mech. Anal. **81** (1983), 181–197.

10. S. I. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. **5** (1965), 1408–1411.

11. L. Zhang, *Uniqueness of positive solutions of* $\Delta u + f(u) = 0$ *in* $\mathbb{R}^n$, preprint.

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