Sadek Gala

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UNIQUENESS OF WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

Sadek Gala, Mostaganem

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Abstract. Consider the Navier-Stokes equation with the initial data $a \in L^2_\sigma(\mathbb{R}^d)$. Let u and v be two weak solutions with the same initial value a. If u satisfies the usual energy inequality and if $\nabla v \in L^2((0,T); \dot{X}_1(\mathbb{R}^d)^d)$ where $\dot{X}_1(\mathbb{R}^d)$ is the multiplier space, then we have $u = v$.

Keywords: Navier-Stokes equations, solution uniqueness, weak Leray-Hopf solution, multiplier space

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1. Introduction

Consider the Navier-Stokes equation in $(0,T) \times \mathbb{R}^d$ with $0 < T < \infty$ and $d \geqslant 3$:

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p &= 0, \qquad (x,t) \in \mathbb{R}^d \times (0,\infty), \\ \nabla \cdot u &= 0, \qquad (x,t) \in \mathbb{R}^d \times (0,\infty), \\ u(x,0) &= a(x), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$ is the velocity field, $p = p(x,t)$ is the scalar pressure and $a(x)$ with $\operatorname{div} a = 0$ in the sense of distributions is the initial velocity field. For simplicity, we assume that the external force has a scalar potential and is included in the pressure gradient.

In their famous paper, Leray [11] and Hopf [5] constructed a weak solution u of (1.1) for arbitrary $a \in L^2_\sigma$. The solution is called the Leray-Hopf weak solution. In the general case the problem of uniqueness of Leray-Hopf's weak solutions is still an open question. Masuda [13] extended Serrin's class for uniqueness of weak solutions and made it clear that the class $L^\infty((0,T);L^d(\mathbb{R}^d))$ plays an important role for the uniqueness of weak solutions. Kozono-Sohr [7] showed that uniqueness holds in $L^\infty((0,T);L^d)$.

Foias [3] and Serrin [14] introduced the class $L^\alpha((0,\infty);L^q)$ and showed that under the additional assumption

$$u \in L^\alpha((0,\infty);L^q) \quad \text{for } \frac{2}{\alpha}+\frac{d}{q}=1 \quad \text{with } q>d,$$

u is the only weak solution.

The purpose of this paper is to improve the criterion on uniqueness of weak solutions to the class $L^2((0,T);\dot{X}_1(\mathbb{R}^d)^d)$ (Definition 2). We know that for every $a \in L^2_\sigma(\mathbb{R}^d)$ there is at least one weak solution u of (1.1) satisfying the energy inequality. Here we mean by the weak solution a function $u \in L^\infty((0,T);L^2_\sigma) \cap L^2((0,T);\dot{H}^1_\sigma)$ satisfying (1.1) in the sense of distributions (Definition 4). For more facts concerning uniqueness of weak solutions we refer to the celebrated paper of Kozono and Sohr [7].

1.1. BMO and Hardy space $\mathcal{H}^1(\mathbb{R}^d)$

We recall that a locally summable function g on $\mathbb{R}^d$ is said to have bounded mean oscillation if

$$\|g\|_{BMO} = \sup_{x,R} \frac{1}{|B(x,R)|}\int_{B(x,R)} |g(y)-g_{B(x,R)}|\,\mathrm{d}y < \infty,$$

where

$$g_{B(x,R)} = \frac{1}{|B(x,R)|}\int_{B(x,R)} g(y)\,\mathrm{d}y.$$

The class of functions of bounded mean oscillation is denoted by BMO and often is referred to as the John-Nirenberg space.

Note that

$$\|g\|_{BMO}=0 \quad \text{if and only if} \quad g=\text{const}.$$

It is thus natural to consider the quotient space $BMO/\mathbb{R}$ with the norm induced by $\|\cdot\|_{BMO}$. Then $BMO/\mathbb{R}$ is a Banach space, which will also be denoted BMO for simplicity. We easily see that $L^\infty \subset BMO$ with continuous injection. For $f(x)=\log|x|$ we have $f\in BMO$ but $f \notin L^\infty$, so BMO is strictly larger than L^∞.

Next, we recall the definition and some of the main properties of Hardy spaces $\mathcal{H}^p(\mathbb{R}^d)$ introduced by E. Stein and G. Weiss [16] (for more facts on these spaces see C. Fefferman and E. Stein [4]).

Definition 1 ([4]). Let $0 < p < \infty$ and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^n} \varphi \, \mathrm{d}x = 1$. A tempered distribution f belongs to the Hardy space $\mathcal{H}^p(\mathbb{R}^d)$ if

$$f^*(x) = \sup_{t>0} |(\varphi_t * f)(x)| \in L^p(\mathbb{R}^d), \tag{1.2}$$

where $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$.

It is known that if $f \in \mathcal{H}^p(\mathbb{R}^d)$, then (1.2) holds for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \varphi \, \mathrm{d}x = 1$. The (quasi)-norm of $\mathcal{H}^p(\mathbb{R}^d)$ is defined, up to equivalence, by

$$\|f\|_{\mathcal{H}^p(\mathbb{R}^d)} = \|f^*(x)\|_{L^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f^*(x)|^p \, \mathrm{d}x\right)^{1/p}.$$

We know by [4], [15] that if $1 \leqslant p < \infty$, then $\mathcal{H}^p$ is a Banach space:

$$\begin{aligned} \mathcal{H}^p(\mathbb{R}^d) &= L^p(\mathbb{R}^d) \quad \text{for } 1 < p < \infty, \\ \mathcal{H}^1(\mathbb{R}^d) &\subset L^1(\mathbb{R}^d) \quad \text{with continuous injection,} \end{aligned}$$

and that $\mathcal{H}^p(\mathbb{R}^d)$, $0 < p < 1$, are quasi-Banach spaces in the quasi-norm $\|\cdot\|_{\mathcal{H}^p(\mathbb{R}^d)}$.

The crucial fact for our purpose is the boundedness of the Riesz transforms R_j on all of the spaces $\mathcal{H}^p$. Furthermore, an L^1-function f on $\mathbb{R}^d$ belongs to $\mathcal{H}^1(\mathbb{R}^d)$ if and only if its Riesz transforms $R_j f$ all belong to $L^1(\mathbb{R}^d)$ and

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^d)} \cong \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|R_j f\|_{L^1(\mathbb{R}^d)} \quad \text{(equivalent norms).}$$

Notice that all functions $f \in \mathcal{H}^1(\mathbb{R}^d)$ satisfy

$$\int_{\mathbb{R}^d} f(x) \, \mathrm{d}x = 0. \tag{1.3}$$

Indeed, the assumption $f \in \mathcal{H}^1(\mathbb{R}^d)$ implies that the Fourier transforms

$$\hat{f}(\xi) = \int f(x) \mathrm{e}^{-\mathrm{i}x\xi} \, \mathrm{d}x \quad \text{and} \quad \widehat{R_j f}(\xi) = \frac{\mathrm{i}\xi_j}{|\xi|} \hat{f}(\xi) \quad (j = 1, \ldots, d),$$

are all continuous on $\mathbb{R}^d$, so $\hat{f}(0) = 0$ and (1.3) is proved.

A fundamental theorem in the theory of Hardy spaces $\mathcal{H}^1(\mathbb{R}^d)$ developed by C. Fefferman and E. Stein [4] asserts:

Theorem 1 (Fefferman). *The dual space of $\mathcal{H}^1(\mathbb{R}^d)$ is BMO. More precisely, L is a continuous linear functional on $\mathcal{H}^1(\mathbb{R}^d)$ if and only if it can be represented as*

$$L(f) = \int_{\mathbb{R}^d} fg$$

for some function g in BMO. Moreover, for any $g \in BMO$ and any $f \in \mathcal{H}^1(\mathbb{R}^d)$ we have

$$\left| \int_{\mathbb{R}^d} fg \, \mathrm{d}x \right| \leqslant c(d) \|f\|_{\mathcal{H}^1} \|g\|_{BMO}. \tag{1.4}$$

Let $\gamma > 1$. We define the maximal function of f depending on γ,

$$M_\gamma f(x) = \sup_{t>0} \left(\frac{1}{|B_t(x)|} \int_{B_t(x)} |f(y)|^\gamma \, \mathrm{d}y \right)^{1/\gamma}.$$

We begin by establishing the following result which is a variant of the Hardy-Littlewood maximal theorem. We need

Lemma 1. *If $\gamma < p \leqslant \infty$, then*

$$M_\gamma \colon L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \quad \textit{is bounded.}$$

P r o o f. See [15]. □

In [1], Coifman, Lions, Meyer and Semmes showed that the Hardy spaces can be used to analyze the regularity of various nonlinear quantities by the compensated compactness theory due to L. Murat [12] and F. Tartar [17]. Since then, these spaces play an important role in studying the regularity of solutions to partial differential equations. In particular, it was shown that for exponents p, q with $1 < p < \infty$, $1/p + 1/q = 1$, and vector fields $u \in L^p(\mathbb{R}^d)^d$, $v \in L^q(\mathbb{R}^d)^d$ with $\operatorname{div} u = 0$, $\operatorname{curl} v = 0$ in the sense of distributions, the scalar product $u \cdot v$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^d)$. Moreover, there exists a positive constant C such that

$$\|u \cdot v\|_{\mathcal{H}^1(\mathbb{R}^d)} \leqslant C \|u\|_{L^p} \|v\|_{L^q}.$$

The main purpose of this subsection is to prove two facts about the div-curl lemma without imposing any a priori assumptions on exact cancellation, namely, the divergence and curl need not be zero. Our results will lead to $\operatorname{div}(uv)$ being in the Hardy space $\mathcal{H}^1(\mathbb{R}^d)$.

The proof will be divided into two parts. In part 1, we consider the case u and v being supported on the ball $|x| \leqslant R_0$, where $R_0 > 1$ is a positive constant to be determined later, while in Part 2, the general case follows by partition of unity. In order to simplify the presentation, we take $p = q = 2$.

The Sobolev space $H^1_p(\mathbb{R}^d)$, $1 \leqslant p < \infty$, consists of functions $f \in L^p(\mathbb{R}^d)$ such that $|\nabla f| \in L^p(\mathbb{R}^d)$. It is a Banach space with respect to the norm

$$\|f\|_{H^1_p} = \|f\|_{L^p} + \|\nabla f\|_{L^p}.$$

Specifically, we will prove

Theorem 2. *Let $u \in H^1_p(\mathbb{R}^d)^d$ and $v \in H^1_q(\mathbb{R}^d)$, $p > 1$, $1/p + 1/q = 1$. Then there exists a positive constant $C(d)$ such that*

$$\|\mathrm{div}(uv)\|_{\mathcal{H}^1(\mathbb{R}^d)} \leqslant C(\|u\|_{L^p}\|\nabla v\|_{L^q} + \|\mathrm{div}\, u\|_{L^p}\|v\|_{L^q}). \tag{1.5}$$

Remark 1. Such inequalities and their generalizations are useful in hydrodynamics. The reader is referred, in particular, to [1], [2].

Theorem 2 is a generalized version of the "div-curl" lemma ([1], Theorem II.1). Observe that when $\mathrm{div}\, u = 0$, Theorem 2 reduces to the classical div-curl lemma [1].

The following result due to [1] shows the importance of the Hardy space theory in estimating the non-linear term $u \cdot \nabla v$ attached to the Navier-Stokes equations. This produces a useful tool for PDE.

Lemma 2. *Let $1 < p < \infty$, $1 < q < d$ and $1/r = 1/p + 1/q < 1/d + 1$. If $u \in L^p(\mathbb{R}^d)^d$ with $\nabla \cdot u = 0$ and $\nabla v \in L^q(\mathbb{R}^d)$, then*

$$u \cdot \nabla v \in \mathcal{H}^r(\mathbb{R}^d)$$

and

$$\|u \cdot \nabla v\|_{\mathcal{H}^r(\mathbb{R}^d)} \leqslant C\|u\|_{L^p}\|\nabla v\|_{L^q}.$$

Proof. The result is due to [1]; but we give here a detailed proof for the reader's convenience. Observe that

$$f = u \cdot \nabla v = \nabla \cdot (u \otimes (v - c))$$

for an arbitrary constant vector c. So we get

$$(\varphi_t * f)(x) = t^{-d-1} \int_{B_t(x)} (\nabla\varphi)(t^{-1}(x-y))u(y)(v(y) - m_B(v))\,\mathrm{d}y,$$

where

$$m_B(v) = \frac{1}{|B_t(x)|} \int_{B_t(x)} v(y)\,\mathrm{d}y.$$

Taking

$$1 < \gamma < \infty, \quad 1 < \beta < d \quad \text{with} \quad \frac{1}{\gamma} + \frac{1}{\beta} = 1 + \frac{1}{d}$$

and writing

$$\frac{1}{\beta^*} = \frac{1}{\beta} - \frac{1}{d},$$

we see by the Poincaré-Sobolev inequality that

$$\begin{aligned}
|(\varphi_t * f)(x)| &\leqslant \frac{C}{t^{d+1}} \left(\int_{B_t(x)} |u(y)|^\gamma \,\mathrm{d}y \right)^{1/\gamma} \left(\int_{B_t(x)} |v(y) - m_B(v)|^{\beta^*} \,\mathrm{d}y \right)^{1/\beta^*} \\
&\leqslant \frac{C}{t^{d+1}} \left(\int_{B_t(x)} |u(y)|^\gamma \,\mathrm{d}y \right)^{1/\gamma} \left(\int_{B_t(x)} |\nabla v(y)|^\beta \,\mathrm{d}y \right)^{1/\beta} \\
&= C \left(\frac{1}{|B_t(x)|} \int_{B_t(x)} |u(y)|^\gamma \,\mathrm{d}y \right)^{1/\gamma} \left(\frac{1}{|B_t(x)|} \int_{B_t(x)} |\nabla v(y)|^\beta \,\mathrm{d}y \right)^{1/\beta} \\
&\leqslant C (M_\gamma u)(x) \cdot (M_\beta(\nabla v))(x).
\end{aligned}$$

We thus obtain

$$\sup_{t>0} |(\varphi_t * f)(x)| \leqslant C (M_\gamma u)(x) \cdot (M_\beta(\nabla v))(x).$$

Since we can take γ and β such that

$$1 < \gamma < p, \quad 1 < \beta < q < d,$$

it follows from Lemma 1 that

$$\|M_\gamma u\|_{L^p} \leqslant C \|u\|_{L^p}, \quad \|M_\beta(\nabla v)\|_{L^q} \leqslant C \|\nabla v\|_{L^q}.$$

Lemma 2 now follows from Hölder's inequality

$$\|f \cdot g\|_{L^r} \leqslant \|f\|_{L^p} \|g\|_{L^q} \quad \Big(0 < p < \infty,\ 0 < q < \infty,\ \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \Big).$$

This completes the proof of the lemma. □

We are now in position to prove Theorem 2.

P r o o f of Theorem 2. We distinguish three cases.

Case A. Let us assume first that

$$\nabla \cdot u = 0.$$

In this case we get

$$\operatorname{div}(vu) = (\nabla v) \cdot u + v \operatorname{div} u = u \cdot \nabla v.$$

Then we have $u \in L^p(\mathbb{R}^d)^d$, $\nabla v \in L^q(\mathbb{R}^d)$ with $\operatorname{div} u = 0$, $\operatorname{curl}(\nabla v) = 0$ in the sense of distributions. It follows from Lemma 2 that

$$u \cdot \nabla v \in \mathcal{H}^1(\mathbb{R}^d)$$

and there exists an absolute constant C such that

$$\|\operatorname{div}(vu)\|_{\mathcal{H}^1(\mathbb{R}^d)} \leqslant C\|u\|_{L^p}\|\nabla v\|_{L^q}.$$

Case B. We may of course introduce an additional assumption that u and v are supported on the ball $|x| \leqslant R_0$. In order to simplify the presentation, we take $p = q = 2$. We shall write Ω for the ball in $\mathbb{R}^d$ of radius R_0 centered at the origin. By $H_0^1(\Omega)$ we denote the closed subspace of $H^1(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ in the H^1 norm. Let

$$g = \operatorname{div} u \in L^2(\mathbb{R}^d).$$

By the classical result (see e.g. [18]) we know that

$$g = \partial_1 g_1 + \ldots + \partial_d g_d,$$

where $g_1, \ldots, g_d$ belong to $H_0^1(\Omega)$. Set

$$G = (g_1, \ldots, g_d) \quad \text{and} \quad r = u - G.$$

Then it follows that

$$\operatorname{div} r = 0 \quad \text{and} \quad r \in L^2(\Omega).$$

Using Lemma 2 we infer

$$\operatorname{div}(rv) \in \mathcal{H}^1(\mathbb{R}^d).$$

Further we set

$$f = \operatorname{div}(Gv).$$

Using Lemma 3 below, we conclude that $f \in \mathcal{H}^1(\mathbb{R}^d)$.

Case C. The general case. We call φ a smooth bump function with compact support provided

$$1 = \sum_{k \in \mathbb{Z}^d} \varphi^2(x-k).$$

If f and g are two functions, we thus have

$$f(x)g(x) = \sum_{k \in \mathbb{Z}^d} f(x)\varphi^2(x-k)g(x) = \sum_{k \in \mathbb{Z}^d} f_k(x)g_k(x),$$

where

$$f_k(x) = \varphi(x-k)f(x) \quad \text{and} \quad g_k(x) = \varphi(x-k)g(x).$$

Now set

$$u_k(x) = \varphi(x-k)u(x) \quad \text{and} \quad v_k(x) = \varphi(x-k)v(x)$$

for $k \in \mathbb{Z}^d$. We then have

$$\operatorname{div}(uv) = \sum_{k \in \mathbb{Z}^d} (u_k v_k) = \sum_{k \in \mathbb{Z}^d} w_k, \quad w_k = \operatorname{div}(u_k v_k).$$

We are going to check that

$$\sum_{k \in \mathbb{Z}^d} \|w_k\|_{\mathcal{H}^1(\mathbb{R}^d)} < \infty.$$

To do this, we apply the local version (Case A). It follows that

$$\begin{aligned} \|w_k\|_{\mathcal{H}^1(\mathbb{R}^n)} &\leqslant C(\|u_k\|_{L^2} + \|\operatorname{div} u_k\|_{L^2})(\|v_k\|_{L^2} + \|\operatorname{div} v_k\|_{L^2}) \\ &= \varepsilon_k \in l^1(\mathbb{Z}^d), \end{aligned}$$

where

$$\varepsilon_k = C(\|u_k\|_{L^2} + \|\operatorname{div} u_k\|_{L^2})(\|v_k\|_{L^2} + \|\operatorname{div} v_k\|_{L^2}).$$

Up to now we have proved

$$\|\operatorname{div}(uv)\|_{\mathcal{H}^1(\mathbb{R}^d)} \leqslant C(\|u\|_{L^2} + \|\operatorname{div} u\|_{L^2})(\|v\|_{L^2} + \|\operatorname{div} v\|_{L^2}). \tag{1.6}$$

This automatically yields the estimate

$$\|\operatorname{div}(uv)\|_{\mathcal{H}^1(\mathbb{R}^d)} \leqslant C(\|u\|_{L^2}\|\nabla v\|_{L^2} + \|v\|_{L^2}\|\operatorname{div} u\|_{L^2}). \tag{1.7}$$

To see this, we may replace u in the above inequality by

$$u = \delta^{1/2-d/2} u\Big(\frac{x}{\delta}\Big) \quad \text{whenever } 0 < \delta < \infty,$$

and similarly v by

$$v_\delta = \delta^{1/2-d/2} v\Big(\frac{x}{\delta}\Big) \quad \text{whenever } 0 < \delta < \infty.$$

Thus the left-hand side of (1.6) fortunately does not change, while on the right-hand side we get rid of the undesirable terms by letting δ to be equal either to 0 or to $+\infty$. This completes the proof. □

Now we turn to the proof of Lemma 3. One can show that every function $f \in L^p(\mathbb{R}^d)$, $p \in (1, +\infty]$, with compact support and $\int f \,\mathrm{d}x = 0$ belongs to $\mathcal{H}^1(\mathbb{R}^d)$. In particular, we have

Lemma 3. *If $d^* = d/(d-1)$, $f \in L^{d^*}$, $\operatorname{supp} f \subset \overline{\Omega}$ and*

$$\int f \,\mathrm{d}x = 0,$$

then $f \in \mathcal{H}^1(\mathbb{R}^d)$.

Proof. We have

$$f = \operatorname{div}(G)v + G \cdot \nabla v$$

and we have to prove that both the terms belong to L^{d^*}. We consider the first term on the right-hand side. Since $\nabla v \in L^2$, we have

$$\operatorname{div}(G) \in L^2 \quad \text{and} \quad v \in L^q \quad \text{where } \frac{1}{2} - \frac{1}{q} = \frac{1}{d}.$$

Thus,

$$v \operatorname{div}(G) \in L^{d^*}.$$

A similar argument works on the second term and this completes the proof of the lemma. □

1.2. Multipliers and Morrey-Campanato spaces

In this section we give a description of the multiplier space $\dot{X}_r$ introduced recently by P. G. Lemarié-Rieusset in his work [9] (see also [10]). The space $\dot{X}_r$ of pointwise multipliers which map L^2 into $\dot{H}^{-r}$ is defined in the following way.

Definition 2. For $0 \leqslant r < d/2$ we define the homogeneous space $\dot{X}_r$ by

$$\dot{X}_r = \{f \in L^2_{\mathrm{loc}} \colon \forall\, g \in \dot{H}^r \ fg \in L^2\},$$

where we denote by $\dot{H}^r(\mathbb{R}^d)$ the completion of the space $\mathcal{D}(\mathbb{R}^d)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{r/2} u\|_{L^2}$.

The norm of $\dot{X}_r$ is given by the operator norm of pointwise multiplication

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \leqslant 1} \|fg\|_{L^2}.$$

Similarly, we define the nonhomogeneous space X_r for $0 \leqslant r < d/2$ equipped with the norm

$$\|f\|_{X_r} = \sup_{\|g\|_{H^r} \leqslant 1} \|fg\|_{L^2}.$$

We have the homogeneity properties : $\forall\, x_0 \in \mathbb{R}^d$

$$\begin{aligned} \|f(x+x_0)\|_{X_r} &= \|f\|_{X_r}, \\ \|f(x+x_0)\|_{\dot{X}_r} &= \|f\|_{\dot{X}_r}, \\ \|f(\lambda x)\|_{X_r} &\leqslant \frac{1}{\lambda^r}\|f\|_{X_r}, \quad 0 < \lambda \leqslant 1, \\ \|f(\lambda x)\|_{\dot{X}_r} &\leqslant \frac{1}{\lambda^r}\|f\|_{\dot{X}_r}, \quad \lambda > 0. \end{aligned}$$

The imbeddings

$$\begin{aligned} &L^{d/t} \subset X_r, \quad 0 \leqslant r < \frac{d}{2}, \ 0 \leqslant t \leqslant r, \\ &L^{d/r} \subset \dot{X}_r, \quad 0 \leqslant r < \frac{d}{2} \end{aligned}$$

hold.

E x a m p l e 1. If $u(x) \in \mathcal{D}(\mathbb{R}^d)$, $\varphi(x) = \Big(\sum\limits_{k=1}^{d} |x_k|^{\gamma_k}\Big)^{-1}$, $\gamma_k > 0$, $d > 2$ and $\sum\limits_{k=1}^{d} \gamma_k^{-1} = d/2$, then

$$\int_{\mathbb{R}^d} \varphi(x)|u(x)|^2 \,\mathrm{d}x \leqslant C \int_{\mathbb{R}^d} |\nabla u(x)|^2 \,\mathrm{d}x$$

and $\varphi \in X_1 t(\mathbb{R}^d)$.

Indeed, the inequality

$$\begin{aligned} &\int_{\lambda<|x|<2\lambda} \varphi(x)|u(x)|^2 \,\mathrm{d}x \\ &\qquad \leqslant \Big[\int_{\lambda<|x|<2\lambda} |u(x)|^{2d/(d-2)} \,\mathrm{d}x\Big]^{(d-2)/2} \Big[\int_{\lambda<|x|<2\lambda} \varphi(x)^{d/2} \,\mathrm{d}x\Big]^{2/d} \end{aligned}$$

and the Sobolev theorem imply that for $\lambda > 0$

$$\int_{\lambda<|x|<2\lambda} \varphi(x)|u(x)|^2\,\mathrm{d}x \leqslant C\Big[\int_{\lambda<|x|<2\lambda} |\nabla u(x)|^2\,\mathrm{d}x + \int_{\lambda<|x|<2\lambda} \frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x\Big] \times \Big[\int_{\lambda<|x|<2\lambda} \varphi(x)^{d/2}\,\mathrm{d}x\Big]^{2/d},$$

where C does not depend on λ. Let us estimate the integral

$$S(\lambda) = \int_{\lambda<|x|<2\lambda} \varphi(x)^{d/2}\,\mathrm{d}x.$$

The domain $\lambda < |x| < 2\lambda$ can be represented as a finite sum of domains $\Omega_{j\lambda}$ such that $|x_j| > \frac{1}{2}\lambda$ if $x \in \Omega_{j\lambda}$ for $j = 1, \ldots, d$. Let for instance $|x_1| > \frac{1}{2}\lambda$. Then

$$\int_{\Omega_{j\lambda}} \varphi(x)^{d/2}\,\mathrm{d}x \leqslant \frac{3}{2}\lambda \int_{\lambda<|x|<2\lambda} \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_d}{((\frac{1}{2}\lambda)^{\gamma_1} + |x_2|^{\gamma_2} + \ldots + |x_d|^{\gamma_d})^{d/2}}.$$

The substitution $x_j = t_j(\frac{1}{2}\lambda)^{\gamma_1/\gamma_j}$ gives the relations

$$S(\lambda) \leqslant C \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_d}{(1 + |t_2|^{\gamma_2} + \ldots + |t_d|^{\gamma_d})^{d/2}} \leqslant C,$$

since the integral is converging. To see this, set $t_s = \tau_s^{1/\gamma_s}$. Then

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}} & \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_d}{(1 + |t_2|^{\gamma_2} + \ldots + |t_d|^{\gamma_d})^{d/2}} \\
& \leqslant C \int_{\mathbb{R}^{d-1}} \frac{|\tau|^{1/\gamma_2+\ldots+1/\gamma_d-(d-1)}}{(1+|\tau|)^{d/2}}\,\mathrm{d}\tau_1 \ldots \mathrm{d}\tau_d \\
& \leqslant C \int_0^\infty \frac{\mathrm{d}|\tau|}{(1+|\tau|)^{1/\gamma+1}} < \infty.
\end{aligned}$$

Therefore,

$$\int_{\lambda<|x|<2\lambda} \varphi(x)|u(x)|^2\,\mathrm{d}x \leqslant C_5\Big[\int_{\lambda<|x|<2\lambda} |u(x)|^2\,\mathrm{d}x + \int_{\lambda<|x|<2\lambda} \frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x\Big].$$

Setting $\lambda = 2^m$, $m \in \mathbb{Z}$ and assuming these inequalities for all m, we obtain that

$$\int_{\mathbb{R}^d} \varphi(x)|u(x)|^2\,\mathrm{d}x \leqslant C\Big(\int_{\mathbb{R}^d} |\nabla u(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x\Big).$$

By Hardy's inequality in $\mathbb{R}^d$, $d \geqslant 3$, we have

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x \leqslant \frac{4}{(d-2)^2}\int_{\mathbb{R}^d} |\nabla u(x)|^2\,\mathrm{d}x, \quad u(x) \in \mathcal{D}(\mathbb{R}^d),$$

and hence

$$\int_{\mathbb{R}^d} \varphi(x)|u(x)|^2\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^d} |\nabla u(x)|^2\,\mathrm{d}x.$$

Now we recall the definition of the Morrey-Campanato spaces ([6], [19]):

Definition 3. For $1 < p \leqslant q \leqslant +\infty$, the Morrey-Campanato space $\mathcal{M}_{p,q}$ is defined by

$$(1.8)\quad \mathcal{M}_{p,q} = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)\colon \|f\|_{\mathcal{M}_{p,q}} = \sup_{x\in\mathbb{R}^d}\sup_{0<R\leqslant 1} R^{d/q-d/p}\|f(y)1_{B(x,R)}(y)\|_{L^p(\mathrm{d}y)} < \infty\}.$$

Let us define the homogeneous Morrey-Campanato spaces $\dot{\mathcal{M}}_{p,q}$ for $1 < p \leqslant q \leqslant +\infty$ by

$$(1.9)\qquad \|f\|_{\dot{\mathcal{M}}_{p,q}} = \sup_{x\in\mathbb{R}^d}\sup_{R>0} R^{d/q-d/p}\left(\int_{B(x,R)} |f(y)|^p\,\mathrm{d}y\right)^{1/p}.$$

It is easy to check the following properties:

$$\|f(\lambda x)\|_{\mathcal{M}_{p,q}} = \frac{1}{\lambda^{d/q}\|f\|_{\mathcal{M}_{p,q}}}, \quad 0 < \lambda \leqslant 1,$$
$$\|f(\lambda x)\|_{\dot{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^{d/q}}\|f\|_{\dot{\mathcal{M}}_{p,q}}, \quad \lambda > 0.$$

We shall use the following classical results [6].

a) For $1 \leqslant p \leqslant p'$, $p \leqslant q \leqslant +\infty$ and for all functions f such that $f \in \dot{\mathcal{M}}_{p,q} \cap L^\infty$ we have

$$\|f\|_{\dot{\mathcal{M}}_{p',q\cdot(p'/p)}} \leqslant \|f\|_{L^\infty}^{1-p/p'}\|f\|_{\dot{\mathcal{M}}_{p,q}}^{p/p'}.$$

b) For p, q, p', q' such that $1/p + 1/p' \leqslant 1$, $1/q + 1/q' \leqslant 1$, $f \in \dot{\mathcal{M}}_{p,q}$, $g \in \dot{\mathcal{M}}_{p',q'}$ we have

$$fg \in \dot{\mathcal{M}}_{p'',q''} \quad \text{with} \quad 1/p + 1/p' = 1/p'',\ 1/q + 1/q' = 1/q''.$$

c) For $1 \leqslant p \leqslant d$ we have

$$\forall\, \lambda > 0,\ \|\lambda f(\lambda x)\|_{\dot{\mathcal{M}}_{p,d}} = \|f\|_{\dot{\mathcal{M}}_{p,d}}.$$

d) If $p' < p$ then

$$\dot{\mathcal{M}}_{p,q} \subset \mathcal{M}_{p,q},$$
$$\dot{\mathcal{M}}_{p,q} \subset \mathcal{M}_{p',q}.$$

e) If $q_2 < q_1$ then

$$\mathcal{M}_{p,q_1} \subset \mathcal{M}_{p,q_2},$$
$$L^q = \dot{\mathcal{M}}_{q,q} \subset \dot{\mathcal{M}}_{p,q}, \quad p \leqslant q.$$

We have the following comparison between multipliers and Morrey-Campanato spaces:

Proposition 1. *For $0 \leqslant r < d/2$, we have*

$$X_r \subseteq \mathcal{M}_{2,d/r},$$
$$\dot{X}_r \subseteq \dot{\mathcal{M}}_{2,d/r}.$$

Proof. Let $f \in X_r$, $0 < R \leqslant 1$, $x_0 \in \mathbb{R}^d$ and $\phi \in \mathcal{D}$, $\phi \equiv 1$ on $B(x_0/R, 1)$. We have

$$\begin{aligned} R^{r-d/2}\left(\int_{|x-x_0|\leqslant R} |f(x)|^2 \,\mathrm{d}x\right)^{1/2} &= R^r\left(\int_{|y-x_0/R|\leqslant 1} |f(Ry)|^2 \,\mathrm{d}y\right)^{1/2} \\ &\leqslant R^r\left(\int_{y\in\mathbb{R}^d} |f(Ry)\phi(y)|^2 \,\mathrm{d}y\right)^{1/2} \\ &\leqslant R^r \|f(Ry)\|_{X_r} \|\phi\|_{H^r} \\ &\leqslant \|f(y)\|_{X_r} \|\phi\|_{H^r} \leqslant C\|f(y)\|_{X_r}. \end{aligned}$$

We observe that the same proof is also valid for homogeneous spaces. □

Additionally, for $2 < p \leqslant d/r$ and $0 \leqslant r < d/2$ we have the following inclusion relations:

$$L^{d/r}(\mathbb{R}^d) \subset L^{d/r,\infty}(\mathbb{R}^d) \subset \dot{\mathcal{M}}_{p,d/r}(\mathbb{R}^d) \subset \dot{X}_r(\mathbb{R}^d) \subset \dot{\mathcal{M}}_{2,d/r}(\mathbb{R}^d),$$

where $L^{p,\infty}$ denotes the usual Lorentz (weak L^p) space. For the definition and basic properties of Lorentz spaces $L^{p,q}$ we refer to [16].

2. Uniqueness theorem

Before turning our attention to uniqueness issues, we start with some prerequisites for our main result. Let

$$C^\infty_{0,\sigma}(\mathbb{R}^d)=\{\varphi\in(C^\infty_0(\mathbb{R}^d))^d\colon\ \operatorname{div}\varphi=0\}\subseteq(C^\infty_0(\mathbb{R}^d))^d.$$

The subspace

$$L^2_\sigma(\mathbb{R}^d)=\overline{C^\infty_{0,\sigma}(\mathbb{R}^d)}^{\|\cdot\|_{L^2}}=\{u\in L^2(\mathbb{R}^d)^d\colon\ \operatorname{div}u=0\}$$

is obtained as the closure of $C^\infty_{0,\sigma}$ with respect to the L^2-norm $\|\cdot\|_{L^2}$. H^r_σ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the norm

$$\|u\|_{H^r}=\|u\|_{L^2}+\|(1-\Delta)^{r/2}u\|_{L^2}\quad\text{for } r\geqslant 0.$$

Our definition of the Leray-Hopf weak solutions (see e.g., [8], [7]) now reads:

Definition 4 (weak solutions). Let $a\in L^2_\sigma$ and $T>0$. A measurable function u is called a weak solution of (1.1) on $(0,T)$ if u has the following properties:

1. $u\in L^\infty((0,T);L^2_\sigma)\cap L^2((0,T);\dot H^1_\sigma)$ for all $T>0$;
2. $u(t)$ is continuous in time in the weak topology of L^2_σ with

$$\langle u(t),\phi\rangle\to\langle a,\phi\rangle\quad\text{as } t\to 0^+$$

for all $\phi\in L^2_\sigma$;

3. for any $0\leqslant s\leqslant t\leqslant T$, u satisfies the identity

$$\int_s^t\{-\langle u,\partial_\tau\phi\rangle+\langle u\cdot\nabla u,\phi\rangle+\langle\nabla u,\nabla\phi\rangle\}\,\mathrm{d}\tau=-\langle u(t),\phi(t)\rangle+\langle u(s),\phi(s)\rangle \tag{2.1}$$

for all $\phi\in H^1((s,t);H^1_\sigma)$. Here $\langle\cdot,\cdot\rangle$ denotes the scalar product and $\|\cdot\|_{L^2}$ denotes the norm in $L^2(\mathbb{R}^d)^d$.

Remark 2. For u and ϕ as above, the integral

$$\int_0^T\langle u\cdot\nabla u,\phi\rangle\,\mathrm{d}\tau$$

is well defined since by the Sobolev inequality we have

$$\|u\|_{L^{2d/(d-2)}}\leqslant C\|\nabla u\|_{L^2}$$

so that

$$\left|\int_0^T \langle u\cdot\nabla u,\phi\rangle\,\mathrm{d}\tau\right| \leqslant \int_0^T \|u\|_{L^{2d/(d-2)}}\|\nabla u\|_{L^2}\|\phi\|_{L^d}\,\mathrm{d}\tau$$
$$\leqslant C\sup_{0<t<T}\|\phi\|_{L^d}\int_0^T \|\nabla u\|_{L^2}^2\,\mathrm{d}\tau.$$

Existence of weak solutions has been established by Leray in [11] for the initial velocity in $L^2_\sigma(\mathbb{R}^d)$. The result is the following

Theorem 3 (Leray-Hopf). *Let $T>0$. Then, for any given $a\in L^2_\sigma(\mathbb{R}^d)$, there exits at least one weak solution u to* (1.1) *on* $(0,T)$ *such that*

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2\,\mathrm{d}s \leqslant \|a\|_{L^2}^2,\quad 0\leqslant t<T \tag{2.2}$$

and

$$\|u(t)-a\|_{L^2}\to 0\quad \text{as } t\to +0.$$

Let us introduce the class $L^s((0,T);L^\gamma)$ with the norm $\|\cdot\|_{L^s((0,T);L^\gamma)}$ by

$$\|u\|_{L^s((0,T);L^\gamma)} = \left(\int_0^T \|u(t)\|_{L^\gamma}^s\,\mathrm{d}t\right)^{1/s}.$$

The classical result on uniqueness of weak solutions in the class $L^s((0,T);L^\gamma)$ was given by Foias, Serrin and Masuda [3], [14], [13].

Theorem 4 (Foias-Serrin-Masuda). *Let $a\in L^2_\sigma(\mathbb{R}^d)$. Let u and v be two weak solutions of* (1.1) *on* $(0,T)$. *Suppose that u satisfies*

$$u\in L^s((0,T);L^\gamma)\quad \text{for } \frac{2}{s}+\frac{d}{\gamma}=1\quad \text{with } d<\gamma<\infty. \tag{2.3}$$

Assume that v fulfils the energy inequality (2.2) *for $0\leqslant t<T$. Then we have $u=v$ on $[0,T)$.*

R e m a r k 3. In Theorem 4, v need not belong to the class (2.3). On the other hand, every weak solution u with (2.3) fulfils the energy identity

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2\,\mathrm{d}s = \|a\|_{L^2}^2,\quad 0\leqslant t\leqslant T. \tag{2.4}$$

It seems to be an interesting question whether every weak solution satisfies the energy inequality (2.2).

R e m a r k 4. The class (2.3) is important from the view point of scaling invariance for the Navier-Stokes equations. It can be easily seen that if (u, p) is a pair of the solution to (1.1) on $\mathbb{R}^d \times (0,T)$, then so is the family $\{u_\lambda, p_\lambda\}_{\lambda>0}$ where

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t).$$

Scaling invariance means that

$$\|u_\lambda\|_{L^s((0,\infty);L^\gamma)} = (\lambda^{1-(2/s+d/\gamma)}\|u\|_{L^s((0,\infty);L^\gamma)}) = \|u\|_{L^s((0,\infty);L^\gamma)} \quad \text{for all } \lambda > 0$$

holds if and only if

$$\frac{2}{s} + \frac{d}{\gamma} = 1.$$

We shall next deal with the critical case with $s = \infty$ and $\gamma = d$ in (2.3).

Theorem 5 (Masuda [13], Kozono-Sohr [7]). *Let $a \in L^2_\sigma(\mathbb{R}^d)$. Let u and v be two weak solutions of* (1.1) *on* $(0,T)$. *Suppose that*

$$u \in L^\infty((0,T); L^d) \tag{2.5}$$

and that v fulfils the energy inequality (2.2) *for all* $0 \leqslant t < T$. *Then we have $u = v$ on* $[0,T)$.

R e m a r k 5. Masuda [13] proved that if $u \in L^\infty((0,T); L^d)$ is continuous from the right on $[0,T)$ in the norm of L^d, then $u = v$ holds on $[0,T)$. Later on, Kozono-Sohr [7] showed that every weak solution in $L^\infty((0,T); L^d)$ of (1.1) on $(0,T)$ becomes necessarily continuous from the right in the norm of L^d.

The same result holds when for $\gamma = +\infty$ we replace the assumption

$$u \in L^2((0,T); L^\infty)$$

by the weaker assumption

$$u \in L^2((0,T); BMO(\mathbb{R}^d)^d).$$

The replacement of the hypothesis $u \in L^2((0,T); L^\infty)$ by $u \in L^2((0,T); BMO(\mathbb{R}^d)^d)$ was recently discussed in a similar context by Kozono and Taniuchi [8]. Moreover, we have

Theorem 6 (Kozono-Taniuchi). *Let $a \in L^2_\sigma(\mathbb{R}^d)$ and let u, v be two weak solutions of* (1.1) *on* $(0,T)$. *Suppose that*

$$u \in L^2((0,T); BMO(\mathbb{R}^d)^d) \tag{2.6}$$

and that v fulfils the energy inequality (2.2) *for $0 \leqslant t < T$. Then we have $u = v$ on $[0,T]$.*

R e m a r k 6. By Theorem 4, every weak solution in $L^2((0,T); L^\infty)$ is unique.

Our result on uniqueness of the weak solution now reads:

Theorem 7. *Let $a \in L^2(\mathbb{R}^d)^d$ with $\nabla \cdot a = 0$. Assume that there exists a solution u for the Navier-Stokes equation on $(0,T) \times \mathbb{R}^d$ (for some $T \in (0,+\infty]$) with the initial data a so that*

$$u \in L^\infty((0,T); L^2_\sigma(\mathbb{R}^d)^d) \cap L^2((0,T); \dot{H}^1_\sigma(\mathbb{R}^d)^d)$$

and

$$\nabla u \in L^2((0,T); \dot{X}_1(\mathbb{R}^d)^d).$$

Then u is the unique Leray-Hopf solution associated with a on $[0,T)$.

The same result holds when the assumption $\nabla u \in L^2((0,T); \dot{X}_1(\mathbb{R}^d)^d)$ is replaced by $u \in L^2((0,T); BMO(\mathbb{R}^d)^d)$.

The following corollary, which is an immediate consequence of Theorem 7, gives a simpler sufficient condition in terms of the Lorentz spaces.

Corollary 1. *Let $a \in L^2(\mathbb{R}^d)^d$ with $\nabla \cdot a = 0$. Assume that there exists a solution u for the Navier-Stokes equation on $(0,T) \times \mathbb{R}^d$ (for some $T \in (0,+\infty]$) with initial data a so that*

$$u \in L^\infty((0,T); L^2_\sigma(\mathbb{R}^d)^d) \cap L^2((0,T); \dot{H}^1_\sigma(\mathbb{R}^d)^d)$$

and

$$\nabla u \in L^2((0,T); L^{d,\infty}(\mathbb{R}^d)^d),$$

where $L^{p,\infty}$ denotes the usual Lorentz (weak L^p) space. Then u is the unique Leray-Hopf solution associated with a on $[0,T)$.

The same result again holds when the assumption $\nabla u \in L^2((0,T); L^{d,\infty}(\mathbb{R}^d)^d)$ is replaced by $u \in L^2((0,T); L^d(\mathbb{R}^d)^d)$.

The following lemmas play a fundamental role in estimating the nonlinear term.

Lemma 4. *Let $f \in H^1(\mathbb{R}^d)$, $g(x) = (g_i(x))_{i=1}^d$ with $\nabla \cdot g = 0$ and $g \in L^2(\mathbb{R}^d)^d$. Furthermore, we assume that $\nabla h \in \dot{X}_1(\mathbb{R}^d)$. Then there exists a constant $C(d) > 0$ independent of f, g and h such that*

$$\left|\int_{\mathbb{R}^d} fg \cdot \nabla h \,\mathrm{d}x\right| \leqslant C\|\nabla f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)^d}\|\nabla h\|_{\dot{X}_1(\mathbb{R}^d)} \tag{2.7}$$

and

$$\left|\int_{\mathbb{R}^d} \nabla f \cdot gh \,\mathrm{d}x\right| \leqslant C\|\nabla f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)^d}\|\nabla h\|_{\dot{X}_1(\mathbb{R}^d)}. \tag{2.8}$$

Proof. The proof is easy, due to the definition of $\dot{X}_1(\mathbb{R}^d)$. Supposing that $\nabla h \in \dot{X}_1(\mathbb{R}^d)$ and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^d} fg \cdot \nabla h \,\mathrm{d}x\right| &\leqslant \left(\int_{\mathbb{R}^d} |f|^2|\nabla h|^2 \,\mathrm{d}x\right)^{1/2} \|g\|_{L^2(\mathbb{R}^d)^d} \\ &\leqslant C\|\nabla h\|_{\dot{X}_1(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d} |\nabla f|^2 \,\mathrm{d}x\right)^{1/2} \|g\|_{L^2(\mathbb{R}^d)^d},\end{aligned}$$

where the constant C is independent of f, g and h. Thus the lemma is proved in the case of (2.7). The proof is similar in the case of (2.8). □

The same result holds when we replace the assumption $\nabla h \in \dot{X}_1(\mathbb{R}^d)$ by the assumption $h \in H^1(\mathbb{R}^d) \cap BMO(\mathbb{R}^d)$. Indeed, we known that

$$h(x) = \log|x| \in BMO$$

and

$$|\nabla h|^2 \leqslant \frac{1}{|x|^2},$$

hence by Hardy's inequality in $\mathbb{R}^d$ $(d \geqslant 3)$ we have

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \,\mathrm{d}x \leqslant C(d) \int_{\mathbb{R}^d} |\nabla f|^2 \,\mathrm{d}x, \quad \forall f \in H^1(\mathbb{R}^d).$$

This remark suggests that the lemma will also hold when we replace the $\dot{X}_1(\mathbb{R}^d)$-norm of ∇h by the BMO-norm of h. In fact, the following is a combination of the compensated compactness results of Coifman, Lions, Meyer and Semmes [1] and the duality of the space BMO.

Lemma 5. *Let $f \in H^1(\mathbb{R}^d)$, $g = (g_i(x))_{i=1}^d$ with $\nabla \cdot g = 0$ and $g \in L^2(\mathbb{R}^d)^d$ and let $h \in H^1(\mathbb{R}^d) \cap BMO(\mathbb{R}^d)$. Then there exists a constant $C(d) > 0$ independent of f, g and h such that*

$$|\langle g \cdot \nabla f, h\rangle| \leqslant C\|\nabla f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)^d}\|h\|_{BMO(\mathbb{R}^d)}. \tag{2.9}$$

P r o o f. It is an immediate consequence of Lemma 2 and the duality inequality (1.4):

$$\begin{aligned}|\langle g \cdot \nabla f, h\rangle| &\leqslant C\|g \cdot \nabla f\|_{\mathcal{H}^1(\mathbb{R}^d)}\|h\|_{BMO(\mathbb{R}^d)}\\ &\leqslant C\|\nabla f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)^d}\|h\|_{BMO(\mathbb{R}^d)}.\end{aligned}$$

□

Next we recall the following well-known result:

Lemma 6 (Poincaré inequality). *Suppose Q is a cube in $\mathbb{R}^d$ of side length ϱ and f is C^2 on Q with $\nabla f \in L^2(Q)$. There exists c independent of f such that*

$$\int_Q |f - m_Q f|^2 \,\mathrm{d}y \leqslant c\varrho^2 \int_Q |\nabla f(y)|^2 \,\mathrm{d}y, \tag{2.10}$$

where $m_Q f = 1/|Q| \int_Q f(y)\,\mathrm{d}y$ is the integral mean of f on Q.

Combining this result with Proposition 1 gives

Proposition 2. *If $f \in H^1(\mathbb{R}^d)$ and $\nabla f \in \dot{X}_1(\mathbb{R}^d)$, then*

$$f \in BMO(\mathbb{R}^d).$$

P r o o f. Since $\dot{X}_1(\mathbb{R}^d) \subset \dot{\mathcal{M}}_{2,d}(\mathbb{R}^d)$, it follows that

$$\nabla f \in \dot{\mathcal{M}}_{2,d}(\mathbb{R}^d).$$

By the classical Poincaré inequality (2.10) we have

$$\begin{aligned}\int_{B(x,R)} |f(y) - m_{B(x,R)} f(y)|^2 \,\mathrm{d}y &\leqslant CR^2 \int_{B(x,R)} |\nabla f(y)|^2 \,\mathrm{d}y\\ &\leqslant CR^d \|\nabla f\|^2_{\dot{\mathcal{M}}_{2,d}}\end{aligned}$$

for every ball $B(x,R)$ of any radius R and

$$\begin{aligned}\|f\|^2_{BMO} &= \sup_{x\in\mathbb{R}^d} \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y) - m_{B(x,R)} f(y)|^2 \,\mathrm{d}y\\ &\leqslant C\|\nabla f\|^2_{\dot{\mathcal{M}}_{2,d}} \leqslant C\|\nabla f\|^2_{\dot{X}_1(\mathbb{R}^d)}.\end{aligned}$$

□

We are now in position to prove the main result.

Proof of Theorem 7. Let v be another weak solution of (1.1) associated to a on $(0,T)$ (with the associated pressure p) such that

$$v \in L^\infty((0,T); L^2_\sigma(\mathbb{R}^d)^d) \cap L^2((0,T); \dot H^1_\sigma(\mathbb{R}^d)^d)$$

and

$$\nabla v \in L^2((0,T); \dot X_1(\mathbb{R}^d)^d).$$

We consider the difference $w = u - v$ and we obtain

$$\begin{aligned} \partial_t w - \Delta w + \nabla p_w &= -[w\cdot\nabla v + u\cdot\nabla w], \\ \operatorname{div} w &= 0, \\ w(x,0) &= 0. \end{aligned} \tag{2.11}$$

If we take the inner product $\langle\cdot,\cdot\rangle$ of L^2 with w, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{L^2} + \|\nabla w\|^2_{L^2} = -\langle w\cdot\nabla v, w\rangle.$$

Integration by parts followed by integration over time then lead to

$$\|w(t)\|^2_{L^2} + 2\int_0^t \|\nabla w\|^2_{L^2}\,\mathrm{d}\tau = -2\int_0^t \langle w\cdot\nabla v, w\rangle\,\mathrm{d}\tau = 2\int_0^t \langle w\cdot\nabla w, v\rangle\,\mathrm{d}\tau \tag{2.12}$$

for all $0 \leqslant t < T$. Lemma 4 with

$$g = w, \quad \nabla f = \nabla w \quad \text{and} \quad h = v$$

yields directly

$$|\langle w.\nabla w, v\rangle| \leqslant C\|\nabla w\|_{L^2(\mathbb{R}^d)}\|w\|_{L^2(\mathbb{R}^d)^d}\|\nabla v\|_{\dot X_1(\mathbb{R}^d)}.$$

We thus observe that by the Young inequality ($ab \leqslant a^2/2 + b^2/2$, $a,b \geqslant 0$) it follows that

$$\left|\int_0^t \langle w\cdot\nabla w, v\rangle\right|\mathrm{d}\tau \leqslant \frac{1}{2}\int_0^t \|\nabla w\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}\tau + \frac{C}{2}\int_0^t \|w\|^2_{L^2(\mathbb{R}^d)^d}\|\nabla v\|^2_{\dot X_1(\mathbb{R}^d)}\,\mathrm{d}\tau.$$

Hence by (2.12)

$$\|w(t)\|^2_{L^2} + \int_0^t \|\nabla w\|^2_{L^2}\,\mathrm{d}\tau \leqslant C\int_0^t \|w\|^2_{L^2(\mathbb{R}^d)^d}\|\nabla v\|^2_{\dot X_1(\mathbb{R}^d)}\,\mathrm{d}\tau$$

for all $t > 0$. Since $\nabla v \in L^2((0,T); \dot{X}_1(\mathbb{R}^d)^d)$ and $w(0) = 0$, it follows from the Gronwall inequality that

$$\|w(t)\|_{L^2}^2 \leqslant \|w(0)\|_{L^2}^2 \exp\left(C \int_0^t \|\nabla v\|_{\dot{X}_1(\mathbb{R}^d)}^2 \, \mathrm{d}\tau \right)$$

and thus

$$\|w(t)\|_{L^2}^2 = 0, \quad 0 \leqslant t < T,$$

which implies the uniqueness of weak solutions.

The proof when

$$u \in L^2((0,T); BMO(\mathbb{R}^d)^d)$$

is quite similar. We apply Lemma 5 with

$$g = w \quad \nabla f = \nabla w \quad \text{and} \quad h = v$$

which yields directly

$$|\langle w \cdot \nabla w, v \rangle| \leqslant C \|\nabla w\|_{L^2(\mathbb{R}^d)} \|w\|_{L^2(\mathbb{R}^d)^d} \|v\|_{BMO(\mathbb{R}^d)}.$$

Using again Young' s inequality, we get

$$\left| \int_0^t \langle w \cdot \nabla w, v \rangle \right| \mathrm{d}\tau \leqslant \frac{1}{2} \int_0^t \|\nabla w\|_{L^2(\mathbb{R}^d)}^2 \, \mathrm{d}\tau + \frac{C}{2} \int_0^t \|w\|_{L^2(\mathbb{R}^d)^d}^2 \|v\|_{BMO(\mathbb{R}^d)}^2 \, \mathrm{d}\tau.$$

Hence it follows from (2.12) that

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2 \, \mathrm{d}\tau \leqslant C \int_0^t \|w\|_{L^2(\mathbb{R}^d)^d}^2 \|v\|_{BMO(\mathbb{R}^d)}^2 \, \mathrm{d}\tau$$

for all $0 \leqslant t < T$. Since $v \in L^2((0,T); BMO(\mathbb{R}^d)^d)$ and $w(0) = 0$, the Gronwall inequality yields

$$\|w(t)\|_{L^2}^2 = 0, \quad 0 \leqslant t < T,$$

from which we get the desired uniqueness. □

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Author's address: *S. Gala*, University of Mostaganem, Department of Mathematics, Box 227, Mostaganem (27000), Algeria, e-mail: `sadek.gala@gmail.com`.