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DOI: 10.2478/awutm-2023-0004

Annals of West University of Timisoara
Mathematics and Computer Science
59 (2023), 35–44

Uniqueness Results for Fractional Integro-differential Equations with State-Dependent Nonlocal Conditions in Fréchet Spaces

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Abstract The aim of this paper is to study the existence of the unique mild solution for non-linear fractional integro-differential equations with state-dependent nonlocal condition. The result was obtained by using nonlinear alternative of Granas-Frigon for contraction in Fréchet spaces. To illustrate the result, an example is provided.

AMS Subject Classification (2020) 34G20
Keywords Fractional integro-differential equations; solution operator; fixed point; mild solution; state-dependent nonlocal condition; nonlinear alternative; Fréchet spaces

1 Introduction

Differential equations having non-local conditions are encountered often in applications. Non-local conditions can be viewed as feedback controls in mathematical modeling of real processes, through which a certain qualitative feature or magnitude of the solution along its evolution matches its original state. In this work, we discuss some existence results for fractional integro-differential equations with state dependent non local conditions. For the importance of nonlocal conditions in different fields, we refer to [10, 11] and the reference therein. Hernandez and O'Regan were the first to establish a new sort of nonlocal condition in [19], which we now refer to as state dependent nonlocal conditions. More information can be found in the paper [18]. For basic results and recent development on differential equations, one can refer to [7–9, 13, 17, 21, 23, 30–32].

The problem of existence solutions for Cauchy problem of fractional integro-differential equations was investigated in numerous works; we refer the reader to books by Kilbas et al. [20], Lakshmikantham et al. [22], and the papers by Agarwal et al. [1], Anguraj et al. [3], Balachandran et al. [5]. Cuevas et al. [12], studied S-asymptotically w-periodic solutions.

In this paper, we consider the existence and uniqueness of mild solutions defined on unbounded interval for semilinear integro-differential equations of fractional order of the form

$$\mathfrak{p}'(\vartheta) - \int_0^{\vartheta} \frac{(\vartheta-\theta)^{\zeta-2}}{\Gamma(\zeta-1)} \Phi \mathfrak{p}_{\mathfrak{p}}(\theta) d\theta = \psi(\vartheta, \mathfrak{p}_{\varrho(\vartheta,\mathfrak{p}_\vartheta)}), \quad \text{a.e. } \vartheta \in \mathbb{R}_+ := [0, +\infty), \tag{1.1}$$

$$\mathfrak{p}_0 = \Psi(\sigma(\mathfrak{p}), \mathfrak{p}) \in \mathcal{C} = C([-\kappa, 0], \Xi), \tag{1.2}$$

where $1 < \zeta < 2$, $\Phi : D(\Phi) \subset \Xi \to \Xi$ is a closed linear operator, and $(\Xi, \|\cdot\|)$ is a Banach space. $\psi : \mathbb{R}_+ \times \mathcal{C} \to \Xi$, $\sigma : C([-\kappa, +\infty), \Xi) \to \mathbb{R}_+$, $\Psi : \mathbb{R}_+ \times C([-\kappa, +\infty), \Xi) \to \Xi$ and $\varrho : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}$, are suitable functions. If $\mathfrak{p} \in C([-\kappa, +\infty), \Xi)$, then for any $\vartheta \in \mathbb{R}_+$, define $\mathfrak{p}_\vartheta$ by $\mathfrak{p}_\vartheta(\varkappa) = \mathfrak{p}(\vartheta + \varkappa)$ for $\varkappa \in [-\kappa, 0]$. Here we establish sufficient conditions to get the existence of the unique mild solution for fractional integro-differential equations with state dependent nonlocal conditions in Fréchet spaces. A nonlinear alternative of Leray-Schauder type for contraction maps in Fréchet spaces due to Frigon-Granas is employed in our study. Many authors have employed this approach to explore different types of differential problems; for additional details, see the works [6, 26, 27] and the references therein.

The following is the paper planning: In Section 2, we will review some fundamental information that will be used all through the remainder of the sections. The main results are given in Section 3, where we demonstrate the existence of mild solutions to the problem (1.1)–(1.2). whereas the final part is an illustrative example

2 Preliminaries

First, we introduce and explain the notations and concepts used in this study.

Let $C([-\kappa, 0]; \Xi)$ be the Banach space of continuous functions with the norm

$$\|\mathfrak{p}\| = \sup\{\|\mathfrak{p}(\vartheta)\| : \vartheta \in [-\kappa, 0]\}.$$

Let $\Omega(\Xi)$ be the space of all bounded linear operators from Ξ into Ξ with the norm

$$\|\mathcal{Y}\|_{\Omega(\Xi)} = \sup_{\|\mathfrak{p}\|=1} \|\mathcal{Y}(\mathfrak{p})\|.$$

Denote $L^1(\mathbb{R}_+, \Xi)$ the Banach space of measurable functions $\mathfrak{p} : \mathbb{R}_+ \to \Xi$ which are Bochner integrable normed by

$$\|\mathfrak{p}\|_{L^1} = \int_0^{+\infty} \|\mathfrak{p}(\vartheta)\| \, d\vartheta.$$

The Laplace transformation of a function $\psi \in L^1(\mathbb{R}_+, \Xi)$ is defined by

$$\mathcal{L}(\psi)(\gamma) :=: \widehat{\psi}(\gamma) = \int_0^\infty e^{-\gamma\vartheta}\psi(\vartheta)d\vartheta, \quad Re(\gamma) > \varpi,$$

if the integral is absolutely convergent for $Re(\gamma) > \varpi$.

Definition 2.1. *Let Φ be a closed and linear operator with a dense domain $D(\Phi)$ defined on a Banach space Ξ. Φ is the generator of a solution operator if there exists $\varpi > 0$ and a strongly continuous function $\mathcal{G} : \mathbb{R}_+ \to \Omega(\Xi)$ where*

$$\{\gamma^\zeta : Re(\gamma) > \varpi\} \subset \varrho(\Phi),$$

and

$$\gamma^{\zeta-1}(\gamma^\zeta - \Phi)^{-1}\mathfrak{q} = \int_0^\infty e^{-\gamma\vartheta}\mathcal{G}(\vartheta)\mathfrak{q}d\vartheta, \; Re\gamma > \varpi, \mathfrak{q} \in \Xi.$$

where $\mathcal{G}(\vartheta)$ is the solution operator generated by Φ.

Proposition 2.1 ([24]). *Let $\{\mathcal{G}(\vartheta)\}_{\vartheta\geq 0} \subset \Omega(\Xi)$ be the solution operator with generator Φ. Then the requirements listed below are met.*

a) $\mathcal{G}(\vartheta)$ is strongly continuous for $\vartheta \geq 0$ and $\mathcal{G}(0) = I$;

b) $\mathcal{G}(\vartheta)D(\Phi) \subset D(\Phi)$ and $\Phi\mathcal{G}(\vartheta)\mathfrak{q} = \mathcal{G}(\vartheta)\Phi\mathfrak{q}$ for all $\mathfrak{q} \in D(\Phi)$, $\vartheta \geq 0$;

c) for every $\mathfrak{q} \in D(\Phi)$ and $\vartheta \geq 0$,

$$\mathcal{G}(\vartheta)\mathfrak{q} = \mathfrak{q} + \int_0^\vartheta \frac{(\vartheta-\theta)^{\zeta-1}}{\Gamma(\zeta)}\Phi\mathcal{G}(\theta)\mathfrak{q}d\theta.$$

d) Let $\mathfrak{q} \in D(\Phi)$. Then $\int_0^\vartheta \frac{(\vartheta-\theta)^{\zeta-1}}{\Gamma(\zeta)}\mathcal{G}(\theta)\mathfrak{q}d\theta \in D(\Phi)$ and

$$\mathcal{G}(\vartheta)\mathfrak{q} = \mathfrak{q} + \Phi\int_0^\vartheta \frac{(\vartheta-\theta)^{\zeta-1}}{\Gamma(\zeta)}\mathcal{G}(\theta)\mathfrak{q}d\theta.$$

In [4, 15, 28, 29], one can find further information on C_0-semigroups and sine and cosine families.

Definition 2.2. *A solution operator $\{\mathcal{G}(\vartheta)\}_{\vartheta>0}$ is called uniformly continuous if*

$$\lim_{\vartheta\to\theta}\|\mathcal{G}(\vartheta) - \mathcal{G}(\theta)\|_{\Omega(\Xi)} = 0.$$

Definition 2.3. *A function $\psi : \mathbb{R}_+ \times \Xi \to \Xi$ is L^1-Carathéodory if it verifies:*

(i) for each $\vartheta \in \mathbb{R}_+$, $\psi(\vartheta,\cdot) : \Xi \to \Xi$ is continuous;

(ii) for each $\mathfrak{p} \in \Xi$, $\psi(\cdot,\mathfrak{p}) : \mathbb{R}_+ \to \Xi$ is measurable;

(iii) for every positive integer ι there exists $\varsigma_\iota \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ where

$$\|\psi(\vartheta, \mathfrak{p})\| \leq \varsigma_\iota(\vartheta)$$

for all $\|\mathfrak{p}\| \leq \iota$ and almost each $\vartheta \in \mathbb{R}_+$.

Let Θ_1 be a Fréchet space with a family of semi-norms $\{\|\cdot\|_\tau\}_{\tau\in\mathbb{N}}$ and let $\Theta_2 \subset \Theta_1$. For every $\tau \in \mathbb{N}$, let the equivalence relation $\sim_\tau$ given by : $\mathfrak{q} \sim_\tau \mathfrak{p}$ if and only if $\|\mathfrak{q} - \mathfrak{p}\|_\tau = 0$ for all $\mathfrak{q}, \mathfrak{p} \in \Theta_1$. We denote ${\Theta_1}^\tau = (\Theta_1|_{\sim_\tau}, \|\cdot\|_\tau)$ the quotient space, the completion of ${\Theta_1}^\tau$ with respect to $\|\cdot\|_\tau$. To every $\Theta_2 \subset \Theta_1$, we associate a sequence the $\{{\Theta_2}^\tau\}$ of subsets ${\Theta_2}^\tau \subset {\Theta_1}^\tau$ as follows: For every $\mathfrak{q} \in \Theta_1$, we denote $[\mathfrak{q}]_\tau$ the equivalence class of $\mathfrak{q}$ of subset ${\Theta_1}^\tau$ and we define ${\Theta_2}^\tau = \{[\mathfrak{q}]_\tau : \mathfrak{q} \in \Theta_2\}$. We denote $\overline{{\Theta_2}^\tau}$, $int_\tau({\Theta_2}^\tau)$ and $\partial_\tau {\Theta_2}^\tau$, respectively, the closure, the interior and the boundary of ${\Theta_2}^\tau$ with respect to $\|\cdot\|$ in ${\Theta_1}^\tau$. We suppose that the family of semi-norms $\{\|\cdot\|_\tau\}$ satisfies :

$$\|\mathfrak{q}\|_1 \leq \|\mathfrak{q}\|_2 \leq \|\mathfrak{q}\|_3 \leq \dots \text{ for every } \mathfrak{q} \in \Theta_1.$$

Definition 2.4 ([16]). *A function $\psi : \Theta_1 \to \Theta_1$ is said to be* a contraction *if for each $\tau \in \mathbb{N}$ there exists $\iota_\tau \in (0,1)$ where :*

$$\|\psi(\mathfrak{q}) - \psi(\mathfrak{p})\|_\tau \leq \iota_\tau \ \|\mathfrak{q} - \mathfrak{p}\|_\tau \ \textit{for all } \mathfrak{q}, \mathfrak{p} \in \Theta_1.$$

Theorem 2.2 ([14]). *Let Θ_1 be a Fréchet space and $\Theta_2 \in \Theta_1$ a closed subset in Θ_2 and let $\mathcal{Y} : \Theta_2 \to \Theta_1$ be a contraction such that $\mathcal{Y}(\Theta_2)$ is bounded. Then one of the following statements holds:*

(C1) $\mathcal{Y}$ has a unique fixed point;

(C2) There exists $\gamma \in [0,1)$, $\tau \in \mathbb{N}$ and $\mathfrak{q} \in \partial_\tau {\Theta_2}^\tau$ such that $\|\mathfrak{q} - \gamma\mathcal{Y}(\mathfrak{q})\|_\tau = 0$.

3 Existence and uniqueness of mild solution

Definition 3.1. *A function $\mathfrak{p} \in C([-\kappa, +\infty], \Xi)$ is said to be a mild solution of (1.1) − (1.2) if $\mathfrak{p}_0 = \Psi(\sigma(\mathfrak{p}), \mathfrak{p})$ for all $\vartheta \in [-\kappa, 0]$ and $\mathfrak{p}$ satisfies*

$$\mathfrak{p}(\vartheta) = \mathcal{G}(\vartheta)\Psi(\sigma(\mathfrak{p}), \mathfrak{p})(0) + \int_0^\vartheta \mathcal{G}(\vartheta - \theta) \ \psi(\theta, \mathfrak{p}_{\varrho(\theta, \mathfrak{p}_\theta)}) \ d\theta \quad \textit{for each } \vartheta \in \mathbb{R}_+. \tag{3.1}$$

The hypotheses:

$(H1)$ There exists a constant $\lambda_1 > 1$ such that

$$\|\mathcal{G}(\vartheta)\|_{\Omega(\Xi)} \leq \lambda_1 \text{ for every } \vartheta \in \mathbb{R}_+;$$

$(H2)$ There exist a continuous nondecreasing function $\chi : \mathbb{R}_+ \to (0, \infty)$ and $\delta \in L^1_{Loc}(\mathbb{R}_+; \mathbb{R}_+)$ where

$$\|\psi(\vartheta, \mathfrak{p})\| \leq \delta(\vartheta) \ \chi(\|\mathfrak{p}\|) \text{ for a.e. } \vartheta \in \mathbb{R}_+ \text{ and each } \mathfrak{p} \in \mathcal{C}.$$

$(H3)$ There exists $\lambda_2 > 0$ such that

$$\|\Psi(\sigma(\mathfrak{p}),\mathfrak{p})\| \leq \lambda_2(1+\|\mathfrak{p}\|) \quad \text{for each } \mathfrak{p} \in \mathcal{C}$$

$(H4)$ For all $\Lambda > 0$, there exists $\beta_\Lambda \in L^1_{Loc}([-\kappa,+\infty);\mathbb{R}_+)$ such that

$$\|\psi(\vartheta,\mathfrak{p}) - \psi(\vartheta,\mathfrak{q})\| \leq \beta_\Lambda(\vartheta)\|\mathfrak{p}-\mathfrak{q}\| \quad \text{for all } \mathfrak{p},\mathfrak{q} \in \mathcal{C};$$

$(H5)$ There exists $\lambda_3 > 0$ such that

$$\|\Psi(\sigma(\mathfrak{p}),\mathfrak{p}) - \Psi(\sigma(\mathfrak{q}),\mathfrak{q})\| \leq \lambda_3\|\mathfrak{p}-\mathfrak{q}\| \quad \text{for all } \mathfrak{p},\mathfrak{q} \in \mathcal{C}.$$

Set

$$\lambda^*_\tau = \int_0^\tau \lambda_1\beta_\tau(\theta)d\theta.$$

Theorem 3.1. *Assume* $(H1)-(H5)$ *are verified. If*

$$\lambda_1\lambda_3 + \lambda^*_\tau < 1,$$

then the problem $(1.1)-(1.2)$ *has a unique mild solution.*

Proof. For every $\tau \in \mathbb{N}$, we define in $\mathcal{C}([-\kappa,+\infty),\Xi)$ the semi-norms by:

$$\|\mathfrak{p}\|_\tau := \sup\{\|\mathfrak{p}(\vartheta)\| : \vartheta \in [0,\tau]\}.$$

Transform the problem $(1.1)-(1.2)$ into fixed-point problem. Consider the operator $\mathcal{Y} : \mathcal{C}([-\kappa,+\infty),\Xi) \to \mathcal{C}([-\kappa,+\infty),\Xi)$ defined by:

$$(\mathcal{Y}\mathfrak{p})(\vartheta) = \begin{cases} \Psi(\sigma(\mathfrak{p}),\mathfrak{p}), & \text{if } \vartheta \in [-\kappa,0]; \\ \mathcal{G}(\vartheta)\Psi(\sigma(\mathfrak{p}),\mathfrak{p})(0) + \displaystyle\int_0^\vartheta \mathcal{G}(\vartheta-\theta)\ \psi(\theta,\mathfrak{p}_{\varrho(\theta,\mathfrak{p}_\theta)})\ d\theta & \text{if } \vartheta \in \mathbb{R}_+. \end{cases} \tag{3.2}$$

It is clear that the fixed points of $\mathcal{Y}$ are mild solutions of the problem $(1.1)-(1.2)$. By $(H1)-(H3)$, we have for each $\vartheta \in [0,\tau]$

$$\begin{aligned} |\mathfrak{p}(\vartheta)| &\leq |\mathcal{G}(\vartheta)||\Psi(\sigma(\mathfrak{p}),\mathfrak{p})(0)| + \int_0^\vartheta \|\mathcal{G}(\vartheta-\theta)\|_{\Omega(\Xi)}|\psi(\theta,\mathfrak{p}_{\varrho(\theta,\mathfrak{p}_\theta)})|\ d\theta \\ &\leq \lambda_1\lambda_2(1+\|\mathfrak{p}\|) + \lambda_1\int_0^\vartheta \delta(\theta)\chi(\|\mathfrak{p}\|_\tau)d\theta. \end{aligned}$$

Then

$$\|\mathfrak{p}\|_\tau \leq \frac{\lambda_1\lambda_2}{1-\lambda_1\lambda_2} + \frac{\lambda_1}{1-\lambda_1\lambda_2}\int_0^\vartheta \delta(\theta)\chi(\|\mathfrak{p}\|_\tau)d\theta.$$

Consider the function ϕ given by

$$\phi(\vartheta) := \sup\{\|\mathfrak{p}(\theta)\| : 0 \leq \theta \leq \vartheta\},\ 0 \leq \vartheta \leq +\infty.$$

Let $\vartheta^* \in [-\kappa, \vartheta]$ be such that $\phi(\vartheta) = \|\mathfrak{p}(\vartheta^*)\|$.
If $\vartheta^* \in [0, \tau]$, we have

$$\phi(\vartheta) \leq \frac{\lambda_1\lambda_2}{1-\lambda_1\lambda_2} + \frac{\lambda_1}{1-\lambda_1\lambda_2}\int_0^{\vartheta} \delta(\theta)\chi(\|\phi\|_\tau)d\theta, \quad \vartheta \in [0, \tau].$$

If $\vartheta^* \in [-\kappa, 0]$, then $\phi(\vartheta) = \|\mathfrak{p}_0\|$ and the previous inequality holds.
Let us take the right-hand side of the above inequality as $\mathfrak{q}(\vartheta)$. Then we have

$$\phi(\vartheta) \leq \mathfrak{q}(\vartheta) \text{ for all } \vartheta \in [0, \tau].$$

From the definition of $\mathfrak{q}$, we have

$$c := \mathfrak{q}(0) = \frac{\lambda_1\lambda_2}{1-\lambda_1\lambda_2}$$

and

$$\mathfrak{q}'(\vartheta) = \frac{\lambda_1}{1-\lambda_1\lambda_2}\delta(\theta)\chi(\phi(\vartheta)) \quad a.e \ \vartheta \in [0, \tau].$$

Using the nondecreasing character of χ, we obtain

$$\mathfrak{q}'(\vartheta) \leq \frac{\lambda_1}{1-\lambda_1\lambda_2}\delta(\vartheta)\chi(\mathfrak{q}(\vartheta)) \quad a.e \ \vartheta \in [0, \tau].$$

By integration, we get

$$\mathfrak{q}(\vartheta) - \mathfrak{q}(0) \leq \frac{\lambda_1}{1-\lambda_1\lambda_2}\int_0^{\vartheta} \delta(\theta)\chi(\mathfrak{q}(\theta))d\theta.$$

Thus

$$\mathfrak{q}(\vartheta) \leq \frac{\lambda_1\lambda_2}{1-\lambda_1\lambda_2} + \frac{\lambda_1}{1-\lambda_1\lambda_2}\int_0^{\vartheta} \delta(\theta)\chi(\mathfrak{q}(\theta))d\theta.$$

Bihari's inequality implies that

$$\mathfrak{q}(\vartheta) \leq \Gamma^{-1}\left(\frac{\lambda_1}{1-\lambda_1\lambda_2}\int_0^{\tau} \delta(\theta)d\theta\right),$$

where

$$\Gamma(\vartheta) = \int_{\mathfrak{q}(0)}^{\vartheta} \frac{dx}{\chi(x)}.$$

Then there exists a constant Δ_τ such that $\mathfrak{q}(\vartheta) \leq \Delta_\tau, \vartheta \in [0, \tau]$ and thus $\phi(\vartheta) \leq \Delta_\tau, \vartheta \in [0, \tau]$. Since for every $\vartheta \in [0, \tau]$, $\|\mathfrak{p}_\vartheta\| \leq \phi(\vartheta)$, we have

$$\|\mathfrak{p}\|_\tau \leq \max\{\|\mathfrak{p}_0\|, \Delta_\tau\} := \lambda_{1\tau}.$$

Set

$$\Theta = \{\mathfrak{p} \in \mathcal{C}([-\kappa, +\infty); \Xi) : \sup\{|\mathfrak{p}(\vartheta)| : 0 \leq \vartheta \leq \tau\} \leq \lambda_{1\tau} + 1 \text{ for all } \tau \in \mathbb{N}\}.$$

Clearly, Θ is a closed subset of $\mathcal{C}([-\kappa, +\infty); \Xi)$.

We will demonstrate that $\mathcal{Y} : \Theta \to \mathcal{C}([-\kappa, +\infty); \Xi)$ is a contraction operator.
Consider $\mathfrak{p}, \overline{\mathfrak{p}} \in \mathcal{C}([-\kappa, +\infty); \Xi)$, thus using $(H1)$, $(H4)$ and $(H5)$ for each $\vartheta \in [0, \tau]$ and $\tau \in \mathbb{N}$

$$\begin{aligned}
|\mathcal{Y}(\mathfrak{p})(\vartheta) - \mathcal{Y}(\overline{\mathfrak{p}})(\vartheta)| &= \Big| \mathcal{G}(\vartheta)[\Psi(\sigma(\mathfrak{p}), \mathfrak{p})(0) - \Psi(\sigma(\overline{\mathfrak{p}}), \overline{\mathfrak{p}})(0)] \\
&\quad + \int_0^\vartheta \mathcal{G}(\vartheta - \theta)[\psi(\theta, \mathfrak{p}_{\varrho(\theta, \mathfrak{p}_\theta)}) - \psi(\theta, \overline{\mathfrak{p}}_{\varrho(\theta, \overline{\mathfrak{p}}_\theta)})]d\theta \Big| \\
&\le \lambda_1 \|\Psi(\sigma(\mathfrak{p}), \mathfrak{p})(0) - \Psi(\sigma(\overline{\mathfrak{p}}), \overline{\mathfrak{p}})(0)\| \\
&\quad + \int_0^\vartheta \lambda_1 |\psi(\theta, \mathfrak{p}_{\varrho(\theta, \mathfrak{p}_\theta)}) - \psi(\theta, \overline{\mathfrak{p}}_{\varrho(\theta, \overline{\mathfrak{p}}_\theta)})| d\theta \\
&\le \lambda_1 \lambda_3 \|\mathfrak{p} - \overline{\mathfrak{p}}\|_\tau + \int_0^\vartheta \lambda_1 \beta_\tau \|\mathfrak{p}(\theta) - \overline{\mathfrak{p}}(\theta)\| d\theta \\
&\le \lambda_1 \lambda_3 \|\mathfrak{p} - \overline{\mathfrak{p}}\|_\tau + \lambda_\tau^* \|\mathfrak{p} - \overline{\mathfrak{p}}\|_\tau \\
&\le (\lambda_1 \lambda_3 + \lambda_\tau^*) \|\mathfrak{p} - \overline{\mathfrak{p}}\|_\tau.
\end{aligned}$$

Therefore,

$$\|\mathcal{Y}(\mathfrak{p}) - \mathcal{Y}(\overline{\mathfrak{p}})\|_\tau \le (\lambda_1 \lambda_3 + \lambda_\tau^*) \|\mathfrak{p} - \overline{\mathfrak{p}}\|_\tau.$$

Consequently, $\mathcal{Y}$ is contraction for all $\tau \in \mathbb{N}$. By the choice of Θ there is no $\mathfrak{p} \in \partial\Theta^\tau$ where $\mathfrak{p} = \gamma \mathcal{Y}(\mathfrak{p})$ for some $\gamma \in (0, 1)$. Then the statement $(C2)$ in Theorem 2.2 is not met. As a result of the nonlinear alternative of Frigon and Granas that $(C1)$ is met. We conclude that $\mathcal{Y}$ has unique fixed-point which is the unique mild solution of the problem (1.1) – (1.2). □

4 An Example

To illustrate our result, we consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial \vartheta}(\vartheta, \upsilon) - \dfrac{1}{\Gamma(\zeta - 1)} \displaystyle\int_\vartheta^0 (\vartheta - \theta)^{\zeta - 2} L_\upsilon u(\theta, \upsilon) d\theta \\ \qquad = Q(\vartheta) |u(\vartheta - \eta u(\vartheta, \upsilon), \upsilon)|^\delta, \qquad \vartheta \in \mathbb{R}_+, \ \upsilon \in [0, \pi], \\ \\ u_0(\varkappa, \upsilon) = \zeta(u_{\sigma(u)}(\varkappa, \upsilon)), \varkappa \in [-\kappa, 0], \ \upsilon \in [0, \pi], \end{cases} \tag{4.1}$$

where $1 < \zeta < 2$, $\eta \in C(\mathbb{R}, [0, \infty))$, $\zeta \in C(\mathbb{R}, \mathbb{R})$, $\sigma \in C((C[-\kappa, +\infty), \Xi), \mathbb{R}_+)$, $Q : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function and λ_υ is the operator with respect to the spatial variable υ which is given by:

$$\lambda_\upsilon = \frac{\partial^2}{\partial \upsilon^2} - \kappa, \quad \text{with} \quad \kappa > 0.$$

Consider $\Xi = L^2([0, \pi], \mathbb{R})$ and the operator $\Phi := \lambda_\upsilon : D(\Phi) \subset \Xi \to \Xi$ with domain

$$D(\Phi) := \{ \ u \in \Xi \ : u, \ u' \text{ are absolutely continuous}, \ u'' \in \Xi, \ u(0) = u(\pi) = 0 \ \}.$$

The operator Φ is densely defined in Ξ and is sectorial. Thus Φ is a generator of a solution operator on Ξ.

Set

$$\mathfrak{p}(\vartheta)(\upsilon) = u(\vartheta,\upsilon),\ \vartheta \in \mathbb{R}_+, \upsilon \in [0,\pi],$$
$$\Psi(\vartheta,\upsilon) = \zeta(\upsilon_\vartheta(\varkappa)),\ \vartheta \in \mathbb{R}_+,\ \varkappa \in [-\kappa,0],$$
$$\psi(\vartheta,\mu)(\upsilon) = Q(\vartheta)|u(\upsilon)|^\delta,\ \text{for } \vartheta \in \mathbb{R}_+, \upsilon \in [0,\pi], \mu \in \Xi.$$

Thus, under the above definitions of ψ and $\Phi(\cdot)$, the system (4.1) can be represented by the problem (1.1)-(1.2). Furthermore, more appropriate conditions on Q ensure the existence of unique mild solution for (4.1) by Theorem 3.1.

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