Tohoku Math. J.
64 (2012), 453–468

UNIQUENESS OF SASAKI-EINSTEIN METRICS

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(Received September 1, 2010, revised February 9, 2012)

Abstract. In this paper, we shall prove the uniqueness of Sasaki-Einstein metrics on compact Sasaki manifolds modulo the action of the identity component of the automorphism group for the transverse holomorphic structure. This generalizes the result of Cho, Futaki and Ono [5] for compact toric Sasaki manifolds.

1. Introduction. The aim of this paper is to show the uniqueness of Sasaki-Einstein metrics up to the action of the identity component of the automorphism group for the transverse holomorphic structure. A *Sasaki manifold* is a Riemannian manifold (S, g) whose cone metric $\bar{g} = dr^2 + r^2 g$ on $C(S) = S \times \boldsymbol{R}_+$ is Kählerian. Sasakian geometry sits naturally in two aspects of Kähler geometry, since for one thing, (S, g) is the base of the *cone manifold* $(C(S), \bar{g})$ which is Kählerian, and for another thing any Sasaki manifold is contact, and the one dimensional foliation associated to the characteristic Reeb vector field admits a *transverse Kähler structure*. A Sasaki-Einstein manifold then admits a one dimensional Reeb foliation with a transverse Kähler-Einstein metric, which is studied from viewpoints of geometry and mathematical physics. Boyer, Galicki, Kollár and Thomas obtained Sasaki-Einstein metrics on a family of the links of hypersurfaces of Brieskorn-Pham type, which include exotic spheres [3, 4]. Gauntlett, Martelli, Sparks and Waldram discovered that there exist irregular toric Sasaki-Einstein manifolds which are not obtained as total spaces of line orbibundles on Kähler-Einstein orbifolds [8, 9]. These toric examples are much explored by Futaki, Ono and Wang [7], who showed that, for any compact toric Sasaki manifold with positive basic first Chern class and trivial first Chern class of the contact bundle, one can find a deformed Sasakian structure on which a Sasaki-Einstein metric exists. Furthermore, Cho, Futaki and Ono proved in [5] the uniqueness of Sasaki-Einstein metrics on compact toric Sasaki manifolds up to the action of the identity component of the automorphism group for the transverse holomorphic structure by showing that the argument of Guan [10] is valid also for the space of Kähler potentials for the transverse Kähler structure. In the present paper, we shall prove such uniqueness without toric assumption:

THEOREM A. *Let (S, g) be a compact Sasaki manifold with a Sasakian structure $\mathcal{S} = \{g, \xi, \eta, \Phi\}$. Assume that the set $\mathscr{E}$ of all Sasaki-Einstein metrics compatible with g is non-empty. Then the identity component of the automorphism group for the transverse holomorphic structure acts transitively on $\mathscr{E}$.*

2000 *Mathematics Subject Classification.* Primary 53C55; Secondary 53D10.
Key words and phrases. Sasaki manifolds, Einstein metrics, transverse Kähler geometry.
The first named author is partly supported by Grant-in-Aid for Young Scientists (B) (23740063).

Our proof of Theorem A is based on a generalization of the argument of Bando and Mabuchi in [1] to Sasakian geometry. The key point is to show an a priori C^0-estimate for solutions of the transverse Monge-Ampère equations (cf. (3.1) and (3.2)) and an intriguing point is an estimate of the infimum for the solutions (cf. Proposition 3.6). The difficulty which we encounter is that the transverse Monge-Ampère equations only give lower bounds for the transverse Ricci curvature, which do not lead lower bounds for the Ricci curvature by positive constants. Therefore we cannot apply Myers' theorem directly to obtain an estimate of the diameter of (S, g). To overcome the difficulty, we introduce a family of Sasakian structures $g_{\varphi,\mu}$ whose contact forms are $\mu^{-1}\eta_\varphi$. Under a suitable choice of μ, it follows that the Ricci curvature of $g_{\varphi,\mu}$ is bounded from below by a positive constant. Thus we can control their diameters by Myers' theorem. The estimate of their volumes together with their diameters gives rise to the desired estimate of the solutions by using the estimate of the Green functions (see Proposition 3.6 for more details). Our method of the estimate is simple and effective in transverse Kähler metrics, which is slightly different from the ordinary argument in Kähler geometry as basic Kähler classes of the family $g_{\varphi,\mu}$ are changing.

Acknowledgments. We would like to express our gratitude to Professors Ryushi Goto and Toshiki Mabuchi for valuable comments and encouragements. Many thanks are due also to the referee for his careful reading of the paper and for his numerous suggestions.

2. Brief review of Sasakian geometry.

2.1. Sasaki manifolds. We recall the basic theory of Sasaki manifolds. For the details, see [2] or [7]. Throughout this paper, we assume that all manifolds are connected. Let (S, g) be a Riemannian manifold and $(C(S), \bar{g}) = (S \times \boldsymbol{R}_+, dr^2 + r^2 g)$ be its cone manifold, where $\boldsymbol{R}_+ = \{x \in \boldsymbol{R}\ ;\ x > 0\}$ and r is the standard coordinate on $\boldsymbol{R}_+$.

DEFINITION 2.1. (S, g) is called a *Sasaki manifold* if the cone manifold $(C(S), \bar{g})$ is a Kähler manifold.

A Sasaki manifold S is often identified with the submanifold $\{r = 1\} \subset (C(S), \bar{g})$ and hence the dimension of S must be odd. Let $\dim S = 2n + 1$. Then, of course, $\dim_{\boldsymbol{C}} C(S) = n + 1$. Let J be a complex structure of the cone $C(S)$ such that the triple $(C(S), J, \bar{g})$ is a Kähler manifold, and define $\tilde{\xi} := J(r\partial/\partial r)$. Then the restriction $\xi := \tilde{\xi}|_{\{r=1\}}$ of $\tilde{\xi}$ to the submanifold $\{r = 1\}$ gives a vector field on S. The vector field ξ is called the *Reeb vector field*. The 1-dimensional foliation $\mathcal{F}_\xi$ generated by ξ is called the *Reeb foliation*. Define a differential 1-form η on S by $\eta := g(\xi, \cdot)$. Then, one can see that

(1) $\tilde{\xi}$ is a Killing vector field and satisfies $L_{\tilde{\xi}} J = 0$,
(2) $\nabla_\xi \xi = 0$,
(3) $\eta(\xi) = 1$, $\iota_\xi d\eta = 0$.

In particular, ξ is a Killing vector field on S. The 1-form η gives a $2n$-dimensional subbundle D of the tangent bundle TS by

$$D = \ker \eta\,.$$

The subbundle D is a contact structure of S and there is an orthogonal decomposition

$$TS = D \oplus \boldsymbol{R}\xi ,$$

where $\boldsymbol{R}\xi$ is the 1-dimensional trivial bundle generated by ξ.

Next we define a section Φ of the endomorphism bundle $\mathrm{End}(TS)$ of the tangent bundle TS by $\Phi = \nabla\xi$. Then it satisfies

$$\Phi^2 = -\mathrm{id} + \eta \otimes \xi$$

and $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$. Furthermore, we have $\Phi|_D = J|_D$ and $\Phi|_{\boldsymbol{R}\xi} = 0$, and this shows that Φ gives a complex structure of D. We call the quadruple $\mathcal{S} = \{g, \xi, \eta, \Phi\}$ a *Sasakian structure* of S. From these description, the restriction $g_D := g|_{D\times D}$ of the metric g to D is a Hermitian metric on D and the associated 2-form of the Hermitian metric is equal to $(1/2)d\eta|_{D\times D}$, that is,

$$d\eta(X, Y) = 2g(\Phi X, Y)$$

for each $X, Y \in D$. Since η is a contact form, $((1/2)d\eta)^n \wedge \eta$ is a non-vanishing $(2n+1)$-form. The covariant differentiation of Φ can be written as, in the language of curvature,

$$(\nabla_X \Phi)(Y) = R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi$$

for any $X, Y \in TS$. The Ricci curvature along the Reeb foliation is given by

$$\mathrm{Ric}(X, \xi) = 2n\eta(X) \tag{2.1}$$

for each $X \in TS$.

2.2. Transverse holomorphic structures and transverse Kähler structures. As we saw in the last subsection, $\tilde{\xi} - \sqrt{-1}J\tilde{\xi}$ is a holomorphic vector field on $C(S)$. Hence there is an action of the holomorphic flow generated by $\tilde{\xi} - \sqrt{-1}J\tilde{\xi}$ on $C(S)$. The local orbits of this action define a *transverse holomorphic structure* on the Reeb foliation $\mathcal{F}_\xi$ in the following sense. There is an open covering $\{U_\alpha\}_{\alpha\in A}$ of S and submersions $\pi_\alpha : U_\alpha \to V_\alpha \subset \boldsymbol{C}^n$ from U_α onto an open subset V_α in $\boldsymbol{C}^n$ such that

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$$

is biholomorphic whenever $U_\alpha \cap U_\beta \neq \emptyset$. Let $(z^1, z^2, \dots, z^n)$ be the local holomorphic coordinates on V_α. We pull back these to U_α and still write them as $(z^1, z^2, \dots, z^n)$. Let x be the coordinate along the leaves with $\xi = \partial/\partial x$. Then $(x, z^1, z^2, \dots, z^n)$ forms a local coordinate system on U_α. We call the coordinate system given above a *foliation chart*. On each open set $V_\alpha \subset \boldsymbol{C}^n$ we can give a Kähler structure as follows. First note that there is a canonical isomorphism $((\pi_\alpha)_*)_p|_{D_p} : D_p \to T_{\pi_\alpha(p)}V_\alpha$ for any $p \in U_\alpha$. Since ξ generates isometries of (S, g), the restriction g_D of the Sasaki metric g to D gives a well-defined Hermitian metric g_α^T on V_α. This Hermitian structure is in fact Kählerian because the pull-back of the fundamental 2-form ω_α^T of g_α^T to U_α is the same as the restriction of $(1/2)d\eta$ to U_α. Hence we see that $\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$ gives an isometry of Kähler manifolds. The collection of Kähler metrics $\{g_\alpha^T\}_{\alpha\in A}$ on $\{V_\alpha\}_{\alpha\in A}$ is called a *transverse Kähler metric*. Since they are isometric over the overlaps we simply denote by g^T. The collection of Kähler forms $\{\omega_\alpha^T\}_{\alpha\in A}$ is called a *transverse Kähler form.* We also write $\nabla^T, R^T, \mathrm{Ric}^T, s^T$ for its Levi-Civita connection, the curvature, the Ricci tensor and the scalar curvature, respectively. By identifying D_p with $T_{\pi_\alpha(p)}V_\alpha$, we have formulas for curvature

$$
\begin{aligned}
(2.2)\qquad R(X,Y,Z,W) = R^T(X,Y,Z,W) &+ g(\Phi(X),Z)g(\Phi(Y),W)\\
&- g(\Phi(X),W)g(\Phi(Y),Z) + 2g(\Phi(X),Y)g(\Phi(Z),W)\,,
\end{aligned}
$$

$$(2.3)\qquad \mathrm{Ric}^T(X,Y) = \mathrm{Ric}(X,Y) + 2g(X,Y)$$

for any local sections X, Y, Z, W of D.

2.3. Basic forms. In this subsection, we assume that the Sasaki manifold (S, g) is compact.

DEFINITION 2.2. A differential k-form α on S is said to be *basic* if

$$\iota_\xi\alpha = 0 \quad\text{and}\quad L_\xi\alpha = 0.$$

Let Λ_B^k be the sheaf of germs of basic k-forms and Ω_B^k the set of all basic k-forms.

Consider a complex basic form α which can be written as

$$\alpha = \sum \alpha_{i_1,\dots,i_p,j_1,\dots,j_q} dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar z^{j_1}\wedge\cdots\wedge d\bar z^{j_q}$$

for a foliation chart $(x, z^1, \dots, z^n)$. We call such α a *basic* (p,q)*-form.* It is easy to see that the definition of basic (p,q)-forms is independent of choice of foliation charts. Let $\Lambda_B^{p,q}$ be the sheaf of germs of basic (p,q)-forms and $\Omega_B^{p,q}$ the set of all basic (p,q)-forms. Then for each k, $\Lambda_B^k\otimes \boldsymbol{C}$ (resp. $\Omega_B^k\otimes\boldsymbol{C}$) can be decomposed as

$$\Lambda_B^k\otimes\boldsymbol{C} = \bigoplus_{p+q=k}\Lambda_B^{p,q} \qquad \left(\text{resp. } \Omega_B^k\otimes\boldsymbol{C} = \bigoplus_{p+q=k}\Omega_B^{p,q}\right).$$

Since the exterior derivative d preserves basic forms, we have the *basic exterior derivative* $d_B := d|_{\Lambda_B^k} : \Lambda_B^k \to \Lambda_B^{k+1}$. Then d_B can be decomposed into $d_B = \partial_B + \bar\partial_B$ by well-defined operators

$$\partial_B : \Lambda_B^{p,q} \to \Lambda_B^{p+1,q} \quad\text{and}\quad \bar\partial_B : \Lambda_B^{p,q}\to\Lambda_B^{p,q+1}\,.$$

Then it is clear that

$$d_B^2 = 0\,,\quad \partial_B^2 = 0 \quad\text{and}\quad \bar\partial_B^2 = 0\,.$$

Let d_B^*, ∂_B^* and $\bar\partial_B^*$ be the formal adjoint operators of d_B, ∂_B and $\bar\partial_B$, respectively, and define

$$\Delta_B := d_B^*d_B + d_Bd_B^*\,,\quad \Box_B := \partial_B^*\partial_B + \partial_B\partial_B^*\,,\quad \bar\Box_B := \bar\partial_B^*\bar\partial_B + \bar\partial_B\bar\partial_B^*\,.$$

As in the cases of compact Kähler manifolds, both $\Box_B$ and $\bar\Box_B$ are real operators and satisfy $2\Delta_B = \Box_B = \bar\Box_B$ (see [6]). Moreover, as shown later Δ_B coincides with Riemannian Laplacian Δ on the space of basic functions. Now we can consider the *basic de Rham complex* (Ω_B^*, d_B) and the *basic Dolbeault complex* $(\Omega_B^{p,*}, \bar\partial_B)$. Their cohomology groups are called the *basic cohomology groups.* Similarly, we can consider the *basic harmonic forms.* El-Kacimi-Alaoui showed in [6] that there is an isomorphism between the basic cohomology groups and the space of basic harmonic forms. Moreover, it was proved in [6] a Sasakian geometry analogue of the $\partial\bar{\partial}$-lemma still holds. We denote by $C_B^\infty(S)$ the set of all real-valued smooth basic functions on S.

PROPOSITION 2.3 ([6]). *Let α be a real d_B-exact basic form of type $(1,1)$ on a compact Sasaki manifold (S, g). Then there exists a basic function $\varphi \in C_B^\infty(S)$ such that*

$$\alpha = \sqrt{-1}\partial_B\bar{\partial}_B\varphi\,,$$

which is unique up to an additive constant.

For arbitrary basic function $\varphi \in C_B^\infty(S)$, define

$$\eta_\varphi := \eta + 2d_B^c\varphi\,,$$

where $d_B^c = (\sqrt{-1}/2)(\bar{\partial}_B - \partial_B)$. Then we have

$$\frac{1}{2}d\eta_\varphi = \frac{1}{2}d\eta + d_B d_B^c\varphi = \frac{1}{2}d\eta + \sqrt{-1}\partial_B\bar{\partial}_B\varphi\,. \tag{2.4}$$

Thus, for small φ, $((1/2)d\eta_\varphi)^n \wedge \eta_\varphi$ is nowhere vanishing and the 1-form η_φ gives a new Sasakian structure $\mathcal{S}_\varphi = \{g_\varphi, \xi, \eta_\varphi, \Phi_\varphi\}$, where

$$\Phi_\varphi = \Phi - \xi \otimes (2d_B^c\varphi)\circ\Phi\,, \quad g_\varphi = \frac{1}{2}d\eta_\varphi \circ (\mathrm{id}\otimes\Phi_\varphi) + \eta_\varphi\otimes\eta_\varphi$$

(see [2]). By construction, $\mathcal{S}_\varphi$ defines the same transverse holomorphic structure with that of $\mathcal{S}$. Note that the contact bundle may be changed under the deformation. As we saw in the last subsection, the transverse Kähler form $\{\omega_\alpha^T\}_{\alpha\in A}$ satisfies

$$\pi_\alpha^*\omega_\alpha^T = \frac{1}{2}d\eta|_{U_\alpha}\,.$$

Thus they are glued together and give a d_B-closed basic $(1,1)$-form $(1/2)d\eta$ on S. We also call $\omega^T = (1/2)d\eta$ the transverse Kähler form. Under such a deformation, the transverse Kähler form is deformed in the same basic $(1,1)$-class $[(1/2)d\eta]_B$ by (2.4). We call this class the *basic Kähler class*. Similarly, we see that the Ricci forms of the transverse Kähler metric $\{\rho_\alpha^T\}_{\alpha\in A}$,

$$\rho_\alpha^T = -\sqrt{-1}\partial_B\bar{\partial}_B\log\det(g_\alpha^T)\,,$$

are glued together and give a d_B-closed basic $(1,1)$-form ρ^T on S. ρ^T is called the *transverse Ricci form*. Of course, the transverse Ricci form ρ^T depends on Sasaki metrics g. Nevertheless, its basic de Rham cohomology class is invariant under deformations of the Sasakian structure by basic functions. The basic de Rham cohomology class $[\rho^T/2\pi]_B$ is called the *basic first Chern class* and denoted by $c_1^B(S)$.

2.4. Basic first Chern classes and the transverse Monge-Ampère equations. Let (S, g) be a $(2n+1)$-dimensional compact Sasaki manifold.

DEFINITION 2.4. A *Sasaki-Einstein manifold* is a Sasaki manifold (S, g) with $\mathrm{Ric} = 2ng$.

The Einstein condition of a Sasaki manifold is translated into Einstein conditions of the Riemannian cone $(C(S), \bar{g})$ and the transverse Kähler structure. In fact, it is known that the following conditions are equivalent (see [2]):

(1) g is a Sasaki-Einstein metric.

(2) The cone metric $\bar{g}$ on $C(S)$ is a Ricci-flat Kähler metric.

(3) The transverse Kähler metric g^T satisfies $\mathrm{Ric}^T = (2n+2)g^T$.

We say that the basic first Chern class $c_1^B(S)$ is *positive* if $c_1^B(S)$ is represented by a transverse Kähler form, and we express this condition by $c_1^B(S) > 0$. If there exists a Sasaki-Einstein metric g, then the transverse Kähler metric g^T satisfies $\mathrm{Ric}^T = (2n+2)g^T$. Hence the basic Kähler class satisfies $2\pi c_1^B(S) = (2n+2)[\omega^T]_B$ and, in particular, the basic first Chern class is positive. It is known that there is a further necessary condition for the existence of positive or negative transverse Kähler-Einstein metric.

PROPOSITION 2.5 (Futaki-Ono-Wang [7]). *The basic first Chern class is represented by $\tau d\eta$ for some constant τ if and only if $c_1(D) = 0$.*

We now consider a condition for the existence of Sasaki-Einstein metrics and set up the transverse Monge-Ampère equation. Let (S, g) be a compact Sasaki manifold with $2\pi c_1^B(S) = (2n+2)[\omega^T]_B$ (in particular $c_1^B(S) > 0$ and $c_1(D) = 0$). Then there is a unique basic function $h \in C_B^\infty(S)$ such that

$$\rho^T - (2n+2)\omega^T = \sqrt{-1}\partial_B\bar{\partial}_B h\,, \qquad \int_S (e^h - 1)\left(\frac{1}{2}d\eta\right)^n \wedge \eta = 0\,.$$

Assume that we can get a Sasaki-Einstein metric g_φ for some basic function φ. Then the associated transverse Kähler form $\omega_\varphi^T = (1/2)d\eta + \sqrt{-1}\partial_B\bar{\partial}_B\varphi$ satisfies

$$\rho_\varphi^T = (2n+2)\omega_\varphi^T\,.$$

This leads the transverse Kähler-Einstein (or equivalently Sasaki-Einstein) equation

$$\frac{\det(g_{i\bar{j}}^T + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}}^T)} = \exp(-(2n+2)\varphi + h) \tag{2.5}$$

with $(g_{i\bar{j}}^T + \varphi_{i\bar{j}})$ positive definite, where $\varphi_{i\bar{j}} := \partial^2\varphi/\partial z^i\partial\bar{z}^j$ for $i, j \in \{1, 2, \ldots, n\}$.

In [5] and [7], the existence and the uniqueness of Sasaki-Einstein metrics on compact toric Sasaki manifolds are studied. In [7], it was proved that for any compact toric Sasaki manifold (S, g) with $c_1^B(S) > 0$ and $c_1(D) = 0$, one can get a Sasaki-Einstein metric by deforming the Sasakian structure varying the Reeb vector field (cf. [7, Theorem 1.2]). Uniqueness of such Einstein metrics up to a connected group action was proved in [5]. Given a Sasaki manifold (S, g), we say that another Sasaki metric g' on S is *compatible* with g if g and g' have the same Reeb vector field and the transverse holomorphic structure. Note that g and g' have the same basic Kähler class. Indeed, for corresponding Sasakian structure $\mathcal{S}' = \{g', \xi', \eta', \Phi'\}$, $\zeta := \eta - \eta'$ is basic because $\xi = \xi'$. This shows that $d\eta - d\eta' = d\zeta$ and in particular $[d\eta]_B = [d\eta']_B$.

DEFINITION 2.6. The *automorphism group of a transverse holomorphic structure of* (S, g) is the set of all biholomorphic automorphisms of $C(S)$ which commute with the holomorphic flow generated by $\tilde{\xi} - \sqrt{-1}J\tilde{\xi}$.

We denote by $\mathrm{Aut}(C(S), \tilde{\xi})$ the group of automorphisms of the transverse holomorphic structure and by G its identity component $\mathrm{Aut}(C(S), \tilde{\xi})_0$. It is known that the action of $\mathrm{Aut}(C(S), \tilde{\xi})$ on $C(S)$ descends to an action on S preserving the Reeb vector field and the transverse holomorphic structure of the Reeb foliation. In particular, G acts on the space of all Sasaki metrics on S which is compatible with g. The Lie algebra of $\mathrm{Aut}(C(S), \tilde{\xi})$ is explained as follows.

DEFINITION 2.7 (Futaki-Ono-Wang, [7]). A complex vector field X on S is called a *Hamiltonian holomorphic vector field* if

(1) for each $\alpha \in A$, $(\pi_\alpha)_* X$ is a holomorphic vector field on V_α,
(2) the complex valued function $u_X := \sqrt{-1}\eta(X)$ satisfies

$$\bar{\partial}_B u_X = -\frac{\sqrt{-1}}{2}\iota_X d\eta\,.$$

By definition, every Hamiltonian holomorphic vector field is supposed to commute with ξ. We denote by $\mathfrak{h}$ the set of all Hamiltonian holomorphic vector fields. One can check easily that $\mathfrak{h}$ is in fact a Lie algebra. Then it was proved in [5] that the Lie algebra of $\mathrm{Aut}(C(S), \tilde{\xi})$ is isomorphic to $\mathfrak{h}$ (for detailed descriptions, see also [7]). Under the notations and conventions, it was proved in [5] that, for toric cases, G acts transitively on the space of all Sasaki-Einstein metrics compatible with g.

2.5. Basic Laplacians for Sasaki manifolds. In Subsection 2.3, we introduced the notion of a basic Laplacian, which is defined on the space of basic forms. Here we shall show that the basic Laplacian Δ_B and the Riemannian Laplacian Δ coincide on the space $C_B^\infty(S)$ of basic functions. Let $T \subset \mathrm{Isom}(S, g)$ be the compact torus defined as the closure of the one-parameter subgroup generated by ξ in $\mathrm{Isom}(S, g)$, and let $d\theta$ be the normalized Haar measure on T. For any smooth function $\varphi \in C^\infty(S)$, define

$$B(\varphi) := \int_T \theta^*\varphi d\theta\,.$$

Then B defines a linear operator on $C^\infty(S)$. It is clear that $B(\varphi) \in C_B^\infty(S)$ for any $\varphi \in C^\infty(S)$ and that $B(\varphi) = \varphi$ if and only if $\varphi \in C_B^\infty(S)$. Furthermore, one can show that B is symmetric with respect to the L^2-inner product on $C^\infty(S)$ by Fubini theorem and the symmetry of T. Hence we obtain a orthogonal decomposition

$$C^\infty(S) = C_B^\infty(S) \oplus C_B^\infty(S)^\perp\,,$$

where $C_B^\infty(S)^\perp$ is the orthogonal complement of $C_B^\infty(S)$ with respect to L^2-inner product. Under the description, B coincides with the orthogonal projection from $C^\infty(S)$ onto $C_B^\infty(S)$.

We denote by d^* the formal adjoint operator of d. For each $\varphi \in C_B^\infty(S)$ and $\alpha \in \Omega_B^1(S)$, we have

$$(d_B\varphi, \alpha) = (d\varphi, \alpha) = (\varphi, d^*\alpha) = (\varphi, Bd^*\alpha)\,,$$

where $(\cdot, \cdot)$ is the L^2-inner product on the space of smooth differential forms. This shows that $d_B^* = B \circ d^*$ and hence

$$\Delta_B\varphi = d_B^* d_B\varphi = Bd^*d\varphi = B\Delta\varphi\,. \tag{2.6}$$

Furthermore, for each $\varphi \in C_B^\infty(S)$ and $\theta \in T$, $\theta^*\Delta\varphi = \Delta\theta^*\varphi = \Delta\varphi$ since θ acts on (S, g) as an isometry. Therefore we obtain

$$B\Delta\varphi = \int_T \theta^*\Delta\varphi d\theta = \int_T \Delta\varphi d\theta = \Delta\varphi\,. \tag{2.7}$$

By combining the equalities (2.6) and (2.7), we have the following proposition.

PROPOSITION 2.8. *For each $\varphi \in C_B^\infty(S)$ we have $\Delta_B\varphi = \Delta\varphi$.*

Using a foliation chart, we obtain an explicit formula for the basic complex Laplacian $\Box_B = (1/2)\Delta_B$ by a similar calculation in Kähler geometry.

PROPOSITION 2.9. *Let U be an open neighborhood of S and $(x, z^1, \ldots, z^n)$ a foliation chart on U. Then*

$$\Box_B\varphi = -(g^T)^{i\bar{j}}\varphi_{i\bar{j}} \text{ on } U$$

for any $\varphi \in C_B^\infty(S)$.

3. A proof of Theorem A.

3.1. Generalized Aubin's equations. In this section, we shall give a proof of Theorem A. Our proof of Theorem A is based on a generalization of the argument of Bando and Mabuchi [1] to Sasakian geometry. The celebrated result on a basic version of Hodge theory due to El-Kacimi-Alaoui [6] (including the basic $\partial\bar{\partial}$-lemma) allows us to imitate Kähler geometry on Sasaki manifolds. Hence we can translate the language of analysis for the complex Monge-Ampère equation on Kähler manifolds into that for the transverse Monge-Ampère equation (2.5) for smooth basic functions on Sasaki manifolds. In particular, by replacing the geometry of Kähler manifolds with the transverse Kähler geometry of Sasaki manifolds, the most of the arguments and techniques in [1] can be generalized directly to our settings (except for Proposition 3.6, as described later) as follows.

Throughout this section, we denote by (S, g) a $(2n+1)$-dimensional compact Sasaki manifold with Sasakian structure $\mathcal{S} = \{g, \xi, \eta, \Phi\}$. By assumption, it follows that $2\pi c_1^B(S) = (2n+2)[\omega^T]_B$. Define $\mathscr{S}(g)$ to be the set of all Sasaki metric on S which is compatible with g, and set $\mathscr{H} := \{\varphi \in C_B^\infty(S)\ ;\ (g^T_{i\bar{j}} + \varphi_{i\bar{j}})$ is positive definite$\}$. Then $g_\varphi \in \mathscr{S}(g)$ for any $\varphi \in \mathscr{H}$.

Let V be the volume of S with respect to $((1/2)d\eta)^n \wedge \eta$, and define the functionals L_η, M_η, I_η and J_η on $\mathscr{H}$ by

$$
\begin{aligned}
L_\eta(\varphi) &:= \frac{1}{V}\int_a^b dt \int_S \dot{\varphi}_t \Big(\frac{1}{2} d\eta_{\varphi_t}\Big)^n \wedge \eta_{\varphi_t}\,,\\
M_\eta(\varphi) &:= -\frac{1}{V}\int_a^b dt \int_S \dot{\varphi}_t (s^T(\varphi_t) - n(2n+2))\Big(\frac{1}{2} d\eta_{\varphi_t}\Big)^n \wedge \eta_{\varphi_t}\,,\\
I_\eta(\varphi) &:= \frac{1}{V}\int_S \varphi\Big(\Big(\frac{1}{2} d\eta\Big)^n \wedge \eta - \Big(\frac{1}{2} d\eta_{\varphi}\Big)^n \wedge \eta_{\varphi}\Big),\\
J_\eta(\varphi) &:= \frac{1}{V}\int_a^b dt \int_S \dot{\varphi}_t\Big(\Big(\frac{1}{2} d\eta\Big)^n \wedge \eta - \Big(\frac{1}{2} d\eta_{\varphi_t}\Big)^n \wedge \eta_{\varphi_t}\Big),
\end{aligned}
$$

where $\{\varphi_t\ ;\ t \in [a,b]\}$ is an arbitrary piecewise smooth path in $\mathscr{H}$ such that $\varphi_a = 0$ and $\varphi_b = \varphi$. These are the "Sasakian geometry version" of the functionals defined on the space of Kähler potentials in [1] and have the similar properties to those. The precise definitions and basic properties can be seen in [11, Appendix A].

Since $[\rho^T]_B = (2n+2)[\omega^T]_B$, there exists a unique basic function $h \in C_B^\infty(S)$ which satisfies $\rho^T - (2n+2)\omega^T = \sqrt{-1}\partial_B\bar{\partial}_B h$ and $\int_S (e^h - 1)((1/2)d\eta)^n \wedge \eta = 0$. Consider one-parameter families of equations, which are Sasakian geometry analogues of (generalized) Aubin's equations,

$$
\frac{\det(g^T_{i\bar{j}} + (\psi_t)_{i\bar{j}})}{\det(g^T_{i\bar{j}})} = \exp(-t(2n+2)\psi_t + h)\,; \quad t \in [0,1]\,, \tag{3.1}
$$

$$
\frac{\det(g^T_{i\bar{j}} + (\varphi_t)_{i\bar{j}})}{\det(g^T_{i\bar{j}})} = \exp(-t(2n+2)\varphi_t - L_\eta(\varphi_t) + h)\,; \quad t \in [0,1]\,, \tag{3.2}
$$

where solutions ψ_t and φ_t are both required to belong to $\mathscr{H}$. Note that, for both equations, these are just the transverse Kähler-Einstein equations at $t = 1$. As a remark in [1], there is no difference between (3.1) and (3.2) in finding solutions for $t \neq 0$.

REMARK 3.1. Choose an arbitrary $t \in [0,1]$. Let ψ_t (resp. φ_t) be a solution of (3.1) (resp. (3.2)) and g_t be the Sasaki metric corresponding to the Sasakian structure η_{ψ_t} (resp. η_{φ_t}). Then g_t satisfies $\rho^T_t = t(2n+2)\omega^T_t + (1-t)(2n+2)\omega^T$, and in particular $\rho^T_t - t(2n+2)\omega^T_t \geq 0$. Furthermore, if $t \neq 1$, then $\rho^T_t - t(2n+2)\omega^T_t$ is strictly positive.

We first discuss the existence and uniqueness of the solutions of the equation (3.2) for $t \in [0,1)$. For the equation (3.1), a result of El-Kacimi-Alaoui [6] guarantees the existence of a solution at $t = 0$. Then the existence and uniqueness of a solution of the equation (3.2) follows immediately, which corresponds to [1, Corollary (4.3.2)].

THEOREM 3.2 (El Kacimi-Alaoui [6]). *If $t = 0$, then the equation* (3.1) *has a solution which is unique up to an additive constant.*

COROLLARY 3.3. *The equation* (3.2) *has a unique solution* φ_0 *at* $t = 0$. *The solution* φ_0 *satisfies* $L_\eta(\varphi_0) = 0$.

The local extension property of solutions of (3.2) for $t \in [0, 1)$ can be represented as follows (cf. [1, Proposition (4.4.1)]).

PROPOSITION 3.4. *Let* $0 \leq \tau < 1$. *Suppose that the equation* (3.2) *has a solution* φ_τ *at* $t = \tau$. *Then for some* $\varepsilon > 0$, φ_τ *uniquely extends to a smooth one parameter family* $\{\varphi_t\ ;\ t \in [0, 1] \cap [\tau - \varepsilon, \tau + \varepsilon]\}$ *of solutions of* (3.2).

REMARK 3.5. A Hamiltonian holomorphic vector field X is said to be *normalized* if the Hamiltonian function u_X satisfies

$$\int_S u_X e^h \left(\frac{1}{2} d\eta\right)^n \wedge \eta = 0 .$$

For any $X \in \mathfrak{h}$, there exists a constant c such that $X + c\xi$ is a normalized Hamiltonian holomorphic vector field. We denote by $\mathfrak{h}_0$ the set of all normalized Hamiltonian holomorphic vector fields. If $\mathfrak{h}_0 = \{0\}$ and $\tau = 1$, the result of Futaki, Ono and Wang (cf. [7]) tells us that $\ker(\Box_{\varphi_1} - (2n+2)) \cong \mathfrak{h}_0 = \{0\}$ and the first positive eigenvalue of $\Box_{\varphi_1}$ is greater than $2n+2$, where $\Box_{\varphi_1}$ is the basic complex Laplacian with respect to g_1. This shows that Proposition 3.4 still holds when $\mathfrak{h}_0 = \{0\}$ and $\tau = 1$.

Next we discuss a bound for solutions of (3.2). Choose $\alpha \in (0, 1)$ arbitrary. Then, by El Kacimi-Alaoui's generalization of Yau's estimate [12] for transverse Monge-Ampère equations, the C^0-estimate for solutions of (3.2) implies the $C^{2,\alpha}$-estimate for them.

First of all, we give a bound of the oscillation $\mathrm{osc}_S\varphi := \sup_S \varphi - \inf_S \varphi$ for $\varphi \in \mathscr{H}$ in terms of the functional I_η (cf. [1, Proposition (3.6)]). There is a difficulty to translate the techniques in [1] to our cases. Their proof of [1, Proposition (3.6)] is based on the estimate of a lower bound of the Green function, which follows from a lower bound of the Ricci curvature. The (generalized) Aubin's equation (cf. [1, (4.1.1)]) gives a lower bound of the Ricci curvature by a positive constant. By applying Myers' theorem, we can obtain a lower bound of the Green function by a universal constant. However, in our cases, the equation (3.2) only gives a lower bound of the transverse Ricci curvature (cf. Remark 3.1), which does not lead a lower bound of the Ricci curvature by a positive constant. Hence we cannot obtain a lower bound of the Green function directly.

Later in Subsection 3.3, we shall overcome such difficulty and show the following proposition.

PROPOSITION 3.6. *Let* G *be the Green function of the initial metric* g *and* K *a real constant which satisfies* $\inf G \geq -K$. *Let* $\varphi \in \mathscr{H}$ *be a basic function which satisfies* $\rho^T_\varphi \geq t(2n + 2)\omega^T_\varphi$ *for some* $t \in (0, 1]$. *Then there exists a positive constant* $C > 0$ *which is independent of* t *and satisfies*

$$\mathrm{osc}_S\varphi \leq I_\eta(\varphi) + 2n\left(\frac{KV}{n!} + \frac{C}{t}\right) .$$

Proposition 3.6 tells us that a bound of I_η for solutions of (3.2) implies a priori C^0-estimate for solutions of (3.2) (see [1, Step 1 of the proof of Theorem (5.3)]).

PROPOSITION 3.7. *If there exists a positive constant $A > 0$ which satisfies*

$$I_\eta(\varphi_t) \leq A$$

for each solution φ_t of (3.2) *at t, then there exists a positive constant $B > 0$ depending only on A, n and the initial metric g which satisfies*

$$\sup_S |\varphi_t| \leq B\,.$$

By combining [11, Lemma 4.9, Proposition A.3] and Proposition 3.7, we can generalize the argument of Bando and Mabuchi to Sasakian geometry to obtain the uniqueness of solutions of the equation (3.2) for $t \in [0, 1)$.

THEOREM 3.8. *Let $\tau \in (0, 1)$. Then any solution φ_τ of* (3.2) *at $t = \tau$ uniquely extends to a smooth family $\{\varphi_t\ ;\ t \in [0, \tau]\}$ of solutions of* (3.2). *In particular the equation* (3.2) *admits at most one solution at $t = \tau$.*

As a straightforward consequence of Theorem 3.8 and Remark 3.5, we have the following corollary.

COROLLARY 3.9. *Let (S, g) be a compact Sasaki manifold with $\mathfrak{h}_0 = \{0\}$. Then there exists at most one Sasaki-Einstein metric on S which is compatible with g.*

3.2. Solutions at $t = 1$. In order to complete the proof of Theorem A, we next refer to the solutions of the equation (3.1) at $t = 1$. By assumption, $\mathscr{E} \neq \emptyset$ and hence the equation (3.1) has a solution at $t = 1$. Consider the G-action on $\mathscr{E}$. Let O be an arbitrary G-orbit in $\mathscr{E}$. For each $g_{\mathrm{SE}} \in \mathscr{E}$, we can uniquely associate a function $\psi = \psi(g_{\mathrm{SE}}) \in \mathscr{H}$ such that $g_{\mathrm{SE}} = g_\psi$ and ψ satisfies the equation (3.1) at $t = 1$. Hence we can regard O as a subset of the set of all solutions of the equation (3.1) at $t = 1$. By the identification, we endow O with the topology induced from the $C^{2,\alpha}$-norm on $C_B^\infty(S)$ for fixed $\alpha \in (0, 1)$. Then the G-action on O is clearly continuous. Hence the topology on O coincides with the natural topology of the homogeneous space $O \cong G/K_{g_{\mathrm{SE}}}$, where $K_{g_{\mathrm{SE}}}$ is the isotropic subgroup of G at g_{SE}. Let $\Box_{\mathrm{SE}}$ be the basic complex Laplacian with respect to g_{SE}. For each $\varphi \in \ker(\Box_{\mathrm{SE}} - (2n + 2))$, we have the associated normalized Hamiltonian holomorphic vector field

$$X_\varphi = \varphi\xi + \nabla^i\varphi\frac{\partial}{\partial z^i} - \eta\left(\nabla^i\varphi\frac{\partial}{\partial z^i}\right)\xi$$

for any foliation chart $(x, z^1, \ldots, z^n)$ (see [7, Theorem 5.1]). Let $f_{\varphi,t}$ be the corresponding one-parameter group $\exp(tX_\varphi^{\boldsymbol{R}})$. Here we denote by $X_\varphi^{\boldsymbol{R}}$ the real part of X_φ. We put $g_{\mathrm{SE}}(t) := f_{\varphi,t}^* g_{\mathrm{SE}}$ and $\psi(t) := \psi(g_{\mathrm{SE}}(t))$. Then we can check easily that $\dot\psi(0) = \varphi + C$ for some $C \in \boldsymbol{R}$. On the other hand, since $\psi(t)$ satisfies the equation (3.1) at $t = 1$, we have $\Box_{\mathrm{SE}}\dot\psi(0) = (2n+2)\dot\psi(0)$ by differentiating the equality (3.1). This shows that $C = 0$ and hence $\dot\psi(0) = \varphi$. Conversely, for each smooth curve $g(t) \in O$ with $g(0) = g_{\mathrm{SE}}$, take the corresponding smooth functions $\psi(t) \in \mathscr{H}$. Then we have $\square_{\mathrm{SE}}\dot{\psi}(0) = (2n+2)\dot{\psi}(0)$ by differentiating the equality (3.1). Thus we obtain

$$T_{g_{\mathrm{SE}}} O \cong \ker(\square_{\mathrm{SE}} - (2n+2)) .$$

Under the notations and conventions we described above, we can completely generalize the arguments in [1] to Sasakian geometry. At first, the lemmas presented in [1, Section 6] can be translated to Sasakian geometry as follows (cf. [1, Lemmas (6.2), (6.3) and (6.4)]).

LEMMA 3.10. *Define* $\iota := (I_\eta - J_\eta)|_O$ (≥ 0) : $O \to \boldsymbol{R}$, *and let* $g_{\mathrm{SE}} \in O$. *Put* $\psi := \psi(g_{\mathrm{SE}})$ *and* $\eta_{\mathrm{SE}} := \eta_\psi$. *Then the following are equivalent.*

(1) g_{SE} *is a critical point of* ι,

(2) $\displaystyle\int_S \varphi\psi\left(\tfrac{1}{2}d\eta_{\mathrm{SE}}\right)^n \wedge \eta_{\mathrm{SE}} = 0$ *for any* $\varphi \in \ker(\square_{\mathrm{SE}} - (2n+2))$.

LEMMA 3.11. *The function* ι *is proper. In particular, its minimum is always attained at some point of the orbit* O.

LEMMA 3.12. *Let* $g_{\mathrm{SE}} \in O$ *be a critical point of* ι. *Then the Hessian* $(\mathrm{Hess}\,\iota)_{g_{\mathrm{SE}}}$ *of* ι *at* g_{SE} *is given by*

$$(\mathrm{Hess}\,\iota)_{g_{\mathrm{SE}}}(\varphi', \varphi'') = \frac{2n+2}{V}\int_S \left(1 - \frac{1}{2}\square_{\mathrm{SE}}\psi\right)\varphi'\varphi''\left(\frac{1}{2}d\eta_\psi\right)^n \wedge \eta_\psi$$

for each $\varphi', \varphi'' \in \ker(\square_{\mathrm{SE}} - (2n+2)) \cong T_{g_{\mathrm{SE}}} O$, *where* $\psi := \psi(g_{\mathrm{SE}})$.

By Lemma 3.12 and using bifurcation technique (see [1, Section 7]), we can prove the local extension property of solutions of (3.1) at $t = 1$ for critical points of ι with non-degenerate Hessian (cf. [1, Theorem (7.3)]). Here we use a Sasakian geometry version of [1, Lemma (7.2)], which can be generalized immediately.

PROPOSITION 3.13. *For each critical point* $g_{\mathrm{SE}} \in O$ *of* ι *with non-degenerate Hessian*, $\psi_1 := \psi(g_{\mathrm{SE}})$ *can be extended to a smooth family* $\{\psi_t\ ;\ t \in [1-\varepsilon, 1]\}$ *of solutions of* (3.1) *for some* $\varepsilon > 0$.

REMARK 3.14. Fix a G-orbit O in $\mathscr{E}$ arbitrary and consider the function $\iota : O \to \boldsymbol{R}$. By Lemma 3.11, ι always has a minimizer $g_{\mathrm{SE}} \in O$, which is a critical point of ι with positive semi-definite Hessian. Then, we can realize a critical point $g^\delta_{\mathrm{SE}} \in O$ with positive definite Hessian by a small change g^δ of the initial metric g, as described in [1, (8.1)]. We denote by ψ^δ_1 the smooth basic function defined by $g^\delta_{\mathrm{SE}} = g^\delta_{\psi^\delta_1}$. Then ψ^δ_1 satisfies the equation (3.1) at $t = 1$ with respect to the initial metric g^δ. By Proposition 3.13, ψ^δ_1 can be extended to a smooth family $\{\psi^\delta_t\ ;\ t \in [1-\varepsilon, 1]\}$ of solutions of (3.1) with respect to the initial metric g^δ.

In view of Remark 3.14, Theorem 3.8 and Proposition 3.13 now enable us to generalize the argument in [1] to complete our proof of Theorem A.

3.3. A proof of Proposition 3.6. Finally in this subsection, we give a proof of Proposition 3.6. As we described before, a difficulty for our proof of Proposition 3.6 is that the positivity of the transverse Ricci curvature does not lead that of the Ricci curvature in general.

This obstructs us to apply Myers' theorem for an estimate of the diameter of (S, g). To overcome the difficulty, we consider the following deformation of the Sasakian structure, which is called the *D-homothetic deformation*. Choose a basic function $\varphi \in \mathscr{H}$ which satisfies

$$\rho_\varphi^T \geq t(2n+2)\omega_\varphi^T \tag{3.3}$$

for some $t > 0$. Then, for any positive constant $\mu > 0$, define

$$\eta_{\varphi,\mu} := \mu^{-1}\eta_\varphi\,, \tag{3.4}$$

$$\xi_\mu := \mu\xi \quad \text{and} \tag{3.5}$$

$$g_{\varphi,\mu} := \mu^{-1}g_\varphi^T + \eta_{\varphi,\mu} \otimes \eta_{\varphi,\mu}\,. \tag{3.6}$$

It is known that $\mathcal{S}_{\varphi,\mu} := \{g_{\varphi,\mu}, \xi_\mu, \eta_{\varphi,\mu}, \Phi_\varphi\}$ gives a Sasakian structure on S (see [2] for example). The transverse metric $g_{\varphi,\mu}^T$ is then given by $g_{\varphi,\mu}^T = \mu^{-1}g_\varphi^T$, and the contact form $\eta_{\varphi,\mu}$ satisfies

$$\left(\frac{1}{2}d\eta_{\varphi,\mu}\right)^n \wedge \eta_{\varphi,\mu} = \mu^{-(n+1)}\left(\frac{1}{2}d\eta_\varphi\right)^n \wedge \eta_\varphi\,. \tag{3.7}$$

Let $\mathrm{Ric}_{\varphi,\mu}$ be the Ricci tensor of $g_{\varphi,\mu}$. To obtain the bound for the oscillation, we need the following estimate on the volume and the diameter of $(S, g_{\varphi,\mu})$. We denote by $V_{\varphi,\mu}$ the volume of S with respect to $((1/2)d\eta_{\varphi,\mu})^n \wedge \eta_{\varphi,\mu}$ and by $D_{\varphi,\mu}$ the diameter of $(S, g_{\varphi,\mu})$.

PROPOSITION 3.15. *Let (S, g) be a $(2n+1)$-dimensional compact Sasaki manifold and $\varphi \in \mathscr{H}$ a basic function satisfying* (3.3) *for some $t > 0$. Put $\mu = t^{-1}$. Then we have the estimates of the volume and the diameter of* $(S, g_{\varphi,\mu})$

$$V_{\varphi,\mu} = t^{n+1}V, \tag{3.8}$$

$$D_{\varphi,\mu} \leq \pi\,. \tag{3.9}$$

PROOF. Since $\mu = t^{-1}$, we have

$$\begin{aligned} V_{\varphi,\mu} &= \int_S \left(\frac{1}{2}d\eta_{\varphi,\mu}\right)^n \wedge \eta_{\varphi,\mu} \\ &= \mu^{-(n+1)} \int_S \left(\frac{1}{2}d\eta_\varphi\right)^n \wedge \eta_\varphi \\ &= t^{n+1} \int_S \left(\frac{1}{2}d\eta\right)^n \wedge \eta \\ &= t^{n+1}V\,. \end{aligned}$$

Furthermore, for each $X, Y \in \ker \eta_{\varphi,\mu}$ we have

$$\mathrm{Ric}_{\varphi,\mu}(X, Y) = \mathrm{Ric}_{\varphi,\mu}^T(X, Y) - 2g_{\varphi,\mu}(X, Y)$$

and

$$\mu g_{\varphi,\mu}(X, Y) = g_\varphi(X, Y)\,.$$

Since the transverse Ricci curvature is invariant under the multiplication by positive constant of a transverse metric, $\mathrm{Ric}^T_{\varphi,\mu} = \mathrm{Ric}^T_{\varphi}$ on $\ker \eta_{\varphi,\mu}$ ($= \ker \eta_\varphi$). Then it follows that

$$\begin{aligned}\mathrm{Ric}^T_{\varphi,\mu}(X, Y) &= \mathrm{Ric}^T_{\varphi}(X, Y)\\ &\geq t(2n+2)g^T_\varphi(X, Y)\\ &= t(2n+2)\mu g^T_{\varphi,\mu}(X, Y)\\ &= (2n+2)g_{\varphi,\mu}(X, Y)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Ric}_{\varphi,\mu}(X, Y) &\geq (2n+2)g_{\varphi,\mu}(X, Y) - 2g_{\varphi,\mu}(X, Y)\\ &= 2n\, g_{\varphi,\mu}(X, Y).\end{aligned}$$

On the other hand, by (2.1) we have

$$\begin{aligned}\mathrm{Ric}_{\varphi,\mu}(X, \xi_\mu) &= 2n\eta_{\varphi,\mu}(X)\\ &= 2n\, g_{\varphi,\mu}(X, \xi_\mu)\end{aligned}$$

for any $X \in TS$. Hence we obtain

$$\mathrm{Ric}_{\varphi,\mu} \geq 2n\, g_{\varphi,\mu}\,. \tag{3.10}$$

Finally, we have $D_{\varphi,\mu} \leq \pi$ by Myers' theorem. □

By Proposition 3.15, we can now give a proof of Proposition 3.6 as follows. First we observe that, by the identity $(1/2)d\eta_\varphi = (1/2)d\eta + \sqrt{-1}\partial_B\bar{\partial}_B\varphi$, we have

$$\Box\varphi \leq n \quad \text{and} \quad \Box_\varphi\varphi \geq -n,$$

where $\Box$ and $\Box_\varphi$ are the basic complex Laplacians with respect to g and g_φ, respectively. Since the basic Laplacian coincides with the restriction of the Riemannian Laplacian to $C^\infty_B(S)$ (cf. Proposition 2.8), we have

$$\begin{aligned}\varphi(p) &= \frac{1}{V}\int_S \varphi\left(\frac{1}{2}d\eta\right)^n \wedge \eta + \frac{1}{n!}\int_S (G(p,q)+K)(\Delta\varphi)(q)\left(\frac{1}{2}d\eta\right)^n \wedge \eta\\ &= \frac{1}{V}\int_S \varphi\left(\frac{1}{2}d\eta\right)^n \wedge \eta + \frac{1}{n!}\int_S (G(p,q)+K)(2\Box\varphi)(q)\left(\frac{1}{2}d\eta\right)^n \wedge \eta\\ &\leq \frac{1}{V}\int_S \varphi\left(\frac{1}{2}d\eta\right)^n \wedge \eta + 2nK\frac{V}{n!}\end{aligned}$$

for any $p \in S$. This leads the estimate for φ

$$\sup_S \varphi \leq \frac{1}{V}\int_S \varphi\left(\frac{1}{2}d\eta\right)^n \wedge \eta + 2n\frac{KV}{n!}. \tag{3.11}$$

To obtain an estimate for the infimum of φ, let $\Delta_{\varphi,\mu}$ be the Laplacian and $G_{\varphi,\mu}$ the Green function with respect to the Sasaki metric $g_{\varphi,\mu}$ defined by (3.6). Put $\mu := t^{-1}$. Then, since

$\mathrm{Ric}_{\varphi,\mu} \geq 2n g_{\varphi,\mu} > 0$, there is a constant $\gamma > 0$ depending only on n such that

$$G_{\varphi,\mu} \geq -\gamma \frac{D^2_{\varphi,\mu}}{V_{\varphi,\mu}} \geq -\gamma \frac{\pi^2}{t^{n+1}V} \tag{3.12}$$

by (3.9) (see also [1, Theorem (3.2)]). We denote by $\Box_{\varphi,\mu}$ the basic complex Laplacian with respect to the transverse Kähler form $(1/2)d\eta_{\varphi,\mu}$. Then it follows that $\Delta_{\varphi,\mu}\varphi = 2\Box_{\varphi,\mu}\varphi$. Since $(1/2)d\eta_{\varphi,\mu} = \mu^{-1}((1/2)d\eta + \sqrt{-1}\partial_B\bar{\partial}_B\varphi)$, we have

$$\Box_{\varphi,\mu}\varphi = \mu\Box_{\varphi,\mu}(\mu^{-1}\varphi) \geq -nt^{-1}\,. \tag{3.13}$$

Now the equality (3.8) gives us that

$$\begin{aligned}
\varphi(p) &= \frac{1}{t^{n+1}V}\int_S \varphi\left(\frac{1}{2}d\eta_{\varphi,\mu}\right)^n \wedge \eta_{\varphi,\mu} + \frac{1}{n!}\int_S G_{\varphi,\mu}(p,q)\Delta_{\varphi,\mu}\varphi\left(\frac{1}{2}d\eta_{\varphi,\mu}\right)^n \wedge \eta_{\varphi,\mu} \\
&= \frac{1}{t^{n+1}V}\int_S \varphi\left(\frac{1}{2}d\eta_{\varphi,\mu}\right)^n \wedge \eta_{\varphi,\mu} \\
&\quad + \frac{1}{n!}\int_S \left(G_{\varphi,\mu}(p,q) + \gamma\frac{\pi^2}{t^{n+1}V}\right)(\Delta_{\varphi,\mu}\varphi)\left(\frac{1}{2}d\eta_{\varphi,\mu}\right)^n \wedge \eta_{\varphi,\mu}.
\end{aligned}$$

The first term is then given by

$$\frac{1}{t^{n+1}V}\int_S \varphi\left(\frac{1}{2}d\eta_{\varphi,\mu}\right)^n \wedge \eta_{\varphi,\mu} = \frac{1}{V}\int_S \varphi\left(\frac{1}{2}d\eta_{\varphi}\right)^n \wedge \eta_{\varphi}.$$

For the second term, we have

$$\begin{aligned}
&\frac{1}{n!}\int_S \left(G_{\varphi,\mu}(p,q) + \gamma\frac{\pi^2}{t^{n+1}V}\right)(\Delta_{\varphi,\mu}\varphi)\left(\frac{1}{2}d\eta_{\varphi,\mu}\right)^n \wedge \eta_{\varphi,\mu} \\
&\quad = \frac{1}{n!}\int_S \left(G_{\varphi,\mu}(p,q) + \gamma\frac{\pi^2}{t^{n+1}V}\right)(2\Box_{\varphi,\mu}\varphi)\left(\frac{1}{2}d\eta_{\varphi,\mu}\right)^n \wedge \eta_{\varphi,\mu} \\
&\quad \geq -\frac{2n}{t}\gamma\frac{\pi^2}{t^{n+1}V}\frac{t^{n+1}V}{n!} \\
&\quad = -2n\gamma\frac{\pi^2}{t(n!)}
\end{aligned}$$

by the inequalities (3.12) and (3.13). Thus we obtain

$$\varphi \geq \frac{1}{V}\int_S \varphi\left(\frac{1}{2}d\eta_{\varphi}\right)^n \wedge \eta_{\varphi} - 2n\gamma\frac{\pi^2}{t(n!)}\,. \tag{3.14}$$

This gives the desired inequality

$$\begin{aligned}
\mathrm{osc}_S\varphi &= \sup_S \varphi - \inf_S \varphi \\
&\leq I_\eta(\varphi) + 2n\left(\frac{KV}{n!} + \frac{C}{t}\right),
\end{aligned} \tag{3.15}$$

where $C = \gamma\pi^2/n!$.

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