Research Article

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Uniqueness of positive solutions for boundary value problems associated with indefinite ϕ-Laplacian-type equations

Abstract: This paper provides a uniqueness result for positive solutions of the Neumann and periodic boundary value problems associated with the ϕ-Laplacian equation

$$(\phi(u'))' + a(t)g(u) = 0,$$

where ϕ is a homeomorphism with $\phi(0) = 0$, $a(t)$ is a stepwise indefinite weight and $g(u)$ is a continuous function. When dealing with the p-Laplacian differential operator $\phi(s) = |s|^{p-2}s$ with $p > 1$, and the nonlinear term $g(u) = u^\gamma$ with $\gamma \in \mathbb{R}$, we prove the existence of a unique positive solution when $\gamma \in \,]-\infty, (1 - 2p)/(p - 1)] \cup \,]p - 1, +\infty[$.

1 Introduction

This paper deals with the ϕ-Laplacian differential equation

$$(\phi(u'))' + a(t)g(u) = 0, \tag{1.1}$$

where ϕ is an increasing homeomorphism defined in an open interval including the origin, with $\phi(0) = 0$, $a(t)$ is a sign-changing L^1-weight function and $g(u)$ is a continuous function with $g(u) > 0$ for $u > 0$.

It is worth noting that the ϕ-Laplacian operator appearing in equation (1.1) includes several classical differential operators, such as the linear operator $\phi(s) = s$, the p-Laplacian operator $\phi(s) = \varphi_p(s) = |s|^{p-2}s$ with $p > 1$ or even the one-dimensional mean-curvature operator in Lorentz-Minkowski spaces $\phi(s) = s/\sqrt{1 - |s|^2}$. Such differential operators have been widely investigated in the literature for their relevance in many mechanical and physical models (cf. [1]).

Starting from [2], it is common to refer to (1.1) as a nonlinear indefinite equation, due to the presence of a sign-changing weight function. The study of indefinite problems, both in the ordinary differential equation setting and in the partial differential equation one, has shown an exceptional interest, from the pioneering

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papers [3–5] to the recent developments dealing with positive solutions of boundary value problems associated with (1.1) (we refer to [6] for a quite comprehensive list of references).

In this paper, we focus our attention on the Neumann and the periodic boundary value problems associated with (1.1) and we analyse both the power-type nonlinearity, that is

$$g(u) = u^q, \quad q > 0, \tag{1.2}$$

and the singular nonlinearity

$$g(u) = \frac{1}{u^\kappa}, \quad \kappa > 0. \tag{1.3}$$

In this framework, lot of work has been done concerning the existence and the multiplicity of positive solutions, see, for instance, [5,7–13] for the power-type nonlinearity (1.2) and [14–21] for the singularity (1.3).

Looking in the above mentioned contributions and, in general, in the literature, we note that the natural question of uniqueness of solutions has received very few attention. More precisely, in the framework of indefinite problems, the uniqueness of positive solution is proved in [7,22] when dealing with a concave $g(u)$, and in [16] when dealing with a singularity of the form (1.3) with $\kappa = 3$. An intermediate situation is studied in [23]; other types of special nonlinearities (convex–concave) are analysed in [24]. We highlight that all these results concerning the uniqueness of the solutions are obtained for the linear differential operator.

As is shown in [25–27], for convex or superlinear nonlinearities, the problem of uniqueness of positive solutions is of greater complexity even when the weight function $a(t)$ is of constant positive sign and apparently it has not yet been completely solved for sign-changing weights. Indeed, for weight functions with multiple changes of the sign, uniqueness is not possible, in view of the results about the multiplicity of positive solutions obtained in [6,11].

The aim of this paper is twofold: on one side, we plan to produce a uniqueness result including both linear and nonlinear differential operators; on the other hand, we investigate a situation allowing the superlinear nonlinearities as a special case. Due to the fact that a weight function $a(t)$ with more than one change of sign allows multiplicity of positive solutions, it is natural to consider a coefficient $a(t)$ with a single change of sign. Similar coefficients have been considered in [12, 15, 16, 18, 28, 29]. In particular, following [16,18], we will focus our attention on a stepwise weight function of the form

$$a(t) = \begin{cases} a_+, & \text{if } t \in [0, \tau[, \\ -a_-, & \text{if } t \in [\tau, T[, \end{cases} \tag{1.4}$$

where $a_+, a_- > 0$ and $0 < \tau < T$. This framework allows us to study the uniqueness question exploiting techniques typical of autonomous system.

The general statement is given in Section 2 (cf. Theorem 2.1); by now, we just present it for some special cases. When dealing with the linear differential operator, i.e. $\phi(s) = s$, it yields the following.

Theorem 1.1. *Let $a \in L^\infty(0, T)$ be as in (1.4). Let $\gamma \in \,]-\infty, -3] \cup \,]1, +\infty[$. Then, the Neumann and the periodic boundary value problems associated with equation*

$$u'' + a(t)u^\gamma = 0 \tag{1.5}$$

have at most one positive solution. Moreover, there exists a unique positive solution if and only if $\gamma \cdot \int_0^T a(t)\mathrm{d}t < 0$.

Note that the case of a singularity with $\gamma = -3$, already solved in [16], is included in the above result. We mention that with our strategy of proof we can also deal with the linear case (i.e. $\gamma = 1$) so as to recover the existence of a simple principal eigenvalue (see Remarks 3.1 and 3.2).

As for the more general case of a p-Laplacian operator $\phi(s) = \varphi_p(s) = |s|^{p-2}s$ with $p > 1$, our main contribution is the following.

Theorem 1.2. *Let* $a \in L^\infty(0, T)$ *be as in* (1.4). *Let* $p > 1$ *and*

$$\gamma \in \left] -\infty, \frac{1 - 2p}{p - 1} \right] \cup \left] p - 1, +\infty \right[.$$

Then, the Neumann and the periodic boundary value problems associated with equation

$$(|u'|^{p-2}u')' + a(t)u^\gamma = 0 \tag{1.6}$$

have at most one positive solution. Moreover, there exists a unique positive solution if and only if $\gamma \cdot \int_0^T a(t)\mathrm{d}t < 0$.

This paper is organized as follows. In Section 2, we present our main abstract uniqueness result for equation (1.1) and, to prove Theorems 1.1 and 1.2, in Section 3 we apply it to the case $\phi(s) = s$ and $\phi(s) = \varphi_p(s) = |s|^{p-2}s$ with $p > 1$. In Section 4, some remarks and open questions are presented, including the presentation of a numerical analysis on the validity of the uniqueness results for wider ranges of γ, as well as a brief discussion for the Minkowski-curvature operator.

2 An abstract uniqueness result

In this section, we aim to present a method to deal with a general class of nonlinear differential problems. Accordingly, we deal with the second-order equation

$$(\phi(u'))' + a(t)g(u) = 0, \tag{2.1}$$

where, for $\Omega \subseteq \mathbb{R}$ an open interval with $0 \in \Omega$, we assume that
$\phi: \Omega \to \phi(\Omega) = \mathbb{R}$ is a homeomorphism with $\phi(0) = 0$, $\phi(s)s > 0$ for all $s \in \Omega \setminus \{0\}$;
$a: [0, T[\to \mathbb{R}$ is a step wise function of the form (1.4);
$g:]0, +\infty[\to]0, +\infty[$ is a continuous function.

In Theorem 2.1, we state a uniqueness result for the Neumann boundary value problem associated with (2.1); a variant is given in Theorem 2.2. In Section 2.2, we discuss the periodic boundary value problem.

Preliminarily, we recall that a positive solution to equation (2.1) is a function $u: [0, T] \to]0, +\infty[$ of class C^1 such that $\phi(u')$ is an absolutely continuous function and equation (2.1) is satisfied for almost every $t \in [0, T]$.

Remark 2.1. When dealing with Neumann and periodic boundary conditions and with $g \in C^1(]0, +\infty[)$, an integration by parts of equation (2.1) divided by $g(u)$ gives

$$\bar{a} := a_+\tau - a_-(T - \tau) = \int_0^T a(t) \, \mathrm{d}t = -\int_0^T \frac{(\phi(u'(t)))'}{g(u(t))} \, \mathrm{d}t$$

$$= -\left[\frac{\phi(u'(t))}{g(u(t))} \right]_0^T - \int_0^T \frac{\phi(u'(t))u'(t)g'(u(t))}{(g(u(t)))^2} \, \mathrm{d}t.$$

As a consequence, we immediately obtain a necessary condition for the existence of positive solutions of (2.1), that is $\bar{a} < 0$ if $g'(u) > 0$ for all $u \in]0, +\infty[$, and $\bar{a} > 0$ if $g'(u) < 0$ for all $u \in]0, +\infty[$.

2.1 The Neumann problem

Let us consider the planar system associated with equation (2.1), that is

$$\begin{cases} x' = \phi^{-1}(y), \\ y' = -a(t)g(x). \end{cases} \tag{2.2}$$

A solution of (2.2) is a couple (x, y) of absolutely continuous functions satisfying (2.2) for almost every t. Throughout the section, we confine ourselves in the half-right part $]0, +\infty[\times \mathbb{R}$ of the phase plane.

According to the assumptions on $\phi(s)$, $a(t)$ and $g(x)$, for every time $t_0 \in [0, T[$ and every initial condition $(x_0, y_0) \in]0, +\infty[\times \mathbb{R}$, system (2.2) admits a unique local noncontinuable solution with $x(t_0) = x_0$ and $y(t_0) = y_0$, denoted by

$$(x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0)).$$

The uniqueness of the solutions of the Cauchy problems is guaranteed by the special choice of the stepwise coefficient $a(t)$; indeed, we enter the setting of the result in [30] concerning planar Hamiltonian systems. Moreover, we remark that $x(t; t_0, x_0, y_0) > 0$ for all t in the maximal interval of existence, where the solution is defined.

In this section, we focus our attention on the Neumann boundary value problem associated with system (2.2). Our goal is to prove that

there exists a unique initial condition $(x_*, y_*) \in]0, +\infty[\times \mathbb{R}$ such that $(x(t; \tau, x_*, y_*), y(t; \tau, x_*, y_*))$ is a solution of system (2.2) defined in $[0, T]$, $x(t; \tau, x_*, y_*) > 0$ for all $t \in [0, T]$, and

$$y(0; \tau, x_*, y_*) = 0, \quad y(T; \tau, x_*, y_*) = 0. \tag{2.3}$$

Setting $h = \phi^{-1}$, we can exploit (1.4) and so define the two autonomous systems

$$(S_+) \quad \begin{cases} x' = h(y), \\ y' = -a_+g(x), \end{cases} \qquad (S_-) \quad \begin{cases} x' = h(y), \\ y' = a_-g(x). \end{cases}$$

Let us assume that there is a hypothetical solution

$$(x(t), y(t)) = (x(t; \tau, x_*, y_*), y(t; \tau, x_*, y_*))$$

of (2.2) defined in $[0, T]$, with $x(t) > 0$ for all $t \in [0, T]$, and satisfying the boundary conditions (2.3). For future convenience, we set

$$\alpha = x(0), \quad \beta = x(T). \tag{2.4}$$

Clearly, $\alpha > 0$ and $\beta > 0$. From (S_+) and $y(0) = 0$, we deduce that

$$y(t) = y(0) + \int_0^t y'(\xi) \, d\xi = - \int_0^t a_+g(x(\xi)) \, d\xi < 0, \quad \text{for every } t \in]0, \tau],$$

and thus $x'(t) = h(y(t)) < 0$ for every $t \in]0, \tau[$. Hence, $x(t)$ is strictly monotone decreasing in $[0, \tau]$ and $x(t) < \alpha$ for every $t \in]0, \tau]$. Analogously, from (S_-) and $y(T) = 0$, we have that $y(t) < 0$ for every $t \in [\tau, T[$, so $x(t)$ is strictly monotone decreasing in $[\tau, T]$ and so $x(t) > \beta$ for every $t \in [\tau, T[$. We conclude that

$$y(t) < 0, \quad \text{for all } t \in]0, T[, \tag{2.5}$$

so the solution $(x(t), y(t))$ is in the fourth quadrant $]0, +\infty[\times]-\infty, 0]$. Moreover, $x(t)$ is strictly monotone decreasing in $[0, T]$, and thus

$$0 < \beta < x_* < \alpha. \tag{2.6}$$

Let H and G be primitives of h and g, respectively. For convenience, we suppose that $H(0) = 0$. When g can be continuously extended to zero, we shall also assume $G(0) = 0$. We denote $G_0 := \lim_{x \to 0} G(x)$ and $G_{+\infty} := \lim_{x \to +\infty} G(x)$. In general, for an arbitrary function $g\colon\,]0, +\infty[\, \to\,]0, +\infty[$ the following four cases are possible:

(i) $G_0 \in \mathbb{R}$ and $G_{+\infty} = +\infty$; in this situation, without loss of generality, we can suppose that $G\colon\,]0, +\infty[\, \to\,]0, +\infty[$ is a strictly monotone increasing surjective function;

(ii) $G_0 \in \mathbb{R}$ and $G_{+\infty} \in \mathbb{R}$; in this situation, without loss of generality, we can suppose that $G\colon\,]0, +\infty[\, \to\,]0, \Lambda[$ is a strictly monotone increasing surjective function, with $\Lambda := \int_0^{+\infty} g(u)\, du$;

(iii) $G_0 = -\infty$ and $G_{+\infty} = +\infty$; in this situation, without loss of generality, we can suppose that $G\colon\,]0, +\infty[\, \to \mathbb{R}$ is a strictly monotone increasing surjective function;

(iv) $G_0 = -\infty$ and $G_{+\infty} \in \mathbb{R}$; in this situation, without loss of generality, we can suppose that $G\colon\,]0, +\infty[\, \to\,]-\infty, 0[$ is a strictly monotone increasing surjective function.

We observe that the quantities $H(y) + a_+ G(x)$ and $H(y) - a_- G(x)$ are constant for all (x, y) solving $(\mathcal{S}_+)$ and $(\mathcal{S}_-)$, respectively. In particular, due to (2.3), (2.4) and $H(0) = 0$, we have that the solution $(x(t), y(t))$ satisfies

$$\begin{aligned} H(y) + a_+ G(x) &= a_+ G(\alpha), \quad \text{on } [0, \tau[, \\ H(y) - a_- G(x) &= -a_- G(\beta), \quad \text{on } [\tau, T[. \end{aligned} \tag{2.7}$$

We notice that the functions $H_l := H|_{]-\infty,\, 0]}$, $H_r := H|_{[0,\, +\infty[}$ and G are invertible since strictly monotone. For sake of simplicity in the notation, we set

$$\mathcal{L}_h = h \circ H_l^{-1}, \quad \mathcal{L}_g = g \circ G^{-1}.$$

We remark that, when h is odd, we find that $H_l^{-1} = -H_r^{-1}$ and, therefore, $\mathcal{L}_h = -h \circ H_r^{-1}$.

Recalling (2.5), from (2.7), $(\mathcal{S}_+)$ and $(\mathcal{S}_-)$, we infer that

$$\begin{aligned} x' = h(y) &= \mathcal{L}_h(a_+ G(\alpha) - a_+ G(x)), \quad \text{on } [0, \tau[, \\ x' = h(y) &= \mathcal{L}_h(a_- G(x) - a_- G(\beta)), \quad \text{on } [\tau, T[. \end{aligned}$$

Recalling (2.6), by an integration in $[0, \tau[$ and in $[\tau, T[$, we obtain

$$\begin{aligned} \int_{x_*}^{\alpha} \frac{dx}{-\mathcal{L}_h(a_+ G(\alpha) - a_+ G(x))} &= \tau, \\ \int_{\beta}^{x_*} \frac{dx}{-\mathcal{L}_h(a_- G(x) - a_- G(\beta))} &= T - \tau, \end{aligned} \tag{2.8}$$

respectively. Conversely, if α, β, x_*, with $0 < \beta < x_* < \alpha$, are such that the aforementioned relations hold, we infer that the solution $(x(t; \tau, x_*, y_*), y(t; \tau, x_*, y_*))$ of the Cauchy problem is defined in $[0, T]$ and is such that $(x(0; \tau, x_*, y_*), y(0; \tau, x_*, y_*)) = (\alpha, 0)$ and $(x(T; \tau, x_*, y_*), y(T; \tau, x_*, y_*)) = (\beta, 0)$. Hence, by the choice of $h = \phi^{-1}$, we deduce that $x(t)$ is a positive decreasing solution of the Neumann boundary value problem associated with (2.1).

Let us perform the change of variable

$$\vartheta = G(x), \quad x = G^{-1}(\vartheta), \quad dx = \frac{d\vartheta}{g(G^{-1}(\vartheta))} = \frac{d\vartheta}{\mathcal{L}_g(\vartheta)},$$

in (2.8), obtaining

$$\int_{G(x_*)}^{G(\alpha)} \frac{d\vartheta}{-\mathcal{L}_h(a_+ G(\alpha) - a_+ \vartheta)\, \mathcal{L}_g(\vartheta)} = \tau \tag{2.9}$$

and

$$\int\limits_{G(\beta)}^{G(x_*)} \frac{\mathrm{d}\vartheta}{-\mathcal{L}_h(a_-\vartheta - a_-G(\beta))\,\mathcal{L}_g(\vartheta)} = T - \tau. \tag{2.10}$$

From (2.7) with $(x, y) = (x_*, y_*)$ one can deduce that

$$G(x_*) = \frac{a_+ G(\alpha) + a_- G(\beta)}{a_+ + a_-},$$

thus $G(x_*)$ is a convex combination of $G(\alpha)$ and $G(\beta)$.

Next, we set

$$\omega := G(\alpha), \quad \sigma := G(\beta), \quad \mu := \frac{a_+}{a_+ + a_-}.$$

Accordingly, (2.9) and (2.10) take the simplified form

$$\int\limits_{\mu\omega+(1-\mu)\sigma}^{\omega} \frac{\mathrm{d}\vartheta}{-\mathcal{L}_h(a_+\omega - a_+\vartheta)\,\mathcal{L}_g(\vartheta)} = \tau \tag{2.11}$$

and

$$\int\limits_{\sigma}^{\mu\omega+(1-\mu)\sigma} \frac{\mathrm{d}\vartheta}{-\mathcal{L}_h(a_-\vartheta - a_-\sigma)\,\mathcal{L}_g(\vartheta)} = T - \tau. \tag{2.12}$$

Let $O := \{(\omega, \sigma) \in G(]0, +\infty[) \times G(]0, +\infty[) : \omega > \sigma\}$ and we introduce the functions $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}} : O \to \,]0, +\infty[$ defined as

$$\mathcal{M}_{\mathrm{I}}(\omega, \sigma) := \int\limits_{\mu\omega+(1-\mu)\sigma}^{\omega} \frac{\mathrm{d}\vartheta}{-\mathcal{L}_h(a_+\omega - a_+\vartheta)\,\mathcal{L}_g(\vartheta)},$$

$$\mathcal{M}_{\mathrm{II}}(\omega, \sigma) := \int\limits_{\sigma}^{\mu\omega+(1-\mu)\sigma} \frac{\mathrm{d}\vartheta}{-\mathcal{L}_h(a_-\vartheta - a_-\sigma)\,\mathcal{L}_g(\vartheta)}.$$

From the above discussion, the following result holds true.

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}$ be an open interval with $0 \in \Omega$. Let $\phi \colon \Omega \to \phi(\Omega) = \mathbb{R}$ be a homeomorphism with $\phi(0) = 0$, $\phi(s)s > 0$ for all $s \in \Omega \setminus \{0\}$. Let $a \in L^\infty(0, T)$ be as in (1.4). Let $g : \,]0, +\infty[\,\to\,]0, +\infty[$ be a continuous function. Then, there exists a pair $(\omega, \sigma) \in O$ which solves*

$$\begin{cases} \mathcal{M}_{\mathrm{I}}(\omega, \sigma) = \tau, \\ \mathcal{M}_{\mathrm{II}}(\omega, \sigma) = T - \tau, \end{cases} \tag{2.13}$$

if and only if the Neumann boundary value problem associated with equation (2.1) has a positive solution. Moreover, the unique solvability of (2.13) is equivalent to the uniqueness of the positive solution of the Neumann problem.

For our applications, we will consider a simplified but equivalent formulation of system (2.13) which can be obtained when $\omega = G(\alpha)$ is of constant sign. This excludes only case (iii) in the aforementioned list.

We introduce the new variable

$$\rho := \frac{a_- G(\beta)}{a_+ G(\alpha)} = \frac{a_-\sigma}{a_+\omega}.$$

Thus, we have

$$\sigma = \frac{a_+}{a_-}\omega\rho, \quad G(x_*) = \mu\omega + (1-\mu)\sigma = \frac{a_+}{a_+ + a_-}\omega(\rho + 1).$$

By performing the change of variable $\vartheta = G(\alpha)\xi = \omega\,\xi$, formulas (2.11) and (2.12) are as follows:

$$\omega \int_{\frac{a_+}{a_+ + a_-}(\rho+1)}^{1} \frac{\mathrm{d}\xi}{-\mathcal{L}_h(a_+\omega(1-\xi))\,\mathcal{L}_g(\omega\xi)} = \tau,$$

$$\omega \int_{\frac{a_+}{a_-}\rho}^{\frac{a_+}{a_+ + a_-}(\rho+1)} \frac{\mathrm{d}\xi}{-\mathcal{L}_h(\omega(a_-\xi - a_+\rho))\,\mathcal{L}_g(\omega\xi)} = T - \tau,$$

respectively.

Let

$$\mathcal{D} := \begin{cases} G(]0, +\infty[) \times \left]0, \dfrac{a_-}{a_+}\right[, & \text{in cases (i) and (ii),} \\[2mm] G(]0, +\infty[) \times \left]\dfrac{a_-}{a_+}, +\infty\right[, & \text{in case (iv),} \end{cases} \tag{2.14}$$

and define the functions $\mathcal{F}_{\mathrm{I}}, \mathcal{F}_{\mathrm{II}} : \mathcal{D} \to]0, +\infty[$ as follows:

$$\mathcal{F}_{\mathrm{I}}(\omega, \rho) := \omega \int_{\frac{a_+}{a_+ + a_-}(\rho+1)}^{1} \frac{\mathrm{d}\xi}{-\mathcal{L}_h(a_+\omega(1-\xi))\,\mathcal{L}_g(\omega\xi)},$$

$$\mathcal{F}_{\mathrm{II}}(\omega, \rho) := \omega \int_{\frac{a_+}{a_-}\rho}^{\frac{a_+}{a_+ + a_-}(\rho+1)} \frac{\mathrm{d}\xi}{-\mathcal{L}_h(\omega(a_-\xi - a_+\rho))\,\mathcal{L}_g(\omega\xi)}.$$

From the above discussion, we have the following uniqueness result.

Corollary 2.1. *Let $\Omega \subseteq \mathbb{R}$ be an open interval with $0 \in \Omega$. Let $\phi: \Omega \to \phi(\Omega) = \mathbb{R}$ be a homeomorphism with $\phi(0) = 0$, $\phi(s)s > 0$ for all $s \in \Omega \setminus \{0\}$. Let $a \in L^\infty(0, T)$ be as in (1.4). Let $g :]0, +\infty[\to]0, +\infty[$ be a continuous function such that (iii) does not hold. Then, there exists a pair $(\omega, \rho) \in \mathcal{D}$ which solves*

$$\begin{cases} \mathcal{F}_{\mathrm{I}}(\omega, \rho) = \tau, \\ \mathcal{F}_{\mathrm{II}}(\omega, \rho) = T - \tau, \end{cases} \tag{2.15}$$

if and only if the Neumann boundary value problem associated with equation (2.1) has a positive solution. Moreover, the unique solvability of (2.15) is equivalent to the uniqueness of the positive solution of the Neumann problem.

We conclude this section by presenting an equivalent version of Theorem 2.1. Instead of (2.7), our new starting point is the fact that the hypothetical solution $(x(t), y(t))$ also satisfies

$$H(y) + a_+G(x) = H(y_*) + a_+G(x_*), \quad \text{on } [0, \tau[,$$
$$H(y) - a_-G(x) = H(y_*) - a_-G(x_*), \quad \text{on } [\tau, T[.$$

We immediately infer that

$$y' = -a_+g(x) = -a_+\mathcal{L}_g\left(G(x_*) + \frac{H(y_*)}{a_+} - \frac{H(y)}{a_+}\right), \quad \text{on } [0, \tau[,$$

$$y' = a_-g(x) = a_-\mathcal{L}_g\left(G(x_*) + \frac{H(y)}{a_-} - \frac{H(y_*)}{a_-}\right), \quad \text{on } [\tau, T[,$$

and integrations in $[0, \tau[$ and, respectively, in $[\tau, T[$ give

$$\begin{cases} \displaystyle\int_{y_*}^{0} \frac{\mathrm{d}y}{a_+ \mathcal{L}_g\left(G(x_*) + \frac{H(y_*)}{a_+} - \frac{H(y)}{a_+}\right)} = \tau, \\[4mm] \displaystyle\int_{y_*}^{0} \frac{\mathrm{d}y}{a_- \mathcal{L}_g\left(G(x_*) + \frac{H(y)}{a_-} - \frac{H(y_*)}{a_-}\right)} = T - \tau. \end{cases} \tag{2.16}$$

The following result holds true.

Theorem 2.2. *Let $\Omega \subseteq \mathbb{R}$ be an open interval with $0 \in \Omega$. Let $\phi: \Omega \to \phi(\Omega) = \mathbb{R}$ be a homeomorphism with $\phi(0) = 0$, $\phi(s)s > 0$ for all $s \in \Omega \setminus \{0\}$. Let $a \in L^{\infty}(0, T)$ be as in (1.4). Let $g :]0, +\infty[\to]0, +\infty[$ be a continuous function. Then, there exists a pair $(x_*, y_*) \in]0, +\infty[\times]-\infty, 0[$ which solves (2.16), if and only if the Neumann boundary value problem associated with equation (2.1) has a positive solution. Moreover, the unique solvability of (2.16) is equivalent to the uniqueness of the positive solution of the Neumann problem.*

2.2 The periodic problem

In this section, we deal with the periodic boundary value problem associated with (2.1) and we show that Theorem 2.1 holds true also in the periodic case. Following a procedure which is standard in this situation, we extend by T-periodicity the weight $a(t)$ as an L^{∞}-stepwise function defined in the whole real line. In this framework, finding a solution of (2.1), satisfying $u(0) = u(T)$ and $u'(0) = u'(T)$, is equivalent to finding a T-periodic solution of (2.1) defined on $\mathbb{R}$.

As in Section 2.1, we analyse the associated planar system (2.2) and we look for periodic solutions $(x(t), y(t))$ of (2.2) such that $x(t) > 0$ for all $t \in \mathbb{R}$. Our purpose is to reduce the study of the periodic problem to the Neumann one, analysed in the previous section. To this aim, we further assume that

$$h = \phi^{-1} : \mathbb{R} \to \mathbb{R} \text{ is odd.}$$

The next two claims relate the existence/uniqueness of positive solutions of the T-periodic problem to the corresponding one for the Neumann problem.

Claim 1. Let $(x(t), y(t))$ be a solution of (2.2) defined in the interval $\left[\frac{\tau}{2}, \frac{T+\tau}{2}\right]$ with $x(t) > 0$ for all $t \in \left[\frac{\tau}{2}, \frac{T+\tau}{2}\right]$ and satisfying the Neumann condition at the boundary, that is

$$y\left(\frac{\tau}{2}\right) = y\left(\frac{T+\tau}{2}\right) = 0. \tag{2.17}$$

Let also $(\hat{x}(t), \hat{y}(t))$ be the T-periodic extension of $(x(t), y(t))$ symmetric with respect to $t = \tau/2$, namely,

$$(\hat{x}(t), \hat{y}(t)) = \begin{cases} (x(t), y(t)), & \text{if } t \in \left[\frac{\tau}{2}, \frac{T+\tau}{2}\right], \\[3mm] (x(\tau - t), -y(\tau - t)), & \text{if } t \in \left[\frac{T-\tau}{2}, \frac{\tau}{2}\right]. \end{cases} \tag{2.18}$$

Then, $(\hat{x}(t), \hat{y}(t))$ is a T-periodic solution of (2.2) with $\hat{x}(t) > 0$ for all $t \in \mathbb{R}$.

Indeed, by construction, $(\hat{x}(t), \hat{y}(t))$ is symmetric with respect to $\frac{\tau}{2}$ and, by direct inspection, one can easily check that it is a solution of system (2.2) on the interval $\left[\frac{T-\tau}{2}, \frac{T+\tau}{2}\right]$; this follows from the fact that the extension of $a(t)$ by T-periodicity is symmetric with respect to $\frac{\tau}{2}$. Moreover, $(\hat{x}(t), \hat{y}(t))$ satisfies the T-periodic condition at the boundary of $\left[\frac{T-\tau}{2}, \frac{T+\tau}{2}\right]$, that is

$$\left(\hat{x}\left(\frac{\tau - T}{2}\right), \hat{y}\left(\frac{\tau - T}{2}\right)\right) = \left(\hat{x}\left(\frac{\tau + T}{2}\right), \hat{y}\left(\frac{\tau + T}{2}\right)\right).$$

Since the weight $a(t)$ has been extended by T-periodicity on the whole real line and $(\hat{x}(t), \hat{y}(t))$ is a T-periodic extension of (2.18), we immediately conclude that $(\hat{x}(t), \hat{y}(t))$ solves (2.2) and is such that $\hat{x}(t) > 0$ for all $t \in \mathbb{R}$, and, by construction, $(\hat{x}(0), \hat{y}(0)) = (\hat{x}(T), \hat{y}(T))$. Then, Claim 1 is proved.

Claim 2. Let $(x(t), y(t))$ be a T-periodic solution of (2.2) with $x(t) > 0$ for all $t \in \mathbb{R}$. Then, the restriction of $(x(t), y(t))$ to the interval $\left[\frac{\tau}{2}, \frac{T+\tau}{2}\right]$ is a solution of (2.2) with $x(t) > 0$ for all $t \in \left[\frac{\tau}{2}, \frac{T+\tau}{2}\right]$ and satisfying the Neumann boundary condition (2.17).

Indeed, for $(x(t), y(t))$ as in the assumption and by the special form (1.4) of the weight function $a(t)$, we have that $y'(t) < 0$ in $]0, \tau[$ and $y'(t) > 0$ in $]\tau, T[$, and so $y(t)$ is strictly monotone decreasing in $[0, \tau]$ and strictly monotone increasing in $[\tau, T]$. Since h is strictly monotone increasing, we also deduce that $x'(t)$ is strictly monotone decreasing in $[0, \tau]$ and strictly monotone increasing in $[\tau, T]$.

We note that if $x'(0) = 0$, then we have $x'(t) < 0$ in $]0, \tau[$ and, since $x'(T) = h(y(T)) = h(y(0)) = x'(0) = 0$, we also have $x'(t) < 0$ in $]\tau, T[$; thus $x(t)$ is strictly monotone decreasing in $[0, T]$, a contradiction with $x(0) = x(T)$. Moreover, if $x'(\tau) = 0$, then $x'(t) > 0$ in $]0, \tau[$ and $x'(t) > 0$ in $]\tau, T[$, a contradiction with $x(0) = x(T)$, as before. Therefore, from the above discussion and Rolle's theorem, we conclude that there exist exactly one critical point in $]0, \tau[$ and exactly one critical point in $]\tau, T[$. Let $\hat{t} \in]0, \tau[$ and $\check{t} \in]\tau, T[$ be such that $x'(\hat{t}) = x'(\check{t}) = 0$. We claim that $\hat{t}$ and $\check{t}$ are the maximum point and, respectively, the minimum point of $x(t)$ in $[0, T]$. This is obvious since $x(t) = x(\hat{t}) + \int_{\hat{t}}^{t} x'(\xi)\, d\xi < x(\hat{t})$ in every neighbourhood of $\hat{t}$ (contained in $]0, \tau[$), while $x(t) = x(\check{t}) + \int_{\check{t}}^{t} x'(\xi)\, d\xi > x(\check{t})$ in every neighbourhood of $\check{t}$ (contained in $]\tau, T[$).

We remark that the function $(x(t), y(t))$ is such that $y(\hat{t}) = y(\check{t}) = 0$, $y(t) < 0$ in $]\hat{t}, \check{t}[$ and $x(t)$ is strictly monotone decreasing in $[\hat{t}, \check{t}]$, $y(t) > 0$ in $]0, \hat{t}[\cup]\check{t}, T[$ and $x(t)$ is strictly monotone increasing in $[0, \hat{t}] \cup [\check{t}, T]$.

We are going to prove that $\hat{t} = \frac{\tau}{2}$ and $\check{t} = \frac{T+\tau}{2}$. Our approach is based on an analysis of the phase-portrait in the (x, y)-plane and is similar to the one performed in [16]. Due to the more general framework and in order to justify the additional hypothesis on h, we give here all the details.

Arguing as in Section 2.1 (cf. (2.7)), we deduce that the solution $(x(t), y(t))$ satisfies

$$H(y) + a_+ G(x) = a_+ G(\alpha), \quad \text{on } [0, \tau], \tag{2.19}$$

$$H(y) - a_- G(x) = -a_- G(\beta), \quad \text{on } [\tau, T], \tag{2.20}$$

with $\alpha = x(\hat{t})$ and $\beta = x(\check{t})$. The trajectory on the time interval $[0, \hat{t}]$ satisfies the relation

$$y = H_r^{-1}(a_+ G(\alpha) - a_+ G(x)), \quad (x, y) \in [x(0), \alpha] \times [0, +\infty[, \tag{2.21}$$

which describes a strictly monotone decreasing curve. Analogously, using the fact that $H_l^{-1} = -H_r^{-1}$, the trajectory on the time interval $[\hat{t}, \tau]$ satisfies the relation

$$y = -H_r^{-1}(a_+ G(\alpha) - a_+ G(x)), \quad (x, y) \in [x(\tau), \alpha] \times]-\infty, 0], \tag{2.22}$$

which describes a strictly monotone increasing curve. On the other hand, the trajectory on the time interval $[\tau, \check{t}]$ satisfies the relation

$$y = -H_r^{-1}(a_- G(x) - a_- G(\beta)), \quad (x, y) \in [\beta, x(\tau)] \times]-\infty, 0], \tag{2.23}$$

which describes a strictly monotone decreasing curve, while the trajectory on the time interval $[\check{t}, T]$ satisfies the relation

$$y = H_r^{-1}(a_- G(x) - a_- G(\beta)), \quad (x, y) \in [\beta, x(T)] \times [0, +\infty[, \tag{2.24}$$

which describes a strictly monotone increasing curve. By the strict monotonicity of the above curves, we infer that there is at most one intersection point between (2.21) and (2.24) in $]\alpha, \beta[\times]0, +\infty[$, and likewise at most one intersection point between (2.22) and (2.23) in $]\alpha, \beta[\times]-\infty, 0[$. Actually, these

intersections exist and are uniquely determined as $(x_+, y_+) = (x(0), y(0)) = (x(T), y(T))$ (in the first quadrant) and $(x_-, y_-) = (x(\tau), y(\tau))$ (in the fourth quadrant). Due to the symmetry of H, it follows that $x_+ = x_-$ and $y_+ = -y_-$. Moreover, from the Abelian-type integrals representing the time mappings, we find that the time necessary to connect (along the level line (2.19)) (x_+, y_+) to $(\alpha, 0)$ coincides with the time necessary to connect (along the same level line) $(\alpha, 0)$ to (x_-, y_-). Hence, $\hat{t} = \frac{\tau}{2}$. The same argument, with respect to the level line (2.20), shows that the times necessary to connect (x_-, y_-) to $(\beta, 0)$ and $(\beta, 0)$ to (x_+, y_+) are equal. Then, $\check{t} = \frac{\tau + T}{2}$. This concludes the proof of Claim 2.

From the above discussion, we deduce that we can reduce the problem of proving the existence and uniqueness of a positive solution of the T-periodic problem to the study of a Neumann problem defined in $\left[\frac{\tau}{2}, \frac{\tau+T}{2}\right]$ for a stepwise function. Clearly, this latter problem is equivalent to the original problem studied in the previous section. In particular, observe also that systems (2.13) and (2.15) would be formally changed to the corresponding new systems in which the target vector $(\tau, T - \tau)$ should be replaced by $\left(\frac{\tau}{2}, \frac{T-\tau}{2}\right)$. With reference to the result obtained in the subsequent section, no relevant point has to be changed also due to the elementary fact that the ratio of the two components of the target vector $(\tau, T - \tau)$ or, respectively, $\left(\frac{\tau}{2}, \frac{T-\tau}{2}\right)$ remains unchanged (see the second equations in (3.5) and (3.9)).

Finally, we conclude that Theorem 2.1 and Corollary 2.1 are both valid also in the periodic case, with the additional assumption that h is odd.

3 Proofs of Theorems 1.1 and 1.2

In this section, we apply Corollary 2.1 when

$$g(u) = u^\gamma, \quad u > 0, \quad \gamma \in \mathbb{R}, \tag{3.1}$$

and for $\phi(s) = s$ in Section 3.1, thus proving Theorem 1.1, while for $\phi(s) = |s|^{p-2}s$ with $p > 1$ in Section 3.2, thus proving Theorem 1.2. We focus our analysis on the Neumann boundary conditions; the result for the periodic problem follows from this as explained in Section 2.2 (indeed, the functions $h = \phi^{-1}$ will be odd).

3.1 The case $\phi(s) = s$

We deal first with the simpler case

$$\phi(s) = s.$$

As a first step, we prove that the Neumann boundary value problem associated with (1.5) has *at most one* positive solution. As a second step, to conclude the proof, we show that there exists *at least one* positive solution.

In this special framework, we have

$$h(y) = \phi^{-1}(y) = y, \quad H(y) = \frac{y^2}{2}, \quad y \in \mathbb{R},$$

$$H_l^{-1}(\xi) = -\sqrt{2\xi}, \quad H_r^{-1}(\xi) = \sqrt{2\xi}, \quad \xi \in [0, +\infty[,$$

which leads to

$$\mathcal{L}_h(\xi) = -\sqrt{2\xi}, \quad \xi \in [0, +\infty[.$$

Moreover, concerning the nonlinear term (3.1), for $\gamma \in \mathbb{R} \setminus \{-1\}$, we deduce that

$$G(x) = \frac{x^{\gamma+1}}{\gamma + 1}, \quad x \in]0, +\infty[, \tag{3.2}$$

$$G^{-1}(\xi) = |\gamma + 1|^{\frac{1}{\gamma+1}} |\xi|^{\frac{1}{\gamma+1}}, \quad \xi \in \text{sign}(\gamma + 1) \cdot]0, +\infty[, \tag{3.3}$$

and thus

$$\mathcal{L}_g(\xi) = |\gamma + 1|^{\frac{\gamma}{\gamma+1}} |\xi|^{\frac{\gamma}{\gamma+1}}, \quad \xi \in \text{sign}(\gamma + 1) \cdot]0, +\infty[. \tag{3.4}$$

Incidentally, note that in this situation case (iii) listed in Section 2.1 does not hold, so we are allowed to use Corollary 2.1.

We study the unique solvability of system (2.15) for $(\omega, \rho) \in \mathcal{D}$, where $\mathcal{D}$ is defined in (2.14). In this special framework, we have

$$\mathcal{F}_{\mathrm{I}}(\omega, \rho) = \frac{|\gamma + 1|^{-\frac{\gamma}{\gamma+1}}}{\sqrt{2}} \omega |\omega|^{-\frac{\gamma}{\gamma+1} - \frac{1}{2}} \int_{\frac{a_+}{a_+ + a_-}(\rho+1)}^{1} \frac{\mathrm{d}\xi}{|a_+ - a_+\xi|^{\frac{1}{2}} |\xi|^{\frac{\gamma}{\gamma+1}}}$$

and

$$\mathcal{F}_{\mathrm{II}}(\omega, \rho) = \frac{|\gamma + 1|^{-\frac{\gamma}{\gamma+1}}}{\sqrt{2}} \omega |\omega|^{-\frac{\gamma}{\gamma+1} - \frac{1}{2}} \int_{\frac{a_+}{a_-}\rho}^{\frac{a_+}{a_+ + a_-}(\rho+1)} \frac{\mathrm{d}\zeta}{|a_-\zeta - a_+\rho|^{\frac{1}{2}} |\zeta|^{\frac{\gamma}{\gamma+1}}}$$

$$= \frac{|\gamma + 1|^{-\frac{\gamma}{\gamma+1}}}{\sqrt{2}} \omega |\omega|^{-\frac{\gamma}{\gamma+1} - \frac{1}{2}} \rho^{1 - \frac{\gamma}{\gamma+1} - \frac{1}{2}} \int_{\frac{a_+}{a_-}}^{\frac{a_+}{a_+ + a_-} \frac{\rho+1}{\rho}} \frac{\mathrm{d}\xi}{|a_-\xi - a_+|^{\frac{1}{2}} |\xi|^{\frac{\gamma}{\gamma+1}}}.$$

For simplicity in notation, we introduce the functions $I_1, I_2 :]0, +\infty[\to \mathbb{R}$ defined as

$$I_1(\rho) = \int_{\frac{a_+}{a_+ + a_-}(\rho+1)}^{1} \frac{\mathrm{d}\xi}{|a_+ - a_+\xi|^{\frac{1}{2}} |\xi|^{\frac{\gamma}{\gamma+1}}},$$

$$I_2(\rho) = \int_{\frac{a_+}{a_-}}^{\frac{a_+}{a_+ + a_-} \frac{\rho+1}{\rho}} \frac{\mathrm{d}\xi}{|a_-\xi - a_+|^{\frac{1}{2}} |\xi|^{\frac{\gamma}{\gamma+1}}}.$$

The functions I_1 and I_2 are formally defined for $\rho \in]0, +\infty[$; however, in the applications we will study them on the intervals $]0, a_-/a_+[$ and $]a_-/a_+, +\infty[$ depending on the choice of γ. With this position, proving the unique solvability of system (2.15) for $(\omega, \rho) \in \mathcal{D}$ is equivalent to prove that there exists a unique pair $(\omega, \rho) \in \mathcal{D}$ which solves

$$\begin{cases} \dfrac{|\gamma + 1|^{-\frac{\gamma}{\gamma+1}}}{\sqrt{2}} \omega |\omega|^{-\frac{\gamma}{\gamma+1} - \frac{1}{2}} I_1(\rho) = \tau, \\ \rho^{-1 + \frac{\gamma}{\gamma+1} + \frac{1}{2}} \dfrac{I_1(\rho)}{I_2(\rho)} = \dfrac{\tau}{T - \tau}, \end{cases} \tag{3.5}$$

where the second equation is the quotient between the two equations in (2.15).

We now introduce the function

$$F_2(\rho) := \rho^{\frac{\gamma-1}{2(\gamma+1)}} \frac{I_1(\rho)}{I_2(\rho)}$$

defined in $]0, a_-/a_+[$, for $\gamma > -1$, and in $]a_-/a_+, +\infty[$, for $\gamma < -1$. We claim that $F_2(\rho)$ is strictly monotone increasing for $\gamma \in]-\infty, -3] \cup [1, +\infty[$.

In the following computations, the prime symbol $'$ denotes the derivative with respect to ρ. We compute

$$I_1'(\rho) = \frac{-\left(\frac{a_+}{a_+ + a_-}(\rho + 1)\right)'}{\left|a_+ - a_+\frac{a_+}{a_+ + a_-}(\rho + 1)\right|^{\frac{1}{2}} \left(\frac{a_+}{a_+ + a_-}(\rho + 1)\right)^{\frac{\gamma}{\gamma+1}}} = -\left(\frac{a_+}{a_+ + a_-}\right)^{1 - \frac{\gamma}{\gamma+1} - \frac{1}{2}} \frac{1}{|a_- - a_+\rho|^{\frac{1}{2}} (\rho + 1)^{\frac{\gamma}{\gamma+1}}}$$

and

$$I_2'(\rho) = \frac{\left(\frac{a_+}{a_+ + a_-}\frac{\rho + 1}{\rho}\right)'}{\left|a_-\frac{a_+}{a_+ + a_-}\frac{\rho + 1}{\rho} - a_+\right|^{\frac{1}{2}} \left(\frac{a_+}{a_+ + a_-}\frac{\rho + 1}{\rho}\right)^{\frac{\gamma}{\gamma+1}}} = -\rho^{-2 + \frac{\gamma}{\gamma+1} + \frac{1}{2}} \left(\frac{a_+}{a_+ + a_-}\right)^{1 - \frac{\gamma}{\gamma+1} - \frac{1}{2}} \frac{1}{|a_-(\rho + 1) - \rho(a_+ + a_-)|^{\frac{1}{2}}(\rho + 1)^{\frac{\gamma}{\gamma+1}}}$$

$$= \rho^{-2 + \frac{\gamma}{\gamma+1} + \frac{1}{2}} I_1'(\rho) = \rho^{-\frac{\gamma+3}{2(\gamma+1)}} I_1'(\rho).$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{I_1(\rho)}{I_2(\rho)}\right) = \frac{I_1'(\rho) I_2(\rho) - I_1(\rho) I_2'(\rho)}{(I_2(\rho))^2} = \frac{I_1'(\rho)}{(I_2(\rho))^2}\left(I_2(\rho) - \rho^{-\frac{\gamma+3}{2(\gamma+1)}} I_1(\rho)\right).$$

We observe that

$$\lim_{\rho \to \frac{a_-}{a_+}} \left(I_2(\rho) - \rho^{-\frac{\gamma+3}{2(\gamma+1)}} I_1(\rho)\right) = 0 \tag{3.6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(I_2(\rho) - \rho^{-\frac{\gamma+3}{2(\gamma+1)}} I_1(\rho)\right) = I_2'(\rho) + \frac{\gamma + 3}{2(\gamma + 1)}\rho^{-\frac{\gamma+3}{2(\gamma+1)} - 1} I_1(\rho) - \rho^{-\frac{\gamma+3}{2(\gamma+1)}} I_1'(\rho) = \frac{\gamma + 3}{2(\gamma + 1)}\rho^{-\frac{\gamma+3}{2(\gamma+1)} - 1} I_1(\rho).$$

We remark that $I_1(\rho)$ is defined in $]0, a_-/a_+[$ and is positive when $\gamma > -1$, while $I_1(\rho)$ is defined in $]a_-/a_+, +\infty[$ and is negative when $\gamma < -1$. Therefore, the derivative of the function $I_2(\rho) - \rho^{-\frac{\gamma+3}{2(\gamma+1)}} I_1(\rho)$ (in the interval of definition) has constant sign equal to the sign of $(\gamma + 3)/(\gamma + 1)$ on $]0, a_-/a_+[$ and to the sign of $-(\gamma + 3)/(\gamma + 1)$ on $]a_-/a_+, +\infty[$. Using (3.6), we have that the function $I_2(\rho) - \rho^{-\frac{\gamma+3}{2(\gamma+1)}} I_1(\rho)$ (in the interval of definition) has constant sign equal to the sign of $-(\gamma + 3)/(\gamma + 1)$. Finally, recalling that $I_1'(\rho)$ is negative, we infer that the (positive) function $I_1(\rho)/I_2(\rho)$ is strictly monotone increasing if $(\gamma + 3)/(\gamma + 1) > 0$ and, respectively, is strictly monotone decreasing if $(\gamma + 3)/(\gamma + 1) < 0$.

Summing up, since $F_2(\rho)$ is a product of two positive functions, we have that $F_2(\rho)$ is strictly monotone if

$$\frac{\gamma - 1}{\gamma + 1} \cdot \frac{\gamma + 3}{\gamma + 1} > 0,$$

thus, if

$$\gamma < -3 \quad \text{or} \quad \gamma > 1.$$

We stress that in these cases $F_2(\rho)$ is strictly monotone increasing. Moreover, if $\gamma = -3$ we obtain that the function $I_1(\rho)/I_2(\rho)$ is constant and so $F_2(\rho) = \rho$ is strictly monotone increasing, while if $\gamma = 1$, then $F_2(\rho) = I_1(\rho)/I_2(\rho)$ is strictly monotone increasing.

From the above discussion, we conclude that if $\gamma \in]-\infty, -3] \cup [1, +\infty[$, then the function $F_2(\rho)$ is strictly monotone increasing. The claim is proved.

Hence, we have shown that there exists at most one value of ρ such that the second equation in (3.5) holds. When $\gamma \neq 1$ the first equation in (3.5) is uniquely solvable, and thus for $\gamma \in]-\infty, -3] \cup]1, +\infty[$ there exists at most one pair (ω, ρ) which solves (3.5).

To conclude the proof of Theorem 1.1, in view of Remark 2.1, we only need to prove that if $\gamma \cdot \bar{a} < 0$, then there exists at least one pair (ω, ρ) which solves (3.5). Preliminarily, we recall that if the second equation (which depends only on ρ) has a solution $\hat{\rho}$, then the first equation is solvable, since its first member (with $\rho = \hat{\rho}$) is surjective on $]0, +\infty[$.

We observe that the function $F_2(\rho)$ is continuous and strictly monotone increasing in $]0, a_-/a_+[$, for $\gamma > 1$, and, respectively, in $]a_-/a_+, +\infty[$, for $\gamma \leq -3$. We compute the limits as ρ approaches the endpoints. By L'Hôpital's rule, we have

$$\lim_{\rho \to \frac{a_-}{a_+}} \frac{I_1(\rho)}{I_2(\rho)} = \lim_{\rho \to \frac{a_-}{a_+}} \frac{I_1'(\rho)}{I_2'(\rho)} = \lim_{\rho \to \frac{a_-}{a_+}} \rho^{-\frac{\gamma-1}{2(\gamma+1)}+1} = \left(\frac{a_-}{a_+}\right)^{-\frac{\gamma-1}{2(\gamma+1)}+1}$$

and thus

$$\lim_{\rho \to \frac{a_-}{a_+}} F_2(\rho) = \frac{a_-}{a_+}. \tag{3.7}$$

If $\gamma > 1$, then

$$\lim_{\rho \to 0^+} F_2(\rho) = 0. \tag{3.8}$$

Indeed, as $\rho \to 0^+$, $I_1(\rho)$ and $I_2(\rho)$ tend to positive constants, and $\rho^{\frac{\gamma-1}{2(\gamma+1)}} \to 0^+$. Therefore, we conclude that the second equation in (3.5) has a solution if and only if $\frac{a_-}{a_+} > \frac{\tau}{T-\tau}$, which is equivalent to assume that $\bar{a} < 0$ (cf. Remark 2.1). While, if $\gamma \leq -3$, then

$$\lim_{\rho \to +\infty} F_2(\rho) = +\infty.$$

Indeed, as $\rho \to +\infty$, $I_1(\rho)$ and $I_2(\rho)$ tend to negative constants, and $\rho^{\frac{\gamma-1}{2(\gamma+1)}} \to +\infty$. Therefore, we conclude that the second equation in (3.5) has a solution if and only if $\frac{a_-}{a_+} < \frac{\tau}{T-\tau}$, which is equivalent to assume that $\bar{a} > 0$.

The proof of Theorem 1.1 is completed. □

Remark 3.1. (The case $\gamma = 1$) The proof of Theorem 1.1 in particular states that, for $\gamma = 1$, the function $F_2(\rho)$ is strictly monotone increasing. Arguing as in the second part of the proof, one can also show that (3.7) and (3.8) are still valid, using L'Hôpital's rule and, respectively, the fact that, when $\gamma = 1$, as $\rho \to 0^+$, $I_1(\rho)$ tends to a positive constant, while $I_2(\rho) \to +\infty$. We can conclude that the second equation in (3.5) is uniquely solved. Let $\hat{\rho}$ be the solution, then the first equation in (3.5) reads as $I_1(\hat{\rho})/2 = \tau$, which either is not solvable, or holds for all $\omega \in]0, +\infty[$. This is in agreement with the well-known fact that the eigenspace of Neumann/periodic one-signed solutions of the linear equation $u'' + a(t)u = 0$ has dimension less than or equal to 1 (cf. [31]).

Remark 3.2. (The linear/nonlinear eigenvalue problem) Let us consider the nonlinear eigenvalue problem

$$u'' + \lambda a(t)u^\gamma = 0$$

together with the Neumann/periodic boundary conditions. In this case, in the above analysis, system (3.5) should be modified to

$$\begin{cases} \dfrac{|\gamma + 1|^{-\frac{\gamma}{\gamma+1}}}{\sqrt{2}} \omega |\omega|^{-\frac{\gamma}{\gamma+1}-\frac{1}{2}} I_1(\rho) = \sqrt{\lambda}\tau, \\ F_2(\rho) = \dfrac{\tau}{T-\tau}. \end{cases}$$

The study of the second equation of this system is the same as above and, for $\gamma \in]-\infty, -3] \cup [1, +\infty[$, it provides a unique solution $\hat{\rho}$. By inserting this value in the first equation, for $\gamma \neq 1$, we obtain the curve

$$\omega = G(u(0)) = \left(\frac{\sqrt{2}\tau}{|\gamma + 1|^{-\frac{\gamma}{\gamma+1}} I_1(\hat{\rho})}\right)^{\frac{2(\gamma+1)}{1-\gamma}} \lambda^{\frac{\gamma+1}{1-\gamma}}$$

and thus the classical bifurcation diagrams appearing in similar situations (cf. [32, Case 1, p. 446]). On the other hand, if $\gamma = 1$, we have

$$\lambda = \lambda_1 := \left(\frac{I_1(\hat{\rho})}{2\tau}\right)^2,$$

which ensures the existence of a unique positive principal eigenvalue consistently with the classical linear theory (cf. [33–35] and, concerning the p-Laplacian, see [36] and references therein).

3.2 The case $\phi(s) = |s|^{p-2}s$ with $p > 1$

We deal with the more general case

$$\phi(s) = \varphi_p(s) = |s|^{p-2}s, \quad p > 1.$$

The proof of Theorem 1.2 follows similar steps to the ones in the proof of Theorem 1.1 in Section 3.1; for this reason, in order to avoid unnecessary repetitions, we focus only on the main differences and skip the verification of some computations.

Recalling that the inverse of φ_p is $\varphi_q(s) = |s|^{q-2}s$, where $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$h(y) = \varphi_q(y) = |y|^{\frac{2-p}{p-1}}y, \quad H(y) = \frac{p-1}{p}|y|^{\frac{p}{p-1}-1}y, \quad y \in \mathbb{R},$$

$$H_l^{-1}(\xi) = -\left(\frac{p}{p-1}\right)^{\frac{p-1}{p}}|\xi|^{\frac{p-1}{p}}, \quad \xi \in \,]-\infty, 0],$$

$$H_r^{-1}(\xi) = \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}}\xi^{\frac{p-1}{p}}, \quad \xi \in [0, +\infty[,$$

which leads to

$$\mathcal{L}_h(\xi) = -\left(\frac{p}{p-1}\right)^{\frac{1}{p}}|\xi|^{\frac{1}{p}}, \quad \xi \in [0, +\infty[.$$

The functions $g(x)$, $G(x)$ and $G^{-1}(\xi)$ are defined as in (3.1)–(3.3), respectively. Hence, $\mathcal{L}_g$ is defined as in (3.4).

With these positions, proceeding similarly as in Section 3.1, we introduce the functions

$$I_1(\rho) = \int_{\frac{a_+}{a_+ + a_-}(\rho+1)}^{1} \frac{\mathrm{d}\xi}{|a_+ - a_+\xi|^{\frac{1}{p}}\,|\xi|^{\frac{\gamma}{\gamma+1}}},$$

$$I_2(\rho) = \int_{\frac{a_+}{a_-}}^{\frac{a_+}{a_+ + a_-}\frac{\rho+1}{\rho}} \frac{\mathrm{d}\xi}{|a_-\xi - a_+|^{\frac{1}{p}}\,|\xi|^{\frac{\gamma}{\gamma+1}}},$$

and, by Corollary 2.1, we infer that proving that system (2.15) admits a unique solution $(\omega, \rho) \in \mathcal{D}$ is equivalent to prove that there exists a unique pair $(\omega, \rho) \in \mathcal{D}$ which solves

$$\begin{cases} \dfrac{|\gamma + 1|^{-\frac{\gamma}{\gamma+1}}}{\left(\dfrac{p}{p-1}\right)^{\frac{1}{p}}}\omega|\omega|^{-\frac{\gamma}{\gamma+1}-\frac{1}{p}}I_1(\rho) = \tau, \\[6mm] \rho^{-1+\frac{\gamma}{\gamma+1}+\frac{1}{p}}\dfrac{I_1(\rho)}{I_2(\rho)} = \dfrac{\tau}{T - \tau}, \end{cases} \tag{3.9}$$

which is exactly system (3.5) when $p = 2$.

As above, we study the auxiliary function

$$F_p(\rho) = \rho^{-1+\frac{\gamma}{\gamma+1}+\frac{1}{p}}\frac{I_1(\rho)}{I_2(\rho)}.$$

We compute

$$I_1'(\rho) = \left(\frac{a_+}{a_+ + a_-}\right)^{1 - \frac{\gamma}{\gamma+1} - \frac{1}{p}} \frac{1}{|a_- - a_+\rho|^{\frac{1}{p}} (\rho + 1)^{\frac{\gamma}{\gamma+1}}},$$

$$I_2'(\rho) = \rho^{-2 + \frac{\gamma}{\gamma+1} + \frac{1}{p}} I_1'(\rho)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{I_1(\rho)}{I_2(\rho)}\right) = \frac{I_1'(\rho)}{(I_2(\rho))^2}\left(I_2(\rho) - \rho^{-2 + \frac{\gamma}{\gamma+1} + \frac{1}{p}} I_1(\rho)\right).$$

Moreover, we have

$$\lim_{\rho \to \frac{a_-}{a_+}} \left(I_2(\rho) - \rho^{-2 + \frac{\gamma}{\gamma+1} + \frac{1}{p}} I_1(\rho)\right) = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(I_2(\rho) - \rho^{-2 + \frac{\gamma}{\gamma+1} + \frac{1}{p}} I_1(\rho)\right) = \left(2 - \frac{\gamma}{\gamma+1} - \frac{1}{p}\right)\rho^{-3 + \frac{\gamma}{\gamma+1} + \frac{1}{p}} I_1(\rho).$$

Reasoning as in the proof of Theorem 1.1, we conclude that $F_p(\rho)$ is strictly monotone if

$$\left(-1 + \frac{\gamma}{\gamma+1} + \frac{1}{p}\right) \cdot \left(2 - \frac{\gamma}{\gamma+1} - \frac{1}{p}\right) > 0,$$

which is equivalent to

$$(p-1)\left(\gamma - \frac{1-2p}{p-1}\right)(\gamma - (p-1)) > 0.$$

Moreover, if $\gamma = (1 - 2p)/(p-1)$ we obtain that the function $I_1(\rho)/I_2(\rho)$ is constant and so F_p is strictly monotone increasing, while if $\gamma = p - 1$, then $F_p(\rho) = I_1(\rho)/I_2(\rho)$ is strictly monotone increasing.

For the part concerning the existence of solutions, we can complete the proof of Theorem 1.2 by arguing exactly as in Section 3.1.   □

Remark 3.3. In this context, the case $\gamma = p - 1$ plays a similar role to the linear case $u'' + a(t)u = 0$. Similar considerations to those in Remark 3.1 are in order.

4 Final remarks

In this final section, we collect some complementary results and open questions motivated by our approach.

4.1 The "excluded" values of γ

Even if Theorems 1.1 and 1.2 cover a wide range of values of the real exponent γ, some interesting cases are not investigated. In this section, we present some comments concerning these situations. In order to simplify the exposition, we focus our attention on the linear differential operator, namely $\phi(s) = s$; similar analysis could be performed for the p-Laplacian operator.

It is convenient to split the analysis for $\gamma \in \;]-3, 1[\,\backslash\, \{-1\}$ into the two cases $\gamma \in \;]-1, 1[$ and $\gamma \in \;]-3, -1[$. In the first situation, there may be non-negative solutions which vanish (when $0 < \gamma < 1$) or hit the singularity in finite time (when $-1 < \gamma < 0$) in the interval $[\tau, T[$ where the weight is negative. In the second case, the singularity satisfies the strong force condition (cf. [37,38]) and this prevents the possibility that solutions reach the singularity.

Case 1. $\gamma \in \,]-1, 1[$. Recalling system (3.5), we consider the function $F_2(\rho)$ with $\rho \in \,]0, a_-/a_+[$. Preliminarily, we notice that for $\gamma = 0$ an easy computation shows that $F_2(\rho) \equiv a_-/a_+$ for all $\rho \in \,]0, a_-/a_+[$. Thus, from now on, we exclude this trivial situation.

By L'Hôpital's rule, we observe that

$$\lim_{\rho \to 0^+} \frac{\rho^{\frac{\gamma-1}{2(\gamma+1)}}}{I_2(\rho)} = \lim_{\rho \to 0^+} \frac{\gamma - 1}{2(\gamma+1)} \frac{\rho^{-\frac{\gamma+3}{2(\gamma+1)}}}{I_2'(\rho)} = \frac{\gamma - 1}{2(\gamma+1)} \lim_{\rho \to 0^+} \frac{\rho^{-\frac{\gamma+3}{2(\gamma+1)}}}{\rho^{-\frac{\gamma+3}{2(\gamma+1)}} I_1'(\rho)}$$

$$= \frac{\gamma - 1}{2(\gamma+1)} \lim_{\rho \to 0^+} \frac{1}{I_1'(\rho)} = \frac{1 - \gamma}{2(\gamma+1)} \left(\frac{a_+ + a_-}{a_+} \right)^{1 - \frac{\gamma}{\gamma+1} - \frac{1}{2}} (a_-)^{\frac{1}{2}}.$$

Therefore, we deduce that

$$\lim_{\rho \to 0^+} F_2(\rho) = \frac{1 - \gamma}{2(\gamma+1)} \left(\frac{a_+ + a_-}{a_+} \right)^{\frac{1-\gamma}{2(\gamma+1)}} \left(\frac{a_-}{a_+} \right)^{\frac{1}{2}} \int_{\frac{a_+}{a_+ + a_-}}^{1} \frac{d\xi}{|1 - \xi|^{\frac{1}{2}} \, |\xi|^{\frac{\gamma}{\gamma+1}}} =: K_0(\gamma).$$

On the other hand, the limit of $F_2(\rho)$ as $\rho \to (a_-/a_+)^-$ is the same as obtained in (3.7).

The fact that $K_0(\gamma) > 0$ for all $\gamma \in \,]-1, 1[$ implies that the function $F_2(\rho)$ is not surjective on $]0, a_-/a_+[$. As a consequence, the condition $\gamma \cdot \bar{a} < 0$ is not sufficient for the existence of positive solutions. This fact is not surprising and actually it is consistent with the previous observation that in the interval where the weight is negative we have the possibility of solutions vanishing in finite time, due to the lack of uniqueness for the Cauchy problems, or hitting the singularity, due to the absence of the strong force condition in zero.

For further convenience, we claim that the function $K_0 : \,]-1, 1[\, \to \,]0, +\infty[$ is a strictly monotone decreasing function such that

$$\lim_{\gamma \to (-1)^+} K_0(\gamma) = +\infty, \quad \lim_{\gamma \to 1^-} K_0(\gamma) = 0, \quad K_0(0) = \frac{a_-}{a_+}. \tag{4.1}$$

Indeed, equalities in (4.1) can be easily checked by a direct inspection. Thus, it is sufficient to check the strict monotonicity of the map $\psi : \,]-1, 1[\, \to \,]0, +\infty[$ defined as

$$\psi(\gamma) = \frac{1 - \gamma}{2(\gamma+1)} L^{\frac{1-\gamma}{2(\gamma+1)}} \int_{\frac{1}{L}}^{1} \frac{d\xi}{(1 - \xi)^{\frac{1}{2}} \, \xi^{\frac{\gamma}{\gamma+1}}},$$

where $L = (a_+ + a_-)/a_- > 1$. Setting $\vartheta = (1 - \gamma)/(2(\gamma+1))$, which maps $]-1, 1[$ to $]0, +\infty[$ as a strictly monotone decreasing function, we are led to prove that the function $\eta : \,]0, +\infty[\, \to \,]0, +\infty[$ defined as

$$\eta(\vartheta) = \vartheta \int_{\frac{1}{L}}^{1} \frac{(L\xi)^{\vartheta}}{(1 - \xi)^{\frac{1}{2}} \, \xi^{\frac{1}{2}}} \, d\xi$$

is strictly monotone increasing. Observing that the map $\vartheta \mapsto (L\xi)^{\vartheta}$ is strictly monotone increasing for every fixed $\xi \in \,]1/L, 1[$, the claim is thus proved.

We are now ready to discuss the solvability of the boundary value problems. In the case $\gamma \in \,]0, 1[$, the nonlinearity is concave and smooth in $]0, +\infty[$ and we can apply [7, Lemma 3.1] (see also [22]) to ensure the fact that there exists at most one positive solution of the problem. This implies that the function $F_2(\rho)$ is strictly monotone. More precisely, taking into account the monotonicity of $K_0(\gamma)$ and $K_0(0) = a_-/a_+$, we have

$$\lim_{\rho \to 0^+} F_2(\rho) = K_0(\gamma) < \frac{a_-}{a_+} = \lim_{\rho \to \frac{a_-}{a_+}} F_2(\rho).$$

Therefore, $F_2(\rho)$ is strictly monotone increasing. Hence, by the continuity of $F_2(\rho)$ on $]0, a_-/a_+[$, we can state the following.

Proposition 4.1. *Let $a \in L^\infty(0, T)$ be as in (1.4). Let $\gamma \in \,]0, 1[$. Then, the Neumann and the periodic boundary value problems associated with equations (1.5) and (1.6) have a positive solution if and only if*

$$K_0(\gamma) < \frac{\tau}{T - \tau} < \frac{a_-}{a_+}. \tag{4.2}$$

Moreover, in such a case, the solution is unique.

The lack of the Lipschitz condition in zero for the nonlinearity $g(u) = u^\gamma$ with $\gamma \in \,]0, 1[$ has as a consequence the fact that nonnegative solutions of some boundary value problems associated with (1.5) may vanish together with their derivatives at some points of the domain where a is negative. In this manner, we can find solutions which are positive on some part of their domain and identically zero elsewhere. This fact, already observed in [39] for the periodic problem, and in [40,7] for the Dirichlet and the Neumann problems associated with

$$\Delta u + a(x)g(u) = 0, \quad x \in \Omega, \tag{4.3}$$

where Ω is a bounded smooth domain of $\mathbb{R}^n$, has raised the interesting question of finding conditions ensuring the existence of *strictly positive* solutions. A recent contribution to this problem is obtained in [41]. In particular, in [41, Corollary 1.8] the following result is proved: *if $g(u) = u^\gamma$ with $\gamma \in \,]0, 1[$, and $\int_\Omega a < 0$, then there exists $q_0 = q_0(a) \in \,]0, 1[$ such that there is a positive solution for the Neumann problem associated with (4.3), provided that $q_0 < \gamma < 1$.* Proposition 4.1, for the special one-dimensional case $\Omega = \,]0, T[$ and $a(x) = a(t)$ a stepwise function, provides a sharp estimate for the constant q_0 in the above result. Indeed, $\int_\Omega a < 0$ corresponds to the second inequality in (4.2). Then, using the strict monotonicity of $K_0(\gamma)$ with $\lim_{\gamma \to 1^-} K_0(\gamma) = 0$, we can define q_0 as the unique value in $]0, 1[$ such that $K_0(q_0) = \tau/(T - \tau)$ and thus reestablish [41, Corollary 1.8] in our special case with a precise value of q_0. For further results in this direction see also [42,43] and references therein.

In the case $\gamma \in \,]-1, 0[$, by the previous argument as before, we have instead

$$\lim_{\rho \to \frac{a_-}{a_+}} F_2(\rho) = \frac{a_-}{a_+} < K_0(\gamma) = \lim_{\rho \to 0^+} F_2(\rho).$$

Even without information of the monotonicity, the continuity of $F_2(\rho)$ implies that its range covers the open interval $]a_-/a_+, K_0(\gamma)[$. Hence, we have the following.

Proposition 4.2. *Let $a \in L^\infty(0, T)$ be as in (1.4). Let $\gamma \in \,]-1, 0[$. Then, the Neumann and the periodic boundary value problems associated with equations (1.5) and (1.6) have a positive solution if*

$$\frac{a_-}{a_+} < \frac{\tau}{T - \tau} < K_0(\gamma). \tag{4.4}$$

To the best of our knowledge, there are no general uniqueness results available for this range of the exponent. Numerical simulations suggest that the function $F_2(\rho)$ is strictly monotone decreasing (Figure 1). This would guarantee that the positive solution is unique also in this situation and consequently condition (4.4) would be sharp, likewise condition (4.2) is sharp in the case $\gamma \in \,]0, 1[$.

To compare our result with a similar case previously studied, we recall that in [17, Corollary 3] the authors investigate a singular equation with a stepwise indefinite function and obtain a range for the existence of positive solutions, that for instance in the case $\gamma = -1/2$ and $a_- = a_+$ would read like

$$1 < \frac{\tau}{T - \tau} \le \sqrt{\frac{35}{27}}.$$

On the other hand, an easy computation shows that $K_0(\gamma) = 5$ and thus (4.4) reads as

$$1 < \frac{\tau}{T - \tau} < 5.$$

This shows that, at least in this case, the range provided by our result is better.

Case 2. $\gamma \in]-3, -1[$. We consider the function $F_2(\rho)$ with $\rho \in]a_-/a_+, +\infty[$ and compute the limits at the endpoints. The limit of $F_2(\rho)$ as $\rho \to (a_-/a_+)^-$ is the same as obtained in (3.7). As $\rho \to +\infty$, $I_1(\rho)$ and $I_2(\rho)$ tend to negative constants and, thus, $F_2(\rho) \to +\infty$ as $\rho \to +\infty$. The continuity of $F_2(\rho)$ on $]a_-/a_+, +\infty[$ implies that its range covers the open interval $]a_-/a_+, +\infty[$. Therefore, we have the following.

Proposition 4.3. *Let $a \in L^\infty(0, T)$ be as in (1.4). Let $\gamma \in]-3, -1[$. Then, the Neumann and the periodic boundary value problems associated with equations (1.5) and (1.6) have a positive solution if and only if* $\int_0^T a(t) \, dt > 0$.

We observe that Proposition 4.3 is already contained in [38, Theorem 1]. Our contribution is just in providing a different proof. Concerning the problem of uniqueness, numerical simulations suggest that $F_2(\rho)$ is strictly monotone increasing in $]a_-/a_+, +\infty[$ (Figure 2). This would guarantee that the positive solution is unique.

At this point, it is natural to raise the following conjecture.

Conjecture 4.1. Let $a \in L^\infty(0, T)$ be as in (1.4). Let $\gamma \in \mathbb{R} \setminus \{-1, 0, 1\}$. Then, the Neumann and the periodic boundary value problems associated with equations (1.5) and (1.6) have at most one positive solution.

4.2 The Minkowski-curvature operator

In this paper, we have applied Theorem 2.1 to the case of homogeneous differential operators $\phi(s)$ and homogeneous nonlinearities $g(u)$. This choice has the advantage to simplify (2.13) to (3.5) or (3.9), and the consequent analysis of the functions $F_2(\rho)$ or, respectively, $F_p(\rho)$. However, our technique appears useful to study more general situations, especially when the function $\mathcal{L}_h$ can be computed explicitly. For example, in the case of the Minkowski-curvature equation

$$\left(\frac{u'}{\sqrt{1-(u')^2}}\right)' + a(t)g(u) = 0, \tag{4.5}$$
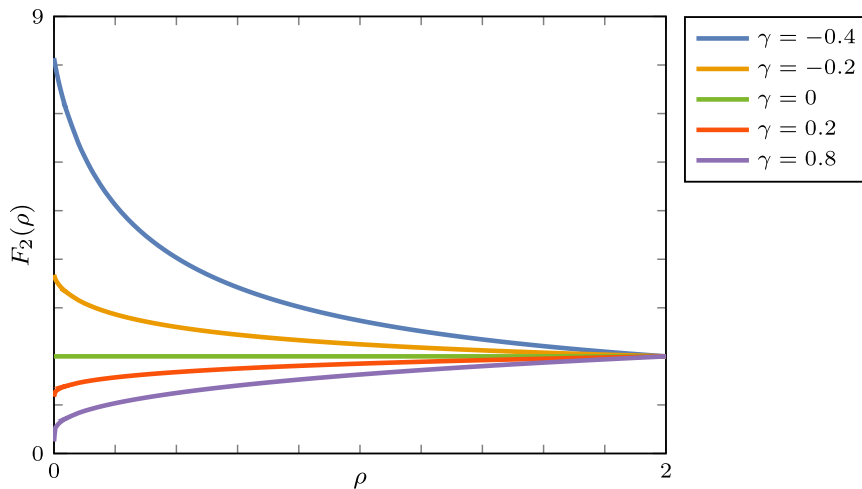
we have

$$\phi(s) = \frac{s}{\sqrt{1-s^2}}$$

Figure 1: Graphs of $F_2(\rho)$ for $\gamma \in \{-0.4, -0.2, 0, 0.2, 0.8\}$, with $a_+ = 1$ and $a_- = 2$.
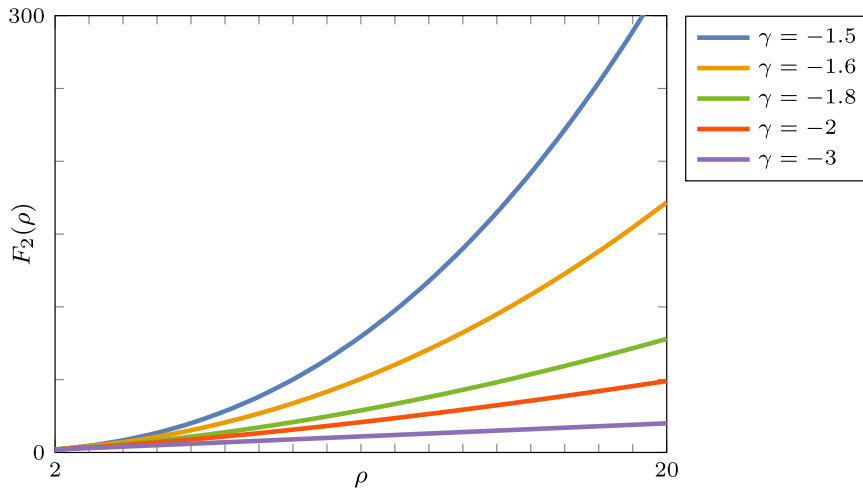
Figure 2: Graphs of $F_2(\rho)$ for $\gamma \in \{-3, -2, -1.8, -1.6, -1.5\}$, with $a_+ = 1$ and $a_- = 2$.

and therefore, for $h = \phi^{-1}$, we find

$$h(y) = \frac{y}{\sqrt{1 + y^2}}, \quad H(y) = \sqrt{1 + y^2} - 1, \quad \mathcal{L}_h(\xi) = -\frac{\sqrt{\xi^2 + 2\xi}}{1 + \xi}.$$

As a consequence, the analysis of system (2.13) could be simplified for some special choices of $g(u)$. We do not pursue here this investigation which can be the topic of future research.

Recent contributions for the Neumann and periodic problems associated with (4.5) show that for $g(u) = u^\gamma$ with $\gamma > 1$ multiple positive solutions do exist also in the case of a weight with a single change of sign as in (1.4) (see [9,44]). On the other hand, numerical simulations suggest the possibility of uniqueness results when $g(u)$ is a strictly increasing function with "super-exponential" growth at infinity (cf. [45]).

Conflict of interest: Authors state no conflict of interest.

References

[1] A. Kristály, V. D. Rădulescu, and C. G. Varga, *Variational principles in mathematical physics, geometry, and economics. Qualitative analysis of nonlinear equations and unilateral problems*, Vol. 136 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2010.

[2] P. Hess and T. Kato, *On some linear and nonlinear eigenvalue problems with an indefinite weight function*, Comm. Partial Differ. Equ. **5** (1980), 999–1030.

[3] S. Alama and G. Tarantello, *Elliptic problems with nonlinearities indefinite in sign*, J. Funct. Anal. **141** (1996), 159–215.

[4] H. Amann and J. López-Gómez, *A priori bounds and multiple solutions for superlinear indefinite elliptic problems*, J. Differ. Equ. **146** (1998), 336–374.

[5] H. Berestycki, I. Capuzzo-Dolcetta, and L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic problems*, NoDEA Nonlinear Differ. Equ. Appl. **2** (1995), 553–572.

[6] G. Feltrin, *Positive Solutions to Indefinite Problems. A Topological Approach*, Frontiers in Mathematics, Birkhäuser/Springer, Cham, 2018.

[7] C. Bandle, M. A. Pozio, and A. Tesei, *Existence and uniqueness of solutions of nonlinear Neumann problems*, Math. Z. **199** (1988), 257–278.

[8] A. Boscaggin, *A note on a superlinear indefinite Neumann problem with multiple positive solutions*, J. Math. Anal. Appl. **377** (2011), 259–268.

[9] A. Boscaggin and G. Feltrin, *Positive periodic solutions to an indefinite Minkowski-curvature equation*, J. Differ. Equ. **269** (2020), 5595–5645.

[10] G. Feltrin and F. Zanolin, *Existence of positive solutions in the superlinear case via coincidence degree: the Neumann and the periodic boundary value problems*, Adv. Differ. Equ. **20** (2015), 937–982.

[11] G. Feltrin and F. Zanolin, *Multiplicity of positive periodic solutions in the superlinear indefinite case via coincidence degree*, J. Differ. Equ. **262** (2017), 4255–4291.

[12] J. López-Gómez and P. Omari, *Characterizing the formation of singularities in a superlinear indefinite problem related to the mean curvature operator*, J. Differ. Equ. **269** (2020), 1544–1570.

[13] J. López-Gómez, P. Omari, and S. Rivetti, *Positive solutions of a one-dimensional indefinite capillarity-type problem: a variational approach*, J. Differ. Equ. **262** (2017), 2335–2392.

[14] C. Bereanu and M. Zamora, *Periodic solutions for indefinite singular perturbations of the relativistic acceleration*, Proc. Roy. Soc. Edinburgh Sect. A **148** (2018), 703–712.

[15] A. Boscaggin and F. Zanolin, *Second-order ordinary differential equations with indefinite weight: the Neumann boundary value problem*, Ann. Mat. Pura Appl. **194** (2015), 451–478.

[16] J. L. Bravo and P. J. Torres, *Periodic solutions of a singular equation with indefinite weight*, Adv. Nonlinear Stud. **10** (2010), 927–938.

[17] J. Godoy and M. Zamora, *A general result to the existence of a periodic solution to an indefinite equation with a weak singularity*, J. Dynam. Differ. Equ. **31** (2019), 451–468.

[18] J. Godoy and M. Zamora, *Periodic solutions for a second-order differential equation with indefinite weak singularity*, Proc. Roy. Soc. Edinburgh Sect. A **149** (2019), 1135–1152.

[19] R. Hakl and M. Zamora, *Periodic solutions to second-order indefinite singular equations*, J. Differ. Equ. **263** (2017), 451–469.

[20] A. J. Ureña, *A counterexample for singular equations with indefinite weight*, Adv. Nonlinear Stud. **17** (2017), 497–516.

[21] T. Zhou, B. Du, and H. Du, *Positive periodic solution for indefinite singular Liénard equation with p-Laplacian*, Adv. Differ. Equ. **2019** (2019), art. 158.

[22] K. J. Brown and P. Hess, *Stability and uniqueness of positive solutions for a semi-linear elliptic boundary value problem*, Differ. Integral Equ. **3** (1990), 201–207.

[23] R. Hakl and M. Zamora, *Existence and uniqueness of a periodic solution to an indefinite attractive singular equation*, Ann. Mat. Pura Appl. **195** (2016), 995–1009.

[24] K. Nakashima, *The uniqueness of indefinite nonlinear diffusion problem in population genetics, part I*, J. Differ. Equ. **261** (2016), 6233–6282.

[25] C. V. Coffman and M. Marcus, *Existence and uniqueness results for semi-linear Dirichlet problems in annuli*, Arch. Rational Mech. Anal. **108** (1989), 293–307.

[26] L. Erbe and M. Tang, *Uniqueness theorems for positive radial solutions of quasilinear elliptic equations in a ball*, J. Differ. Equ. **138** (1997), 351–379.

[27] L. Erbe and M. Tang, *Uniqueness of positive radial solutions of $\Delta u + K(|x|)\gamma(u) = 0$*, Differ. Integral Equ. **11** (1998), 663–678.

[28] J. López-Gómez, P. Omari, and S. Rivetti, *Bifurcation of positive solutions for a one-dimensional indefinite quasilinear Neumann problem*, Nonlinear Anal. **155** (2017), 1–51.

[29] E. Sovrano and F. Zanolin, *Indefinite weight nonlinear problems with Neumann boundary conditions*, J. Math. Anal. Appl. **452** (2017), 126–147.

[30] C. Rebelo, *A note on uniqueness of Cauchy problems associated to planar Hamiltonian systems*, Portugal. Math. **57** (2000), 415–419.

[31] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[32] P.-L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev. **24** (1982), 441–467.

[33] M. Bôcher, *The smallest characteristic numbers in a certain exceptional case*, Bull. Amer. Math. Soc. **21** (1914), 6–9.

[34] K. J. Brown and S. S. Lin, *On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function*, J. Math. Anal. Appl. **75** (1980), 112–120.

[35] A. Zettl, *Sturm-Liouville Theory*, vol. 121 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2005.

[36] G. Meng, P. Yan, and M. Zhang, *Spectrum of one-dimensional p-Laplacian with an indefinite integrable weight*, Mediterr. J. Math. **7** (2010), 225–248.

[37] P. J. Torres, *Mathematical Models with Singularities: A Zoo of Singular Creatures*, vol. 1 of Atlantis Briefs in Differential Equations, Atlantis Press, Paris, 2015.

[38] A. J. Ureña, *Periodic solutions of singular equations*, Topol. Methods Nonlinear Anal. **47** (2016), 55–72.

[39] G. J. Butler, *Periodic solutions of sublinear second order differential equations*, J. Math. Anal. Appl. **62** (1978), 676–690.

[40] C. Bandle, M. A. Pozio, and A. Tesei, *The asymptotic behavior of the solutions of degenerate parabolic equations*, Trans. Amer. Math. Soc. **303** (1987), 487–501.

[41] U. Kaufmann, H. Ramos Quoirin, and K. Umezu, *Positivity results for indefinite sublinear elliptic problems via a continuity argument*, J. Differ. Equ. **263** (2017), 4481–4502.

[42] U. Kaufmann, H. Ramos Quoirin, and K. Umezu, *Positive solutions of an elliptic Neumann problem with a sublinear indefinite nonlinearity*, NoDEA Nonlinear Differ. Equ. Appl. **25** (2018), art. 12.

[43] U. Kaufmann, H. Ramos Quoirin, and K. Umezu, *Nonnegative solutions of an indefinite sublinear Robin problem I: positivity, exact multiplicity, and existence of a subcontinuum*, Ann. Mat. Pura Appl. **199** (2020), 2015–2038.

[44] A. Boscaggin and G. Feltrin, *Pairs of positive radial solutions for a Minkowski-curvature Neumann problem with indefinite weight*, Nonlinear Anal. **196** (2020), art. 111807.

[45] A. Boscaggin, G. Feltrin, and F. Zanolin, *Positive solutions for a Minkowski-curvature equation with indefinite weight and super-exponential nonlinearity*, arXiv:2007.00338, (2020).