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UNIQUENESS OF SYMPLECTIC CANONICAL CLASS, SURFACE CONE AND SYMPLECTIC CONE OF 4-MANIFOLDS WITH $B^+ = 1$

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Abstract

Let M be a closed oriented smooth 4-manifold admitting symplectic structures. If M is minimal and has $b^+ = 1$, we prove that there is a unique symplectic canonical class up to sign, and any real second cohomology class of positive square is represented by symplectic forms. Similar results hold when M is not minimal.

1. Introduction

Let M be a smooth, closed oriented 4-manifold. An orientation-compatible symplectic structure on M is a closed 2-form ω such that $\omega \wedge \omega$ is nowhere vanishing and agrees with the orientation. Let Ω_M be the moduli space of such 2-forms. In the first part of this paper, we devote ourselves to the understanding of the topology of this moduli space, which can be studied in three ways.

First of all, on Ω_M, there is a natural equivalence relation, the deformation equivalence. ω_1 and ω_2 in Ω_M are said to be deformation equivalent if there is an orientation-preserving diffeomorphism ϕ such that $\phi^*\omega_1$ and ω_2 are connected by a path of symplectic forms. Clearly, the group of orientation-preserving diffeomorphisms acts on the set of connected components of Ω_M, and the number of deformation classes of symplectic structures is just the number of the orbits of this action.

Secondly, there is a map of canonical class $K : \Omega_M \longrightarrow H^2(M;\mathbf{Z})$. Any symplectic structure determines a homotopy class of compatible almost complex structures on the cotangent bundle, whose first Chern

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class is called the symplectic canonical class. For each symplectic canonical class K, if we let $\Omega_{M,K}$ be the subset of Ω_M, whose elements have K as the symplectic canonical class, then Ω_M is the disjoint union of the $\Omega_{M,K}$. There is also a natural equivalence relation on the set of symplectic canonical classes. We say two symplectic canonical classes K_1 and K_2 are equivalent if there is an orientation-preserving diffeomorphism ϕ such that $\phi^* K_1 = \pm K_2$. Symplectic structures in a connected component have the same symplectic canonical class. Moreover if two symplectic structures are related by an orientation-preserving diffeomorphism, so are their symplectic canonical classes. Therefore the set of deformation equivalence classes of orientation-compatible symplectic structures maps onto the set of equivalence classes of symplectic canonical classes, and can be understood via the latter.

Thirdly, by taking the cohomology class, we have a projection $CC : \Omega_M \longrightarrow H^2(M;\mathbf{R})$. The image of this projection is called the symplectic cone of M, and is denoted by $\mathcal{C}_M$. For each symplectic canonical class, if we define the K-symplectic cone

$$\mathcal{C}_{M,K} = \{e \in \mathcal{C}_M \mid e = [\omega] \text{ for some } \omega \in \Omega_K \}.$$

then $\mathcal{C}_M$ is just the union of the $\mathcal{C}_{M,K}$ (this union is in fact a disjoint union, see Proposition 4.1).

It turns out that the determination of the number of deformation classes of symplectic structures is a very hard problem. The first breakthrough was made by Taubes [50], who showed that there is one deformation class on CP^2. We [25] later showed that the uniqueness holds for all S^2-bundles and their blow ups. However, for other 4-manifolds, it is not even known whether the number of deformation classes is always finite.

Here we are able to provide some evidence for the finiteness by showing that the the image of the map K is always finite. When $b^+(M) > 1$ ($b^+(M)$ is the dimension of a maximal positive subspace of $H^2(M;\mathbf{R})$), it follows from [48] and [54] that there are finitely many symplectic canonical classes. In addition, any 4-manifold with Kähler structures has only one equivalent class of symplectic canonical classes (see [4] and [11]). More recently, examples of 4-manifolds with $b^+ > 1$ and inequivalent classes of symplectic canonical classes were first obtained in [36], and later in [19], [47] and [53] (see also [5], [15] and [37] for the recent results for the moduli space of complex structures on 4-manifolds). In this paper, our first main result completely settles this issue for the case $b^+ = 1$.

Theorem 1. *Let M be a smooth, closed oriented 4-manifold with $b^+ = 1$ and suppose Ω_M is not empty. Then the number of equivalent classes of the symplectic canonical classes is one. Furthermore, if M is minimal, there is a unique symplectic canonical class up to sign.*

Here M is said to be (smoothly) minimal if $\mathcal{E}_M$ is the empty set, where $\mathcal{E}_M$ is the set of cohomology classes whose Poincaré duals are represented by smoothly embedded spheres of square -1. Consequently we obtain:

Corllary 1. *Let M be a smooth, closed oriented 4-manifold. The number of equivalence classes of the symplectic canonical classes is finite.*

To the contrary, in higher dimensions, there can be an infinite number of equivalence classes. Let us briefly indicate the construction of such a manifold. Using their knot surgery, Fintushel and Stern [9] have constructed infinite number of symplectic 4-manifolds homeomorphic to $K3$, whose diffeomorphism types are distinguished by the number of their Seiberg-Witten basic classes. As can be shown with the methods in [42], the products of those manifolds with S^2 are diffeomorphic, and it carries infinite number of equivalent classes.

We also have some substantial results about the map CC. More precisely, for a 4-manifold with $b^+ = 1$ and nonempty symplectic cone, we can characterize the K-symplectic cones $\mathcal{C}_{M,K}$, and from which, obtain a characterization of the symplectic cone $\mathcal{C}_M$ itself.

To determine the K-symplectic cone, the first step is to investigate the set of 'K-stable' classes of symplectic surfaces, $A_{M,K}$, which we define now. Let $A_{M,\omega}$ be the set of $s \in H^2(M; \mathbf{Z})$ whose Poincaré duals can be represented by embedded ω-symplectic surfaces. Define

$$A_{M,K} = \{e \in H^2(M; \mathbf{Z}) \,|\, e \in A_{M,\omega} \text{ for all } \omega \in \Omega_{M,K}\}.$$

Built on results in [50], [24], we are able to compute the symplectic Seiberg-Witten invariants on those 4-manifolds and use them to probe a large part of $A_{M,K}$. To state the result, we need to introduce yet another concept, the forward cone associated to a symplectic canonical class. On a closed oriented 4-manifold with $b^+ = 1$, the set of classes of positive square $\mathcal{P}$ fall into two connected components. Given an orientation-compatible symplectic form ω, we will call the component containing $[\omega]$ the forward cone $\mathcal{FP}$ associated to ω. Similarly, given a symplectic canonical class K, we will call the component containing $\mathcal{C}_K$ the forward cone associated to K and denote it by $\mathcal{FP}(K)$.

For a minimal 4-manifold with $b^+ = 1$ and a symplectic canonical class K, the result we need about $A_{M,K}$ is that (see Proposition 4.2) it contains large multiples of any class in the forward cone $\mathcal{FP}(K)$.

Comparing with a result of Donaldson (see [6]), which states that the Poincaré dual of the class of a sufficiently high multiple of an integral symplectic form can be represented by symplectic submanifolds, it is natural to expect that any class in the forward cone is represented by symplectic forms. Using the inflation process of Lalonde and McDuff (see [17], [18], [31] and [2]), we can show that this is indeed the case.

Theorem 2. *Let M be a minimal closed, oriented 4-manifold with $b^+ = 1$ and K be a symplectic canonical class. Then*

$$\mathcal{C}_{M,K} = \mathcal{FP}(K).$$

Consequently, any real cohomology class of positive square is represented by an orientation-compatible symplectic form.

For non-minimal 4-manifolds, it is no longer true that any real cohomology class of positive square is represented by an orientation-compatible symplectic form, due to the presence of the set $\mathcal{E}$. In fact what is relevant for the K-symplectic cone is the subset of K-exceptional spheres

$$\mathcal{E}_{M,K} = \{E \in \mathcal{E}_M \,|\, E \cdot K = -1\}.$$

An important fact from [25] is that $\mathcal{E}_{M,K}$ lies in $A_{M,K}$. From this, together with the blow up formula for Gromov-Taubes invariants in [27], we can obtain:

Theorem 3. *Let M be a minimal closed, oriented 4-manifold with $b^+ = 1$ and K be a symplectic canonical class. Then*

$$\mathcal{C}_{M,K} = \{e \in \mathcal{FP}(K) \,|\, e \cdot E > 0 \textit{ for any } E \in \mathcal{E}_{M,K}\}.$$

Moreover, for the symplectic cone $\mathcal{C}_M$, we have:

Theorem 4. *Let M be a closed, oriented 4-manifold with $b^+ = 1$ and $\mathcal{C}_M$ nonempty. Then*

$$C_M = \{e \in \mathcal{P} \,|\, 0 < |e \cdot E| \textit{ for all } E \in \mathcal{E}\ \}.$$

Let us remark that, on a complex surface, our K-symplectic cone is similar to (often larger than) the Kähler cone. A priori the Kähler cone is convex because the sum of two Kähler forms is still a Kähler form, while our K-symplectic cone may not have this property because the sum of two symplectic forms may fail to be a symplectic form. However, for 4-manifolds with $b^+ = 1$, it is easy to see from Theorem 3 that our K-symplectic cone turns out to be convex.

As we mentioned, the knowledge of the set $A_{M,K}$ is crucial in the understanding of the moduli space of symplectic forms. We gradually realize that the set itself is an important invariant of M. In the last part of the paper, we devote ourselves to studying it.

For the same class of 4-manifolds above, we are able to determine which multiples of K are in $A_{M,K}$. As a pleasant byproduct, we obtain an analogue of Casteluovo's criterion of rational surfaces.

Corllary 2. *Let M be a minimal symplectic 4-manifold with symplectic canonical class K. If $b_1 = 0$ and $2K \neq 0$, and $2K$ is not in $A_{M,K}$, then M is CP^2 or $S^2 \times S^2$.*

We also define the rational K-surface cone $\mathcal{S}^{\mathbf{Q}}_{M,K}$ to be the convex subset of $H^2(M;\mathbf{Q})$ generated by elements in $A_{M,K}$. The rational K-surface cone is a less refined object but easier to study. Our main result about the K-surface cone is

Theorem 5. *Let M be a closed, oriented 4-manifold with $b^+ = 1$ and symplectic canonical class K. Then*

$$\mathcal{FP}^{\mathbf{Q}}(K) + \sum_{E_i \in \mathcal{E}_K} \mathbf{Q}^+ E_i \subset \mathcal{S}^{\mathbf{Q}}_{M,K} \subset \overline{\mathcal{FP}}^{\mathbf{Q}}(K) + \sum_{E_i \in \mathcal{E}_K} \mathbf{Q}^+ E_i,$$

where $\mathcal{FP}^{\mathbf{Q}}(K)$ and $\overline{\mathcal{FP}}^{\mathbf{Q}}(K)$ are the sets of rational classes in $\mathcal{FP}(K)$ and $\overline{\mathcal{FP}}(K)$ respectively.

On a complex surface, by the Nakai-Moishezon criterion, a rational type $(1,1)$ cohomology class with positive square is represented by a Kähler form if and only if it is positive on any irreducible holomorphic curve. We notice that, for the 4-manifolds in Theorems 2, 3 and 5, these theorems imply that a rational cohomology class with positive square is represented by a symplectic form if and only if it is positive on any class in $A_{M,K}$, thus providing a sort of symplectic analogue.

Actually, if we define the rational K-symplectic cone $\mathcal{C}^{\mathbf{Q}}_K$ to be the set of rational classes in the K-symplectic cone, then immediate from the definitions, then inside $H^2(M;\mathbf{Q})$, the rational K-surface cone is contained in the dual of the rational K-symplectic cone and vice versa. Moreover, Theorems 2, 3 and 5 suggest that there is the following strong duality:

Duality Conjecture. Let M be a closed, oriented 4-manifold with $b^+ = 1$. Suppose K is a symplectic canonical class, then the rational K-surface cone and the rational K-symplectic cone are dual to each other.

For a minimal 4-manifold, this conjecture simply asks whether the rational K-surface cone is the closure of the forward cone. We are able to confirm it for several classes of minimal 4-manifolds (see Proposition 6.5).

After the first draft of this paper appeared, we were informed by P. Biran that weaker versions of our Theorems 2 and 3 appeared in [2] and [3] (see Theorem 3.2 in page 297 in [3] and Theorem 3.2 B in pages 9–10 in [2]). The set 'Class C' that appears in his theorems is a rather general notion. In fact, following from the results in our paper, it is precisely the set of 4-manifolds M with $b^+ = 1$ and $\mathcal{C}_M$ nonempty.

The organization of this paper is as follows. In §2, we review the Seiberg-Witten theory and the Gromov-Taubes theory on symplectic 4-manifolds, in particular those with $b^+ = 1$. In §3, we present the proof of Theorem 1. The K-symplectic cone and the symplectic cone are studied in Section 4. In §4.1, we deal with minimal 4-manifolds and prove Theorem 2. In §4.2, we deal with non-minimal 4-manifolds and prove Theorems 3–4. We would like to mention that some of the results in §4 appeared in [31] in slightly weaker form (see the remark after Proposition 4.3). In §5, we prove Corollary 2. In §6, we study the Duality conjecture.

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Conventions. In the rest of the paper, we will make the following simplifications. An integral cohomology class is identified with its Poincaré dual, and a complex line bundle is identified with its first Chern class. A symplectic 4-manifold refers to a pair consisting of a closed oriented 4-manifold M with an orientation-compatible symplectic form ω. All the symplectic forms are orientation-compatible and all the diffeomorphisms are orientation preserving. All the surfaces are embedded unless specified otherwise. We will often drop M from the notations like Ω_M, $\Omega_{M,K}$, $\mathcal{C}_{M,K}$, $A_{M,\omega}$, $A_{M,K}$ and $\mathcal{S}_{M,K}$ where there is no confusion.

2. The Seiberg-Witten invariants and the Gromov-Taubes invariants of symplectic 4-manifolds

In this section we first review the Seiberg-Witten theory on symplectic 4-manifolds, in particular those with $b^+ = 1$. Then we review the Gromov-Taubes theory of counting embedded symplectic surfaces in symplectic 4-manifolds. Finally we review the equivalence between the two theories on symplectic 4-manifolds with $b^+ = 1$.

2.1 The Seiberg-Witten invariants of symplectic 4-manifolds

In this subsection we review the Seiberg-Witen invariants. For more details, see e.g., [38] and [46], and for the Seiberg-Witten invariants on symplectic 4-manifolds, see [48].

The Seiberg-Witten invariant SW is defined on the set of Spin^c structures $\mathcal{SP}$. Associated to each Spin^c structure $\mathcal{L}$ is a rank 2 complex vector bundle, whose determinant line bundle $c_1(\mathcal{L})$ is called the determinant bundle of $\mathcal{L}$. A useful fact to keep in mind is that $\mathcal{SP}$ is an affine space modelled on $H^2(M;\mathbf{Z})$. So, when fixing $\mathcal{L}$, any nonzero class $e \in H^2(M;\mathbf{Z})$ gives rise to a Spin^c structure, denoted by $\mathcal{L} \otimes e$. And any Spin^c structure is of this form. The determinant line bundle of $\mathcal{L} \otimes e$ is related to that of $\mathcal{L}$ by $c_1(\mathcal{L} \otimes e) = c_1(\mathcal{L}) + 2e$.

Fix a riemannian metric g and a real self-dual 2-form μ on M. The Seiberg-Witten equations can be written down for any Spin^c structure $\mathcal{L}$ (we refer the readers to [38] for the actual equations). Let $\mathcal{M}(M,\mathcal{L})$ denote the moduli space of the Seiberg-Witten equations. For generic pairs (g,μ), $\mathcal{M}(M,\mathcal{L})$ has the nice property that it is a compact manifold of real dimension

$$2d(\mathcal{L}) = \frac{1}{4}(2\chi(M) + 3\sigma(M)) - \frac{1}{4}(c_1(\mathcal{L}) \cdot c_1(\mathcal{L})).$$

Here $\chi(M)$ and $\sigma(M)$ are the Euler characteristic and the signature of M respectively. Furthermore, an orientation of the real line $\det^+ = H^0(M;\mathbf{R}) \otimes \Lambda^{b_1} H^1(M;\mathbf{R}) \otimes \Lambda^{b^+} H^+(M;\mathbf{R})$ naturally orients $\mathcal{M}(M,\mathcal{L})$. Here b_1 is the first Betti number of M and $H^+(M;\mathbf{R})$ is a maximal positive subspace of $H^2(M;\mathbf{R})$. In addition, there is a naturally defined circle bundle on $M \times \mathcal{M}(M,\mathcal{L})$, whose Euler class induces a map ϕ from $H_*(M;\mathbf{Z})$ to $H^{2-*}(\mathcal{M}(M,\mathcal{L});\mathbf{Z})$.

Now we give the definition of SW:

Definition 2.1. Fix an orientation for the line $\det^+$ and a Spin^c structure $\mathcal{L}$. Fix also a generic pair (g,μ) with the salient properties above. Let $\gamma_1\wedge\cdots\wedge\gamma_p$ be a decomposable element in $\Lambda^p(H_1(M;\mathbf{Z})/\text{Torsion})$. Then the value of $SW(M,\mathcal{L})\in\Lambda^*H^1(M;\mathbf{Z})$ is defined as follows:

(2.1)
$$SW(M,\mathcal{L};\gamma_1\wedge\cdots\wedge\gamma_p)=\int_{\mathcal{M}(M,\mathcal{L})}\phi(\gamma_1)\wedge\cdots\wedge\phi(\gamma_p)\wedge\phi(pt)^{(2d-p)/2},$$

where pt is the class of a point in $H_0(M;\mathbf{Z})$.

When $b^+>1$, the value of SW is independent of the choice of the generic pair (g,μ). Hence SW can be viewed as a map from the set of Spin^c structures $\mathcal{SP}$ to $\Lambda^*H^1(M;\mathbf{Z})$. Let $SW^i(M,\mathcal{L})$ denote the part of $SW(M,\mathcal{L})$ in $\Lambda^iH^1(M;\mathbf{Z})$. Suppose η is the positive generator of $\Lambda^0H^1(M;\mathbf{Z})\equiv\mathbf{Z}$. Then the integer $SW^0(M,\mathcal{L};\eta)$ will be simply denoted by $SW^0(M,\mathcal{L})$.

When $b^+=1$, SW also depends on the choice of a chamber. On a 4-manifold with $b_2^+=1$, the set of real second cohomology classes with positive square $\mathcal{P}$ is a cone with two connected components. Pick one of them and call it the forward cone. Given a metric g, there is a unique self-dual harmonic 2-form ω_g for g in the forward cone with $\omega_g^2=1$. For a pair (g,μ) and a Spin^c structure $\mathcal{L}$, define the discriminant $\triangle_{\mathcal{L}}(g,\mu)=\int(c_1(\mathcal{L})-\mu)\omega_g$. The set of pairs (g,μ) with positive and negative discriminant are called the positive and negative $\mathcal{L}$ chamber respectively. The map $SW(M,\mathcal{L})$ is constant on any $\mathcal{L}$ chamber. So in the case $b^+=1$, given a choice of the the forward cone and a Spin^c structure, we can define $SW_+(M,\mathcal{L})$ and $SW_-(M,\mathcal{L})$ for the positive and negative $\mathcal{L}$ chambers respectively. $SW^i_\pm(M,\mathcal{L})$ are similarly defined.

From now on, when there is no confusion, we will often drop M from $SW^i(M,\mathcal{L}_{K^{-1}}\otimes e)$ or $SW^i_\pm(M,\mathcal{L}_{K^{-1}}\otimes e)$.

Recall that a symplectic 4-manifold is a closed oriented 4-manifold with an orientation-compatible symplectic form ω. When M is a symplectic 4-manifold, we have the following facts.

1. As mentioned in §1, the symplectic form ω (actually, the deformation class of ω) determines a unique homotopy class of ω-compatible almost complex structures, and hence a canonical line bundle K. These almost complex structures induce a Spin^c structure $\mathcal{L}_{K^{-1}}$ with K^{-1} as its determinant line bundle.

2. Since $K\cdot K=2\chi(M)+2\sigma(M)$, the formal dimension of the Seiberg-

Witten moduli space of $\mathcal{L}_{K^{-1}} \otimes e$ is then give by

$$2d = e \cdot e - K \cdot e. \tag{2.2}$$

3. There is a natural orientation of the line det^+.

4. When $b^+ = 1$, as mentioned in §1, ω determines the choice of the forward cone. With respect to this choice, the negative chamber is called the symplectic chamber.

Hence we will use the symplectic structure to orient the Seiberg-Witten moduli spaces. In the case $b^+ > 1$, the SW invariant so defined will be denoted by SW_ω; and, in the case $b^+ = 1$, use the symplectic structure to define $SW_{\omega,+}$ and $SW_{\omega,-}$ as well.

A fundamental result of Taubes is:

Theorem 2.2 (see [48]). *Let M be a symplectic 4-manifold with symplectic canonical class K. Then $SW^0_\omega(\mathcal{L}_{K^{-1}}) = 1$ when $b^+ > 1$, and $SW^0_{\omega,-}(\mathcal{L}_{K^{-1}}) = 1$ when $b^+ = 1$.*

An involution on the set of Spin^c structure induces the following symmetry of the Seiberg-Witten invariants:

Symmetry Lemma 2.3 ([54]). *Let M be a symplectic 4-manifold with $b^+ = 1$ and symplectic canonical class K. Then, for any integral class e,*

$$SW^i_{\omega,+}(\mathcal{L}_{K^{-1}} \otimes e) = (-1)^{(1-b_1-i+b^+)/2} SW^i_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes (K - e)).$$

Notice that, by the symmetry lemma and Theorem 2.2, we have $SW^0_+(\mathcal{L}_{K^{-1}} \otimes K) = \pm 1$.

The following inequalities constrain the range of symplectic forms:

Theorem 2.4. *Let M be a smooth, closed oriented 4-manifold.*

1. ([49]) *Suppose M has $b^+ > 1$ and e is represented by a symplectic form ω with K as its symplectic canonical class. If L is an integral class such that $SW_\omega(\mathcal{L}_{K^{-1}} \otimes L)$ is nontrivial, then $K \cdot e \geq L \cdot e$ with equality only if $K = L$.*

2. ([25]) *Suppose M has $b^+ = 1$. Suppose e_1 and e_2 are represented by symplectic forms with K_1 and K_2 as their symplectic canonical classes respectively, and satisfies $e_1 \cdot e_2 > 0$. Then $K_1 \cdot e_1 \geq K_2 \cdot e_1$, and equality holds only if $K_1 - K_2$ is a torsion class.*

2.2 Gromov-Taubes invariants of symplectic 4-manifolds

Let M, ω be a closed symplectic 4-manifold with symplectic canonical class K. Like the Seiberg-Witten invariants, the Gromov-Taubes invariants, introduced by Taubes in [50], are defined for any class $e \in H^2(M; \mathbf{Z})$ and take values in $\oplus \Lambda^* H^1(M; \mathbf{Z})$.

When e is a nonzero integral class, introduce the Gromov-Taubes dimension:

$$d(e) = e \cdot e - K \cdot e.$$

$d(e)$ is the expected maximal complex dimension of the components of pseudo-holomorphic curves (the domain of the curves can be any Riemann surfaces with arbitrary number of connected components) representing e.

If $d(e) \geq 0$, fix an integer $p \in \{0, 1, \cdots, d\}$ and then fix $\gamma_1 \wedge \cdots \wedge \gamma_{2p} \in \Lambda^{2p} H_1(M; \mathbf{Z})$. As in the case of Seiberg-Witten invariants, it suffices to define $Gr_\omega(M, e)(\gamma_1 \wedge \cdots \wedge \gamma_{2p})$. Fixing a compatible almost complex structure J, let $\mathcal{H}(e, J, Z, \Gamma)$ be the set of J-holomorphic curves representing e, and passing through a set of $d - p$ points Z, and intersecting Γ, a set of $2p$ disjoint circles representing $\gamma_1, \ldots, \gamma_{2p}$.

For generic J, Z and Γ, $\mathcal{H}(e, J, Z, \Gamma)$ has the following properties:

1. $\mathcal{H}(e, J, Z, \Gamma)$ is a finite set.
2. Let $h \in \mathcal{H}(J, Z, \Gamma)$. Let $C_1, \cdots, C_k$ be the irreducible components of h which represent the classes $e_1, \cdots, e_k$ and have multiplicities $m_1, \cdots, m_k$. Then $C_1, \cdots, C_k$ are embedded and disjoint.
3. $m_j = 1$ unless C_j is a torus and $e_j^2 = 0$.
4. $e_j^2 \geq -1$, and $e_j^2 = -1$ only if C_k is a sphere.
5. To each $h \in \mathcal{H}(e, J, Z, \Gamma)$, an integer $q(h)$ can be assigned in a delicate way.

Definition 2.5. When e is the zero class, $Gr_\omega(M, 0)$ is simply defined to be 1. When e is not zero, fix $\gamma_1 \wedge \cdots \wedge \gamma_{2p} \in \Lambda^{2p} H_1(M; \mathbf{Z})$. Choose generic (J, Z, Γ) with the salient properties above. $Gr_\omega(M, e)(\gamma_1 \wedge \cdots \wedge \gamma_{2p})$ is then defined to be $\sum q(h)$.

When there is no confusion, we will freely drop M from $Gr_\omega(M, e)$.

The Gromov-Taubes invariants are independent of the choice of the generic (J, Z, Γ). In fact they only depend on the deformation class of the symplectic form and it is natural with respect to diffeomorphisms. A version of Gromov-Taubes invariants was introduced by Ruan in [44]. It was also shown in [14] that the Gromov-Taubes invariants can be constructed from the Ruan-Tian invariants in [45].

Suppose e is a class with nontrivial Gromov-Taubes invariant. Then for generic $J, Z, \Gamma)$, there exists a J-holomorphic curve h representing e satisfying Properties 2–4 above. Since any embedded J-holomorphic curve is an embedded symplectic surface, it is clear that, if $m_j = 1$ for all j, then e is represented by an embedded symplectic surface. If $m_j > 1$ for some j, then C_j is a torus with square zero. In this case, we have the following simple fact.

Lemma 2.6. *Let M be a symplectic 4-manifold and e be an integral class with square zero. If e can be represented by connected embedded symplectic surfaces, then a positive multiple of e can also be represented by embedded symplectic surfaces.*

Proof. Let C be a connected embedded symplectic surface representing e. By the symplectic neighborhood theorem, the tubular neighborhood of C is symplectically a product $C \times D^2$, where D^2 is a 2-disk. For any positive integer n, pick n points in D^2 and we get n disjoint embedded symplectic surfaces whose disjoint union represents ne. The proof is complete. q.e.d.

Thus by Properties 1–4 and Lemma 2.6 we have:

Theorem 2.7 ([51])**.** *Let M be a symplectic 4-manifold. If $e \in H^2(M; \mathbf{Z})$ and $Gr_\omega(e)$ is nontrivial, then e can be represented by an embedded symplectic surface whose only components with negative square are spheres with square -1.*

Given a symplectic sphere Σ with square -1 in M, one can replace a neighborhood of Σ by a symplectic ball to obtain a new symplectic 4-manifold. This process is called (symplectic) blowing down. The reverse process is called (symplectic) blowing up (at a point in the symplectic ball).

Let $\mathcal{E}_{M,\omega}$ be the set of classes which are represented by symplectic -1 spheres. M is called symplectically minimal if $\mathcal{E}_{M,\omega}$ is empty.

Since every symplectic 4-manifold M can be obtained by blowing up a minimal symplectic 4-manifold N at a number of points (see [32]), the following blow up formula is useful to compute the Gromov-Taubes invariants of non-minimal symplectic 4-manifolds.

Theorem 2.8 (see [27]). *Let M, ω be a symplectic 4-manifold with $b^+ = 1$. Suppose $E \in \mathcal{E}_{M,\omega}$ is represented by the symplectic sphere Σ. Let N, β be the symplectic manifold obtained by blowing down Σ. If $v \in H^2(N; \mathbf{Z})$ and $u = v - lE$ for some integer l. Then, under the canonical identification between $\Lambda^* H^1(N; \mathbf{Z})$ and $\Lambda^* H^1(M; \mathbf{Z})$,*

$$Gr_\omega(M, u) = Pr(2d(u)) Gr_\beta(N, v).$$

Here $pr(2d(u))$ is the projection from $\Lambda^* H^1(M; \mathbf{Z})$ to $\oplus_{i=0}^{2d(u)} \Lambda^i H^1(N; \mathbf{Z})$.

2.3 Equivalence between SW and Gr in the case $b^+ = 1$

In this subsection, M is a symplectic 4-manifold with $b^+ = 1$. Another fundamental and deep result of Taubes ([50]) is

Theorem 2.9. *Let M, ω be a symplectic 4-manifold with $b^+ = 1$.*

1. *If $SW_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes e)$ is nontrivial, then there are positive integers n_i and classes e_i such that e can be written as $e = \sum_i n_i e_i$ and e_i is represented by an ω-symplectic surface. In particular, if $SW_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes e)$ is nonzero, then $e \cdot \omega \geq 0$ and $e \cdot \omega = 0$ only if $e = 0$.*

2. *If M is minimal, then $SW_{\omega,-} = Gr_\omega$. For a non-minimal symplectic 4-manifold, the above conclusion holds with the additional assumption that*

$$e \cdot E \geq -1 \text{ for each } E \in \mathcal{E}_\omega. \tag{2.3}$$

For classes violating the condition (2.3), $SW_{\omega,-}$ and Gr_ω are different. To extend the equivalence to those classes McDuff introduces a map Gr'_ω (see [30]). It is a variation of the map Gr which takes into account multiply covered -1 spheres, and coincides with Gr_ω for all classes satisfying (2.3). It was shown in [27] that indeed $Gr'_\omega = SW_{\omega,-}$ for all classes not satisfying (2.3).

3. Uniqueness of symplectic canonical class

In this section we will present the proof of Theorem 1 and Corollary 1, which rely on the computation of the Seiberg-Witten invariants.

We start with citing the following simple, but useful lemma concerning the intersection pairing of a 4-manifold with $b^+ = 1$ (see [25]).

Lemma 3.1 (Light Cone Lemma)**.** *Suppose M is a manifold with $b^+ = 1$. Let A and B be two classes in $H^2(M;\mathbf{R})$ with $A^2 \geq 0$.*

1. *If $B \cdot A = 0$, then $B^2 \leq 0$. And $B^2 = 0$ iff $A^2 = 0$ and $B = rA$ up to torsion.*

2. *If A and B are both in the closure of the forward cone, then $A \cdot B \geq 0$.*

Now we turn to the computation of the SW invariants.

Lemma 3.2. *Let M, ω be a minimal symplectic manifold with $b^+ = 1$ and the symplectic canonical class K. Let e be a class in $H^2(M;\mathbf{Z})$.*

I. *Suppose $K^2 \geq 0$ and $K \cdot \omega \geq 0$.*
If $(K^{-1} + 2e) \cdot \omega \geq 0$ and $SW_{\omega,+}(\mathcal{L}_{K^{-1}} \otimes e)$ is nontrivial, then:

- I.1. *$e = K$ if $K^2 > 0$ or K is a torsion class.*
- I.2. *$e = rK$ for some rational number $r \in [1/2, 1]$ or $2e - K$ is a torsion class if $K^2 = 0$.*

II. *Similarly, if $(K^{-1}+2e)\cdot\omega \leq 0$ and $SW_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes e)$ is nontrivial, then:*

- II.1. *e is the zero class if $K^2 > 0$ or K is a torsion class.*
- II.2. *$e = rK$ for some rational number $r \in [0, 1/2]$ or $2e - K$ is a torsion class if $K^2 = 0$.*

Proof. Since $K^2 \geq 0$ and $K\cdot\omega \geq 0$, K is in the closure of the forward cone.

Case I: $(\mathcal{L}_{K^{-1}} + 2e) \cdot \omega \geq 0$ and $SW_{\omega,+}(\mathcal{L}_{K^{-1}} \otimes e)$ is nontrivial.
By the Symmetry Lemma 2.3,

$$SW^i_{\omega,+}(\mathcal{L}_{K^{-1}} \otimes e) = \pm SW^i_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes 2(K - e)).$$

Thus $SW_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes 2(K-e))$ is nontrivial. By Theorem 2.9, if $K \neq e$, then $(K - e) \cdot \omega > 0$ and $Gr_\omega(K - e)$ is nontrivial. Furthermore, since M is minimal, by Theorem 2.7, $K - e$ is represented by an embedded symplectic surface whose components all have nonnegative square. So $(K - e)^2 \geq 0$ and $K - e$ is in the forward cone.

Let us first deal with the case that K is a torsion class. Since $K \cdot \omega = 0$, we see that $e \cdot \omega \geq 0$. And if $e \neq K$, $e \cdot \omega = (e - K) \cdot \omega < 0$. This is a contradiction. Therefore, in this case, e must be equal to K.

Now let us treat the case that K is not a torsion class. Since $K - e$ is in the forward cone,

$$(K - e) \cdot K \geq 0. \tag{3.1}$$

Suppose $(K^{-1} - 2e) \cdot \omega = 0$, then $K - 2e$ is a torsion class. The Seiberg-Witten dimension of $\mathcal{L}_{K^{-1}} \otimes e$, which should be positive, is then given by $e \cdot e - K \cdot e = -K^2/2$. Since K^2 is assumed to be nonnegative, we must have $K^2 = 0$.

Now assume that $(K^{-1} + 2e) \cdot \omega$ is strictly positive. So there is a $c \geq 1$ such that $(K - c(K^{-1} + 2e)) \cdot \omega = 0$ and hence $(K - c(K^{-1} + 2e))^2 \leq 0$, with equality only if $K - c(K^{-1} + 2e) = 0$. However,

$$\begin{aligned}(K - c(K^{-1} + 2e))^2 &= ((1 + c)K - 2ce)^2 \\ &= (1 - c)^2 K^2 + 4c(K^2 - e \cdot K) + 4c^2(e^2 - e \cdot K).\end{aligned}$$

All three terms are nonnegative, the first by assumption, the second by (3.1), the third being the Seiberg-Witten dimension of $\mathcal{L}_{K^{-1}} \otimes e$.

When $K^2 > 0$, equality holds if and only if $c = 1$, and thus $(K - e) \cdot \omega = 0$.

When $K^2 = 0$, equality holds if and only if $e^2 = e \cdot K = 0$. Since K is not a torsion class, by the light Cone Lemma 3.1, $e = rK$. r is no bigger than one since $(K - e) \cdot \omega \geq 0$ and r is no less than $1/2$ since $(K^{-1} + 2e) \cdot \omega \geq 0$. We have thus finished the proof of Case I.

Case II: $(K^{-1} + 2e) \cdot \omega \leq 0$ and $SW_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes e)$ is nontrivial.

Then $(K^{-1} + 2(K - e)) \cdot \omega \geq 0$ and $SW_{\omega,+}(\mathcal{L}_{K^{-1}} \otimes (K - e))$ is nontrivial. Thus we can similarly determine $K - e$ and hence e itself. The proof of the lemma is complete.

We will need the wall crossing formulas in [24], [27] (see also [40] and [41]).

Lemma 3.3. *Let M, ω be a symplectic 4-manifold with $b^+ = 1$ and symplectic canonical class K. Let e be a class in $H^2(M; \mathbf{Z})$. When $b^+ = 1$, the choice of the forward cone is just an orientation of the line $H^+ = \Lambda^{b^+} H^+$, so it induces an orientation of $\Lambda^{b_1} H^1$.*

1. *Suppose $d(e) \geq b_1$. Let $\gamma_1 \cdots, \gamma_{b_1}$ be a basis of $H_1(M; \mathbf{Z})/Torsion$ such that $\gamma_1 \wedge \cdots \wedge \gamma_p$ is the dual orientation of the symplectic*

orientation on $\Lambda^{b_1}H^1(M;\mathbf{Z})$*. Then*

$$SW^{b_1}_{\omega,-}(\mathcal{L}_{K^{-1}}\otimes e;\gamma_1\wedge\cdots\wedge\gamma_p)-SW^{b_1}_{\omega,+}(\mathcal{L}_{K^{-1}}\otimes e;\gamma_1\wedge\cdots\wedge\gamma_p)=1.$$

2. *Suppose* $d(e)\geq 0$ *and* $\Lambda^2H^1(M;\mathbf{Z})$ *is of rank* 1*. Let* γ *be the generator of* $\Lambda^2H^1(M;\mathbf{Z})\subset H^2(M;\mathbf{Z})$ *such that* $\omega\cdot\gamma>0$*, then*

$$SW^0_{\omega,-}(\mathcal{L}_{K^{-1}}\otimes e)-SW^0_{\omega,+}(\mathcal{L}_{K^{-1}}\otimes e)=\frac{(K^{-1}+2e)\cdot\gamma}{2}.$$

Lemma 3.4. *Let* M *be a rational or irrational ruled symplectic 4-manifold with symplectic canonical class* K*. For any class* $e\in H^2(M;\mathbf{Z})$*,* $SW^0_{\omega,-}(\mathcal{L}_{K^{-1}}\otimes e)$ *is nontrivial if* $(K^{-1}+2e)$ *is in the forward cone and* $d(e)\geq 0$*.*

Proof. Choose a metric g of positive scalar curvature on M. With such a metric and the zero self-dual 2-form, the Seiberg-Witten moduli space of each Spin^c structure is empty ([54]). Since $K^{-1}+2e$ is assumed to be in the forward cone, $(K^{-1}+2e)\cdot\omega_g>0$, where ω_g is the unique self-dual harmonic 2-form in the forward cone. So the pair $(g,0)$ is in the positive chamber for the Spin^c structure $\mathcal{L}_{K^{-1}}\otimes e$. We thus find that $SW_{\omega,+}(\mathcal{L}_{K^{-1}}\otimes e)=0$. For a rational 4-manifold, $b_1=0$, the conclusion then follows from part one of Lemma 3.3. For an irrational ruled 4-manifold, $b_1\geq 2$ and γ is nonzero. Since $(K^{-1}+2e)$ is in the forward cone and γ has square zero, $(K^{-1}+2e)\cdot\gamma$ is nonzero. Hence the conclusion in this case follows from part two of Lemma 3.3.

Now we need to review the notion of minimal reduction. Recall we defined $\mathcal{E}_M$ in §1, and M is said to be (smoothly) minimal if $\mathcal{E}_M$ is empty. Obviously, if M is a symplectic 4-manifold, $\mathcal{E}_{M,\omega}$ is a subset of $\mathcal{E}_M$, and so M is symplectically minimal if it is (smoothly) minimal. In fact it follows from the results in [22] that the reverse is also true.

Any 4-manifold M can be decomposed as a connected sum of a minimal manifold N with some number of $\overline{CP}^2$. Such a decomposition is called a (smooth) minimal reduction of M, and N is a minimal model of M. M is said to be rational if one of its minimal models is CP^2 or $S^2\times S^2$; and irrational ruled if one of its minimal models is an S^2-bundle over a Riemann surface of positive genus. When M has nonempty symplectic cone and is not rational nor irrational ruled, M has a unique minimal reduction (see [22] and also [31]).

The only minimal rational manifolds are CP^2 and $S^2\times S^2$. And a non-minimal rational manifold has two kinds of decompositions. It is either decomposed as $CP^2\#l\overline{CP}^2$ or as $S^2\times S^2\#(l-1)\overline{CP}^2$. We will always use the first decomposition and call it a standard decomposition. The picture for irrational ruled manifolds is similar. S^2-bundles over a Riemann surface of positive genus are the only minimal irrational ruled manifolds. Fix the base surface Σ_g, there are two S^2-bundles over it, the trivial one $S^2\times\Sigma_g$ and the unique nontrivial one $S^2\tilde{\times}\Sigma_g$. A non-minimal irrational ruled manifold also has two types of decompositions. It is either decomposed as $S^2\times\Sigma_g\#l\overline{CP}^2$ or as $S^2\tilde{\times}\Sigma_g\#l\overline{CP}^2$. We will use the first decomposition and call it a standard decomposition.

Let H be a generator of $H^2(CP^2;\mathbf{Z})$ and $F_1,\dots,F_l$ be the generators of H^2 of the $\overline{CP}^2$. Let U and T be classes in $S^2\times\Sigma_g$ represented by $\{pt\}\times\Sigma_g$ and $S^2\times\{pt\}$ respectively. $H,F_1,\dots,F_l$ are naturally considered as classes in $H^2(CP^2\#l\overline{CP}^2;\mathbf{Z})$ and form a basis. We will call such a basis a standard basis. Similarly, $U,T,F_1,\dots,F_l$ are naturally considered as classes in $H^2(S^2\times\Sigma_g\#l\overline{CP}^2;\mathbf{Z})$ and form a basis. Such a basis is also called a standard basis. On $CP^2\#l\overline{CP}^2$, let $K_0=-3H+\sum_i F_i$; and on $S^2\times\Sigma_g\#l\overline{CP}^2$, let $K_0=2U+(2-2g)T+\sum_i F_l$. By the blow up construction (see e.g., [29]) K_0 is a symplectic canonical class.

For a 4-manifold M and a choice of symplectic canonical class K, recall the set of K-exceptional spheres introduced in §1 is $\mathcal{E}_K=\{E\in\mathcal{E}|K\cdot E=-1\}$. We will need the following facts about $\mathcal{E}_K$.

Lemma 3.5. *Let M,ω be a symplectic 4-manifold with K as its symplectic canonical class.*

1. *Let f be a diffeomorphism. Then $\mathcal{E}_{f^*K}=f^*\mathcal{E}_K$.*
2. *$\mathcal{E}_K=\mathcal{E}_\omega$, i.e., every class in $\mathcal{E}_K$ is represented by an ω-symplectic exceptional sphere. Moreover, if $E_1,\dots,E_p\in\mathcal{E}_K$ are pairwise orthogonal, then they are represented by p disjoint ω-symplectic spheres.*
3. *If E_1 and E_2 are two distinct classes in $\mathcal{E}_K$, then $E_1\cdot E_2\geq 0$.*
4. *Suppose M is not rational nor irrational ruled and it has a smooth minimal reduction $N\#l\overline{CP}^2$, with F_i a generator of H^2 of the i-th $\overline{CP}^2$. If we let $\delta_i=K\cdot F_i$, then $\delta_i=\pm1$ and*
$$\mathcal{E}_K=\{-\delta_1F_1,\dots,-\delta_lF_l\}.$$

Proof. C is a -1 sphere symplectic with respect to ω if and only if $f^{-1}(C)$ is a -1 sphere symplectic with respect to $f^*\omega$. Since the (cohomology) class represented by $f^{-1}(C)$ is the pull back of the (cohomology) class represented C, and the symplectic canonical class of $f^*\omega$ is f^*K, we have $f^*\mathcal{E}_K = \mathcal{E}_{f^*K}$.

Part 2 was proved in [25] and [50], and Part 3 follows from Part 2 and the fact proved in [29] that, if E_1 and E_2 are in $\mathcal{E}_\omega$ and distinct, then $E_1 \cdot E_2 \geq 0$. Finally, Part 4 was proved in [22].

Proposition 3.6. *Suppose M, ω is a symplectic 4-manifold with symplectic canonical class K. Suppose it has a (smooth) minimal reduction $N\#l\overline{CP}^2$. Let F_i be a generator of H^2 of the i-th $\overline{CP}^2$. Then there is a symplectic form β on N, such that (M, ω) is obtained by blowing up (N, β) at l points. Moreover, there is a diffeomorphism carrying K to $V \pm F_1 + \cdots \pm F_l$, where V is a symplectic canonical class of N.*

Proof. When there are l disjoint ω-symplectic -1 spheres in M, we can simultaneously blow them down to obtain a symplectic 4-manifold N'. Suppose these ω-symplectic spheres represent $F_1', \dots, F_l'$ and V' is the symplectic canonical class of N'. Then $K = V' + F_1' + \cdots + F_l'$. And if there is a diffeomorphism ϕ such that $\phi^* F_i' = \pm F_i$ for each i, then N' is diffeomorphic to N.

We start with the easier case when M is not rational nor ruled. By Lemma 3.5, $\mathcal{E}_\omega = \{\delta_1 F_1, \dots, \delta_l F_l\}$, where $\delta_i = \pm 1$. Since $F_i \cdot F_j = 0$ for each pair $i \neq j$, we can find l disjoint ω-symplectic spheres representing $\delta_1 F_1, \dots, \delta_l F_l$. Blowing down these spheres, we obtain a symplectic 4-manifold N, β. If V is the symplectic canonical class of β, then $K = V + \delta_1 F_1 + \cdots + \delta_l F_l$.

Suppose M is rational or irrational ruled. The F_i are represented by disjoint smoothly embedded -1 spheres. By the proof of Theorem 1 in [22], there exists a diffeomorphism ϕ_l such that $F_l' = \phi_l^* F_l$ satisfies $F_l' \cdot K = -1$.

To describe the diffeomorphism, recall that if α is a class with square -1, then one can define an automorphism of H^2, the reflection $R(\alpha)$ along α, as follows:

$$R(\alpha)\beta = \beta + 2(\beta \cdot \alpha)\alpha.$$

And if $\alpha \in \mathcal{E}$, then the reflection is realized by a diffeomorphism.

ϕ_l is in fact constructed as a composition of reflections along a series of classes $Y_1, \dots, Y_p$ represented by ω-symplectic -1 spheres. Moreover, if we go through the proof carefully and use the fact, mentioned in the proof of Lemma 3.5, that two ω-symplectic -1 spheres in two distinct classes have nonnegative intersection, we find that the classes Y_i have the following property: If A is represented by an ω-symplectic -1 sphere and $A \cdot F_l = 0$, then $A \cdot Y_i = 0$ for each i. Therefore A is invariant under ϕ_l.

The class $\phi_l^* E_{l-1}$ is still represented by a smoothly embedded -1 sphere. By repeating the above process, we find a diffeomorphism ϕ_{l-1} such that $\phi_{l-1}^* \phi_l^* E_{l-1} \cdot K = -1$. Moreover, since $F_l' = \phi_l^* F_l$ is orthogonal to $\phi^* E_{l-1}$ and represented by an ω-symplectic -1 sphere, $\phi_{l-1}^* F_l' = F_l'$. Repeating this process $l-2$ more times, we get $l-2$ more diffeomorphisms $\phi_{l-2}, \dots, \phi_1$ such that the diffeomorphism $\phi = \phi_1 \circ \cdots \circ \phi_l$ has the property that $\phi^* F_i \cdot K = -1$ for each i. Therefore we find that the symplectic 4-manifold M, ω can indeed be blown down to a symplectic 4-manifold diffeomorphic to N.

To prove the last statement, consider $\omega' = \phi^* \omega$. Its symplectic canonical class is $\phi^* K$ and $F_i \cdot \phi^* K = -1$. So we can blow down l disjoint ω'-symplectic -1 spheres in M, ω' to obtain a symplectic 4-manifold diffeomorphic to N. If V is the symplectic canonical class, then $\phi^* K = V + F_1 + \cdots + F_l$, i.e., K is carried by ϕ to $V = F_1 + . \cdots + F_l$. The proof is complete.

We are ready to prove Theorem 1.

Theorem 1. *Let M be a smooth, closed oriented manifold with $b^+ = 1$. There is exactly one equivalence class of symplectic canonical classes. In fact, if M is minimal, the symplectic canonical class is unique up to sign.*

Proof. We first prove that, when M is minimal, if K is a symplectic canonical class, then the only other symplectic canonical class is $-K$.

Case 1. *M not rational nor irrational ruled.*

Fix an orientation-compatible symplectic structure ω and let K be its symplectic canonical class. We have $K \cdot \omega \geq 0$ and $K^2 \geq 0$ by [28]. Suppose $\widetilde{\omega}$ is another orientation-compatible symplectic form and $\widetilde{K}$ is its symplectic canonical class. Since the symplectic canonical class of $-\widetilde{\omega}$ is just $-\widetilde{K}$, we can assume that $\widetilde{\omega}$ and ω are in the same component of the positive cone so they determine the same forward cone. Again we have $\widetilde{K} \cdot \omega \geq 0$ and $SW_-^0(\mathcal{L}_{\widetilde{K}^{-1}}) = 1$. Thus $SW_+^0(\mathcal{L}_{\widetilde{K}^{-1}} \otimes \widetilde{K}) = \pm 1$ by the Symmetry Lemma 2.3.

The Spin^c structure $\mathcal{L}_{\widetilde{K}^{-1}} \otimes \widetilde{K}$ is of the form $\mathcal{L}_{K^{-1}} \otimes e$ for some class e. Comparing the determinant line bundles, we have

$$\widetilde{K}^{-1} + 2\widetilde{K} = K^{-1} + 2e,$$

and so $2e = K + \widetilde{K}$. To prove that $K = \widetilde{K}$, it suffices to show that $e = K$.

Since $\widetilde{K}$ is in the closure of the forward cone, we have $(K^{-1}+2e)\cdot\omega \geq 0$. By the first part of Lemma 3.2, if $K^2 > 0$ or K is a torsion class, e is equal to K. Therefore $K = \widetilde{K}$ in this case.

By the second part of Lemma 3.2, in the remaining case when $K^2 = 0$ and K is not a torsion class, either $e = rK$ for some rational number $r \leq 1$ or $2e - K$ is a torsion class. Since $2e = K + \widetilde{K}$, we have either $\widetilde{K} = (2r-1)K$ for some rational number $r \in [1/2, 1]$, or $2\widetilde{K} - K$ is a torsion class. In both cases, it is easy to see that $\widetilde{K}^2 = 0$ and $\widetilde{K}$ is not a torsion class. Now if we start with $\widetilde{\omega}$ and $\widetilde{K}$ and repeat the argument above, we conclude that either $K = (2\widetilde{r}-1)\widetilde{K}$ for some rational number $\widetilde{r} \in [1/2, 1]$, or $2K - \widetilde{K}$ is a torsion class. Comparing the two sets of relations, we find that, since K and $\widetilde{K}$ are not torsion classes, the only possibility is $\widetilde{K} = (2r-1)K$ and $\widetilde{K} = (2\widetilde{r}-1)K$ with $r = \widetilde{r} = 1$. Therefore $K = \widetilde{K}$.

Case 2. *M rational or irrational ruled.*

In this case it is proved in [25]. For the convenience of readers, we briefly present the argument here. Let us recall that a classical theorem of Wu states that a class c is the first Chern class of an almost complex structure on M only if $c^2 = 2\chi(M) + 3\sigma(M)$ and its mod 2 reduction is $w_2(M)$. This fact alone determines the choice of the symplectic canonical classes up to sign when M is CP^2 or an S^2-bundle over S^2 via a simple calculation.

When M is irrational ruled, the main observation is that, because M has a metric of positive scalar curvature, by the above mentioned result in [54] and Theorem 2.2, one concludes that the wall crossing number of K^{-1} must be one. By Lemma 3.3,

$$K^{-1}\cdot\gamma = 2SW^0_-(\mathcal{L}_{K^{-1}}) - 2SW^0_+(\mathcal{L}_{K^{-1}}) = 2. \tag{3.2}$$

Invoking Wu's theorem, we again find the choice of K is unique up to sign.

Now suppose M is non-minimal. Let $N\#l\overline{CP}^2$ be a minimal reduction of M. Suppose V is a symplectic canonical class of N, then we have just proved that the only other symplectic canonical class on N is $-V$. Given any symplectic canonical class K of M, by Proposition 3.5, there is a diffeomorphism carrying K to $V \pm F_1 + \cdots \pm F_l$ or $-V \pm F_1 + \cdots \pm F_l$, Using reflections along F_i, we see all the symplectic canonical classes of

M can be carried to either $V + F_1 + \cdots + F_l$ or $-(V + F_1 + \cdots + F_l)$. Therefore they are all equivalent. The proof of Theorem 1 is finished.

Let $\mathcal{K}$ be the set of the symplectic canonical classes. We can often give a concrete description of $\mathcal{K}$.

When M is rational and $b^- \leq 9$ (b^- is the dimension of a maximal subspace of $H^2(M; \mathbf{R})$), $\mathcal{K}$ is the set of characteristic classes with square $2\chi(M) + 3\sigma(M)$, since it was shown in [25] that the group of diffeomorphism acts transitively on this set.

When M is not rational nor ruled, let V be a symplectic canonical class of N, it is easy to see from Theorem 1 and Proposition 3.6 that

$$\mathcal{K} = \{\pm V \pm F_1 \cdots \pm F_l\}.$$

Proposition 3.7. *Suppose M is irrational ruled and is given a standard minimal reduction $(S^2 \times \Sigma_h)\# l\overline{CP}^2$ and a standard basis. Let $\mathbf{D}$ be the set of $(l+1)$-tuples $\mathbf{d} = \{\epsilon, c_1, \ldots, c_l\}$ with $\epsilon = \pm 2$ and c_i odd. For each $\mathbf{d} \in \mathbf{D}$ define*

$$K_{\mathbf{d}} = \epsilon U + \frac{(8 - 8g - l + \sum_i c_i^2)}{2\epsilon} T + \sum c_i F_i.$$

Then $\mathcal{K} = \{K_{\mathbf{d}} | \mathbf{d} \in \mathbf{D}\}$.

Proof. In this case $\gamma = T$. By Equation (3.2), if $K = aU + bT + \sum_i c_i F_i$ is in $\mathcal{K}$, then $a = \pm 2$. And since K is characteristic with square $8 - 8g - l$, K must be of the form $K_{\mathbf{d}}$ for some $\mathbf{d} \in \mathbf{D}$.

To show that $K_{\mathbf{d}} \in \mathcal{K}$, it suffices to show that K_0 can be carried to $K_{\mathbf{d}}$ by a composition of reflections along classes in $\mathcal{E}$. Since T and F_i are represented by disjoint spheres, and T is of square zero, it is easy to see that $-kT - F_i$ is in $\mathcal{E}$ for any k and i. Let $e = aU + bT + \sum_i c_i F_i$. Under the reflection r_{-1}^i along F_i,

$$c_i \longrightarrow -c_i, a \longrightarrow a, b \longrightarrow b, c_j \longrightarrow c_j \text{ if } j \neq i.$$

Under the reflection f_i^k along $-kT - F_i$,

$$c_i \longrightarrow 2ka - c_i, b \longrightarrow b + 2(-ka + c_i)k, a \longrightarrow a, c_j \longrightarrow c_j \text{ if } j \neq i.$$

Let $\mathbf{d} \in \mathbf{D}$ be a sequence with $\epsilon = 2$. Write c_i as $4k_i - \tau_i$ with $\tau_i = \pm 1$. Denote the identity on $H^2(M; \mathbf{Z})$ also by r_1^i for each i. Then

$$K_{\mathbf{d}} = (r_{\tau_1}^1 \circ f_1^{k_1}) \circ \cdots \circ (r_{\tau_l}^l \circ f_l^{k_l})(K_0) \tag{3.3}$$

and thus is in $\mathcal{K}$. For the case $\epsilon = -2$, we just have to observe that $K_{-\mathbf{d}} = -K_{\mathbf{d}}$. In fact we have just proved that the set $\mathcal{K}$ is given by $\{K_{\mathbf{d}}, \mathbf{d} \in \mathbf{D}\}$.

Corollary 1. *Let M be a smooth, closed oriented 4-manifold. The number of equivalence classes of the symplectic canonical classes is finite.*

Proof of Corollary 1. If M is a smooth, closed oriented 4-manifold with $b^+ > 1$, the number of Spin^c structures $\mathcal{L}$ such that $SW(\mathcal{L})$ is nontrivial is finite ([54]). This corollary simply follows from this fact, Theorem 2.2 and Theorem 1.

Corollary 3.8. *Let M be a minimal 4-manifold with $b^+ = 1$ and nonempty symplectic cone. Suppose K is a symplectic canonical class. If ϕ is a diffeomorphism, then $\phi^* K = \pm K$.*

Proof. Let ϕ be a diffeomorphism. Since $\phi^* K$ is the symplectic canonical class of $\phi^*\omega$, This corollary is immediate from the proof of Theorem 1.

It follows from Corollary 3.8, up to sign, the action of ϕ on H^2 is determined by its restriction to L, the orthogonal complement of K in $H^2(M;\mathbf{Z})$. If $K^2 > 0$, then L is negative definite, and therefore it has only finitely many automorphisms. Since $K^2 = 2\chi(M) + s\sigma(M) > 0$, we have the following generalization of Corollary 4.8 in [11]:

Corollary 3.9. *Let M be a minimal closed, oriented 4-manifold with $b^+ = 1$ and nonempty symplectic cone. Let $D(M)$ be the image of diffeomorphisms of M in the automorphisms of $H^2(M;\mathbf{Z})$. Then $D(M)$ is finite if $2\chi(M) + 3\sigma(M) > 0$.*

4. $\mathcal{C}$ and $\mathcal{C}_K$

In this section, we are going to determine the K-symplectic cone and the symplectic cone for a closed oriented 4-manifold with $b^+ = 1$ and nonempty symplectic cone.

We start with some general properties of the K-symplectic cone.

Proposition 4.1. *Let M be a closed, oriented 4-manifold and K be a symplectic canonical class.*

1. *If K' is another symplectic canonical class, then $\mathcal{C}_K \cap \mathcal{C}_{K'}$ is empty.*
2. *Let f be a diffeomorphism, then $f^*\mathcal{C}_K = \mathcal{C}_{f^*K}$.*

3. *Suppose M has $b^+ > 1$ and $b_1, \dots, b_l$ are the classes such that $SW(\mathcal{L}_{K^{-1}} \otimes b_i)$ is nontrivial. Then*

$$\mathcal{C}_K \subset \{e \in \mathcal{P} \,|\, e \cdot K \geq 0 \text{ and } e \cdot K > |e \cdot b_i| \text{for all } b_i \neq \pm K \ \}.$$

Proof. We first claim that $K - K'$ can not be a nonzero torsion class. In the case $b^+ > 1$, this follows from Theorem 2.4.1; in the case $b^+ = 1$, this is due to Theorem 1. Thus if $e \in \mathcal{C}_K$ and $e' \in \mathcal{C}_{K'}$, we have by Theorem 2.4

$$K \cdot e > K' \cdot e \text{ and } K \cdot e' < K' \cdot e'.$$

Consequently $e \neq e'$ and the first part is proved.

If ω is a symplectic form with K as its symplectic canonical class, then $f^*\omega$ is symplectic with f^*K as its symplectic canonical class. Therefore the second part holds.

The last part follows directly from Theorem 2.4. The proof of the proposition is complete.

We would like to remark that, for all the known manifolds with trivial K, which is either the $K3$ surface, or a T^2-bundle over T^2, the third part is in fact an equality. Indeed if K is trivial, then the only Seiberg-Witten basic class is 0 and the righthand side is just $\mathcal{FP}$. When M is a T^2-bundle over T^2, K is trivial and it has been shown explicitly in [12] that all classes in $\mathcal{P}$ can indeed be represented by symplectic forms. For $K3$ surface, this is also the case.

4.1 When M is minimal

In this subsection, M is a minimal closed, oriented 4-manifold with $b^+ = 1$. We will first describe the set A_K. The knowledge of A_K is then used to provide a complete description of $\mathcal{C}_K$.

Proposition 4.2. *Let M be a minimal symplectic 4-manifold with $b^+ = 1$. Let $e \in H^2(M)$ be a class in the forward cone. If $e - K$ is in the closure of the forward cone and is not equal to zero, then e is represented by connected symplectic surfaces. In particular for N big, Ne is represented by connected symplectic surfaces.*

Proof. The assumption that e being in the forward cone and $e - K$ in the closure of the forward cone implies that $(e - K) \cdot e > 0$. Since $d(e) = (e - K) \cdot e$ is even, we have $d(e) \geq 2$. It also implies that $2e - K = e + (e - K)$ is in the forward cone, thus $(K^{-1} + 2e) \cdot \omega > 0$. By

Lemmas 3.2–3.4, $SW^0_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes e)$ or $SW^2_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes e)$ is nontrivial if M is ruled or $d(e) \geq b_1$. But if M is non-ruled, then $b_1(M) \leq 2$ by [28]. Therefore, under the assumption, $SW_{\omega,-}(K^{-1} \otimes e)$ is nontrivial. By Theorems 2.9(1) and 2.7, e is represented by an embedded symplectic surface.

Finally we prove this surface is connected. Since M is assumed to be minimal, every component has nonnegative square. Since e is in the forward cone, e^2 is positive. Therefore at least one component has positive square. If the surface has more than one component, then it violates the Light Cone Lemma 3.1. The proof is complete. q.e.d.

Theorem 2. *Let M be a minimal closed, oriented 4-manifold with $b^+ = 1$ and K be a symplectic canonical class. Then*

$$\mathcal{C}_{M,K} = \mathcal{FP}(K).$$

Consequently, any real cohomology class of positive square is represented by an orientation-compatible symplectic form.

Proof. Fix a symplectic form ω whose symplectic canonical class is K. Since being symplectic is an open condition, we can assume that $[\omega]$ is an integral class.

We first show that any integral e in the forward cone is in the K-symplectic cone. The first step is to show that, for a large integer l, $le - [\omega]$ is represented by a symplectic surface. Indeed if l is large, since $\omega \cdot e > 0$, $le - [\omega]$ is in the forward cone and $(le - [\omega]) - K$ is in the closure of the forward cone. Thus by Proposition 4.2, $le - [\omega]$ can be represented by a symplectic surface. Given a symplectic surface C with nonnegative self-intersection, the inflation process of Lalonde and McDuff constructs a judicious Thom form ρ, representing the Poincáre dual to the class of C and supported in an arbitrarily small neighborhood of C, such that $\omega + \kappa\rho$ remains symplectic for all positive number κ. Thus $le = [\omega] + (le - [\omega])$ is represented by a symplectic form.

Since any positive real multiple of a symplectic form is a symplectic form with the same canonical class, we have shown that any real multiple of an integral class is in the K-symplectic cone. To show this is true for a general class α in the forward cone, we use a trick in [1]: α can be written as $\alpha = \sum_{i=1}^{p} \alpha_i$, where the rays of α_i are arbitrarily close to that of α and each $\alpha_i = s_i\beta_i$ for some positive real number s_i and an integral class β_i. Fix such a decomposition such that each $beta_i$ is in the forward cone. Our strategy is to show inductively that for any q, $\sum_{i=1}^{q} \alpha_i$ is in the K-symplectic cone.

Since we know α_1 is in the K-symplectic cone, we can choose a symplectic form ω_1 with $\alpha_1 = [\omega_1]$. For a large integer l, since $\beta_2 \cdot \omega_1 > 0$, by Proposition 4.2, $l\beta_2$ is represented by an ω_1-symplectic surface. By the inflation process, $\alpha_1 + \kappa(l\beta_2)$ is represented by a symplectic form for any real number κ. If we choose $\kappa = s_i/l$, then we find that $\alpha_1 + \alpha_2$ is in the K-symplectic cone. Now choose a symplectic form ω_2 with $[\omega_2] = \alpha_1 + \alpha_2$. It can be shown in the same way that $l\beta_3$ is represented by a ω_2-symplectic surface and $(\alpha_1 + \alpha_2) + \alpha_3$ is in the K-symplectic cone. Repeating this process, we find that α is in the K-symplectic cone. Thus we have shown that $\mathcal{C}_K = \mathcal{FP}$.

The last statement is clear now because any nonzero real multiple of an orientation-compatible symplectic form is still such a form. The proof is complete.

Let us remark that Donaldson's result mentioned in §1 fits nicely with our results. Indeed, if $e \in \mathcal{FP}$, Theorem 2 tells us that e can be represented by a symplectic form ω. Donaldson's construction then produces ω-symplectic surfaces representing large multiples of e.

4.2 When M is not minimal

In this subsection M is a non-minimal 4-manifold with $b^+ = 1$ and nonempty symplectic cone.

The following result is the analogue of Proposition 4.2.

Proposition 4.3. *Let M be a symplectic 4-manifold with $b^+ = 1$ and symplectic canonical class K. Let $e \in H^2(M)$ be a class in the forward cone. Assume that $e - K$ is in the closure of forward cone and $e - K \neq 0$. Further assume that $e \cdot E \geq -1$ for all $E \in \mathcal{E}_\omega$. Then e can be represented by a symplectic surface. Furthermore, if $e \cdot E \geq 0$ for all $E \in \mathcal{E}_\omega$, then the symplectic surface is connected.*

Proof. Let N be a (symplectic) minimal reduction of M, i.e., N is minimal and M is obtained from N by blowing up some number of points. Let $F_1, \dots, E_k$ be the exceptional classes for the blow down map $M \longrightarrow N$ and V be the symplectic canonical class of N. Thus $F_i \in \mathcal{E}_\omega$ and $K = V + \sum F_i$. Write e as $e' - \sum n_i F_i$. Then

$$e' \cdot e' = e \cdot e + \sum n_i^2$$
$$e' \cdot V = e \cdot K - \sum n_i$$
$$e' - V = (e - K) + \sum (n_i + 1) F_i.$$

So $e' \cdot e' > 0$ and $e' - K_N$ is in the closure of the forward cone. If we use $Gr^i(M,e)$ to denote the part of $Gr_\omega(M,e)$ in $\Lambda^i H^1(M;\mathbf{Z})$, then, by Proposition 4.2, $Gr^0_\omega(N,e')$ or $Gr^2(N,e')$ is nonzero. Since $d(e) \geq 2$, by Theorem 2.8, $Gr^0_\omega(M,e)$ or $Gr^2_\omega(M,e)$ is nonzero. Now the proposition follows from Theorems 2.9(1) and 2.7.

A slightly weaker version of Propositions 4.2–4.3 appeared in [35] and [31] (see Lemma 2.2 in [31] and the proof of Proposition 4.11 in this paper). I. Smith informed us that via a construction using Lefschetz fibrations in [7], he could obtain similar results about the existence of symplectic surfaces.

For any two subsets U and F of $H^2(M)$, we define

$$U_F = \{e \in U \mid e \cdot f > 0 \text{ for any } f \in F\ \}.$$

Theorem 3. *Let M be a smooth, closed oriented 4-manifold with $b^+ = 1$ and K be a symplectic canonical class. Use any class in Ω_K to define the forward cone. Then*

$$\mathcal{C}_K = \mathcal{FP}_{\mathcal{E}_K}.$$

Proof. By Lemma 3.5(1) $\mathcal{C}_K$ is contained in $\mathcal{FP}_{\mathcal{E}_K}$. To prove the inclusion in the other direction, we fix any integral symplectic form ω on M with K as its symplectic canonical class. Let e be an integral class in $\mathcal{FP}_{\mathcal{E}_K}$. Since for a large integer l, $le - [\omega]$ and $(le - [\omega] - K$ are both in the forward cone, and $le \cdot E > 0$ for any $E \in \mathcal{E}_\omega$ by Lemma 3.5, $le - [\omega]$ is represented by a connected ω-symplectic surface by Proposition 4.3. By the inflation process, we find that e is in the K-symplectic cone.

To pass to a general class $\alpha \in \mathcal{FP}_{\mathcal{E}_K}$, we notice that Biran's trick still works here. Since $\mathcal{FP}_{\mathcal{E}_K}$ is open, α can be written as a sum $\sum_i \alpha_i$, where $\alpha_i = s_i\beta_i$ for some positive real number and β_i is integral and lies in $\mathcal{FP}_{\mathcal{E}_K}$. The rest of the proof is exactly the same as that in Theorem 2. The proof is complete.

We would like to point out that an amusing fact from Theorems 2 and 3 is that, in the case $b^+ = 1$, $\mathcal{C}_K$ is a convex set, which is not obvious at all from its definition.

We would also like to speculate on the relation between the Kähler cone and the K-symplectic cone for Kähler surfaces with $b^+ = 1$ (or equivalently zero geometric genus).

In the case of a rational surface, it is not known whether the Kähler cone can be as big as the symplectic cone. By Nakai-Moishezon's criterion for ampleness, for a rational surface, the Kähler cone is simply the set of real cohomology classes with positive square, which are positive on any irreducible curves with negative squares. In light of Theorem 5, this is true if one can show that for any integer s, there are s generic points on the projective plane such that, on the rational surface obtained by blowing up these points, the only irreducible curves with negative squares are smooth rational curves with square -1. It was observed in [34] that this kind of question is related to Nagata conjecture on the existence of equisingular plane curves. The connection was also pointed out to the authors by Kollár. In fact some very interesting results addressing the connection has been obtained by Biran (see [3]). In [10] good generic rational surfaces are studied. For those surfaces, the only irreducible curves with negative squares are the exceptional curves and those representing K^{-1}. Thus by Nakai-Moishezon's criterion for ampleness, the Kähler cone is given by

$$\{e \in \mathcal{FP} \,|\, e \cdot C > 0 \text{ for any holomorphic } -1 \text{ curve } C \text{ and } e \cdot K^{-1} > 0\ \}.$$

For general symplectic 4-manifold with $b^+ = 1$, for an almost complex structure J with K as the canonical class, we can introduce the J-symplectic cone

$$\mathcal{C}_J = \{e \in H^2(M;\mathbf{R}) \,|\, e \ \text{ has a symplectic representative compatible with } \ J\}.$$

When J is integrable, $\mathcal{C}_J$ is just the Kähler cone. It would be nice to know when there exists J such that $\mathcal{C}_J$=$\mathcal{C}_K$.

Now we will prove Theorem 4, which characterizes $\mathcal{C}$ in terms of the set $\mathcal{E}$. For this purpose we summarize some facts about $\mathcal{E}$ in the next lemma.

Lemma 4.4. *Let M be a closed oriented 4-manifold with $\mathcal{C}$ nonempty. Suppose M has a (smooth) minimal reduction $N\#l\overline{CP}^2$. Let $F_1, \dots, F_l$ be the generators of H^2 of $\overline{CP}^2$.*

1. *$\mathcal{E} = \cup_{K\in\mathcal{K}}\mathcal{E}_K$, where $\mathcal{K}$ is the set of symplectic canonical classes.*
2. *If $e \in \mathcal{C}$ and $E \in \mathcal{E}$, then $e \cdot E \neq 0$.*
3. *When N is not CP^2 nor an S^2-bundle, $\mathcal{E} = \{\pm F_1, \dots, \pm F_l\}$.*

Proof. Parts 1 and 3 were proved in [22]. We only have to prove Part 2. When $b^+ > 1$, it is due to Taubes [48]. The case $b^+ = 1$

is implicitly in [22] and we will make it clear here. Suppose ω is a symplectic form in the class of e and K is its symplectic canonical class. Since K is characteristic and $E^2 = -1$, $K \cdot E$ is odd. In particular $K \cdot E = l \neq 0$. Then by Lemma 2.2 and Theorem T1 in [22], we find that lE or $-lE$ is represented by a symplectic surface (not necessarily embedded). This implies that $e \cdot E \neq 0$. The proof is finished.

When M is rational or irrational ruled with $b^- \geq 2$, $\mathcal{E}$ is in fact an infinite set. It is not hard to write down it explicitly when M is irrational ruled. When M is rational, it is determined in [21] (it was also shown in [24] that every class with square -1 is in $\mathcal{E}$ if $b^- \leq 9$).

We start the proof of Theorem 4 in the easier case when M is not rational nor irrational ruled.

Proposition 4.5. *Let M be a smooth, closed oriented 4-manifold with $b^+ = 1$ and $\mathcal{C}_M$ nonempty. If M is not rational nor irrational ruled, then*

$$\mathcal{C}_M = \{e \in \mathcal{P} \,|\, e \cdot E \neq 0 \text{ for any } E \in \mathcal{E}\}.$$

Proof. Suppose M is given a minimal reduction $N \# l\overline{CP}^2$. Let $F_1, \ldots, F_l$ be the generators of H^2 of the $\overline{CP}^2$. By Lemma 4.4, $\mathcal{E} = \{\pm F_1, \ldots, \pm F_l\}$. If $e \in \mathcal{C}_M$, $e \in \mathcal{C}_K$ for some K. By Lemma 3.5 and Theorem 3, and $e \cdot F_i \neq 0$ for any i. Thus we have shown

$$\mathcal{C}_M \subset \{e \in \mathcal{P} \,|\, e \cdot E \neq 0 \text{ for any } E \in \mathcal{E}\}.$$

Now let us prove the inclusion in the reverse direction. Suppose $e \in \mathcal{P}$ and $e \cdot F_i \neq 0$. If V is one of the two symplectic canonical classes of N, as mention in §3, any symplectic canonical class of M is of the form $K = \pm V \pm F_1 \pm \cdots \pm F_l$. By possibly changing e to $-e$, we can assume that e is in the forward cone determined by any symplectic canonical class of the form $V + \pm F_1 \pm \cdots \pm F_l$. Let $\epsilon_i = (e \cdot F_i)/|e \cdot F_i|$ and $K = V + \sum \epsilon_i F_i$. Then $\mathcal{E}_K = \{\epsilon_1 F_1, \ldots, \epsilon_l F_l\}$, and $e \cdot \epsilon_i F_i > 0$. Therefore $e \in \mathcal{C}_K$ by Theorem 3. Thus we have shown that

$$\{e \in \mathcal{P} \,|\, e \cdot E \neq 0 \text{ for any } E \in \mathcal{E}\} \subset \mathcal{C}_M.$$

The proof of Proposition 4.5 is complete. q.e.d.

To deal with the non-minimal rational and irrational ruled 4-manifolds, we need to introduce the notion of a reduced class.

Definition 4.6. For a non-minimal rational manifold with a standard decomposition $CP^2 \# l\overline{CP}^2$ and a standard basis $\{H, F_1, \ldots, F_l\}$, a class $\xi = aH - \sum_{i=1}^n b_i F_i$ is called reduced if

$$\begin{cases} b_1 \geq b_2 \geq \cdots \geq b_l \geq 0 \\ a \geq b_1 + b_2 + b_3. \end{cases}$$

For a non-minimal irrational ruled manifold with a standard decomposition $S^2 \times \Sigma_g \# l\overline{CP}^2$ and a standard basis $\{U, T, F_1, \ldots, F_l\}$, a class $e = aU + bT - \sum c_i F_i$ is called reduced if

$$\begin{cases} c_1 \geq c_2 \geq \cdots \geq c_l \geq 0 \\ a \geq c_i \text{ for any } i. \end{cases}$$

The following facts about reduced classes in [21] are crucial.

Lemma 4.7. *Let M be a non-minimal rational or irrational ruled 4-manifold with a standard decomposition and a standard basis.*

1. *There is a simple algorithm to transform any class of positive square to a reduced class via a diffeomorphism.*
2. *Suppose e is a reduced class and $E \in \mathcal{E}_{K_0}$. Then $e \cdot E \geq 0$, and $e \cdot E > 0$ if E is not one of the F_i.*

Proposition 4.8. *Let M be a non-minimal rational or irrational ruled 4-manifold. Then*

$$\{e \in \mathcal{P} \mid 0 < |e \cdot E| \textit{ for any } E \in \mathcal{E}\} = \mathcal{C}_M.$$

Proof. Let M be given a standard decomposition and standard basis, so we can define reduced classes. Let e be a class in $\mathcal{P}$ such that $e \cdot E \neq 0$ for any $E \in \mathcal{E}$. By Lemma 4.7(1) e can be transformed to a reduced class by some diffeomorphism, say ϕ. On one hand, since the set $\mathcal{E}$ is preserved by the group of diffeomorphisms, $\phi^* e \cdot E \neq 0$ for any $E \in \mathcal{E}$. On the other hand, since $\phi^* e$ is a reduced class, by Lemma 4.7(2), $\phi^* e \cdot E \geq 0$ for any $E \in \mathcal{E}_{K_0}$. Therefore $\phi^* e \cdot E > 0$ for any $E \in \mathcal{E}_{K_0}$. By Theorem 3, we see that $\phi^* e \in \mathcal{C}_{K_0}$. Therefore by Proposition 4.1, $e \in \phi^* \mathcal{C}_{K_0} = \mathcal{C}_{\phi^* K_0} \subset \mathcal{C}$. Thus we have proved

$$\{e \in \mathcal{P} \mid 0 < |e \cdot E| \text{ for any } E \in \mathcal{E}\} \subset \mathcal{C}_M.$$

The inclusion in the reverse direction follows from Lemma 4.4(2). The proof of Proposition 4.8 is complete. q.e.d.

Propositions 4.5 and 4.8 immediately give us:

Theorem 4. *Let M be a smooth, closed oriented 4-manifold with $b^+ = 1$ and $\mathcal{C}_M$ nonempty. Then*

$$\mathcal{C}_M = \{e \in \mathcal{P} \,|\, e \cdot E \neq 0 \text{ for any } E \in \mathcal{E}\}.$$

When M is given a minimal reduction, we can give a more explicit presentation of $\mathcal{C}$.

Proposition 4.9. *Suppose M is as in Theorem 4 and is given a minimal reduction $N\#l\overline{CP}^2$. Let F_i be the generators of H^2 of the $\overline{CP}^2$. When M is rational or irrational ruled, further assume the minimal reduction is standard and a standard basis is given.*

1. *When M is not rational nor irrational ruled, A class e with positive square is represented by a symplectic form if and only if $e \cdot F_i \neq 0$ for any i.*
2. *When M is irrational ruled, a class $e = aU + bT + \sum_i c_i F_i$ with positive square is represented by a symplectic form if and only if C_i/A is not an integer for any i.*
3. *When M is rational, a reduced class $e = aH - \sum_{i=1}^n b_i F_i$ with positive square is in $\mathcal{C}_{K_0}$ if and only if $b_i > 0$ for each i. For a general class e, transform it to a reduced class e'. Then e is represented by a symplectic form if and only if e' is thus represented.*

Proof. The first part is contained in the proof of Proposition 4.5. If M is irrational ruled, the conclusion follows immediately from:

Lemma 4.10. *If $Y = \{s_1 T \pm F_1, \dots, s_l T \pm F_l, s_i \in \mathbf{Z}\}$, then $\mathcal{E} = Y$.*

Proof. We have shown in course of the proof of Proposition 3.7 that $Y \subset \mathcal{E}$. To prove the inclusion in the reverse direction, by Lemma 4.4(1), it suffices to show that for every symplectic canonical class K, $\mathcal{E}_K \subset Y$.

Consider K_0 first. If the symplectic form ω is obtained by blowing up a product symplectic form on $S^2 \times \Sigma_h$ which pairs positively with U and T, then it has K_0 as the canonical class. For such a form, it is not hard to show (see [1]) that

$$\mathcal{E}_{K_0} = \{F_1, \dots, F_l, T - F_1, \dots, T - F_l\}.$$

So $\mathcal{E}_{K_0} \subset Y$. By Proposition 3.7, each symplectic canonical class is of the form $K_{\mathbf{d}}$. By Equation (3.3) and Lemma 3.5,

$$\begin{aligned}\mathcal{E}_{K_{\mathbf{d}}} &= (r^1_{\tau_1} \circ f_1^{k_1}) \circ \cdots \circ (r^l_{\tau_l} \circ f_l^{k_l})\mathcal{E}_{K_0} \\ &= \{-2k_1T + \tau_1F_1, (1+2k_1)T - \tau_1F_1, \dots, \\ &\qquad -2k_lT + \tau_lF_l, (1+2k_l)T - \tau_lF_l\}.\end{aligned}$$

Notice that $\tau_i = \pm 1$ for each i, $\mathcal{E}_{K_{\mathbf{d}}} \subset Y$. Lemma 4.10 is proved. q.e.d.

Suppose M is rational. Let $e = aH - \sum_{i=1}^l b_iF_i$ be a reduced class with positive square. If it is in $\mathcal{C}_{K_0}$, then since $F_i \in \mathcal{E}_{K_0}$, it is necessary that $b_i > 0$ for each i. Conversely, if $b_i > 0$ for each i, by Lemma 4.7(2) and Theorem 3, it is in $\mathcal{C}_{K_0}$. The first statement is thus proved. The last statement is obvious, so the proof of Proposition 4.9 is finished.

q.e.d.

As we previously remarked, when M is rational, though $\mathcal{E}$ can be determined, it is hard to write down it explicitly. So we do not have a as nice presentation of $\mathcal{C}$ as those in the other cases. However, in light of Lemma 4.7(1), it is still a very effective one.

Having determined the image of the map $CC : \Omega_M \longrightarrow H^2(M; \mathbf{R})$, we are also able to say something about its inverse image by generalizing a result of McDuff.

Proposition 4.11. *Let M be closed oriented 4-manifold with $b^+ = 1$. Let ω_1 and ω_2 be two cohomologous symplectic forms. If they can be joined by a path of symplectic forms then they can be joined by a path of cohomologous symplectic forms.*

Proof. This result was proved as Theorem 1.2 in [31] under the assumption that M is not of 'Seiberg-Witten simple type'. In [31], a symplectic 4-manifold is said to be of 'SW simple type' if its only nonzero Gr^0 invariant occur in classes with zero Gromov-Taubes dimension. For a symplectic 4-manifold M not of 'SW simple type', what is used in the proof in [31] is the following fact: assuming $[\omega]$ is rational, then there is a basis of $H^2(M; \mathbf{Q})$ formed by rational classes $n[\omega], e_1, \dots, e_k$ with $e_j^2 < 0$ for all j such that $l(n[\omega] \pm e_j)$ is represented by a connected ω-symplectic surface for all j and large l.

By Propositions 4.1–4.3, all symplectic 4-manifolds with $b^+ = 1$ satisfy this property. So the proof of this Proposition is identical to that of Theorem 1.2 in [31].

In terms of the map CC, the result above can be interpreted as saying that, when restricted to a connected component of Ω_M, the inverse image of a point of CC is connected. This result is useful in §6 for the duality conjecture.

5. Symplectic Casteluovo's criterion

In this and the following sections we will study further A_K, the set of 'K-stable' classes of symplectic surfaces. Here we will determine, for a minimal 4-manifold, which integral multiples of K is in A_K. As an interesting corollary, we obtain the sympletic Casteluovo's criterion for rationality.

Lemma 5.1. *Let M be a minimal 4-manifold with $b^+ = 1$ and K a non-torsion symplectic canonical class. A_K contains only non-torsion classes with nonnegative square.*

Proof. We observe that, by definition, if $a \in A_K$ and $e \in \mathcal{C}_K$, then $a \cdot e > 0$. So obviously a can not be a torsion class. Suppose $a \in A_K$ has negative square. The orthogonal complement of a in $H^2(M; \mathbf{R})$ still contains classes of positive square. Let β be such a class. By Theorem 2, either $\beta \in \mathcal{C}_K$ or $-\beta \in \mathcal{C}_K$. However, the fact that $a \cdot \beta = 0$ contradicts with the observation above. The proof is finished.

Notice that, for a fixed symplectic structure, there might be classes of negative square represented by symplectic surfaces. Such examples are easy to find. For any positive integer n, the Hirzebruch surface F_{2n} is a minimal algebraic surface having holomorphic curves (hence symplectic with respect to any Kähler form) of square $-2n$ in F_{2n}.

Proposition 5.2. *Let M be a minimal 4-manifold with $b^+ = 1$ and K a non-torsion symplectic canonical class. Then nK is in A_K in the following cases:*

1. *$n \leq -1$ and M is CP^2, $S^2 \times S^2$ or an S^2-bundle over T^2.*

2. *$n = 1$ and M has $b_1 = 2$ and is not an S^2-bundle over T^2.*

3. *$n \geq 2$ and M is not rational nor irrational ruled.*

Proof. By Lemma 5.1, for any pair M, K with $K^2 < 0$, no multiple of K is in A_K. Such manifolds are S^2-bundles over Riemann surfaces of genera at least 2.

Assume now that M is not an S^2-bundle over a Riemann surface of genus at least 2. Then for any integer n, the Seiberg-Witten dimension of $\mathcal{L}_{K^{-1}} \otimes nK$ is nonnegative, because

$$d(\mathcal{L}_{K^{-1}} \otimes nK) = -K \cdot nK + nK \cdot nK = (n^2 - n)K^2 \geq 0.$$

Let ω be a symplectic form with K as its symplectic canonical class.

The manifolds in Case 1 satisfy $K^2 \geq 0$, $K \cdot \omega < 0$. So $K^{-1} + 2nK$ is in the forward cone for all $n \leq -1$. Thus it follows from Lemma 3.4 that $Gr_\omega(nK)$ is nontrivial for $n \leq -1$ and Case 1 is settled.

The manifolds in Case 2 satisfy $K^2 \geq 0$ and $K \cdot \omega > 0$. Let γ be the generator of $\Lambda^2 H^1(M; \mathbf{Z})$ such that $\omega \cdot \gamma \geq 0$. By Lemma 3.1, $K \cdot \gamma \geq 0$. By Theorem 2.2 and Lemma 2.3 $SW^0_{\omega,+}(\mathcal{L}_{K^{-1}} \otimes K) = 1$. When $b_1 = 2$, by Lemma 3.3,

$$SW^0_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes K) = SW^0_{\omega,+}(\mathcal{L}_{K^{-1}} \otimes K) + \frac{1}{2} K \cdot \gamma \neq 0$$

unless $K \cdot \gamma = -2$. But this is impossible, so by Theorem 2.9 $Gr_\omega(K)$ is nontrivial. Case 2 is settled.

By Lemma 3.2, $SW_{\omega,+}(\mathcal{L}_{K^{-1}} \otimes nK)$ is zero if $n \geq 2$. Since the manifolds in Case 3 satisfy $K \cdot \omega > 0$, $(K^{-1} + 2nK) \cdot \omega > 0$ for $n \geq 1$. When $b_1 = 0$, or when $b_1 = 2$ and $K \cdot \gamma \neq 0$, by Lemma 3.3, $SW^0_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes K)$ is nonzero. Therefore $Gr_\omega(nK)$ is nonzero for $n \geq 2$.

When $b_1 = 2$ and $K \cdot \gamma = 0$, from Case 2 we know $Gr_\omega(K)$ is nontrivial. By Lemma 3.1, K must have square zero, so it is represented by an embedded symplectic torus. Thus, for each $n \geq 2$, nK can be represented by an embedded symplectic torus as well by Lemma 2.6. The proof of Proposition 5.2 is complete.

Now we can give the proof of the symplectic Castlenuova criterion.

Corollary 2. *Let M be a closed, oriented 4-manifold with $b^+ = 1$ and a non-torsion symplectic canonical class K. If $b_1 = 0$ and $2K$ is not in A_K, then M is rational.*

Proof. First assume that M is minimal. By Proposition 5.2, M must be rational or irrationally ruled. Since irrational ruled manifolds have $b_1 > 0$, M is rational as claimed.

6. K-surface cone and duality conjecture

Recall that the rational K-surface cone $\mathcal{S}_K^{\mathbf{Q}}$, introduced in §1, is the cone $\sum_{v \in A_K} \mathbf{Q}^+ v$ in $H^2(M; \mathbf{Q})$. In this section we will study this cone and discuss the duality between it and the rational K-symplectic cone $\mathcal{C}_K^{\mathbf{Q}} = \mathcal{C}_K \cap H^2(M; \mathbf{Q})$ inside $H^2(M; \mathbf{Q})$.

We start with a couple of simple algebraic lemmas. Let W be a subset in $H^2(M;\mathbf{Q})$. Define the dual of W to be

$$W^\wedge = \{v \in H^2(M;\mathbf{Q}) \,|\, v \cdot w > 0 \text{ for any } w \in W\}.$$

Clearly $W^\wedge$ is a convex subset, and $W^{\wedge\wedge} = W$ if W is convex.

Lemma 6.1. *Let M be a symplectic 4-manifold with symplectic canonical class K. Then $\mathcal{S}_K^{\mathbf{Q}}$ is convex, and*

$$\mathcal{S}_K^{\mathbf{Q}} \subset \mathcal{C}^{\mathbf{Q}\wedge}_K \quad \text{and} \quad \mathcal{C}_K^{\mathbf{Q}} \subset \mathcal{S}_K^{\mathbf{Q}\wedge}.$$

Proof. It directly follows from the definitions of the two cones.

Lemma 6.2. *Let M be a closed oriented 4-manifold with $b^+ = 1$. Let $\mathcal{FP}^{\mathbf{Q}}$ be a component of $\mathcal{P}^{\mathbf{Q}}$ and F be a subset of $H^2(M;\mathbf{Q})$. Then*

$$\mathcal{FP}_F^{\mathbf{Q}\wedge} = \overline{\mathcal{FP}}^{\mathbf{Q}} + \sum_{f\in F} \mathbf{Q}^+ f.$$

Proof. Let us first show that $\overline{\mathcal{FP}}^{\mathbf{Q}} = \mathcal{FP}^{\mathbf{Q}\wedge}$. By the Light Cone Lemma 3.1, $\mathcal{FP}^{\mathbf{Q}} \subset \mathcal{FP}^{\mathbf{Q}\wedge}$. By the argument in Lemma 5.1 $\mathcal{FP}^{\mathbf{Q}\wedge} \subset \overline{\mathcal{FP}}^{\mathbf{Q}}$. Since

$$\sum_{f\in F} \mathbf{Q}^+ f \subset \mathcal{FP}_F^{\mathbf{Q}\wedge} \quad \text{and} \quad \overline{\mathcal{FP}}^{\mathbf{Q}} = \mathcal{FP}^{\mathbf{Q}\wedge} \subset \mathcal{FP}_F^{\mathbf{Q}\wedge},$$

we have, by the convexity of $\mathcal{FP}_F^{\mathbf{Q}\wedge}$,

$$\sum_{f\in F} \mathbf{Q}^+ f + \overline{\mathcal{FP}}^{\mathbf{Q}} \subset \mathcal{FP}^{\mathbf{Q}\wedge}_F.$$

Let $p \in \overline{\mathcal{FP}}^{\mathbf{Q}} + \sum_{f\in F} \mathbf{Q}^+ f$. Then $p \cdot e > 0$ for any $e \in \overline{\mathcal{FP}}^{\mathbf{Q}}$, thus $p \in \overline{\mathcal{FP}}^{\mathbf{Q}\wedge} = \mathcal{FP}^{\mathbf{Q}}$. And $p \cdot f > 0$ for any $f \in F$. Therefore

$$\left(\overline{\mathcal{FP}}^{\mathbf{Q}} + \sum_{f\in F} \mathbf{Q}^+ f\right)^\wedge \subset \mathcal{FP}_F^{\mathbf{Q}}.$$

The proof is complete. q.e.d.

Theorem 5. *Let M be a smooth, closed oriented 4-manifold with $b^+ = 1$. Let K be a symplectic canonical class. Use any class in Ω_K to define the forward cone. Then*

$$\mathcal{FP}^{\mathbf{Q}} + \sum_{E_i\in\mathcal{E}_K} \mathbf{Q}^+ E_i \subset \mathcal{S}_K^{\mathbf{Q}} \subset \overline{\mathcal{FP}}^{\mathbf{Q}} + \sum_{E_i\in\mathcal{E}_K} \mathbf{Q}^+ E_i.$$

Proof. We start with the first inclusion. Let ω be a symplectic form with K as its symplectic canonical class. By the same argument as in Propositions 4.2–4.3, we can show that, if e is an integral class in $\mathcal{FP}^{\mathbf{Q}}$, then for a large integer l, $SW_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes le)$ is nontrivial. Therefore $le \in \mathcal{S}_{\omega}^{\mathbf{Q}}$ by Theorem 2.9(1). We thus have shown that $\mathcal{FP}^{\mathbf{Q}} \subset \mathcal{S}_K^{\mathbf{Q}}$. By Lemma 3.5, $\mathcal{E}_K = \mathcal{E}_{\omega}$, so

$$\sum_{E_i \in \mathcal{E}_K} \mathbf{Q}^+ E_i \subset \mathcal{S}_K^{\mathbf{Q}}.$$

By the convexity of $\mathcal{S}_K^{\mathbf{Q}}$, we have

$$\mathcal{FP}^{\mathbf{Q}} + \sum_{E_i \in \mathcal{E}_K} \mathbf{Q}^+ E_i \subset \mathcal{S}_K^{\mathbf{Q}}.$$

The second inclusion follows from Theorem 3 and Lemmas 6.1–6.2. The proof is complete. q.e.d.

Duality Conjecture. Let M be a closed, oriented 4-manifold with $b^+ = 1$. Suppose K is a symplectic canonical class, then the rational K-surface cone and the rational K-symplectic cone are dual to each other.

We remark that this conjecture can be viewed as the analogue of the duality between the Kähler cone and the cone of numerically effective curves on an algebraic surface.

We will now show the Duality Conjecture holds for several classes of minimal 4-manifolds.

Lemma 6.3 *Let M be a closed oriented 4-manifold with $b^+ = 1$ and a symplectic canonical class K. The conjecture holds for M and K if any class $e \in \overline{\mathcal{FP}}^{\mathbf{Q}}(K)$ with square 0 is in $\mathcal{S}_K^{\mathbf{Q}}$.*

Proof. $\mathcal{C}_K^{\mathbf{Q}}$ is convex by Theorem 3, so $\mathcal{C}_K^{\mathbf{Q}\wedge\wedge} = \mathcal{C}_K^{\mathbf{Q}}$. Therefore $\mathcal{C}_K^{\mathbf{Q}} = \mathcal{S}_K^{\mathbf{Q}\wedge}$ if $\mathcal{S}_K^{\mathbf{Q}} = \mathcal{C}_K^{\mathbf{Q}\wedge}$. By Lemmas 6.1–6.2 and Theorems 3 and 5,

$$\mathcal{FP}^{\mathbf{Q}} + \sum_{E_i \in \mathcal{E}_K} \mathbf{Q}^+ E_i \subset \mathcal{S}_K^{\mathbf{Q}} \subset \mathcal{C}_K^{\mathbf{Q}\wedge} = \mathcal{FP}_{\mathcal{E}_K}^{\mathbf{Q}\ \wedge} = \overline{\mathcal{FP}}^{\mathbf{Q}} + \sum_{E_i \in \mathcal{E}_K} \mathbf{Q}^+ E_i.$$

So $\mathcal{S}_K^{\mathbf{Q}} = \mathcal{C}_K^{\mathbf{Q}\wedge}$ if the complement of $\mathcal{FP}^{\mathbf{Q}}(K)$ in $\overline{\mathcal{FP}}^{\mathbf{Q}}(K)$ is in $\mathcal{S}_K^{\mathbf{Q}}$. Since this complement consists of the classes $e \in \overline{\mathcal{FP}}^{\mathbf{Q}}(K)$ with square 0, the lemma is proved.

Two cohomologous symplectic forms are said to be isotopic if they can be joined by a path of cohomologous symplectic forms. For a symplectic form ω, recall that A_ω is the set of integral classes which can be represented by ω-symplectic surfaces.

Lemma 6.4 *Let M be a closed, oriented 4-manifold with $b^+ = 1$. If ω_1 and ω_2 are two symplectic forms which are deformation equivalent and cohomologous, then $A_{\omega_1} = A_{\omega_2}$.*

Proof. By the basic result of Moser [39] it is easy to see that $A_{\omega_1} = A_{\omega_2}$ if ω_1 and ω_2 are isotopic. Now the lemma follows from Proposition 4.11. q.e.d.

Proposition 6.5 *The Duality Conjecture holds for:*

a) *4-manifolds with torsion symplectic canonical classes, $b^+ = 1$ and $b_1 = 0$.*

b) *CP^2 and $S^2 \times S^2$.*

c) *S^2-bundles over T^2.*

d) *$S^2 \times \Sigma$ with Σ a surface with genus at least two.*

e) *T^2-bundles over T^2 with $b^+ = 1$, $\Lambda^2 H^2(M; \mathbf{Z})$ nontrivial, and having a unique deformation class of symplectic forms.*

f) *$S^1 \times X$ with X a fibered 3-manifold with $b_1 = 1$ and the genus of the fiber at least 2, and having a unique deformation class of symplectic forms.*

Proof. For the manifolds in e) and f), symplectic structures can be constructed by the construction in [52]. So all the manifolds listed have symplectic structures. Let K be any one of the two symplectic canonical classes. Let ω be an arbitrary symplectic form with K as its symplectic canonical class. By Lemma 6.3 what we need to show is, for any class e in $\overline{\mathcal{FP}}^{\mathbf{Q}}(K)$ with $e^2 = 0$, some (rational) multiple of it is represented by an ω-symplectic surface. We will show it is the case for manifolds in a)–e).

For manifolds in a), K is a torsion class, so any class e in $\overline{\mathcal{FP}}^{\mathbf{Q}}(K)$ with square zero has $d(e) = 0$ and $(K^{-1} + 2e) \cdot \omega > 0$. If, in addition $b_1 = 0$, by Lemmas 3.1 and 3.2(I), $SW_{\omega,-}(\mathcal{L}_{K^{-1}} \otimes e)$ is nontrivial. By Theorem 2.9, $e \in A_K$.

For manifolds in b), K^{-1} is in the forward cone. Thus $K^{-1}+2e$ is in the forward cone if e is in $\overline{\mathcal{FP}}^{\mathbf{Q}}(K)$. By Lemma 3.4 and Theorem 2.9 $Gr_\omega(e)\neq 0$ for any class $e\in\overline{\mathcal{FP}}^{\mathbf{Q}}$.

For manifolds in c), K^{-1} has square zero and is in $\overline{\mathcal{FP}}^{\mathbf{Q}}(K)$. By the same argument as above, if e is in $\overline{\mathcal{FP}}^{\mathbf{Q}}(K)$ with square zero and $e\neq K^{-1}$, then it is in A_K. The fact that K^{-1} is also in A_K is proved in Proposition 5.2.

For a product irrational ruled 4-manifold $S^2\times\Sigma$, if e has square 0, then it is Poincaré dual to either a multiple of $[S^2]$ or $[\Sigma]$. On such a 4-manifold every symplectic form is isotopic to a product from by [18]. Since any product symplectic form remains a symplectic form on any S^2 and Σ, the conclusion follows from Lemma 6.4.

For the manifolds in e), K is trivial and both b_1 and b_2 are equal to two. If γ is the nontrivial generator of $\Lambda^2H^2(M;\mathbf{Z})$ as in Lemma 3.3, it is the class of the fibers. Since the intersection form on $H^2(M;\mathbf{Z})$ is $(1)\oplus(-1)$, the classes with square zero are (rational) multiples of γ and another integral class which we denote by η. We can assume that $\eta\cdot\omega>0$. Since $K=0$, $d(\eta)$ is equal to zero. Notice that $\gamma\cdot\eta\neq 0$. This, together with Lemmas 3.2–3.3, imply that $SW_{\omega,-}(\mathcal{L}_{K^{-1}}\otimes\eta)$ is nontrivial. Therefore $Gr_\omega(\eta)$ is nontrivial. All these manifolds are geometric, and it is shown in [12] that every class with positive square can be represented by a 'geometric' symplectic form such that is symplectic on the fibers. In particular, the class of ω is represented by such a 'geometric' symplectic form ω'. Under the assumption that there is a unique deformation class of symplectic forms, the conclusion follows from Lemma 6.4.

For the manifolds in f), the Künneth formula tells us that b_1 and b_2 are again equal to two. Since $K^2=2\chi(M)+3\sigma(M)$, K has square 0. Let g be the genus of the fibers of $\pi: X\longrightarrow S^1$. Then $M=S^1\times X$ fibers over the 2-torus with fibers of genus g. Since g is at least 2, the class of the fibers is nontrivial in $H^2(M;\mathbf{Q})$. In fact by the construction in [52], the fibers are symplectic with respect to some symplectic structures on M. By the ajunction formula in [25], $K\cdot\gamma=(2g-2)$. Therefore K is nontrivial in $H^2(M;\mathbf{Q})$ as well and the classes with square zero are (rational) multiples of K and γ. By Proposition 5.2 K is in A_K, so we only have to deal with γ by explicitly constructing a symplectic form ω' in the class of ω such that the fibers are ω'-symplectic. Then, under the assumption that there is a unique deformation class of symplectic forms, the conclusion that γ is represented by an ω-symplectic surface again follows from Lemma 6.4. Choose a metric on X such that $\pi : X \longrightarrow S^1$ is a harmonic map. Let $*_X$ be the star operator on X with respect to this metric. Let $d\theta$ be the volume form of the base circle, and μ be $*_X \pi^* d\theta$, . Let ds be the volume form of the product circle. Then for any pair of positive numbers α and β, the form $\pm(\alpha ds \wedge d\theta + \beta\mu)$ is symplectic and restricts to a symplectic form on each fiber. In particular the class of ω can be thus represented. The proof of Proposition 6.5 is complete. q.e.d.

We finish the paper by mentioning that we can also introduce the real K-surface cone $\mathcal{S}_K$. It is easy to check that the statements in Lemmas 6.1–6.3 and Theorem 5 remain valid for $\mathcal{S}_K$ if the $\mathbf{Q}$ are removed or replaced by $\mathbf{R}$. The real K-surface cone is similar to (often smaller than) the cone of numerically effective curves. In fact, a more precise analogue of the real K-surface cone is the deformed symplectic effective cone introduced by Ruan in [43]. In particular, if Ω_K has only one connected component, then they coincide.

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