PORTUGALIAE MATHEMATICA
Vol. 63 Fasc. 4 – 2006
Nova Série

UNIQUENESS PROPERTIES OF FUNCTIONALS WITH LIPSCHITZIAN DERIVATIVE

Biagio Ricceri

Abstract: In this paper, we prove that if X is a real Hilbert space and if $J\colon X \to \mathbb{R}$ is a C^1 functional whose derivative is Lipschitzian, with Lipschitz constant L, then, for every $x_0 \in X$, with $J'(x_0) \neq 0$, the following alternative holds: either the functional $x \to \frac{1}{2}\,\|x-x_0\|^2 - \frac{1}{L}\,J(x)$ has a global minimum in X, or, for every $r > J(x_0)$, there exists a unique $y_r \in J^{-1}(r)$ such that $\|x_0 - y_r\| = \mathrm{dist}(x_0, J^{-1}(r))$ and, for every $r > 0$, the restriction of the functional J to the sphere $\{x \in X\colon \|x-x_0\| = r\}$ has a unique global maximum.

1 – Introduction

Let X be a real Hilbert space and J a C^1 functional on X. For $x_0 \in X$, $r > 0$, set $S(x_0, r) = \{x \in X\colon \|x - x_0\| = r\}$.

Also on the basis of the beautiful theory developed and applied by Schechter and Tintarev in [2], [3], [4] and [5], it is of particular interest to know when the restriction of J to $S(0, r)$ has a unique maximum.

The aim of the present paper is to offer a contribution along this direction.

We show that such a uniqueness property holds (for suitable r) provided that J' is Lipschitzian and $J'(0) \neq 0$. At the same time, we also show that (for suitable s) the set $J^{-1}(s)$ has a unique element of minimal norm.

After proving the general result (Theorem 1), we present an application to a semilinear Dirichlet problem involving a Lipschitzian nonlinearity (Theorem 2).

Received: August 12, 2005.

2 – The main result

With the usual convention $\inf \emptyset = +\infty$, our main result reads as follows:

Theorem 1. *Let X be a real Hilbert space and let $J\colon X \to \mathbb{R}$ be a C^1 functional, with Lipschitzian derivative. Let L be the Lipschitz constant of J'.*

Then, for each $x_0 \in X$ with $J'(x_0) \neq 0$, if we set

$$\alpha_0 = \inf_{x \in M_{\frac{1}{L}}} J(x)$$

and

$$\beta_0 = \operatorname{dist}(x_0, M_{\frac{1}{L}}) ,$$

where $M_{\frac{1}{L}}$ is the set of all global minima of the functional $x \to \frac{1}{2}\|x - x_0\|^2 - \frac{1}{L} J(x)$, we have $\alpha_0 > J(x_0)$, $\beta_0 > 0$, and the following properties hold:

- **(i)** *for every $r \in \,]J(x_0), \alpha_0[$ there exists a unique $y_r \in J^{-1}(r)$ such that*

$$\|x_0 - y_r\| = \operatorname{dist}\big(x_0, J^{-1}(r)\big) ;$$

- **(ii)** *for every $r \in \,]0, \beta_0[$ the restriction of the functional J to the set $S(x_0, r)$ has a unique global maximum.*

The main tool used to get Theorem 1 is the following particular case of Theorem 3 of [1].

Theorem A. *Let X be a reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval and $\Psi\colon X \times I \to \mathbb{R}$ a function such that $\Psi(x, \cdot)$ is concave and continuous for all $x \in X$, while $\Psi(\cdot, \lambda)$ is sequentially weakly lower semicontinuous and coercive, with a unique local minimum for all $\lambda \in \operatorname{int}(I)$.*

Then, one has

$$\sup_{\lambda \in I} \inf_{x \in X} \Psi(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in I} \Psi(x, \lambda) . \blacksquare$$

We will also use the two propositions below.

Proposition 1. *Let Y be a nonempty set, $f, g\colon Y \to \mathbb{R}$ two functions, and a, b two real numbers, with $a < b$. Let y_a be a global minimum of the function $f - a\,g$ and y_b a global minimum of the function $f - b\,g$.*

Then, one has $g(y_a) \leq g(y_b)$. If either y_a or y_b is strict and $y_a \neq y_b$, then $g(y_a) < g(y_b)$.

Proof: We have

$$f(y_a) - a\,g(y_a) \;\le\; f(y_b) - a\,g(y_b)$$

as well as

$$f(y_b) - b\,g(y_b) \;\le\; f(y_a) - b\,g(y_a)\ .$$

Summing, we get

$$-a\,g(y_a) - b\,g(y_b) \;\le\; -a\,g(y_b) - b\,g(y_a)$$

and so

$$(b-a)\,g(y_a) \;\le\; (b-a)\,g(y_b)$$

from which the first conclusion follows. If either y_a or y_b is strict and $y_a \ne y_b$, then one of the first two inequalities is strict and hence so is the third one. ∎

Proposition 2. *Let Y be a real Hilbert space and let $\varphi\colon Y \to \mathbb{R}$ be a C^1 functional whose derivative is a contraction.*

Then, for every $y_0 \in Y$, the functional $y \to \frac{1}{2}\|y - y_0\|^2 - \varphi(y)$ is coercive and strictly convex, and so has a unique local minimum.

Proof: Let ν be the Lipschitz constant of φ'. So, $\nu < 1$, by assumption. For each $y \in Y$, we have

$$\varphi(y) \;=\; \varphi(0) + \int_0^1 \langle \varphi'(ty), y\rangle\; dt$$

and so

$$\begin{aligned}
|\varphi(y)| \;&\le\; |\varphi(0)| \;+ \int_0^1 |\langle \varphi'(ty), y\rangle|\; dt \\
&\le\; |\varphi(0)| \;+\; \|y\| \int_0^1 \|\varphi'(ty)\|\; dt \\
&\le\; |\varphi(0)| \;+\; \|y\| \left(\int_0^1 \|\varphi'(ty) - \varphi'(0)\|\; dt \;+\; \|\varphi'(0)\| \right) \\
&\le\; |\varphi(0)| \;+\; \frac{\nu}{2}\,\|y\|^2 \;+\; \|\varphi'(0)\|\,\|y\|\ .
\end{aligned}$$

From this, we then get

$$\frac{1}{2}\,\|y - y_0\|^2 - \varphi(y) \;\ge\; \frac{1-\nu}{2}\,\|y\|^2 - \big(\|\varphi'(0)\| + \|y_0\|\big)\,\|y\| + \frac{1}{2}\,\|y_0\|^2 - |\varphi(0)|$$

and hence

$$\lim_{\|y\|\to+\infty} \frac{1}{2}\,\|y-y_0\|^2 - \varphi(y) \;=\; +\infty \;,$$

which yields the coercivity of the functional $y \to \frac{1}{2}\|y-y_0\|^2 - \varphi(y)$. To show that it is also strictly convex, we note that its derivative is strictly monotone. In fact, for each $x, y \in X$, we have

$$\begin{aligned}\big\langle x-\varphi'(x)-y+\varphi'(y),\; x-y\big\rangle \;&=\; \|x-y\|^2 - \big\langle \varphi'(x)-\varphi'(y),\; x-y\big\rangle \\ &\geq\; \|x-y\|^2 - \|\varphi'(x)-\varphi'(y)\|\,\|x-y\| \\ &\geq\; (1-\nu)\,\|x-y\|^2 \;. \blacksquare\end{aligned}$$

Proof of Theorem 1: First, note that, for each $\gamma > L$, the operator $\frac{1}{\gamma}\,J'$ is a contraction, and so, by Proposition 2, the functional $x \to \frac{1}{2}\|x-x_0\|^2 - \frac{1}{\gamma}J(x)$ has a unique global minimum, say $x_{\frac{1}{\gamma}}$. Fix $\gamma > L$. By Proposition 1, we have $J(x_0) \leq J(x_{\frac{1}{\gamma}})$. We claim that $J(x_0) < J(x_{\frac{1}{\gamma}})$. Arguing by contradiction, assume that $J(x_0) = J(x_{\frac{1}{\gamma}})$. Then, by Proposition 1 again, we would have $x_{\frac{1}{\gamma}} = x_0$. Consequently, the derivative of the functional $x \to \frac{1}{2}\|x-x_0\|^2 - \frac{1}{\gamma}\,J(x)$ would vanish at x_0, that is $-\frac{1}{\gamma}\,J'(x_0) = 0$, against one of the hypotheses. Then, we have

$$J(x_0) < J(x_{\frac{1}{\gamma}}) \leq J(x)$$

for all $x \in M_{\frac{1}{L}}$, and so $J(x_0) < \alpha_0$. Clearly, $x_{\frac{1}{\gamma}}$ is the global minimum of the functional $x \to \frac{\gamma}{2}\|x-x_0\|^2 - J(x)$, while any $z \in M_{\frac{1}{L}}$ is a global minimum of the functional $x \to \frac{L}{2}\|x-x_0\|^2 - J(x)$. Consequently, if we apply Proposition 1 again (with $f(x) = -J(x)$, $g(x) = -\|x-x_0\|^2$, $a = \frac{L}{2}$, $b = \frac{\gamma}{2}$), for any $z \in M_{\frac{1}{L}}$, we get

$$-\|z-x_0\|^2 \;\leq\; -\|x_{\frac{1}{\gamma}} - x_0\|^2 \;,$$

and so

$$\beta_0 \;\geq\; \|x_{\frac{1}{\gamma}} - x_0\| \;>\; 0 \;.$$

Now, to prove (i), fix $r \in \,]J(x_0), \alpha_0[$ and consider the function $\Psi\colon\; X \times [0, \frac{1}{L}] \to \mathbb{R}$ defined by

$$\Psi(x,\lambda) \;=\; \frac{1}{2}\,\|x-x_0\|^2 + \lambda\big(r - J(x)\big)$$

for all $(x,\lambda) \in X \times [0, \frac{1}{L}]$. Taken Proposition 2 into account, it is clear that the function Ψ satisfies all the assumptions of Theorem A. Consequently, we have

$$\sup_{\lambda\in[0,\frac{1}{L}]}\,\inf_{x\in X}\,\Psi(x,\lambda) \;=\; \inf_{x\in X}\,\sup_{\lambda\in[0,\frac{1}{L}]}\,\Psi(x,\lambda) \;.$$

The functional $\sup_{\lambda\in[0,\frac{1}{L}]}\Psi(\cdot,\lambda)$ is weakly lower semicontinuous and coercive, and so there exists $x^*\in X$ such that

$$\sup_{\lambda\in[0,\frac{1}{L}]}\Psi(x^*,\lambda) = \inf_{x\in X}\sup_{\lambda\in[0,\frac{1}{L}]}\Psi(x,\lambda)\ .$$

Also, the function $\inf_{x\in X}\Psi(x,\cdot)$ is upper semicontinuous, and so there exists $\lambda^*\in[0,\frac{1}{L}]$ such that

$$\inf_{x\in X}\Psi(x,\lambda^*) = \sup_{\lambda\in[0,\frac{1}{L}]}\inf_{x\in X}\Psi(x,\lambda)\ .$$

Hence, from this it follows that

$$\begin{aligned}\frac{1}{2}\|x^*-x_0\|^2+\lambda^*\big(r-J(x^*)\big) &= \inf_{x\in X}\frac{1}{2}\|x-x_0\|^2+\lambda^*\big(r-J(x)\big)\\ &= \sup_{\lambda\in[0,\frac{1}{L}]}\frac{1}{2}\|x^*-x_0\|^2+\lambda\big(r-J(x^*)\big)\ .\end{aligned}$$

We claim that $J(x^*)=r$. Indeed, if it were $J(x^*)<r$, then we would have $\lambda^*=\frac{1}{L}$, and so $x^*\in M_{\frac{1}{L}}$, against the fact that $r<\alpha_0$. If it were $J(x^*)>r$, then we would have $\lambda^*=0$, and so $x^*=x_0$, against the fact that $J(x_0)<r$. We then have

$$\frac{1}{2}\|x^*-x_0\|^2 = \inf_{x\in X}\frac{1}{2}\|x-x_0\|^2 + \lambda^*\big(r-J(x)\big)\ .$$

This implies, on one hand, that $\lambda^*<\frac{1}{L}$ (since $r<\alpha_0$) and, on the other hand, that each global minimum (and x^* is so) of the restriction to $J^{-1}(r)$ of the functional $x\to\frac{1}{2}\|x-x_0\|^2$ is a global minimum in X of the functional $x\to\frac{1}{2}\|x-x_0\|^2-\lambda^*J(x)$. But this functional (just because $\lambda^*<\frac{1}{L}$) has a unique global minimum, and so (i) follows. Let us now prove (ii). To this end, fix $r\in\,]0,\beta_0[$ and consider the function $\Phi\colon X\times[L,+\infty[\,\to\mathbb{R}$ defined by

$$\Phi(x,\lambda) = \frac{\lambda}{2}\left(\|x-x_0\|^2-r^2\right)-J(x)$$

for all $(x,\lambda)\in X\times[L,+\infty[$. Applying Theorem A, we get

$$\sup_{\lambda\in[L,+\infty[}\inf_{x\in X}\Phi(x,\lambda) = \inf_{x\in X}\sup_{\lambda\in[L,+\infty[}\Phi(x,\lambda)\ .$$

Arguing as before (note, in particular, that $\lim\limits_{\lambda\to+\infty}\inf\limits_{x\in X}\Phi(x,\lambda)=-\infty$), we get $\hat{x}\in X$ and $\hat{\lambda}\in[L,+\infty[$ such that

$$\sup_{\lambda\in[L,+\infty[}\Phi(\hat{x},\lambda) = \inf_{x\in X}\sup_{\lambda\in[L,+\infty[}\Phi(x,\lambda)$$

and

$$\inf_{x\in X} \Phi(x,\hat\lambda) = \sup_{\lambda\in[L,+\infty[} \inf_{x\in X} \Phi(x,\lambda) .$$

So that

$$\begin{aligned}\frac{\hat\lambda}{2}\left(\|\hat x - x_0\|^2 - r^2\right) - J(\hat x) &= \inf_{x\in X} \frac{\hat\lambda}{2}\left(\|x - x_0\|^2 - r^2\right) - J(x)\\ &= \sup_{\lambda\in[L,+\infty[} \frac{\lambda}{2}\left(\|\hat x - x_0\|^2 - r^2\right) - J(\hat x) .\end{aligned}$$

From this it follows at once that $\|\hat x - x_0\|^2 \le r^2$. But, if it were $\|\hat x - x_0\|^2 < r^2$ we would have $\hat\lambda = L$. This, in turn, would imply that $\hat x \in M_{\frac{1}{L}}$, against the fact that $r < \beta_0$. Hence, we have $\|\hat x - x_0\|^2 = r^2$. Consequently

$$-\frac{1}{\hat\lambda} J(\hat x) = \inf_{x\in X} \frac{1}{2}\left(\|x - x_0\|^2 - r^2\right) - \frac{1}{\hat\lambda} J(x) .$$

This implies, on one hand, that $\hat\lambda > L$ (since $r < \beta_0$) and, on the other hand, that each global maximum (and $\hat x$ is so) of the restriction of the functional J to the set $S(x_0, r)$ is a global minimum in X of the functional $x \to \frac{1}{2}\|x - x_0\|^2 - \frac{1}{\hat\lambda} J(x)$. Since $\hat\lambda > L$, this functional has a unique global minimum, and so (ii) follows. ∎

Remark 1. It is clear from the proof that the assumption $J'(x_0) \neq 0$ has been used to prove $\alpha_0 > J(x_0)$ and $\beta_0 > 0$, while it has no role in showing (i) and (ii). However, when $J'(x_0) = 0$, it can happen that $\alpha_0 = J(x_0)$, $\beta_0 = 0$, with (i) (resp. (ii)) holding for no $r > \alpha_0$ (resp. for no $r > 0$). To see this, take, for instance, $X = \mathbb{R}$, $J(x) = \frac{1}{2} x^2$, $x_0 = 0$. □

3 – An application

From now on, Ω is an open, bounded and connected subset of $\mathbb{R}^n$ with sufficiently smooth boundary, and X denotes the space $W_0^{1,2}(\Omega)$, with the usual norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2 \, dx\right)^{\frac{1}{2}} .$$

Moreover, $f\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitzian function, with Lipschitz constant μ.

Let $\lambda \in \mathbb{R}$. As usual, a classical solution of the problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

is any $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, zero on $\partial\Omega$, which satisfies the equation pointwise in Ω.

For each $u \in X$, put

$$J(u) = \int_\Omega \left(\int_0^{u(x)} f(\xi)\, d\xi \right) dx \ .$$

By a classical result, the functional J is continuously Gâteaux differentiable and one has

$$J'(u)(v) = \int_\Omega f(u(x))\, v(x)\ dx$$

for all $u, v \in X$. Moreover, by a standard regularity result, the critical points in X of the functional $u \to \frac{1}{2}\|u\|^2 - \lambda\, J(u)$ are exactly the classical solutions of problem (P_λ).

Denote by λ_1 the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u \quad \text{in } \ \Omega \\ u_{|\partial\Omega} = 0 \ . \end{cases}$$

Recall that $\|u\|_{L^2(\Omega)} \le \lambda_1^{-\frac{1}{2}} \|u\|$ for all $u \in X$.

We are now in a position to state the following

Theorem 2. *Assume that* $f(0) \neq 0$. *For each* $r > 0$, *put*

$$\gamma(r) = \sup_{\|u\|^2 = r} J(u) \ .$$

Further, put

$$\delta_0 = \inf_{u \in M} \|u\|^2$$

where M *is the set of all global minima in* X *of the functional* $u \to \frac{1}{2}\|u\|^2 - \frac{\lambda_1}{\mu} J(u)$.

Then, $\delta_0 > 0$, *the function* γ *is* C^1 *and* γ' *is positive in* $]0, \delta_0[$ *and there exists a continuous function* $\varphi\colon]0, \delta_0[\to X$ *such that, for each* $r \in]0, \delta_0[$, $\varphi(r)$ *is a classical solution of the problem*

$$\begin{cases} -\Delta u = \dfrac{1}{2\,\gamma'(r)}\, f(u) \quad \text{in } \ \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

satisfying $\|\varphi(r)\|^2 = r$ *and* $J(\varphi(r)) = \gamma(r)$.

Proof: Fix $u, v, w \in X$, with $\|w\| = 1$. We have

$$\begin{aligned} \left|J'(u)(w) - J'(v)(w)\right| &\leq \int_\Omega \left|f(u(x)) - f(v(x))\right| |w(x)| \, dx \\ &\leq \mu \|u - v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \\ &\leq \frac{\mu}{\lambda_1} \|u - v\| \,, \end{aligned}$$

and hence

$$\left\|J'(u) - J'(v)\right\| \leq \frac{\mu}{\lambda_1} \|u - v\| \,.$$

That is, J' is Lipschitzian in X, with Lipschitz constant $\frac{\mu}{\lambda_1}$. Moreover, since $f(0) \neq 0$, we have $J'(u) \neq 0$ for all $u \in X$. Then, thanks to Theorem 1, for each $r \in]0, \delta_0[$, the restriction of the functional J to the sphere $S(0, \sqrt{r})$ has a unique maximum. At this point, taken into account that $\gamma(r) > 0$ for all $r > 0$, the conclusion follows directly from Lemma 2.1 and Corollary 2.13 of [2]. ∎

REFERENCES

[**1**] Ricceri, B. – Minimax theorems for limits of parametrized functions having at most one local minimum lying in a certain set, *Topology Appl.,* 153 (2006), 3308–3312.

[**2**] Schechter, M. and Tintarev, K. – Spherical maxima in Hilbert space and semilinear elliptic eigenvalue problems, *Differential Integral Equations,* 3 (1990), 889–899.

[**3**] Schechter, M. and Tintarev, K. – Points of spherical maxima and solvability of semilinear elliptic equations, *Canad. J. Math.,* 43 (1991), 825–831.

[**4**] Schechter, M. and Tintarev, K. – Eigenvalues for semilinear boundary value problems, *Arch. Rational Mech. Anal.,* 113 (1991), 197–208.

[**5**] Schechter, M. and Tintarev, K. – Families of 'first eigenfunctions' for semilinear elliptic eigenvalue problems, *Duke Math. J.,* 62 (1991), 453–465.

Biagio Ricceri,
Department of Mathematics, University of Catania,
Viale A. Doria 6, 95125 Catania — ITALY
E-mail: ricceri@dmi.unict.it