Uniqueness of the Fock quantization of fields with unitary dynamics in nonstationary spacetimes

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The Fock quantization of fields propagating in cosmological spacetimes is not uniquely determined because of several reasons. Apart from the ambiguity in the choice of the quantum representation of the canonical commutation relations, there also exists a certain freedom in the choice of field: one can scale it arbitrarily absorbing background functions, which are spatially homogeneous but depend on time. Each nontrivial scaling turns out into a different dynamics and, in general, into an inequivalent quantum field theory. In this work we analyze this freedom at the quantum level for a scalar field in a nonstationary, homogeneous spacetime whose spatial sections have S^3 topology. A scaling of the configuration variable is introduced as part of a linear, time dependent canonical transformation in phase space. In this context, we prove in full detail a uniqueness result about the Fock quantization requiring that the dynamics be unitary and the spatial symmetries of the field equations have a natural unitary implementation. The main conclusion is that, with those requirements, only one particular canonical transformation is allowed, and thus only one choice of the field-momentum pair (up to irrelevant constant scalings). This complements another previous uniqueness result for scalar fields with a time varying mass on S^3, which selects a specific equivalence class of Fock representations of the canonical commutation relations under the conditions of a unitary evolution and the invariance of the vacuum under the background symmetries. In total, the combination of these two different statements of uniqueness picks up a unique Fock quantization for the system. We also extend our proof of uniqueness to other compact topologies and spacetime dimensions.

I. INTRODUCTION

The unique character of nature is alluded in physics by the uniqueness of the theories employed to describe it. In particular, by imposing appropriate physical criteria, the quantization of a classical system *should* yield a unique quantum description up to unitary equivalence. Since the quantization process involves choices that may lead to inequivalent theories, the specification of a unique description is a nontrivial task.

Even in systems in which one already starts with a *specific* choice of basic canonical variables and an associated set of canonical commutation relations (CCRs), there exists an intrinsic ambiguity in the quantization process because the CCRs can be represented in nonequivalent ways. In the case of linear systems with a finite number of degrees of freedom, these ambiguities are essentially suppressed by the imposition of certain unitarity and continuity conditions on the representation of the algebra of

observables (as stated in the Stone–von Newmann theorem [1]), so that uniqueness follows without misadventures. Nonetheless, the situation changes drastically in the arena of field theory. These systems accept infinite nonequivalent representations of the CCRs [2] and there is no general procedure to select a preferred quantum description. In this situation, *physical results* depend on the representation adopted, a fact that brings into question their significance. It is then necessary to look for additional criteria to warrant uniqueness and regain robustness in the quantum predictions.

The usual procedure to select a preferred representation in field theory for a *given* set of CCRs is to exploit the classical symmetries. For instance, the invariance under the Poincaré group is the criterion imposed to arrive at a unique representation in ordinary quantum field theory. Thus, if the field theory corresponds to a scalar field, Poincaré invariance, adapted to the dynamics of the considered theory, selects a complex structure [2], which is the mathematical object that encodes the ambiguity in the quantization and determines the vacuum state of the Fock representation. For stationary spacetimes, the time translation symmetry is exploited to formulate the so-called energy criterion and

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then select a preferred complex structure [3]. But when the symmetries are severely restricted, as it is the case for generic curved spacetimes or for manifestly nonstationary systems, extra requirements must be imposed to complete the quantization process. For example, in the case of de Sitter space in $1 + 1$ dimensions, it is possible to pick up a unique de Sitter invariant Fock vacuum for a free scalar field by looking for an invariant Gaussian solution to a properly regulated Schrödinger equation [4].

In the context of quantum cosmology, the extra criterion of a unitary implementation of the dynamics has been successfully employed to specify a unique, preferred Fock quantization for the Gowdy spacetimes. These are spacetimes which possess two spacelike Killing isometries and spatial sections of compact topology [5]. In the case of a three-torus topology and a content of linearly polarized gravitational waves, the local gravitational degrees of freedom can be described by a scalar field with a specific time dependent mass and which propagates in an auxiliary, static background with the spatial topology of the circle [6]. For this choice of basic field for the model, one is able to find a unique Fock quantization which incorporates the background symmetries as symmetries of the vacuum and implements the field dynamics as a family of unitary quantum transformations [6–9].

More recently, in a broader context, a unitary equivalence class of Fock representations has been specified for scalar fields with *generic* time varying mass, defined on spheres in three or less dimensions [10,11]. Again, the procedure consists in requiring a unitary dynamics and the vacuum invariance under the symmetries of the field equation. The particularly relevant case of the three-sphere, with the dimensionality observed in our universe, was considered in Ref. [11].

Apart from the inherent ambiguity in choosing the representation of the CCRs, the quantization of fields in curved spacetimes is affected by another kind of ambiguity. It is due to the freedom in choosing a specific field *parametrization* to describe the physical system, namely, the freedom in declaring a particular choice of field (together with its associated dynamics) as the *fundamental* one. Let us concentrate our attention on the case of homogeneous but nonstationary spacetimes, like those encountered as backgrounds in cosmology. In these circumstances, it is most natural to consider field redefinitions which absorb background functions. This leads to a scaling of the field by a time dependent function, such that the linearity of the field equations and of the structures of the system are preserved. If this time dependence is nontrivial, the two fields (i.e., the scaled and the unscaled ones) are governed by different dynamics. Since a change in the dynamics typically calls for inequivalent representations, the construction of a quantum theory clearly depends on the selection of a specific field description for the system among all those related by these scaling transformations.

As commented above, these considerations are crucial for quantum matter fields propagating in inflationary or cosmological backgrounds, which are spatially homogeneous but not stationary. The discussion is also relevant for the quantization of local gravitational degrees of freedom, in contrast with the previous context of quantum matter fields in classical spacetimes that are solutions to the gravitational field equations. This latter class of systems includes, e.g., the already mentioned Gowdy models and the case of gravitational perturbations around cosmological backgrounds. For these gravitational systems, there exists a great freedom in the choice of parametrization of the metric components in terms of fields. In all these situations, the choice of a suitable field parametrization involves a time dependent scaling related to background functions and whose specific form depends on the particular system under study. This choice often leads to fields which effectively propagate in an auxiliary static background, therefore simplifying in part the corresponding dynamics, although there remain (or appear) time dependent potentials which manifest that the scenario is a nonstationary one.

The question immediately arises of whether it is again possible to invoke natural criteria to remove (at least in certain situations) the ambiguity that this freedom in the choice of field introduces at the quantum level. A detailed analysis about this issue was first carried out for the quantization of the linearly polarized Gowdy model with three-torus topology in Ref. [9]. That work studied a family of linear, time dependent canonical transformations that involve a scaling of the field. It was proven that there actually exists no freedom left in performing a transformation of this kind, once the criteria of invariance under the remaining spatial symmetries and the unitary implementation of the dynamics are imposed. More precisely, Ref. [9] shows that the considered transformations lead to new dynamics such that one cannot attain a unitary quantum evolution in a Fock representation while keeping the symmetry invariance of the vacuum.[1] The requirements of unitary evolution and invariance, therefore, suffice to select a specific scaling of the field and a privileged family of equivalent Fock quantizations for it. In other words, the uniqueness is guaranteed both for the choice of fundamental field (with its corresponding dynamics) and for the quantum representation of the corresponding CCRs.

One may wonder whether the uniqueness in the choice of field description can also be guaranteed in other, more general systems than the Gowdy model, and, in particular, for nonstationary settings where there already exist results about the uniqueness of the representation of the CCRs. The case of fields in $1 + 3$ dimensional spacetimes with

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[1]In fact, in this particular linear system one can still introduce a redefinition of the momentum which implies no scaling of the field, but this turns out to be irrelevant inasmuch as no new nonequivalent Fock quantization arises.

compact spatial topology is specially important, owing to its applications, e.g., to cosmology. A summary of the discussion for scalar fields propagating in a nonstationary spacetime with sections of S^3 topology was already presented by us in Ref. [12], anticipating that the answer to the question of uniqueness is in the affirmative. The aim of the present work is to provide full details of the demonstration of this result.

We will consider a scaling of the field by a generic function of time. This scaling can always be completed into a time dependent canonical transformation. We demand such a transformation to be compatible with all linear structures on phase space and with the symmetries of the field equations. Any admissible canonical transformation is then linear and, furthermore, can be divided into two parts. The first one is a linear canonical transformation that is explicitly time independent but takes into account the initial conditions, rendering simple ones for the remaining part, which incorporates then all the time dependence. We will demonstrate that there exists only one possible choice of phase space variables such that the resulting field theory admits a Fock quantization with unitary dynamics and a natural implementation of the symmetries of the field equations. The unique choice which remains available is precisely the one which corresponds to a transformed scalar field that propagates in a static spacetime with S^3 spatial topology, though in the presence of a time varying mass term. Recall that, for this latter field, the uniqueness of the representation of the CCR's was proven in Ref. [11].

As we have already mentioned, the list of scenarios where this result finds direct applications includes the case of inflationary models where a scalar field with constant mass propagates in a Friedmann-Robertson-Walker (FRW) spacetime with compact spatial topology. In this case, one can check that a linear, time dependent canonical transformation allows one to rewrite the field equation as that of a field in a spacetime with identical spatial topology but static, whereas the mass becomes time varying. Another type of situation where our result has implications is given by the quantization of (inhomogeneous) perturbations around nonstationary homogeneous solutions of the Einstein equations, typically cosmological backgrounds [13–17]. Examples are the gauge-invariant energy density perturbation amplitude in a FRW spacetime with S^3 spatial topology filled with a perfect fluid (when the perturbations of the energy-momentum tensor are adiabatic [14,16]) or the matter perturbations around the same FRW spacetime for a massive scalar field [18]. With a suitable scaling (and in an appropriate gauge in the case of the massive field), the corresponding equations of motion can be related to those of a scalar field in a static spacetime with a time dependent mass term (see Ref. [9] for additional details). At this point, it may be worth commenting that, although flat FRW universes receive a special attention in cosmology nowadays, some recent works find reasons to prefer closed FRW models with S^3 topology, for example, from the point of view of perturbation theory in relation with the choice of appropriate gauges which embody Mach's principle [19], or in an attempt to account for a low microwave background quadrupole [20,21]. On the other hand, we will argue later on that our results can be generalized to the case of flat but compact FRW universes.

In summary, the question that we are going to investigate is whether the criteria of unitary dynamics and invariance of the vacuum under the symmetries of the field equation select a unique Fock quantization among all those arising from different time dependent scalings of the field. We will concentrate our discussion on the case that a particular scaling renders the dynamics into that of a scalar field with time varying mass propagating in a static spacetime, with inertial spatial sections that have the topology of a three-sphere. We will study at the quantum level the consequences of local, time dependent canonical transformations which involve a scaling of the field. These transformations must preserve the invariance of the field equation under the group of symmetries and the linearity of the space of solutions. Transformations of this kind consist of a scaling of the configuration variable by a function of time, the inverse scaling of the canonical momentum, and possibly a contribution to the momentum that is linear in the configuration variable, the proportionality factor being time dependent. Our main goal in this work is to provide a full proof demonstrating that the canonical transformation is so severely restricted by our criteria that it turns out to be fixed. In addition, we will argue that the analysis can be generalized to lower dimensions, replacing the three-sphere with S^2 or S^1 (for this last case, see Ref. [9]), as well as to other compact topologies.

The content of the paper is organized as follows. In Sec. II we summarize the results that are already known about the Fock quantization of a scalar field with a time varying mass in a static spacetime whose spatial sections have the topology of S^3. In Sec. III we introduce the linear, time dependent canonical transformation which accounts for the scaling of the field and discuss its consequences at the quantum level. Section IV contains the detailed proof that only one of these canonical transformations leads to a field dynamics which is compatible with our criteria of a quantum unitary evolution and the symmetry invariance of the vacuum. We discuss the results and conclude in Sec. V. Finally, an appendix which deals with some technical parts of our demonstration is added.

II. PRELIMINARIES: THE SYSTEM AND THE FOCK QUANTIZATION OF REFERENCE

Let us start by reviewing some of the key aspects and results about the Fock quantization of a real scalar field ϕ subject to a time dependent potential $V(\phi) = s(t)\phi^2/2$, where $s(t)$ is in principle any regular function of time (conditions on this function will be introduced later on).

The field propagates in a static background in $1 + 3$ dimensions whose Cauchy surfaces are three-spheres, equipped with the standard round metric

$$h_{ab}dx^a dx^b = d\chi^2 + \sin^2(\chi)[d\theta^2 + \sin^2(\theta)d\sigma^2]. \quad (1)$$

Here, χ and θ have a range of π, and $\sigma \in S^1$. The time coordinate t runs over an interval $\mathbb{I}$ of the real line, so that the spacetime has the topology of $\mathbb{I} \times S^3$. Its metric is

$$ds^2 = -dt^2 + h_{ab}dx^a dx^b. \quad (2)$$

In the canonical approach, the dynamics of the system are governed by the equations

$$\dot{P}_\phi = \sqrt{h}[\Delta\phi - s(t)\phi], \qquad \dot{\phi} = \frac{1}{\sqrt{h}}P_\phi, \quad (3)$$

where P_ϕ is the canonical momentum of ϕ, $h = \sin^2(\theta)\sin^4(\chi)$ is the determinant of the metric (1), Δ denotes the Laplace-Beltrami operator on S^3, and the dot stands for the time derivative.

The canonical phase space of the theory is a symplectic linear space Γ coordinatized by the field variables (ϕ, P_ϕ) (evaluated on a particular Cauchy section, e.g., the section $t = t_0$ for a given value of time t_0) and endowed with a symplectic structure Ω such that these variables form a canonical pair, namely, their corresponding Poisson bracket is

$$\{\phi(x), P_\phi(y)\} = \delta^{(3)}(x - y), \quad (4)$$

where the Dirac delta is defined on S^3. The equations of motion (3) amount to the linear wave equation

$$\ddot{\phi} - \Delta\phi + s(t)\phi = 0. \quad (5)$$

Since the Laplace-Beltrami operator on S^3 is invariant under the rotation group SO(4), the above equation is clearly invariant under this group as well. On the other hand, notice by comparison with the Klein-Gordon equation that a nonnegative function $s(t)$ can be interpreted as an effective nonnegative time dependent mass $m(t) = s^{1/2}(t)$.

Owing to the field character of the theory, the system accepts infinite nonequivalent representations of the CCRs. Restricting one's attention to representations of the Fock type, this freedom is encoded in the complex structure, which is a linear symplectic map $j: \Gamma \rightarrow \Gamma$, compatible with the symplectic structure [in the sense that the bilinear map $\Omega(j\cdot, \cdot)$ is positive-definite], and such that $j^2 = -1$ (see, e.g., Refs. [2,22,23]). Different choices of complex structure select distinct, in general not unitarily related, spaces of quantum states for the theory; thus, physical predictions depend on the choice of j.

Actually, as we have mentioned, it has been proven recently that, in the (ϕ, P_ϕ) description, there exists one (and only one) subfamily of equivalent complex structures satisfying the criteria of SO(4) invariance and a unitary

implementation of the dynamics. Let us sketch the main steps of the proof and explain the corresponding quantization [11]. Given the invariance of the field equation under SO(4), it is convenient to expand the field in terms of (hyper-)spherical harmonics Q_{nlm}, where the integer n satisfies $n \geq 0$, and the integers ℓ and m are constrained by $0 \leq \ell \leq n$ and $|m| \leq l$ [13,24,25]. In this basis, the Laplace-Beltrami operator Δ is diagonal with eigenvalues equal to $-n(n + 2)$. Although the (hyper-)spherical harmonics are complex functions, it is straightforward to obtain a real basis from the real and imaginary parts of Q_{nlm}, with which one can directly expand the real field ϕ (see Ref. [11] for details). The degrees of freedom are represented by the coefficients $q_{n\ell m}$ in this expansion, which can be understood as a discrete set of modes. These are functions of time which satisfy the linear equation

$$\ddot{q}_{n\ell m} + [\omega_n^2 + s(t)]q_{n\ell m} = 0, \quad (6)$$

with $\omega_n^2 = n(n + 2)$. Hence, the modes $q_{n\ell m}$ are decoupled from each other. Besides, together with their canonically conjugate momenta $p_{n\ell m} = \dot{q}_{n\ell m}$, they form a complete set of variables in phase space. For each fixed value of n, there exist $g_n = (n + 1)^2$ modes with the same dynamics, because the equation of motion is independent of the labels ℓ and m. Obviously, the quantity g_n is just the dimension of the corresponding eigenspace of the Laplace-Beltrami operator. The canonical phase space Γ can be split then as a direct sum

$$\Gamma = \bigoplus_n \mathcal{Q}_n \oplus \mathcal{P}_n, \quad (7)$$

where $\mathcal{Q}_n$ and $\mathcal{P}_n$ are the respective configuration and momentum subspaces for the modes with fixed n. From now on, we will omit the labels ℓ and m, unless they are necessary in the analysis.

Furthermore, in the following we restrict our study to the inhomogeneous sector, namely, to modes with $n \neq 0$. This does not affect the properties of the system related to its field character, because this is maintained if one removes a finite number of modes.[2] We will introduce the annihilation and creation-like variables

$$\begin{pmatrix} a_n \\ a_n^* \end{pmatrix} = \frac{1}{\sqrt{2\omega_n}}\begin{pmatrix} \omega_n & i \\ \omega_n & -i \end{pmatrix}\begin{pmatrix} q_n \\ p_n \end{pmatrix}. \quad (8)$$

Notice that these are precisely the variables which one would naturally adopt in the case of a free massless scalar field. They form a complete set in the phase space of the inhomogeneous sector. Given a set of initial data $\{a_n(t_0), a_n^*(t_0)\}$ at initial time t_0, it is possible to write the classical evolution to an arbitrary time $t \in \mathbb{I}$ in the form

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$$\begin{pmatrix} a_n(t) \\ a_n^*(t) \end{pmatrix} = \begin{pmatrix} \alpha_n(t,t_0) & \beta_n(t,t_0) \\ \beta_n^*(t,t_0) & \alpha_n^*(t,t_0) \end{pmatrix} \begin{pmatrix} a_n(t_0) \\ a_n^*(t_0) \end{pmatrix}. \qquad (9)$$

Let us call $\mathcal{U}_n(t,t_0)$ the linear evolution operator defined in this way. Since the time functions $\alpha_n(t,t_0)$ and $\beta_n(t,t_0)$ provide a symplectomorphism on Γ, one has

$$|\alpha_n(t,t_0)|^2 - |\beta_n(t,t_0)|^2 = 1, \qquad (10)$$

independently of the particular values of n, t_0, and t. Such Bogoliubov coefficients $\alpha_n(t,t_0)$ and $\beta_n(t,t_0)$ of this evolution map can be determined in the way explained in Ref. [11]. For our present analysis, we only need to employ that their asymptotic behavior when $n \to \infty$ is given by

$$\alpha_n(t,t_0) = e^{-i(n+1)\tau} + O\left(\frac{1}{n}\right), \qquad \beta_n(t,t_0) = O\left(\frac{1}{n^2}\right), \qquad (11)$$

where $\tau = t - t_0$ and the symbol O denotes the asymptotic order. The derivation of this asymptotic behavior makes use of the mild assumption that the function $s(t)$ in Eq. (5) must be differentiable, with a derivative that is integrable in every closed subinterval of $\mathbb{I}$. To simplify the notation, we will omit in the following the reference to the initial time t_0 in the coefficients of the evolution operator and in the initial data, called now $\{a_n, a_n^*\}$.

The SO(4) symmetry of the field equations is imposed at the quantum level by demanding that the complex structure be invariant under this group. We call *invariant* this class of complex structures. By Schur's lemma [26], any invariant complex structure has to be block diagonal with respect to the decomposition of the phase space as the direct sum of the subspaces $\mathcal{Q}_n \oplus \mathcal{P}_n$. In other words, the complex structure can be decomposed as a direct sum $j = \bigoplus_n j_n$, where j_n is an invariant complex structure defined on the n-th subspace of the inhomogeneous sector of Γ. Moreover, a further application of Schur's lemma shows that each of the complex structures j_n is again block diagonal and independent of the labels l and m, so that it can be characterized by a complex structure in two dimensions, describing, e.g., the action on the annihilation and creation-like variables (a_n, a_n^*) for any fixed mode labels l and m (see Ref. [11] for more details).

On the other hand, let us call j_0 the complex structure which in our basis of variables $\{a_n, a_n^*\}$ takes the diagonal form:

$$j_{0n} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \qquad (12)$$

A general complex structure j is related with j_0 via a symplectic transformation $j = \mathcal{K} j_0 \mathcal{K}^{-1}$. Taking into account the form of the invariant complex structures, the symplectic transformation $\mathcal{K}$ must be also block diagonal and independent of the degeneracy labels l and m. We call $\mathcal{K}_n$ the 2×2 block corresponding to the n-th mode, for which we adopt the notation:

$$\mathcal{K}_n = \begin{pmatrix} \kappa_n & \lambda_n \\ \lambda_n^* & \kappa_n^* \end{pmatrix}. \qquad (13)$$

The symbol $*$ denotes again complex conjugation. Here, $|\kappa_n|^2 - |\lambda_n|^2 = 1$ because $\mathcal{K}_n$ is a symplectomorphism. It follows, in particular, that $|\kappa_n| \geq 1 \ \forall \ n \in \mathbb{N}^+$. Note that there exist infinite invariant complex structures. Actually, they are not all unitarily equivalent, so that the imposition of SO(4) symmetry does not eliminate the ambiguity in the Fock quantization on its own.

In order to select a class of equivalent invariant complex structures we need to appeal to additional conditions. A unitary implementation of the classical dynamics at the quantum level turns out to determine a preferred class, and hence a unique Fock quantization up to equivalence. We recall that a symplectic transformation T is implementable as a unitary transformation in the quantum theory for a given complex structure j if and only if $j - TjT^{-1}$ is a Hilbert-Schmidt operator (on the one-particle Hilbert space defined by j, see, e.g., Refs. [3,23]). In the case of the time evolution operator, and choosing the complex structure j_0, this condition is satisfied if and only if

$$\sum_{n\ell m} |\beta_n(t)|^2 = \sum_n g_n |\beta_n(t)|^2 < \infty \quad \forall \ t \in \mathbb{I}, \qquad (14)$$

i.e., if and only if the sequences $\{\sqrt{g_n}\beta_n(t)\}$ are square summable (SQS) for all possible values of time. Then, since $\sqrt{g_n} = n + 1$, the asymptotic behavior (11) of the beta coefficients guarantees the desired summability, ensuring that the dynamics is implemented unitarily in the Fock quantization picked up by j_0, namely, the complex structure associated to the natural choice of annihilation and creation-like variables for the free massless case.

Let us suppose now that we choose a different invariant complex structure j, which can be obtained from j_0 by means of a symplectic transformation $\mathcal{K}$, as we have commented. The unitary implementation of the evolution operator with respect to the new complex structure j is equivalent to the unitary implementation of a transformed evolution operator with respect to the complex structure j_0 [11]. This transformed evolution operator is obtained from the original one by the action of $\mathcal{K}$. Its diagonal blocks are $\mathcal{K}_n \mathcal{U}_n(t) \mathcal{K}_n^{-1}$, with corresponding beta coefficients given by

$$\beta_n^j(t) := (\kappa_n^*)^2 \beta_n(t) - \lambda_n^2 \beta_n^*(t) + 2i\kappa_n^* \lambda_n \Im[\alpha_n(t)]. \qquad (15)$$

Here, the symbol $\Im$ denotes imaginary part. If one assumes that the evolution is unitary in the Fock quantization determined by j, so that the sequences $\{\sqrt{g_n}\beta_n^j(t)\}$ are SQS $\forall \ t \in \mathbb{I}$, one can prove that the sequence $\{\sqrt{g_n}\lambda_n\}$ must be SQS as well [11]. But this summability is precisely the sufficient and necessary condition for the unitary implementation of the symplectic transformation $\mathcal{K}$ in the Fock representation determined by j_0, what amounts to the equivalence of the two complex structures j and j_0.

Therefore, the SO(4) invariance and the requirement of unitary dynamics select a unique equivalence class of complex structures, removing the ambiguity in the choice of representation of the CCRs.

III. THE AMBIGUITY IN THE CHOICE OF FIELD AND THE UNITARITY CRITERION

Although we have succeeded in selecting a preferred representation of the CCRs for the (ϕ, P_ϕ) variables, we can always change from the (ϕ, P_ϕ) description of the phase space to a new canonical description by means of a canonical transformation. Since many canonical transformations fail to be represented by unitary operators quantum mechanically, the classical equivalence of these descriptions may be broken in the quantum arena, originating another type of ambiguity in the quantization. In our case, we are only interested in considering linear, local canonical transformations, which respect the linearity of the field equations and, consequently, the linear nature of the structures of the system. As we have explained in the Introduction, the class of canonical transformations that we want to analyze results in a time dependent scaling of the field. It is this time dependence that makes the transformation nontrivial, otherwise any admissible representation of the original field would provide an admissible one for the transformed field by linearity. But when the canonical transformation is time dependent, the field dynamics changes, affecting the properties of the quantum theory.

A time dependent scaling of the configuration field variable can be regarded as a contact transformation, which can easily be completed into a canonical one. Then, the canonical momentum must experience the inverse scaling and, optionally, may be modified with the addition of a term depending on the configuration field variable, which we restrict to be linear (and local), according to our previous comments. The coefficient in this linear term may vary in time like the rest of coefficients in the linear canonical transformation under consideration. In this way, one obtains a transformation of the form

$$\varphi = F(t)\phi, \qquad P_\varphi = \frac{P_\phi}{F(t)} + G(t)\sqrt{h}\phi. \qquad (16)$$

We recall that the momentum variable is a scalar density of unit weight. This explains the square root of the determinant of the spatial metric appearing in Eq. (16). In order that the transformation does not spoil the differential formulation of the field theory, nor produces singularities, F and G are restricted to be two real and differentiable functions of time, with $F(t)$ different from zero everywhere. Notice also that the homogeneity of F and G preserves the SO(4) invariance of the field dynamics. In the following, we will consider only time dependent canonical transformations of the form (16).

As we have pointed out, different choices of the basic fieldlike variables typically lead to distinct dynamics. For instance, a canonical transformation with $F(t) = 1/a(t)$, where a is a solution of the second order differential equation

$$\frac{\ddot{a}}{a} - m^2 a^2 + s(t) = 0, \qquad (17)$$

leads from the field equation (5) to the dynamics of a Klein-Gordon field with mass m propagating in the FRW background $d\bar{s}^2 = a^2(t)ds^2$ [see Eq. (2)]. This indicates that one can extract information about the dynamics of different field theories by performing a time dependent canonical transformation of the above type. Let us emphasize, however, that with this procedure one is not transforming a given field theory into another one, but rather considering distinct field descriptions of a given physical system, assuming that none of these descriptions is imposed from the start. In this kind of system, one has to address the ambiguity associated with the choice of field parametrization (i.e., with the selection of a fundamental field, together with its associated dynamics). Then it is necessary to invoke additional, physically acceptable criteria to pick up a preferred quantization, otherwise, the significance of the predictions of the quantum theory would be in question. The criteria that we are going to adopt are indeed the same that allow us to select a unique equivalence class of Fock representations in the (ϕ, P_ϕ) representation, that is, the SO(4) invariance and the unitary implementation of the evolution.

For the rest of our analysis, it is convenient to split the time dependent canonical transformation (16) into two parts, one that takes care of the initial conditions on the functions $F(t)$ and $G(t)$, and the other that carries all the time dependence. We fix once and for all an initial reference time t_0, and denote by F_0 and G_0, respectively, the initial values $F(t_0)$ and $G(t_0)$. Then, any transformation of the form (16) can be obtained as the composition of the canonical transformation

$$\tilde{\varphi} = F_0\varphi, \qquad P_{\tilde{\varphi}} = \frac{P_\varphi}{F_0} + G_0\sqrt{h}\varphi, \qquad (18)$$

which does not vary in time, with a linear canonical transformation of the type (16),

$$\varphi = f(t)\phi, \qquad P_\varphi = \frac{P_\phi}{f(t)} + g(t)\sqrt{h}\phi, \qquad (19)$$

but such that the functions $f(t)$ and $g(t)$ now have fixed initial values, namely, $f(t_0) = 1$ and $g(t_0) = 0$. The transformation (18) is just a time independent linear one, with no impact in our discussion, given the linearity of the Fock representations of the CCRs. In fact, if a quantization with SO(4) invariance and unitary dynamics is achieved for the canonical pair (φ, P_φ), one immediately obtains a quantization with the same properties for the transformed pair $(\tilde{\varphi}, P_{\tilde{\varphi}})$. No real ambiguity comes from this kind of transformation, since the quantum representation for the

original and the transformed fields is actually the same (see Ref. [9] for more details).

Thus, we shall restrict our analysis to the family of canonical transformations (19) with fixed initial conditions. We will demonstrate that any such transformation, except the identity, leads to a classical evolution which admits no unitary implementation with respect to any of the Fock representations defined by an SO(4) invariant complex structure. Thus, our criteria fix completely the choice of field description [up to a trivial time independent transformation of the type (18)].

Let us discuss now the form of the new dynamics obtained with the transformation (19), and present the mathematical condition necessary for a unitary implementation of this dynamical evolution. We recall that the linear transformation (19) preserves the SO(4) invariance of the field equations and that we demand that the (real) functions $f(t)$ and $g(t)$ be differentiable. Moreover, [like the functions $F(t)$] the function $f(t)$ is required to differ from zero everywhere. The sign of the function $f(t)$ is therefore constant and, since its initial value has been fixed equal to the unit, in what follows we take $f(t) > 0 \ \forall \ t \in \mathbb{I}$.

As we have already commented, since the canonical transformation (19) depends on time, the classical evolution operator that describes the dynamics of the pair (φ, P_φ) differs from that corresponding to the original pair (ϕ, P_ϕ). In order to describe the new dynamics, we will follow the same procedure adopted in the previous section. Namely, we first expand the field φ and its momentum P_φ in (hyper-)spherical harmonics, extracting in this way their spatial dependence, and then introduce annihilation and creation-like variables, defined in terms of the coefficients of the expansion like in Eq. (8). One can check [using the transformation (19) and the corresponding initial conditions] that, with those variables, the blocks of the original evolution matrix $\mathcal{U}_n(t)$ are replaced by new 2×2 matrices $\tilde{\mathcal{U}}_n(t) = \mathcal{T}_n(t)\mathcal{U}_n(t)$, where[3]

$$\mathcal{T}_n(t) := \begin{pmatrix} f_+(t) + i\frac{g(t)}{2\omega_n} & f_-(t) + i\frac{g(t)}{2\omega_n} \\ f_-(t) - i\frac{g(t)}{2\omega_n} & f_+(t) - i\frac{g(t)}{2\omega_n} \end{pmatrix}, \quad (20)$$

and $2f_\pm(t) := f(t) \pm 1/f(t)$. Finally, a straightforward computation allows us to obtain the Bogoliubov coefficients $\tilde{\alpha}_n(t)$ and $\tilde{\beta}_n(t)$ of the evolution matrices $\tilde{\mathcal{U}}_n(t)$, which are of the form

$$\tilde{\alpha}_n(t) := f_+(t)\alpha_n(t) + f_-(t)\beta_n^*(t) + i\frac{g(t)}{2\omega_n}[\alpha_n(t) + \beta_n^*(t)],$$

$$\tilde{\beta}_n(t) := f_+(t)\beta_n(t) + f_-(t)\alpha_n^*(t) + i\frac{g(t)}{2\omega_n}[\alpha_n^*(t) + \beta_n(t)].$$
$$(21)$$

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[3]While the dependence of $\mathcal{U}_n(t)$ on t_0 is not shown explicitly to simplify the notation, the matrix $\mathcal{T}_n(t)$ actually does not depend on the initial time.

One can now simply follow the procedure explained in Sec. II and write down the condition for a unitary implementation of the dynamics of the transformed canonical pair (φ, P_φ) with respect to a representation of the CCRs defined by an SO(4) invariant complex structure. We again call $\mathcal{K}$ the symplectic transformation that determines the invariant complex structure j under consideration in terms of the complex structure of reference j_0. We also adopt the notation (13) for its coefficients, which do not depend on time. The new dynamics admits a unitary implementation with respect to the representation determined by j if and only if the sequences $\{\sqrt{g_n}\tilde{\beta}_n^J(t)\}$ are SQS $\forall \ t \in \mathbb{I}$, where

$$\tilde{\beta}_n^j(t) := (\kappa_n^*)^2\tilde{\beta}_n(t) - \lambda_n^2\tilde{\beta}_n^*(t) + 2i\kappa_n^*\lambda_n\Im[\tilde{\alpha}_n(t)]. \quad (22)$$

IV. UNIQUENESS IN THE CHOICE OF FIELD DESCRIPTION

We will now present the detailed proof that the unitarity condition introduced in the previous section implies that the transformation (19) must in fact be the identity transformation.

Let us assume that the unitarity condition is satisfied. Then, the sequences $\{\sqrt{g_n}\tilde{\beta}_n^j(t)\}$ are SQS for all values of time in the considered interval $\mathbb{I}$. In particular, this requires that the terms $\sqrt{g_n}\tilde{\beta}_n^j(t)$ of these sequences tend to zero in the limit $n \rightarrow \infty$. Since both g_n and $|\kappa_n|$ are greater than 1, it must be also true that $\tilde{\beta}_n^j(t)/(\kappa_n^*)^2$ tends to zero. Taking into account the asymptotic limits of the Bogoliubov coefficients $\alpha_n(t)$ and $\beta_n(t)$, given in Eq. (11), and introducing for convenience the notation $z_n = \lambda_n/\kappa_n^*$, we arrive at the conclusion that

$$[e^{i(n+1)\tau} - z_n^2e^{-i(n+1)\tau}]f_-(t) - 2iz_n\sin[(n+1)\tau]f_+(t)$$
$$(23)$$

must have a vanishing limit when $n \rightarrow \infty$ for all values of $t \in \mathbb{I}$. Recall that $\tau = t - t_0$.

By considering separately the real and imaginary parts of the above expression, we get that the two sequences given, respectively, by

$$(2\Im[z_n]f_+(t) - \Im[z_n^2]f_-(t))\sin[(n+1)\tau]$$
$$+ (1 - \Re[z_n^2])f_-(t)\cos[(n+1)\tau] \quad (24)$$

and

$$(\{1 + \Re[z_n^2]\}f_-(t) - 2\Re[z_n]f_+(t))\sin[(n+1)\tau]$$
$$- \Im[z_n^2]f_-(t)\cos[(n+1)\tau] \quad (25)$$

have to tend to zero when $n \rightarrow \infty \ \forall \ t \in \mathbb{I}$. Here, the symbol $\Re$ denotes the real part.

We can now apply arguments similar to those presented in Ref. [9] and show that, if it is true that the sequences given in Eq. (25) tend to zero for all values of time, then it is impossible that the two sequences formed by

$$1 - \Re[z_n^2] \quad \text{and} \quad \Im[z_n^2] \qquad (26)$$

have simultaneously a vanishing limit on any (infinite) subsequence of the positive integers $\mathbb{M} \subset \mathbb{N}^+$ (i.e., for $n \in \mathbb{M} \subset \mathbb{N}^+$). Let us see this in more detail.

We first note that $(\Re[z_n])^2$ tends to the unit whenever the two terms in Eq. (26) tend to zero. This can be checked by summing the square of the two terms (26), which gives

$$(1 - |z_n|^2)^2 + 4(\Im[z_n])^2. \qquad (27)$$

By our assumptions, this expression tends to zero on a given subsequence $\mathbb{M}$. Then, we get that $|z_n|$ must tend to the unit and $\Im[z_n]$ to zero on this subsequence, and this implies that the limit of $(\Re[z_n])^2$ is equal to one.

Suppose then that there really exists a particular subsequence $\mathbb{M} \subset \mathbb{N}^+$ such that the terms (26) tend to zero on it for all possible values of time. Since the factor

$$f_-(t) \cos[(n + 1)\tau], \qquad (28)$$

which multiplies $\Im[z_n^2]$ in Eq. (25), is bounded for every particular value of t, it follows that

$$(\{1 + \Re[z_n^2]\}f_-(t) - 2\Re[z_n]f_+(t)) \sin[(n + 1)\tau] \quad (29)$$

must have a vanishing limit on $\mathbb{M} \ \forall \ t \in \mathbb{I}$. Besides, since by hypothesis $1 - \Re[z_n^2]$ tends to zero on $\mathbb{M}$ as well, one further obtains that

$$(f_-(t) - \Re[z_n]f_+(t)) \sin[(n + 1)\tau] \qquad (30)$$

must tend to zero on $\mathbb{M}$ at each possible value of the time t.

We now make use of the result proven above that $(\Re[z_n])^2$ necessarily tends to the unit on $\mathbb{M}$. Then, there exists at least one subsequence $\mathbb{M}' \subset \mathbb{M}$ such that $\Re[z_n]$ tends to 1 or to -1 on $\mathbb{M}'$. In any of these cases, given that $\mathbb{M}'$ is a subsequence of $\mathbb{M}$, and hence expression (30) must tend to zero on $\mathbb{M}'$, we conclude (using the definition of $f_\pm$) that either

$$\sin[(n + 1)\tau]f(t) \quad \text{or} \quad \frac{\sin[(n + 1)\tau]}{f(t)} \qquad (31)$$

(or both) have a vanishing limit on the subsequence $\mathbb{M}' \subset \mathbb{N}^+ \ \forall \ t \in \mathbb{I}$. But, since the function $f(t)$ is continuous and vanishes nowhere, this implies that $\sin[(n + 1)\tau]$ must tend to zero on $\mathbb{M}'$ for all possible values of time in $\mathbb{I}$, or equivalently $\forall \ \tau \in \bar{\mathbb{I}}$, where $\bar{\mathbb{I}}$ is the domain obtained from $\mathbb{I}$ after a shift by the initial time t_0.

Let us finally prove that this limiting behavior is not allowed. Take a positive number L such that $[0, L] \subset \bar{\mathbb{I}}$. We have, in particular, that $\sin^2[(n + 1)\tau]$ tends to zero on $\mathbb{M}' \ \forall \ \tau \in [0, L]$. However, a simple application of the Lebesgue dominated convergence shows that this statement is false. The details are presented in the Appendix. Essentially, one can see that the integral of $\sin^2[(n + 1)\tau]$ over the interval $[0, L]$ is bounded from below by a strictly positive number for large n, something which is incompatible with a vanishing limit for this function in the entire

interval. Therefore, one can exclude the possibility that the two sequences of time independent terms appearing in Eq. (26) can both converge to zero on a subsequence $\mathbb{M}' \subset \mathbb{N}^+$.

We will now use this fact to demonstrate that the function $f(t)$ is necessarily a constant function. Let us study again the real sequences given by Eqs. (24) and (25) which, as we have seen, must necessarily tend to zero in the limit $n \to \infty$ for all possible values of time $t \in \mathbb{I}$ if the dynamics of the (φ, P_φ) canonical pair admits a unitary implementation.

We concentrate our attention on a specific subset of values of the shifted time τ, namely, all values of the form $\tau = 2\pi q/p$ where q and p can be any positive integers, except for the condition that the resulting value of τ belongs to the interval of definition of this variable, $\bar{\mathbb{I}}$. For each value of p, we consider the subsequence of positive integers

$$\mathbb{M}_p := \{n = kp - 1 > 0, k \in \mathbb{N}^+\}. \qquad (32)$$

Given p, the terms (24) and (25) tend to zero on the subsequence $\mathbb{M}_p$ when $n \to \infty$ for all the values of τ reached when q varies. Then, we reach the conclusion that both

$$(1 - \Re[z_{kp-1}^2])f_-\left(t_0 + \frac{2\pi q}{p}\right) \qquad (33)$$

and

$$\Im[z_{kp-1}^2]f_-\left(t_0 + \frac{2\pi q}{p}\right) \qquad (34)$$

must tend to zero as k goes to infinity. The limit must vanish for every possible integer value of p and q. Note, however, that the time independent factors on the left of these expressions are precisely those given in Eq. (26), which we have proven cannot tend simultaneously to zero on any subsequence of the positive integers, e.g., those provided by $\mathbb{M}_p$ for each of the values of p. Therefore, the only possibility left is that the function $f_-(t_0 + 2\pi q/p)$ is equal to zero at all the considered values of p and q. Using the fact that $f(t) > 0 \ \forall \ t \in \mathbb{I}$, the last result amounts to the equality

$$f\left(t_0 + \frac{2\pi q}{p}\right) = 1 \quad \forall \ p, q. \qquad (35)$$

Realizing that the subset of time values $\{t_0 + 2\pi q/p\}$ is dense in $\mathbb{I} \subset \mathbb{R}$ and that the function $f(t)$ is continuous, we are led to the conclusion that $f(t)$ must equal the unit function on its entire domain.

It remains to be proven that the function $g(t)$ in the transformation (19) necessarily vanishes under the condition of unitary dynamics. Note first that the identity $f(t) = 1$ that we have just demonstrated implies that z_n tends to zero when $n \to \infty$. In fact, after introducing this identity in Eq. (23), one sees that the sequences

$\{z_n \sin[(n+1)\tau]\}$ must tend to zero $\forall \tau \in \bar{\mathbb{I}}$. Therefore, in order to avoid again the false conclusion that $\sin^2[(n+1)\tau]$ tends to zero on some subsequence of the positive integers for all values of τ in a compact interval, it is necessary that the complex sequence z_n has a vanishing limit. Taking into account that $|\kappa_n|^2 = |\lambda_n|^2 + 1$, it is straightforward to check that the sequence formed by the coefficients λ_n must tend to zero, and that $1/|\kappa_n|^2$ (and $|\kappa_n|^2$) approaches the unit in the limit of large n, what implies, in particular, that the sequence given by κ_n is bounded.

To complete the proof that $g(t)$ vanishes, we consider again the sequences $\{\sqrt{g_n}\tilde{\beta}_n^j(t)\}$, particularized now to the only value allowed for the function $f(t)$, namely, the identity, so that $f_+(t) = 1$ and $f_-(t) = 0$. Employing the definition of the coefficient $\beta_n^j(t)$, given in Eq. (15), one can check that the leading terms in $\tilde{\beta}_n^j(t)$ are

$$\tilde{\beta}_n^{\,j}(t) \cong \beta_n^j(t) + i\frac{g(t)}{2\omega_n}[(\kappa_n^*)^2\alpha_n^*(t) + \lambda_n^2\alpha_n(t)]. \quad (36)$$

The condition of unitarity demands that $\sqrt{g_n}\tilde{\beta}_n^j(t)$ tend to zero in the limit $n \to \infty$ at all values of time, and therefore the same must happen to the sequences with terms $\sqrt{g_n}\tilde{\beta}_n^j(t)/\kappa_n^{*2}$. Using this condition and taking into account the known asymptotic behavior (11) of $\alpha_n(t)$ and $\beta_n(t)$, as well that λ_n tends to zero and $\sqrt{g_n}/\omega_n$ tends to the unit for large n, a simple calculation leads to the result that the sequences given by

$$g(t) - 4z_n\omega_n \sin[(n+1)\tau]e^{-i(n+1)\tau} \quad (37)$$

must have a vanishing limit $\forall t \in \mathbb{I}$. We then consider the real and imaginary parts of these sequences, namely,

$$g(t) - 4|z_n|\omega_n \sin[(n+1)\tau]\cos[(n+1)\tau - \delta_n] \quad (38)$$

and

$$4|z_n|\omega_n \sin[(n+1)\tau]\sin[(n+1)\tau - \delta_n], \quad (39)$$

where we have written the complex numbers z_n in terms of its phase and complex norm:

$$z_n = |z_n|e^{i\delta_n}. \quad (40)$$

Although we already know that $|z_n|$ tends to zero, the limit of the product $|z_n|\omega_n$ is still undetermined, because ω_n grows like n at infinity.

Let us suppose first that the sequence $\{|z_n|\omega_n\}$ tends to zero. In this case, recalling that the sequences given in Eq. (38) should tend to zero $\forall t \in \mathbb{I}$, it follows immediately that $g(t)$ must be the zero function on $\mathbb{I}$, as we wanted to prove. Finally, let us demonstrate that the alternate possibility, i.e., the hypothesis that $\{|z_n|\omega_n\}$ does not tend to zero, leads to a contradiction. We make use of the fact that the sequences formed by the terms (39) tend to zero $\forall t$. If $\{|z_n|\omega_n\}$ does not tend to zero, there must exists a subsequence $\mathbb{M}$ of the positive integers such that the (posi-

tive) sequence $\{|z_n|\omega_n\}$ is bounded from below on $\mathbb{M}$. Thus, on that subsequence,

$$\sin[(n+1)\tau]\sin[(n+1)\tau - \delta_n] \quad (41)$$

must necessarily have a zero limit $\forall \tau \in \mathbb{I}$. But, as shown also in the Appendix, this last statement can never be true. Again the crucial argument involves the application of the Lebesgue dominated convergence.

As a result, the only function $g(t)$ that is allowed by the condition of unitarity is the zero function. In total, we have demonstrated that the only canonical transformation of the type (19) which is permitted once one accepts the unitarity criterion is the trivial one, i.e., the identity transformation. In this way, the choice of a field parametrization for the system turns out to be completely fixed (up to irrelevant constant scalings) by the requirements of invariance under the symmetry group of the field equations, SO(4), and the unitary implementation of the dynamics. The ambiguity in the selection of a field description is totally removed.

V. CONCLUSIONS AND DISCUSSION

In this work, we have begun our analysis by reviewing the Fock quantization of a scalar field with a time varying mass in a static background, in which the inertial spatial sections have S^3 topology. For this particular scenario, we have seen that the criteria of (i) invariance of the vacuum under the SO(4) symmetry of the field equations; and (ii) unitary implementation of the field dynamics, are sufficient to select a unique Fock representation of the CCR's.

An additional question concerns the possibility of changing the field description if one allows for a scaling of the field by time dependent functions. This is a situation frequently found in cosmology, where it is common to introduce scalings of the fields in order to absorb part of the time dependence of the cosmological background. The prototypical example is that of fields in a FRW spacetime with compact topology (S^3 for our discussion), or the closely related scenario of field perturbations around a FRW background of that kind. There is, therefore, an extra ambiguity affecting the quantization of such systems, namely, the choice of the field description, which necessarily affects the dynamics.

In the above mentioned systems, it is generally the case that a time dependent scaling of the field renders the field equations into a form describing the effective propagation in a static background with a time varying mass, i.e., the model that we considered initially. We have demonstrated here the result that we anticipated in Ref. [12], namely, that our criteria of symmetry invariance and unitary evolution allow only for one admissible field description among all those that can be reached by means of time dependent canonical transformations that include a time dependent scaling of the field. The analyzed canonical transformations are linear, in order to maintain the linearity

of all the structures on phase space, and preserve the symmetry of the field equations.

To arrive at this uniqueness result, very mild requirements have been imposed on the mass function $s(t)$ appearing in the field equation (5). Specifically, the only condition that has been assumed is that the mass function has a first derivative which is integrable in all closed subintervals of the domain of definition. In addition, if one wants that the zero mode of the scalar field (the homogeneous sector) can be quantized consistently in the standard Schrödinger representation with the Lebesgue measure (on $\mathbb{R}$), an extra condition has to be added: the mass $s(t)$ has to be *nonnegative* for all possible values of time.

Let us comment on some key points underlying our uniqueness result. A fundamental question is to understand why one can reach unitarity in the quantum evolution and how this unitarity selects a unique field description as well as a unique equivalence class of complex structures for it, among the set of all symmetry invariant complex structures. In this respect, we first notice that infrared divergences are not an issue to begin with, owing to the fact that the spatial sections have compact topology (leading, in particular, to a discrete spectrum for the Laplace-Beltrami operator). Like for many other considerations in cosmology, the compactness of the spatial sections is essential. When the spatial topology is not compact, the infrared problem appears and changes the scenario drastically.[4] On the other hand, the ultraviolet divergences are absent in the system precisely because we are using an appropriate representation of the CCRs. This representation turns out to be the one naturally associated with a free massless scalar field. The reason is that, in the asymptotic limit of large wave numbers, which are the relevant modes for the ultraviolet regime, the behavior of the system (when the field is properly scaled) approaches sufficiently fast the behavior of a massless field. Only Fock quantizations (with the desired invariance) which are equivalent to the one that we have chosen, keep this good ultraviolet property. In this way, one obtains a single family of unitarily equivalent Fock quantizations which incorporate the symmetries of the field equation and respect the unitarity in the evolution. Concerning the choice of field description, let us also note that nonstationary spacetimes give rise to damping terms (first order time derivatives of the scalar field) in the equations of motion. In relation with our previous comments, such contributions spoil the unitary implementation of the dynamics at the quantum level. Fortunately, a suitable scaling of the field relegates all the information about the nonstationarity of the system to the (effective) mass term.

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[4]For instance, the well-known inequivalence of the quantum representations corresponding to free scalar fields of different masses in Minkowski spacetime is precisely due to the long range behavior of the quantum fields. See Ref. [27] for an account.

Let us see this last point in some more detail. As we have explained, in order to quantize a scalar field in a nonstationary setting along the lines presented in this paper, one generically performs a canonical transformation which involves a time dependent scaling of the field, so that the transformed field effectively propagates in a static background. Let us call φ and $\tilde{s}(t)$, respectively, the field and its time dependent mass previous to the discussed transformation. As commented above, the corresponding field equation contains a damping term, which is linear in $\dot{\varphi}$. We call $r(t)$ the function multiplying $\dot{\varphi}$ in this damping contribution. We now want to give the explicit expressions of the time dependent scaling factor, $F(t)$ [see Eq. (16)], and of the mass function $s(t)$ for the field $\phi = \varphi/F(t)$. A straightforward calculation shows that

$$F(t) = F_0 \exp\left[-\int_{t_0}^{t} d\tau \frac{r(\tau)}{2}\right],$$
$$s(t) = \tilde{s}(t) - \frac{[r^2(t) + 2\dot{r}(t)]}{4}. \tag{42}$$

We also note that the condition imposed on $s(t)$ for the validity of our uniqueness result is met, for instance, if $\tilde{s}(t)$ satisfies the same condition and $r(t)$ has a second derivative which is integrable in all compact subintervals of the time domain $\mathbb{I}$. On the other hand, the positivity of the mass function (for a standard quantization of the homogeneous sector) amounts just to

$$\tilde{s}(t) \geq \frac{[r^2(t) + 2\dot{r}(t)]}{4} \quad \forall\, t \in \mathbb{I}. \tag{43}$$

Let us now address possible generalizations of our results, starting with the case of scalar fields in different compact spatial manifolds. The analysis carried out here, together with the dimensional arguments explained in Ref. [11] in relation to the uniqueness of the representation of the CCRs for the field description selected by our criteria, strongly indicate that the results that we have achieved for the three-sphere can be extended to other compact spatial manifolds provided that the spatial dimension d is equal or smaller than 3. Suppose that, in these cases, the representation of the symmetry group of the field equation is irreducible in each of the eigenspaces of the Laplace-Beltrami operator [like it happens for SO(4) in the case of the three-sphere]. This property is actually *sufficient* (though not necessary) to characterize the invariant complex structures in a block diagonal form similar to that discussed in this work. One can then follow the same kind of steps that have allowed us to complete the proof of uniqueness, reaching analogous conclusions.

In all the cases with $d \leq 3$, our arguments therefore support the expectation that, when one adopts the scalar field description with propagation in a static background, the free massless representation provides the unique (equivalence class of) Fock quantization that satisfies our criteria of symmetry invariance and unitary dynamics [11].

Besides, our criteria are expected to fix again the function $f(t)$ in the canonical transformations of the type (19). This ensures that there is no ambiguity in the scaling of the field, either. The only freedom remaining in the canonical transformation is given by the function $g(t)$. It is not difficult to realize, repeating the arguments discussed here, that whether or not the function $g(t)$ is fixed to vanish depends on the square summability of the sequence $\{\sqrt{g_n}/\omega_n\}$. If the sequence is not SQS, as it happens for the cases of the two-sphere and the three-sphere, the function $g(t)$ must vanish. However, if the sequence is SQS, there exists an arbitrariness and our criteria do not determine the definition of the momentum P_φ completely. For instance, this is the case of the circle S^1 [9]. It is worth pointing out that, nevertheless, this freedom has nothing to do with the scaling of the field, leaving intact the time evolution. If a choice of momentum and of invariant complex structure permits a unitary implementation of the dynamics, the same complex structure leads to a unitary evolution for any other admissible choice of the momentum canonically conjugate to the field. In other words, this freedom to change the momentum by adding a time dependent contribution linear in the configuration field variable, when available, does not allow one to reach a new representation satisfying our criteria.

As we have explained in the Introduction, a framework where our results find a natural application is in the quantization of (inhomogeneous) perturbations around a closed FRW spacetime. In this context, the simplest system is a scalar field coupled to a homogeneous and isotropic universe with compact spatial sections. This system is specially relevant in cosmology. On the one hand, the considered perturbations provide the seeds for structure formation. On the other hand, those perturbations explain the anisotropies imprinted in the power spectrum of the cosmic microwave background. Our criteria to eliminate the quantization ambiguities can now be applied in their quantum treatment and the subsequent analysis of the power spectrum.

Although the discussion that we have carried out has been focused on scalar fields, there does not seem to exist any technical or conceptual obstacle to extend the analysis to other kind of fields, applying to them our criteria in order to pick up a unique Fock quantization. For instance, an interesting case is provided by the traceless and divergenceless tensor perturbations of the metric around a FRW spacetime with compact spatial topology. These tensor perturbations describe gravitational waves. The primordial gravitational waves generated in the early universe can also contribute to the power spectrum of the cosmic microwave background, in the form of tensor modes. In fact, with a convenient scaling and in conformal time, these tensor perturbations satisfy again equations of motion like those for a free field with a time dependent quadratic potential in a static spacetime when the perturbations of the energy-momentum tensor are isotropic (see Ref. [14]). Let us mention also the case of fermionic fields. The study of the perturbations around a closed FRW spacetime produced by fermions of constant mass was carried out in Ref. [28], where a quantization was achieved after expanding the perturbations in spinor harmonics on the three-sphere. Preliminary calculations indicate that the kind of techniques employed here can be extended to deal as well with the uniqueness of the Fock quantization for fermions. It is worth emphasizing that the criteria for this uniqueness are the natural implementation of the symmetries of the field equations and the unitarity of the evolution. For cases other than the scalar field (and gravitational waves, as noticed above), these criteria may not necessarily imply that the selected field description corresponds to a field propagating in a stationary background.

The Fock quantization of fields in the context of modern approaches to quantum cosmology is another interesting framework where our results can have applications. One of the most promising approaches is what nowadays is called Loop Quantum Cosmology [29–31]. LQC employs the techniques of Loop Quantum Gravity [32–34] in the study of models of interest in cosmology, obtained from general relativity by the imposition of certain symmetries. In the specific case of a FRW spacetime coupled to a scalar field (see Refs. [35–37]), where homogeneity and isotropy are imposed, LQC predicts that the classical big bang singularity is replaced by a big bounce, which connects the observed branch of the Universe with a previous branch in the evolution. For semiclassical states with certain properties [38], the evolution is peaked around a trajectory which shows a behavior different from the classical one in Einstein's theory. Then, one could use such a trajectory to define an effective, *quantum corrected* background. If inhomogeneous matter fields are introduced, their scaling by background functions would then provide a different time dependent scaling with respect to the conventional case in general relativity. At the quantum level, the combination of the use of loop techniques for the homogeneous background with a standard Fock quantization of the inhomogeneous fields, which propagate in it, is known in the literature as *hybrid quantization* [39–42]. This quantization procedure assumes that the most relevant quantum geometry effects (characteristic of Loop Quantum Gravity) are those that affect the homogeneous degrees of freedom of the gravitational field. A family of systems in which the application of such a quantization procedure seems most natural is the already commented case of perturbations around FRW spacetimes. Our analysis provides a unique Fock quantization for those perturbations, which ought to be recovered from LQC (either with a hybrid or with a genuinely loop quantization) in regimes in which the behavior of the degrees of freedom of the background can be described satisfactorily by an effective trajectory.

In conclusion, we have completed the demonstration that a scalar field with time dependent mass in a FRW spacetime with compact spatial sections admits (within the considered infinite family of possibilities that respect the linear structure of the theory, and up to equivalence) a unique Fock quantization where the vacuum is invariant under the symmetries of the field equation and the dynamical evolution is unitary. In this sense, it is not only that one does not have to renounce to unitarity in the context of quantum cosmology, but, furthermore, the requirement of unitarity has the remarkable counterpart of selecting a unique Fock description among all those that incorporate the symmetry invariance of the system.

ACKNOWLEDGMENTS

APPENDIX: APPLICATION OF THE LEBESGUE DOMINATED CONVERGENCE

In Sec. IV we used the fact that $\sin^2[(n+1)\tau]$, or more generally $\sin[(n+1)\tau]\sin[(n+1)\tau - \delta_n]$ (from which the previous case is recovered by taking $\delta_n = 0$), cannot tend to zero in the limit $n \to \infty$ on any subsequence of the positive integers $\forall \tau \in \bar{\mathbb{I}}$. In this appendix we are going to prove an even more general result. Let

$$\mathbb{U} = \{u_n, n \in \mathbb{N}^+\} \tag{A1}$$

be a monotonous and diverging sequence of positive real numbers, i.e., $u_{n+1} > u_n \; \forall \, n \in \mathbb{N}^+$, with $\{u_n\}$ unbounded. Let also

$$\mathbb{D} = \{\delta_n, n \in \mathbb{N}^+\} \tag{A2}$$

be a sequence of phases [namely, real numbers, identified modulo 2π], and let $L > 0$ be an arbitrary positive number. Then, the sequences of values

$$x_n(\tau) = \sin(u_n \tau) \sin(u_n \tau - \delta_n) \tag{A3}$$

cannot tend to zero $\forall \, \tau \in [0, L]$.

To prove this, we assume from the start that the sequence formed by $\cos(\delta_n)$ with $\delta_n \in \mathbb{D}$ does not tend to zero when n tends to infinity. We will show below that there is no loss of generality in making this assumption. A straightforward computation shows that

$$\int_0^L x_n(\tau) d\tau = \frac{L}{2} \cos(\delta_n) - \cos(u_n L - \delta_n) \frac{\sin(u_n L)}{2u_n}. \tag{A4}$$

Taking into account the range of the trigonometric functions, we get the following bounds, valid for all positive integers n:

$$\frac{L}{2} \cos(\delta_n) + \frac{1}{2u_n} \geq \int_0^L x_n(\tau) d\tau \geq \frac{L}{2} \cos(\delta_n) - \frac{1}{2u_n}. \tag{A5}$$

Given that $\cos(\delta_n)$ does not tend to zero, there exists a subsequence $\mathbb{M}'' \subset \mathbb{N}^+$ and a number $\Delta > 0$ such that $|\cos(\delta_n)| \geq \Delta, \; \forall \, n \in \mathbb{M}''$. Thus, there exists a subsequence $\mathbb{M}' \subset \mathbb{M}''$ such that

$$\cos(\delta_n) \geq \Delta \qquad \forall \, n \in \mathbb{M}', \tag{A6}$$

or

$$\cos(\delta_n) \leq -\Delta \qquad \forall \, n \in \mathbb{M}', \tag{A7}$$

(both types of sequences may exist).

Let us consider for the moment the first case (A6). Since the sequence of positive numbers $1/u_n$ (with $u_n \in \mathbb{U}$) tends to zero for large n, one can find a positive integer $n_0 \in \mathbb{M}'$ such that $L\Delta > 1/u_{n_0}$. Moreover, since $u_{n+1} > u_n$ in $\mathbb{U}$, one gets from the second inequality in Eq. (A5) that the considered integral is bounded from below on the given sequence by a positive number:

$$\int_0^L x_n(\tau) d\tau \geq \frac{L\Delta}{2} - \frac{1}{2u_{n_0}} > 0, \qquad \forall \, n > n_0, \qquad n \in \mathbb{M}'. \tag{A8}$$

It is clear that the second possibility, i.e., the existence of a sequence $\mathbb{M}'$ such that $\cos(\delta_n) \leq -\Delta$, leads to a negative upper bound by similar arguments. Taking into account the two possibilities, we conclude that [assuming that $\cos(\delta_n)$ does not tend to zero] there exist positive numbers Δ, $n_0 \in \mathbb{N}^+$, and $M = L\Delta - 1/u_{n_0}$, as well as a subsequence $\mathbb{M} \subset \mathbb{N}^+$ such that

$$\left| \int_0^L x_n(\tau) d\tau \right| \geq \frac{M}{2} \quad \forall \, n \in \mathbb{M}. \tag{A9}$$

The sequence $\mathbb{M}$ is formed by those elements of $\mathbb{M}'$ such that $n > n_0$.

Suppose now that the sequence of functions $x_n(\tau)$ converges to the zero function on $[0, L]$. Since the functions $|x_n(\tau)|$ are obviously bounded from above by the constant unit function, we are in the conditions of Lebesgue dominated convergence [1], and it follows that the sequence of integrals $\int_0^L x_n(\tau) d\tau$ must converge to the integral of the zero function, i.e., to zero. But this conclusion is obviously contradicted by the bound obtained in Eq. (A9). Therefore, it is not possible that the values of $x_n(\tau)$ converge to zero $\forall \, \tau \in [0, L]$.

To conclude the proof, it only remains to consider the situation in which $\cos(\delta_n)$ tends to zero for large n. In that case, the vanishing limit of $x_n(\tau)$ implies that $\sin(2u_n\tau)$ must tend to zero, and therefore so must $\sin^2(2u_n\tau)$. But this last situation is covered by the proof presented above.

It suffices to make all the phases δ_n identically null, and identify the sequence $\{2u_n\}$ as the new sequence $\mathbb{U}$, since the positive real numbers $2u_n$ form a monotonous and diverging sequence if the u_n's do as well.

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