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UNIQUENESS OF THE COMPLETE NORM TOPOLOGY AND CONTINUITY OF DERIVATIONS ON BANACH ALGEBRAS*

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1. We are concerned with the following two propositions regarding a complex Banach algebra A:

$\mathcal{N}$ A has unique complete norm topology

$\mathcal{D}$ Derivations on A are necessarily continuous.

Note that if A does not satisfy $\mathcal{N}$ the validity of $\mathcal{D}$ may depend on the choice of topology, cf. the example given below.

It has been known for some time that C^*-algebras satisfy $\mathcal{N}$ [10, Corollary 4.1.18] and $\mathcal{D}$ [11], and also that commutative semi-simple Banach algebras satisfy $\mathcal{N}$ [10, Corollary 2.5.18]. Recently Johnson [6, Theorem 2] has shown that these latter satisfy $\mathcal{D}$ also, so that by [12, Theorem 1] the zero map is the only derivation on such algebras. Subsuming these results Johnson [5] proved that semi-simple Banach algebras satisfy $\mathcal{N}$, and, in the joint paper [7], that these algebras also satisfy $\mathcal{D}$.

Clearly $\mathcal{N}$ and $\mathcal{D}$ are not true in general, since any Banach space can be made into a Banach algebra by defining all products to be zero, and in this situation any norm is an algebra norm, and any linear operator is a derivation. A less extreme example is the following. Let A be the algebra of [4, p. 771] so that $A = l^2 \oplus R$ algebraically, under a norm $\|\cdot\|$ such that l^2 is dense in A, and R is the principal ideal generated by an element r such that $r^2 = rx = 0$, $x \in l^2$. The function $x+\alpha r \mapsto \|x\|' + |\alpha|$, where $\|\cdot\|'$ is the usual l^2 norm is easily seen to be a Banach algebra norm on A, inequivalent to $\|\cdot\|$ since l^2 is $\|\cdot\|'$-closed. Thus A does not satisfy $\mathcal{N}$. Now let D be a derivation on A, so that $D(x)y + xD(y) = D(xy) = D((x+\alpha r)y) = D(x)y + \alpha D(r)y + xD(y)$ for all $x, y \in l^2$, $\alpha \in C$, and so $D(r) = \mu r$ for some constant μ. Thus $D(R) \subseteqq R$ and so D induces a derivation $\overline{D}$ on the semisimple algebra A/R. It follows that $\overline{D}$ is zero and hence that $D(A) \subseteqq R$. Thus if $x, y \in l^2$, $D(xy) = D(x)y + xD(y) = 0$ so

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that D vanishes on $(l^2)^2=l^1$. Defining a linear functional f on l^2 by $D(x)=f(x)r$ we have $D(x+\alpha r)=(f(x)+\mu\alpha)r$ and $f(l^1)=\{0\}$. On the other hand if f is any linear functional on l^2 which vanishes on l^1 (such functionals, non-zero, are easily seen to exist) then the map $x+\alpha r \to (f(x)+\mu\alpha)r$ is a derivation on A for any constant μ. It follows that all non-zero derivations on A are unbounded under $\|\cdot\|$, and are $\|\cdot\|'$-bounded if and only if $f=0$. Thus A does not satisfy $\mathcal{D}$ under either $\|\cdot\|$ or $\|\cdot\|'$. We remark that if B is any Banach algebra with B^2 properly dense in B then the same argument shows the existence of derivations on $B\oplus R$, where R is the one dimensional ideal spanned by an element r such that $r^2=xr=rx=0$ for $x\in B$, which are unbounded under the norm $\|x+\alpha r\|=\|x\|+|\alpha|$.

Two problems thus arise. Firstly, to determine the relationship, if any, between the two propositions $\mathcal{N}$ and $\mathcal{D}$, and, secondly, to find classes of algebras which are not necessarily semisimple and for which $\mathcal{N}$ and/or $\mathcal{D}$ are satisfied. In this paper we consider only the second of these which, though perhaps the more mundane, is the point from which conjectures or counter-examples to the first are to be obtained. Throughout the paper, an algebra will denote any associative algebra, with identity (always denoted e), over the complex field $\mathbf{C}$.

Let A be an algebra, $A[x]$ the polynomial ring over A in an indeterminate x, and let $\alpha(x)=\alpha_0+\alpha_1 x+\cdots+\alpha_{n-1}x^{n-1}+x^n\in A[x]$ be a monic polynomial. In the case when A is commutative and semisimple, Lindberg [8] has recently shown that if $A[x]/(\alpha(x))$ is a Banach algebra under a norm $\|\cdot\|$ then it satisfies $\mathcal{N}$, and in fact any Banach algebra norm is equivalent to the Arens-Hoffman norm

$$\left\|\sum_{i=0}^{n-1} a_i x^i+\alpha(x)\right\|_t=\sum_{i=0}^{n-1}\|a_i\|t^i$$

where t is any positive real with $t^n\geqq\sum_{i=0}^{n-1}\|\alpha_i\|t^i$ (see [2]). The result is not true if A is not semisimple, as is shown by the example at the end of [3], cf. the example give above. In section two we show that $\mathcal{D}$ is satisfied by a class of algebras which include $A[x]/(\alpha(x))$, A semisimple.

By an algebra of power series over an algebra A we mean an algebra $\mathbf{A}$ whose elements are formal power series $\sum_{i\geqq 0} a_i t^i$ over A in an indeterminate t (with $t^0=e$), with term by term vector space operations and Cauchy multiplication. It will also be required that $\mathbf{A}$ contain all polynomials in t and that $et=t$. We identify A with its image in $\mathbf{A}$ under the map $a\to a+0\cdot t+0\cdot t^2+\cdots$. If $\mathbf{A}$ is a Banach algebra under a norm $\|\cdot\|$ then $(\mathbf{A},\|\cdot\|)$ will be termed a Banach algebra of power series over A if the projections $p_j:\Sigma a_i t^i\to a_j$ of $\mathbf{A}$

into A are continuous for all $j \geqq 0$, where the relative topology is taken on A. Since $A = \bigcap_{i>0} \ker p_i$ it follows that A is $\|\cdot\|$-closed in $\mathbf{A}$. In the case $A = \boldsymbol{C}$ the author has shown [9] that Banach algebras of power series over A satisfy $\mathcal{D}$, and exactly the same proof is valid for arbitrary A. The third section of this paper is concerned with the validity of $\mathcal{N}$ for such algebras.

2. Let A be an algebra, $\mathbf{A}$ an algebra over A such that there is a set $\{x_0 = e, x_1, \cdots, x_n\}$ in the commutant of A in $\mathbf{A}$ forming a basis for $\mathbf{A}$ over A, so that, in particular, $\mathbf{A}$ is a finite dimensional A-bimodule. Suppose that $\mathbf{A}$ is a Banach algebra under a norm $\|\cdot\|$, and that the projections $p_j : \Sigma a_i x_i \mapsto a_j$ of $\mathbf{A}$ into A are continuous for $0 \leqq j \leqq n$, where the relative topology is taken on A. Since $A = \bigcap_{1 \leqq i \leqq n} \ker p_i$ it follows that A is itself a Banach algebra under $\|\cdot\|$. In this situation we have the following.

THEOREM 2.1. *If $(A, \|\cdot\|)$ satisfies $\mathcal{D}$ then so does $(\mathbf{A}, \|\cdot\|)$.*

PROOF. Let D be a derivation on $\mathbf{A}$, and for $0 \leqq i \leqq n$ define $f_i = p_i D$. If $x, y \in A$ the derivation equation $D(xy) = D(x)y + xD(y)$ shows that $\sum_i f_i(xy) x_i = \sum_i \{f_i(x) x_i y + x f_i(y) x_i\}$. Using the fact that $\{x_i\}$ is a subset of the commutant of A in $\mathbf{A}$ and then equating coefficients of the $\{x_i\}$ it follows that $f_i(xy) = f_i(x) y + x f_i(y)$, so that each $f_i | A$, $0 \leqq i \leqq n$, is a derivation and hence is continuous by the hypothesis on A. But for $x = \sum_i a_i x_i \in \mathbf{A}$,

$$D(x) = \sum_i \{D(a_i) x_i + a_i D(x_i)\} = \sum_i \left\{ \left[\sum_j f_j(p_i(x)) x_j \right] x_i + p_i(x) D(x_i) \right\},$$

and it follows that D is continuous.

COROLLARY 2.2. *Let A be a commutative Banach algebra which satisfies $\mathcal{D}$. Then $A[x]/(\alpha(x))$, with an Arens-Hoffman norm, satisfies $\mathcal{D}$.*

REMARK. Taking A semisimple and $\alpha(x) = x^2$ we obtain an algebra containing a nilpotent, namely x, and which satisfies both $\mathcal{N}$ and $\mathcal{D}$.

3. We now turn to the consideration of $\mathcal{N}$ for algebras of power series. Let A be an algebra, $\mathbf{A}$ an algebra of power series over A which is a Banach algebra under a norm $\|\cdot\|$. A sequence $\{\sigma_n\}$ of positive real numbers will be termed a growth sequence for $(\mathbf{A}, \|\cdot\|)$ if $\lim_n \sigma_n p_n(x) = 0$ for all $x \in \mathbf{A}$, the limit taken in the $\|\cdot\|$-topology on A.

LEMMA 3.1. *$(\mathbf{A}, \|\cdot\|)$ is a Banach algebra of power series over A if and only if $(\mathbf{A}, \|\cdot\|)$ admits a growth sequence.*

PROOF. If $(\mathbf{A}, \|\cdot\|)$ is a Banach algebra of power series over A it is easily seen that $\sigma_n = (n\|p_n\|)^{-1}$ defines a growth sequence.

For the converse, let $\{\sigma_n\}$ be a growth sequence for $(\mathbf{A}, \|\cdot\|)$ and suppose that at least one of the projections, and hence one of least index, p_k say, is discontinuous. We show that this leads to a contradiction. In order to give a proof valid for all values of k we make the convention that empty sums have value zero.

Since each of (the possibly empty set) $p_0, \cdots, p_{k-1}$ is continuous, define inductively a sequence $\{x_n\}_{n\geqq 0}\subseteqq \mathbf{A}$ such that

$$\text{(i)} \qquad \|x_n\| \leqq \min_{0\leqq i\leqq n}(1, 2^{-n}\|t^i\|^{-1})$$

$$\text{(ii)} \qquad \|p_k(x_n)\| \geqq \sigma_{n+k}^{-1} + \sum_{i=0}^{k-1}\|p_i\| + \sum_{i=1}^{n}\|p_{i+k}(x_{n-i})\| \,.$$

Then for each j, $\sum_{i\geqq j} x_i t^{i-j}$ converges in $\mathbf{A}$, to y_j say. Thus for each n

$$p_n(y_0) = p_n\left\{y_{n+1}t^{n+1} + \sum_{i=0}^{n} x_i t^i\right\} = p_n\left\{\sum_{i=0}^{n} x_i t^i\right\}$$

$$= \sum_{j=0}^{n}\sum_{i=0}^{n} p_j(x_i)p_{n-j}(t^i) = \sum_{i=0}^{n} p_i(x_{n-i}) \,.$$

But then for $n > k$

$$\begin{aligned}\|p_n(y_0)\| &\geqq \|p_k(x_{n-k})\| - \sum_{i=0}^{k-1}\|p_i(x_{n-i})\| - \sum_{i=k+1}^{n}\|p_i(x_{n-i})\| \\ &\geqq \|p_k(x_{n-k})\| - \sum_{i=0}^{k-1}\|p_i\| - \sum_{i=1}^{n-k}\|p_{i+k}(x_{n-k-i})\| \\ &\geqq \sigma_n^{-1} \quad \text{by (ii)}.\end{aligned}$$

Thus $\sigma_n\|p_n(y_0)\| \geqq 1$ for $n > k$, contradicting the supposition that $\{\sigma_n\}$ is a growth sequence.

Using this result we obtain the following, cf. [3, Theorem 2].

THEOREM 3.2. *Let A be a Banach algebra which satisfies $\mathfrak{N}$, $\mathbf{A}$ an* *algebra of power series over A. Suppose that $(\mathbf{A}, \|\cdot\|)$ is a Banach algebra of power series over A for some norm $\|\cdot\|$. Then for $\mathbf{A}$ to satisfy $\mathcal{N}$ it is necessary and sufficient that A be closed in $\mathbf{A}$ under any Banach algebra norm on $\mathbf{A}$.*

PROOF. As noted above, A is $\|\cdot\|$-closed, and so the condition on A is certainly necessary.

For the sufficiency, let $\{\sigma_n\}$ be a growth sequence for $(\mathbf{A}, \|\cdot\|)$, and let $\|\cdot\|'$ be any Banach algebra norm on $\mathbf{A}$. Then A is closed under both $\|\cdot\|$ and $\|\cdot\|'$ and so these are equivalent on A. But this means $\{\sigma_n\}$ is a growth sequence for $(\mathbf{A}, \|\cdot\|')$, so that $\mathbf{A}$ is a Banach algebra of power series over A under $\|\cdot\|'$.

Now let $I: (\mathbf{A}, \|\cdot\|) \rightarrow (\mathbf{A}, \|\cdot\|')$ be the identity map, and suppose $\{x_n\} \subseteqq \mathbf{A}$, $x_n \rightarrow 0$ under $\|\cdot\|$, $I(x_n) = x_n \rightarrow y$ under $\|\cdot\|'$, for some $y \in \mathbf{A}$. Then by the continuity of the projections in both topologies, $p_j(x_n) \rightarrow 0$, $p_j(x_n) \rightarrow p_j(y)$ under $\|\cdot\|$, $\|\cdot\|'$ respectively, for each $j \geqq 0$. But these two norms are equivalent on A, whence $p_j(y)=0$ for $j \geqq 0$, so $y=0$. It follows that I is closed, hence continuous, hence bicontinuous, and the theorem is proved.

The question now arises as to when the hypothesis of Theorem 3.2 is satisfied; an immediate example, of course, is $A=\mathbf{C}$. For a more general result we require two lemmas, cf. [3, Lemma 1] and [8, Theorem 3.4.1].

LEMMA 3.3. *Let A be a commutative semisimple algebra, $\mathbf{A}$ an algebra of power series over A. If $\mathbf{A}$ is a Banach algebra under a norm $\|\cdot\|$ then there is a norm $\|\cdot\|'$ on A under which A is a Banach algebra. Also, $\|\cdot\|'$ minorizes $\|\cdot\|$ restricted to A.*

PROOF. If $x \in A$ is invertible in $\mathbf{A}$ it is easily seen to be invertible in A, and so the set $\{x \in A : \|e-x\| < 1\}$ consists entirely of invertible elements. Thus A is a Q-algebra under $\|\cdot\|$, and so every maximal ideal of A is $\|\cdot\|$-closed (in A). It now follows from the semisimplicity of A that $x \in A$, $\nu(x)=0$ imply $x=0$.

Denote by $M(A)$, $M(\mathbf{A})$ the carrier spaces of A, $\mathbf{A}$ respectively, and for $\varphi \in M(A)$ let $E_\varphi = \{\psi \in M(\mathbf{A}) : \psi | A = \varphi\}$. Note that if $\varphi \in M(A)$ the functional $\tilde{\varphi} : \Sigma a_i t^i \rightarrow \varphi(a_0)$ lies in $M(\mathbf{A})$, and $M(\mathbf{A}) = \cup [E_\varphi : \varphi \in M(A)]$.

Now let R be the radical of $\mathbf{A}$, and denote by $\|\cdot\|$ the quotient norm on $\mathbf{A}/R$. It follows from the above that the map $x \mapsto x+R$ of A into $\mathbf{A}/R$ is an isomorphism, and so we may consider A with the norm $\|\cdot\|'$. Let $\{x_n\} \subseteqq A$ be $\|\cdot\|'$-Cauchy, so that there is $y = \Sigma a_i t^i \in \mathbf{A}$, $a_i \in A$, such that $\|x_n - y + R\|' \rightarrow 0$. Since $\nu(z) \leqq \|z+R\|$ for any $z \in \mathbf{A}$ we have, for $\varphi \in M(A)$, $\psi \in E_\varphi$,

$\psi(x_n-y)\to 0$. But then $\psi(x_n-a_0)=\tilde{\varphi}(x_n-y)\to 0$ and so $\psi(a_0-y)=\psi(x_n-y)-\psi(x_n-a_0)\to 0$. It follows that $a_0-y\in R$ so that $\|x_n-a_0+R\|'=\|x_n-y+R\|'\to 0$. Thus A is complete under $\|\cdot\|$ and the result follows.

LEMMA 3.4. *Let A, $\mathbf{A}$, $\|\cdot\|$ be as in Lemma 3.3. Then A is $\|\cdot\|$-closed.*

PROOF. Let $\bar{A}$ denote the closure of A in the $\|\cdot\|$-topology. The map $\varphi\mapsto\tilde{\varphi}|\bar{A}$ is easily seen to be a homeomorphism between $M(A)$ and $M(\bar{A})$.

Suppose by way of induction that $\bar{A}\subseteqq\bigcap_{i=1}^{k-1}\operatorname{Ker}p_i$ for some integer $k>1$. We show this implies $\bar{A}\subseteqq\operatorname{Ker}p_k$. Using this inductive hypothesis on $\bar{A}$ it is easily seen that the mapping $D: x\mapsto p_k(x)$ is a derivation of $\bar{A}$ into the algebra of bounded functions on $M(\bar{A})$ in the sense that if $x, y\in\bar{A}$ and $\varphi\in M(\bar{A})$ then $D(xy)(\varphi)=D(x)(\varphi)\,\hat{y}(\varphi)+\hat{x}(\varphi)\,D(y)(\varphi)$. But then [6, Theorems 1 and 3] show that the functionals $x\mapsto D(x)(\varphi)$ are continuous for $\varphi\in M(\bar{A})\backslash F$ where F is a finite set each of whose members is isolated in $M(\bar{A})$.

Now let $x\in\bar{A}$, $\{x_n\}\subseteqq A$ with $x_n\to x$ under $\|\cdot\|$. If $\varphi\in M(\bar{A})\backslash F$ then it follows immediately that $\varphi p_k(x)=0$. If $\varphi\in F$ then $\psi=\varphi|A$ is isolated in $M(A)$ and so by Silov's theorem (since A is a Banach algebra under $\|\cdot\|'$ by Lemma 3.3) there is an idempotent $f\in A$ with $\hat{f}$ the characteristic function of $\{\psi\}$. It follows by the semisimplicity of A that $fy=\psi(y)f$ for $y\in A$ so that fA is one-dimensional and hence $\|\cdot\|$-closed. But then $fx=\lim fx_n=fy$ for some $y\in A$ and so, since $fx=\Sigma f p_i(x)t^i$, equating of coefficients of t^k shows $\psi(p_k(x))f=fp_k(x)=0$. Thus $\varphi p_k(x)=0$ for all $\varphi\in M(A)$ and so $p_k(x)=0$, as required.

The proof that $\bar{A}\subseteqq\operatorname{Ker}p_1$ is exactly as the inductive step and so we conclude that $\bar{A}\subseteqq\bigcap_{k\geqq 1}\operatorname{Ker}p_k=A$ and the result follows.

THEOREM 3.5. *Let A be a commutative semisimple algebra, $(\mathbf{A}, \|\cdot\|)$ a Banach algebra of power series over A. Then the algebra $\mathbf{A}$ satisfies $\mathfrak{N}$.*

PROOF. Immediate from Lemma 3.4 and Theorem 3.2.

Since any semisimple Banach algebra satisfies $\mathfrak{N}$, Theorem 3.5 is only of interest when $\mathbf{A}$ is not semisimple. This is trivially so if the element t is quasinilpotent, and it might be hoped that the converse was true. The following result shows this is indeed the case under certain conditions on A.

THEOREM 3.6. *Let A be a commutative semisimple algebra, $\mathbf{A}$ an algebra of power series over A which is a Banach algebra under a norm*

$\|\cdot\|$, *and suppose that* $\limsup_n(\|p_n(x)\|\cdot\|t^n\|)^{1/n}<\infty$ *for each* $x\in\mathbf{A}$. *Then* $(\mathbf{A},\|\cdot\|)$ *is a Banach algebra of power series over* A. *If the stronger condition* $\limsup(\|p_n(x)\|\cdot\|t^n)^{1/n}\leqq 1$ *is satisfied for all* $x\in\mathbf{A}$ *then* $\sigma(t)=\{\lambda:|\lambda|\leqq\nu(t)\}$, *and* $\mathbf{A}$ *is semisimple if and only if* $\nu(t)>0$.

PROOF. Let $\sigma_n=n^{-n}\|t^n\|$. If $x\in\mathbf{A}$ then there is a constant $C=C(x)$ such that $\|p_n(x)\|\leqq C^n\|t^n\|^{-1}$ for all n sufficiently large, and so $\sigma_n\|p_n(x)\|\to 0$ as $n\to\infty$. It follows that $\{\sigma_n\}$ is a growth sequence for $(\mathbf{A},\|\cdot\|)$ and the first statement follows from Lemma 3.1.

Suppose now that $\limsup(\|p_n(x)\|\cdot\|t^n\|)^{1/n}\leqq 1$ for $x\in\mathbf{A}$. For $\lambda\in\rho(t)$, the resolvent set of t, a simple argument of equating coefficients shows that $(t-\lambda e)^{-1}=-\lambda^{-1}\Sigma t^i\lambda^{-i}$. Let $\lambda_0\in\rho(t)$. Then since $\rho(t)$ is open there is $\mu\in\rho(t)$, $|\mu|<|\lambda_0|$, and so $\Sigma t^i\mu^{-i}\in\mathbf{A}$ whence $\limsup\|t^n\|^{1/n}\leqq|\mu|$. Thus if $|\lambda|\geqq|\lambda_0|>|\mu|$, $\Sigma\|t^i\|\cdot|\lambda|^{-i}<\infty$ and so $\Sigma t^i\lambda^{-i}\in\mathbf{A}$, which shows $\lambda\in\rho(t)$. It follows immediately that $\rho(t)=\{\lambda:|\lambda|>\nu(t)\}$, that is, $\sigma(\mathrm{t})=\{\lambda:|\lambda|\leqq\nu(t)\}$.

Finally, $\nu(t)>0$ is clearly necessary for semisimplicity. For the sufficiency, suppose $\nu(t)>0$ and let $|\lambda|<\nu(t)$. Then $\|t^i\|\geqq\nu(t)^i>|\lambda|^i$ for each $i>0$ and so $\Sigma\|p_i(x)\lambda^i\|\leqq\Sigma\|p_i(x)\|\cdot\|t^i\|\cdot|\lambda|^i\nu(t)^{-i}<\infty$ for $x\in\mathbf{A}$. Thus $\Sigma p_i(x)\lambda^i$ converges in $\mathbf{A}$, and so in A, as this is closed in $\mathbf{A}$. If $\varphi\in M(A)$ the map $\varphi_\lambda: x\to\Sigma\varphi p_i(x)\lambda^i$ is thus defined and clearly $\varphi_\lambda\in M(\mathbf{A})$. Suppose then, that $x\in\mathbf{A}$ in quasinilpotent, so that $\varphi_\lambda(x)=0$ certainly, for $\varphi\in M(A)$, $|\lambda|<\nu(t)$. Since $\varphi_\lambda(x)$ is an analytic function of λ on $|\lambda|<\nu(t)$, for each $\varphi\in M(A)$, it follows that $\varphi p_i(x)=0$ for $\varphi\in M(A)$, $i\geqq 0$. The semisimplicity of A now shows that $p_i(x)=0$, $i\geqq 0$, and so $x=0$.

If t is quasinilpotent then t is necessarily a topological divisor of zero. The converse is false even under the strong condition of Theorem 3.6 and the supposition that polynomials in t are dense in $\mathbf{A}$. A counterexample follows from the example on pages 12 and 13 of [1] and Theorem 4.3 of that paper. Thus the following result does not subsume Theorem 3.6. The author is indebted to Dr. B. E. Johnson for suggesting the simple but elegant inductive step in the argument.

THEOREM 3.7. *Let* A *be a commutative semisimple algebra,* $\mathbf{A}$ *an algebra of power series over* A *which is a Banach algebra under a norm* $\|\cdot\|$. *Suppose that polynomials in* t *are dense in* $\mathbf{A}$ *and that* t *is not a topological divisor of zero. Then* $(\mathbf{A},\|\cdot\|)$ *is a semisimple Banach algebra of power series over* A.

PROOF. Note firstly that p_0 is a homomorphism of $\mathbf{A}$ into A and since the latter is a semisimple Banach algebra under $\|\cdot\|$ (Lemma 3.4) p_0 is necessarily continuous by [10, Theorem 2.5.17]. Also, as t is not a topological divisor of zero there is a constant C such that $\|tx\| \geqq C\|x\|$ for $x \in \mathbf{A}$.

Suppose now that $x \in \operatorname{Ker} p_0$, and let $\{x_n\}$ be a sequence of polynomials in t converging to x. Since p_0 is continuous we may suppose, by replacing x_n by $x_n - p_0(x_n)$ if necessary, that $p_0(x_n) = 0$ for all n. But then $x_n = ty_n$ for some polynomial y_n and since $\|y_n - y_m\| \leqq C^{-1}\|x_n - x_m\|$ $\{y_n\}$ converges, to y say, with $x = ty$. Thus $\operatorname{Ker} p_0 \subseteqq t\mathbf{A}$, and the converse inclusion is obvious, so that $\operatorname{Ker} p_0 = t\mathbf{A}$.

Let $\Gamma : \mathbf{A} \to \mathbf{A}$ be the linear mapping of multiplication by t, so that Γ is a continuous linear bijection of $\mathbf{A}$ onto the closed ideal $t\mathbf{A}$. It follows easily from the closed graph theorem that Γ^{-1} is also continuous. If $x \in \mathbf{A}$ then $x - p_0(x) \in t\mathbf{A}$ by the above, and $p_k(x) = p_{k-1}\Gamma^{-1}(x - p_0(x))$, $k \geqq 1$. A simple inductive argument shows that p_k is continuous, $k \geqq 0$, and in fact $\|p_k\| \leqq (1 + \|p_0\|)^k \|\Gamma^{-1}\|^k \|p_0\|$. Thus $(\mathbf{A}, \|\cdot\|)$ is certainly a Banach algebra of power series over A. Also, if $|\lambda| < (1 + \|p_0\|)^{-1}\|\Gamma^{-1}\|^{-1}$ then $\Sigma\|p_i(x)\lambda^i\| < \infty$ for all $x \in \mathbf{A}$, and semisimplicity follows as in the proof of Theorem 3.6.

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