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UNIQUENESS THEOREMS IN AN ANGULAR DOMAIN

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Abstract. There are many papers on the uniqueness theory of meromorphic functions in the whole plane $\boldsymbol{C}$. However, the uniqueness theory concerned with shared sets in an angular domain does not yet seem widely investigated. In this paper, we deal with the problem of uniqueness for meromorphic functions in $\boldsymbol{C}$ under some conditions in an angular domain instead of the whole plane. Moreover, examples show that those conditions are necessary.

1. Introduction and main results. In this paper, unless otherwise stated, by a meromorphic function we mean that the function is defined and meromorphic in the whole plane $\boldsymbol{C}$. We also assume that the reader is familiar with the basic results and notation of Nevanlinna's value distribution theory of meromorphic functions (see [11] or [12]), such as $T(r, f)$, $N(r, f)$ and $m(r, f)$. Meanwhile, the lower order μ and the order λ of a meromorphic function f are defined as follows:

$$\mu := \mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and

$$\lambda := \lambda(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}\,.$$

Let S be a subset of distinct elements in $\hat{\boldsymbol{C}}$ and $X \subseteq \boldsymbol{C}$. Define

$$E_X(S, f) = \bigcup_{a\in S}\{z \in X \mid f_a(z) = 0,\ \text{counting multiplicities}\}\,,$$

$$\bar{E}_X(S, f) = \bigcup_{a\in S}\{z \in X \mid f_a(z) = 0,\ \text{ignoring multiplicities}\}\,,$$

where $f_a(z) = f(z) - a$ if $a \in \boldsymbol{C}$ and $f_\infty(z) = 1/f(z)$.

Let f and g be two non-constant meromorphic functions in $\boldsymbol{C}$. If $E_X(S, f) = E_X(S, g)$, we say f and g share the set S CM (counting multiplicities) in X. If $\bar{E}_X(S, f) = \bar{E}_X(S, g)$, we say f and g share the set S IM (ignoring multiplicities) in X. In particular, when $S = \{a\}$, where $a \in \hat{\boldsymbol{C}}$, we say f and g share the value a CM in X if $E_X(S, f) = E_X(S, g)$, and we

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say f and g share the value a IM in X if $\bar{E}_X(S, f) = \bar{E}_X(S, g)$. When $X = \boldsymbol{C}$, we give the simple notation as before, $E(S, f)$, $\bar{E}(S, f)$ and so on (see [28]).

In [10], Gross proved that there exist three finite sets S_j $(j = 1, 2, 3)$ such that any two non-constant entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$ must be identical, and asked the following question (see [10, Question 6]).

QUESTION A. Can one find two finite sets S_j $(j = 1, 2)$ such that any two entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ $(j = 1, 2)$ must be identical?

Yi seems to have been the first to give the affirmative answer to the above Question A completely (see [23]). Since then, many results have been obtained concerning this question and related topics (see [5–7, 13–15, 19, 21, 22] and [24–27]).

In [10], Gross noted that 'if the answer to Question A is affirmative, it would be interesting to know how large both sets would have to be', namely he also asked the following question.

QUESTION B. What are the smallest cardinalities of S_1 and S_2, where S_1 and S_2 are two finite sets such that any two entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$ must be identical?

In 1998, Yi actually proved the following theorems as the answers to these two questions, respectively.

THEOREM A ([27, Theorem 4]). *Let $S_1 = \{0\}$ and $S_2 = \{w \mid w^n(w + a) - b = 0\}$, where n (≥ 2) is an integer, and a and b are two non-zero constants such that the algebraic equation $w^n(w + a) - b = 0$ has no multiple roots. If f and g are two entire functions satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f \equiv g$.*

THEOREM B ([27, Theorem 2]). *If S_1 and S_2 are two finite sets such that any two entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$ must be identical, then $\max\{\#(S_1), \#(S_2)\} \geq 3$, where $\#(S)$ denotes the cardinality of the set S.*

Zheng [31] and [32] considered the uniqueness of meromorphic functions with shared values in angular domains. Following him, we ask the following question.

QUESTION 1. Does there exist an angular domain $X = X(\alpha, \beta) := \{z \mid \alpha < \arg z < \beta\}$ $(0 \leq \alpha < \beta \leq 2\pi)$ such that $f \equiv g$ is always the case when f and g are two entire functions satisfying $E_X(\{S_j\}, f) = E_X(\{S_j\}, g)$ for $j = 1, 2$ in Theorem A?

Note that Yi and one of the authors of this paper extended Theorem A to the following results on some class of meromorphic functions.

THEOREM C ([29, Theorem 1]). *Let $S_1 = \{0\}$, $S_2 = \{\infty\}$ and $S_3 = \{w \mid w^n(w + a) - b = 0\}$, where n (≥ 3) is an integer, and a and b are two non-zero constants such that the algebraic equation $w^n(w+a)-b = 0$ has no multiple roots. If f and g are two meromorphic functions satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$ and $\Theta(\infty, f) > 0$, then $f \equiv g$.*

In [29, Theorem 2], it is also shown that the same conclusion still holds when the two assumptions '$n \geq 3$' and '$\Theta(\infty, f) > 0$' are replaced by '$n \geq 2$' and '$\Theta(\infty, f) > 1/2$', respectively.

Therefore, we may also ask Question 1 for Theorem C, that is, meromorphic functions and three sets S_j $(j = 1, 2, 3)$ given there.

In this paper, we prove the following theorems from this point of view in Question 1.

THEOREM 1. *Let $S_1 = \{0\}$, $S_2 = \{\infty\}$ and $S_3 = \{w \mid w^n(w + a) - b = 0\}$, where n (≥ 3) is an integer, and a and b are two non-zero constants such that the algebraic equation $w^n(w + a) - b = 0$ has no multiple roots. Assume that f is a meromorphic function of lower order $\mu(f) \in (1/2, \infty)$ in $\boldsymbol{C}$ and $\delta := \delta(\iota, f) > 0$ for some $\iota \in \hat{\boldsymbol{C}} \setminus \{0, -a\}$. Then for each $\sigma < \infty$ with $\mu(f) \leq \sigma \leq \lambda(f)$ there exists an angular domain $X = X(\alpha, \beta)$ with $0 \leq \alpha < \beta$ and*

$$\beta - \alpha > \max\left\{\frac{\pi}{\sigma}, \quad 2\pi - \frac{4}{\sigma} \arcsin\sqrt{\frac{\delta}{2}}\right\}, \tag{1}$$

such that if the conditions $E(S_1, f) = E(S_1, g)$ and $E_X(S_j, f) = E_X(S_j, g)$ $(j = 2, 3)$ hold for a meromorphic function g in $\boldsymbol{C}$ of finite order or more generally with the growth satisfying either $\log T(r, g) = O(\log T(r, f))$ or

$$\lim_{\substack{r \to \infty \\ r \notin E_1}} \frac{\log \log T(r, g)}{\min\{\log r, \log T(r, f)\}} = 0, \tag{$*$}$$

where E_1 is a set of finite linear measure, then $f \equiv g$.

REMARK 1. The following example shows that the inequality (1) cannot be replaced by '$=$', so the condition (1) is best possible.

EXAMPLE 1. Let $f(z) = e^{-iz}$ and $g(z) = e^{-2iz}$. Then $\lambda(f) = \lambda(g) = \mu(f) = \mu(g) = 1$. The algebraic equation $w^4 - (1/2)w^3 + 1/1000 = 0$ has four distinct roots whose absolute values are strictly less than 1. Hence, $f^4 - (1/2)f^3 + 1/1000 \neq 0$ and $g^4 - (1/2)g^3 + 1/1000 \neq 0$ on $\{z \mid \Im z > 0\}$. Obviously, $\delta := \delta(\infty, f) = 1$, $\sigma = 1$ and $E(S_1, f) = E(S_1, g) = E(S_2, f) = E(S_2, g) = \emptyset$. Hence,

$$\max\left\{\frac{\pi}{\sigma},\ 2\pi - \left(\frac{4}{\sigma}\right) \arcsin\sqrt{\frac{\delta}{2}}\right\} = \pi.$$

If we take $X = X\ (0, \pi)$ or even if $X = \{z \mid 0 \leq \arg z \leq \pi,\ \text{or } z = 0\}$, then $E_X(S_3, f) = E_X(S_3, g) = \emptyset$, but $f \not\equiv g$.

On the other hand, for any $\varepsilon > 0$, one can find two different points z_1 and z_2 in $X\ (0, \pi + \varepsilon)$ such that $f(z_1)^4 - (1/2)f(z_1)^3 + 1/1000 = g(z_2)^4 - (1/2)g(z_2)^3 + 1/1000 = 0$, because f and g have a Julia direction on the real axis, respectively.

We note that the condition (1) no longer has any meaning when $\sigma \leq 1/2$, since $2\pi \geq \beta - \alpha > \pi/\sigma$. In this case, however, by taking $X = \boldsymbol{C}$ regarded as the closure of $X(0, 2\pi)$ and $\iota = \infty$, our consideration is reduced to the result of Theorem C, since $\Theta(\infty, f) \geq \delta(\infty, f)$.

REMARK 2. The following example (see [29]) shows that the condition of deficiencies in Theorem 1 is necessary.

EXAMPLE 2. Let

$$f(z) = -\frac{ae^z(e^{nz}-1)}{e^{(n+1)z}-1} = e^z g(z)\,, \quad g(z) = -\frac{a(e^{nz}-1)}{e^{(n+1)z}-1}\,.$$

It is easy to see that $f \not\equiv g$ and they satisfy $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$. For any $\iota \neq 0, -a$, we see that $\delta := \delta(\iota, f) = 0$, so that (1) becomes $\beta - \alpha > 2\pi$, which is impossible. This shows that the assumption '$\delta := \delta(\iota, f) > 0$ for some $\iota \neq 0, -a$' in Theorem 1 cannot be simply deleted.

In fact, the pair of functions f and g in Example 2 is essentially a unique exception in Theorem 1. Concretely, the above deficiency condition can be replaced by the condition that f and g are not of the form

$$f = -\frac{ae^\gamma(e^{n\gamma}-1)}{e^{(n+1)\gamma}-1} = e^\gamma g\,, \quad g = -\frac{a(e^{n\gamma}-1)}{e^{(n+1)\gamma}-1}\,,$$

where γ is an entire function. See the proof of Lemma 6 below.

If we exclude the case where f and g are given by

$$f = -\frac{ah(h^n-1)}{h^{n+1}-1} = hg\,, \quad g = -\frac{a(h^n-1)}{h^{n+1}-1}\,,$$

where h is a meromorphic function in $\boldsymbol{C}$ which is analytic and zero-free in X, the conclusion of Theorem 1 is also true under the conditions:

(i) $E_X(S_j, f) = E_X(S_j, g)$ $(j = 1, 2, 3)$; and

(ii) f attains one of the values 0, ∞ and $-a$ at least once in X;

instead of the conditions:

(i′) $E(S_1, f) = E(S_1, g)$ and $E_X(S_j, f) = E_X(S_j, g)$ $(j = 2, 3)$; and

(ii′) $\delta := \delta(\iota, f) > 0$ for some $\iota \neq 0, -a$.

REMARK 3. It is clear that the constant a as well as b should never be zero. In fact, for any meromorphic function f in $\boldsymbol{C}$ and a primitive $(n+1)$th root of unity ε, two functions f and $g := \varepsilon f$ satisfy $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$ with $a = 0$.

THEOREM 2. *Let $S_1 = \{0\}$ and $S_2 = \{w \mid w^n(w+a) - b = 0\}$, where n (≥ 2) is an integer, and a and b are two non-zero constants such that the algebraic equation $w^n(w+a) - b = 0$ has no multiple roots. Assume that f is an entire function of lower order $\mu(f) \in (1/2, \infty)$. Then, for each $\sigma < \infty$ with $\mu(f) \leq \sigma \leq \lambda(f)$, there exists an angular domain $X = X(\alpha, \beta)$ whose opening $\beta - \alpha$ is larger than π/σ if $\sigma \leq 1$, and $2\pi - \pi/\sigma$ if $\sigma > 1$ with the following property: If the conditions $E(S_1, f) = E(S_1, g)$ and $E_X(S_2, f) = E_X(S_2, g)$ hold for an entire function g satisfying either $\log T(r, g) = O(\log T(r, f))$ or $(*)$ in Theorem 1 holds as $r \to \infty$ possibly outside a set E_1 of finite linear measure, then $f \equiv g$.*

REMARK 4. The condition $n \geq 2$ in Theorem 2 is sharp, since two entire functions $f(z) = e^z - 1$ and $g(z) = e^{-z} - 1$ satisfy $E(S_1, f) = E(S_1, g)$ and $E_X(S_2, f) = E_X(S_2, g)$ for any α and β with $\beta - \alpha > \pi$ when our algebraic equation is given by $w(w + 3) + 2 = 0$.

In the meantime, it seems open whether the assumption $n \geq 3$ is sharp or not in Theorem 1.

Under the condition that $\lambda(f) = \infty$, we obtain the following theorems.

THEOREM 3. *Let S_1, S_2 and S_3 be defined as in Theorem* 1. *Assume that f is a meromorphic function of infinite order, but that it grows not so rapidly that*

$$\limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r} < \infty ,$$

and assume further that $\delta(\iota, f) > 0$ for some $\iota \in \hat{\boldsymbol{C}} \setminus \{0, -a\}$. Then there exists a direction $\arg z = \alpha$ $(0 \leq \alpha < 2\pi)$ *such that for any ε $(0 < \varepsilon < \pi/2)$, if a meromorphic function g satisfies the growth condition*

$$(**) \qquad \log T(r, g) = O(r^\tau \log r T(r, f)) , \quad r \notin E$$

for a constant $\tau > 0$ and a set E of finite linear measure, and $E(S_1, f) = E(S_1, g)$ and $E_X(S_j, f) = E_X(S_j, g)$ for $j = 2, 3$ in the angular domain $X = X(\alpha - \varepsilon, \alpha + \varepsilon)$, then $f \equiv g$.

THEOREM 4. *Let S_1 and S_2 be defined as in Theorem* 2. *Assume that f is an entire function of infinite order but that it satisfies*

$$\limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r} < \infty .$$

Then there exists a direction $\arg z = \alpha$ $(0 \leq \alpha < 2\pi)$ *such that for any ε $(0 < \varepsilon < \pi/2)$, if an entire function g with the growth $(**)$ in Theorem* 3 *satisfies the conditions $E(S_1, f) = E(S_1, g)$ and $E_X(S_2, f) = E_X(S_2, g)$ in the angular domain $X = X(\alpha - \varepsilon, \alpha + \varepsilon)$, then $f \equiv g$.*

2. Some lemmas. We shall prove the theorems by using the Nevanlinna theory of meromorphic functions defined in an angular domain (see [16]). First of all, we recall some notation and definitions. Let $f(z)$ be a meromorphic function on the closed angular domain $\bar{X} := \bar{X}(\alpha, \beta) = \{z \mid \alpha \leq \arg z \leq \beta\} \cup \{0\}$, where $0 < \beta - \alpha \leq 2\pi$. Nevanlinna defined the following notation (also see [9]):

$$A_{\alpha,\beta}(r, f) := \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\{\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|\} \frac{dt}{t} ,$$

$$B_{\alpha,\beta}(r, f) := \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ |f(re^{i\theta})| \sin \omega(\theta - \alpha)\, d\theta ,$$

$$C_{\alpha,\beta}(r, f) := 2 \sum_{1<|b_m|<r} \left(\frac{1}{|b_m|^\omega} - \frac{|b_m|^\omega}{r^{2\omega}}\right) \sin \omega(\theta_m - \alpha) ,$$

where $\omega = \pi/(\beta - \alpha)$, $1 \le r < \infty$ and $b_m = |b_m|e^{i\theta_m}$ are the poles of $f(z)$ on $\bar{X}$ appearing often according to their multiplicities. $C_{\alpha,\beta}(r, f)$ is called the angular counting function of the poles of f on $\bar{X}$ and the Nevanlinna angular characteristic function is defined as follows:

$$\boldsymbol{S}_{\alpha,\beta}(r, f) := A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f)\,.$$

Similarly, for $a \ne \infty$, we can define $A_{\alpha,\beta}(r, f_a)$, $B_{\alpha,\beta}(r, f_a)$, $C_{\alpha,\beta}(r, f_a)$ and $\boldsymbol{S}_{\alpha,\beta}(r, f_a)$ with $f_a = 1/(f - a)$ and so on. For the sake of simplicity, we omit the subscript of all the notation and use the notation $A(r, a)$, $B(r, a)$, $C(r, a)$ and $\boldsymbol{S}(r, a)$ instead of $A_{\alpha,\beta}(r, f_a)$, $B_{\alpha,\beta}(r, f_a)$, $C_{\alpha,\beta}(r, f_a)$ and $\boldsymbol{S}_{\alpha,\beta}(r, f_a)$ if $a \ne \infty$. We shall give some properties of $\boldsymbol{S}(r, f)$ as follows.

LEMMA 1 ([9]). *Let $f(z)$ be a meromorphic function on $\bar{X}(\alpha, \beta)$. Then, for an arbitrary finite complex number a, we have*

$$\boldsymbol{S}(r, a) = \boldsymbol{S}(r, f) + \varepsilon(r, a)\,,$$

where $\varepsilon(r, a) = O(1)$ as $r \to \infty$.

LEMMA 2 ([9]). *Let $P(z)$ be a polynomial of degree $d \ge 1$, and $f(z)$ be a meromorphic function on $\bar{X}(\alpha, \beta)$. Then*

$$\boldsymbol{S}(r, P(f)) = d\boldsymbol{S}(r, f) + O(1)\,.$$

For a meromorphic function f defined in $\boldsymbol{C}$, we denote by $Q(r, f)$ a quantity satisfying:

(i) $Q(r, f) = O(1)$ as $r \to \infty$ if $\lambda(f) < \infty$;

(ii) $Q(r, f) = O(\log rT(r, f))$ as $r \to \infty$ and $r \notin E$ if $\lambda(f) = \infty$, where E is a set of finite linear measure.

LEMMA 3 ([9]). *Let $f(z)$ be a meromorphic function in $\boldsymbol{C}$, and $0 \le \alpha < \beta \le 2\pi$. Then*

$$A\left(r, \frac{f'}{f}\right) + B\left(r, \frac{f'}{f}\right) = Q(r, f)\,.$$

Lemma 3 was first demonstrated by Nevanlinna [16] in the case where the function $f(z)$ is a meromorphic function of finite order in the whole plane, and then it was generalized into the present form by Dufresnoy [2] and Ostrovskii [17] (see also [9, Chapter III]). It was an open question whether for any meromorphic function $f(z)$ defined only on $\bar{X}(\alpha, \beta)$,

$$A\left(r, \frac{f'}{f}\right) + B\left(r, \frac{f'}{f}\right) = o(S(r, f))\,, \tag{2}$$

holds as $r \to \infty$ possibly outside a set of finite linear measure. In 1975, Gol'dberg [8] constructed an unexpected counterexample. He showed that, for any function $\phi(r) \to \infty$, $r \to \infty$, there is an entire function $f(z)$ such that $\boldsymbol{S}(r, f) \equiv 0$ but $A(r, f'/f)/\phi(r) \to \infty$ as $r \to \infty$. Thus, (2) is not valid in general (see [4]).

LEMMA 4. *Let $f(z)$ be a meromorphic function in $\boldsymbol{C}$, and $0 \le \alpha < \beta \le 2\pi$. Then*

$$\boldsymbol{S}(r, f) \le \bar{C}(r, f) + \bar{C}\left(r, \frac{1}{f}\right) + \bar{C}\left(r, \frac{1}{f - c}\right) - C_0\left(r, \frac{1}{f'}\right) + Q(r, f)\,,$$

where, as we noted above, $Q(r,f) = O(\log rT(r,f))$, $r \notin E$, E *is a set of finite linear measure and* $c \neq 0$. *Moreover,* $\bar{C}(r,f)$ *is the reduced counting function of poles of* f, *each distinct pole of* $f(z)$ *in* $X(\alpha,\beta) \cap \{z \mid |z| < r\}$ *being counted only once*; $C_0(r, 1/f')$ *is the counting function of the zeros of* f' *but not the zeros of* f *and* $f - c$ *in* $X(\alpha,\beta) \cap \{z \mid |z| < r\}$.

Lemma 4 can be proved by the same argument as in the proof of Nevanlinna's second fundamental theorem (see [11, Theorem 2.1]).

Next, we introduce some notation for the following main lemma.

Let f be a meromorphic function on a closed angular domain $\bar{X}(\alpha,\beta)$. We denote by $C_2(r,f)$ the counting function of the poles of f in $\{z \in X(\alpha,\beta) \mid |z| < r\}$, where a simple pole is counted once and a multiple pole is counted twice. In the same way, we can define $C_2(r, 1/f)$.

LEMMA 5. *Let* F *and* G *be two non-constant meromorphic functions in* $\boldsymbol{C}$ *such that* F *and* G *share* $1, \infty$ *CM in* $X(\alpha,\beta)$. *Then one of the following three cases holds*:

(i) $\boldsymbol{S}(r) \leq C_2(r, 1/F) + C_2(r, 1/G) + 2\bar{C}(r,F) + Q(r,F) + Q(r,G)$;

(ii) $F \equiv G$;

(iii) $FG \equiv 1$,

where $\boldsymbol{S}(r) = \max\{\boldsymbol{S}(r,F), \boldsymbol{S}(r,G)\}$, $Q(r,F)$ *and* $Q(r,G)$ *are as defined above immediately before Lemma* 3 *was stated.*

PROOF. Set

$$\Phi := \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}. \tag{3}$$

Suppose that $\Phi \not\equiv 0$. Using Lemma 4, we can deduce that $Q(r,F') = Q(r,F)$ and $Q(r,G') = Q(r,G)$. Therefore, we have

$$A(r,\Phi) + B(r,\Phi) = Q(r,F) + Q(r,G). \tag{4}$$

Since F and G share $1, \infty$ CM in $X(\alpha,\beta)$, we have

$$\begin{aligned} C(r,\Phi) \leq \bar{C}_{(2}\left(r, \frac{1}{F}\right) + \bar{C}_{(2}\left(r, \frac{1}{G}\right) + C_0\left(r, \frac{1}{F'}\right) \\ + C_0\left(r, \frac{1}{G'}\right) + Q(r,F) + Q(r,G), \end{aligned} \tag{5}$$

where $C_0(r, 1/F')$ is the same as in Lemma 4, and $\bar{C}_{(2}(r, 1/F)$ denotes the counting function of zeros of F with multiplicity at least 2 in $\{z \in X(\alpha,\beta) \| z| < r\}$ counting twice.

Combining (4) and (5), we have

$$\begin{aligned} \boldsymbol{S}(r,\Phi) \leq \bar{C}_{(2}\left(r, \frac{1}{F}\right) + \bar{C}_{(2}\left(r, \frac{1}{G}\right) + C_0\left(r, \frac{1}{F'}\right) \\ + C_0\left(r, \frac{1}{G'}\right) + Q(r,F) + Q(r,G). \end{aligned} \tag{6}$$

Suppose that z_0 is a simple zero of $F-1$. Then, by an elementary calculation, we obtain that z_0 is also one of the zeros of Φ. Thus, we obtain from (6) that

$$\begin{aligned} C_{1)}\left(r, \frac{1}{F-1}\right) = C_{1)}\left(r, \frac{1}{G-1}\right) &\le C\left(r, \frac{1}{\Phi}\right) \le \boldsymbol{S}(r, \Phi) + O(1) \\ &\le \bar{C}_{(2}\left(r, \frac{1}{F}\right) + \bar{C}_{(2}\left(r, \frac{1}{G}\right) \\ &\quad + C_0\left(r, \frac{1}{F'}\right) + C_0\left(r, \frac{1}{G'}\right) + Q(r, F) + Q(r, G)\,, \end{aligned}$$

where $C_{1)}(r, 1/F)$ denotes the counting function of zeros of F with multiplicity one in $\{z \in X(\alpha, \beta) \mid |z| < r\}$.

It follows that

$$\begin{aligned} \bar{C}\left(r, \frac{1}{F-1}\right) + \bar{C}\left(r, \frac{1}{G-1}\right) = 2\bar{C}\left(r, \frac{1}{F-1}\right) &\le \bar{C}_{1)}\left(r, \frac{1}{F-1}\right) + C\left(r, \frac{1}{F-1}\right) \\ &\le C\left(r, \frac{1}{F-1}\right) + \bar{C}_{(2}\left(r, \frac{1}{F}\right) + \bar{C}_{(2}\left(r, \frac{1}{G}\right) \\ &\quad + C_0\left(r, \frac{1}{F'}\right) + C_0\left(r, \frac{1}{G'}\right) + Q(r, F) + Q(r, G)\,. \end{aligned}$$

Applying again Lemma 4 to F and G, we deduce that

$$\begin{aligned} \boldsymbol{S}(r, F) + \boldsymbol{S}(r, G) &\le C\left(r, \frac{1}{F-1}\right) + C_2\left(r, \frac{1}{F}\right) + C_2\left(r, \frac{1}{G}\right) \\ &\quad + 2\bar{C}(r, F) + Q(r, F) + Q(r, G)\,. \end{aligned}$$

Therefore, (i) holds.

Suppose that $\Phi \equiv 0$. By integration, we have from (3) that

$$\frac{1}{G-1} = \frac{A}{F-1} + B\,,$$

where A ($\neq 0$) and B are constants. It follows that

$$G = \frac{(B+1)F + (A-B-1)}{BF + (A-B)}\,. \tag{7}$$

We distinguish the following three cases.

Case 1. Suppose that $B \neq 0, -1$. If $A - B - 1 \neq 0$, then from (7) we obtain

$$\bar{C}\left(r, \frac{1}{G}\right) = \bar{C}\left(r, \frac{1}{F + (A-B-1)/(B+1)}\right).$$

By Lemma 4, we have

$$\begin{aligned} \boldsymbol{S}(r, F) &\le \bar{C}\left(r, \frac{1}{F}\right) + \bar{C}(r, F) + \bar{C}\left(r, \frac{1}{F + (A-B-1)/(B+1)}\right) + Q(r, F) \\ &= \bar{C}\left(r, \frac{1}{F}\right) + \bar{C}(r, F) + \bar{C}\left(r, \frac{1}{G}\right) + Q(r, F)\,. \end{aligned}$$

Thus, (i) holds. If $A - B - 1 = 0$, we rewrite (7) as

$$G = \frac{(B+1)F}{BF+1}\,.$$

Then

$$\bar{C}\left(r, \frac{1}{G}\right) = \bar{C}\left(r, \frac{1}{F+1/B}\right).$$

Using Lemma 4, we have

$$\begin{aligned}\boldsymbol{S}(r, F) &\le \bar{C}\left(r, \frac{1}{F}\right) + \bar{C}(r, F) + \bar{C}\left(r, \frac{1}{F+1/B}\right) + Q(r, F)\\ &= \bar{C}\left(r, \frac{1}{F}\right) + \bar{C}(r, F) + \bar{C}(r, 1/G) + Q(r, F)\,.\end{aligned}$$

Thus, (i) also holds.

Case 2. Suppose that $B = 0$. We rewrite (7) as

$$G = \frac{F + (A-1)}{A}\,.$$

If $A \neq 1$, then $\bar{C}(r, 1/G) = \bar{C}(r, 1/(F + A - 1))$. By a similar method to Case 1, we also obtain (i). If $A = 1$, it follows that $F \equiv G$. Thus, (ii) holds.

Case 3. Suppose that $B = -1$. We rewrite (7) as

$$G = \frac{A}{-F + (A+1)}\,.$$

If $A + 1 \neq 0$, then $\bar{C}(r, G) = \bar{C}(r, 1/(F - (A+1)))$, and similarly we also obtain (i). If $A + 1 = 0$, we deduce that (iii) holds. This completes the proof of Lemma 5.

Let f and g be two non-constant meromorphic functions in $\boldsymbol{C}$, and $S_3 = \{w \mid w^n(w + a) - b = 0\}$, where n (≥ 2) is an integer, and a and b are two non-zero constants such that the algebraic equation $w^n(w + a) - b = 0$ has no multiple roots. We denote

$$F = \frac{f^n(f+a)}{b}\,, \quad G = \frac{g^n(g+a)}{b}\,. \tag{8}$$

Obviously, if $E_X(S_3, f) = E_X(S_3, g)$ then F and G share 1 CM in X. In the following, we shall give some lemmas relating to F and G.

LEMMA 6. *Suppose that* $\bar{E}(\{0\}, f) = \bar{E}(\{0\}, g)$ *and* $\delta(\iota, f) > 0$ *for* $\iota \in \hat{\boldsymbol{C}} \setminus \{0, -a\}$. *If* $F \equiv G$, *where* F *and* G *are defined as* (8), *then* $f \equiv g$.

In particular, we assume that f is an entire function and $n \geq 3$. Then $F \equiv G$ implies $f \equiv g$ even if the two conditions $\bar{E}(\{0\}, f) = \bar{E}(\{0\}, g)$ and $\delta(\iota, f) > 0$ for $\iota \in \hat{\boldsymbol{C}} \setminus \{0, -a\}$ are replaced by the single condition $\bar{E}_X(\{0\}, f) = \bar{E}_X(\{0\}, g)$ for some domain $X \subset \boldsymbol{C}$. When $n = 2$, we have an exception:

$$f = -\frac{a}{3}e^{-H}(e^H - c)(e^H - c^2) = -\frac{a}{3}(2\cosh H + 1) \quad \text{and}$$
$$g = -\frac{a}{3c}e^{-H}(e^H - 1)(e^H - c^2) = -\frac{a}{3}(2\cosh(H - 2\pi i/3) + 1)\,,$$

where H is an entire function and $c = \exp(2\pi i/3)$.

In general, if $\bar{E}_X(\{0\}, f) = \bar{E}_X(\{0\}, g)$ holds for a domain X in $\boldsymbol{C}$, $F \equiv G$ does imply $f \equiv g$, unless

$$f = -\frac{ah(h^n - 1)}{h^{n+1} - 1} = hg, \quad g = -\frac{a(h^n - 1)}{h^{n+1} - 1},$$

for some meromorphic function h defined in $\boldsymbol{C}$ which is analytic and zero-free in X.

PROOF. Suppose that $f \not\equiv g$. Since $F \equiv G$, we have

$$f^n(f + a) = g^n(g + a), \tag{9}$$

and hence from the assumption $\bar{E}(\{0\}, f) = \bar{E}(\{0\}, g)$ we see that f and g share $0, \infty$ CM. Thus, we may assume that

$$\frac{f}{g} = e^{\gamma}, \tag{10}$$

where γ is an entire function. By $f \not\equiv g$, we obtain that $e^{\gamma} \not\equiv 1$. From (9) and (10) we deduce that

$$f = -\frac{ae^{\gamma}(e^{n\gamma} - 1)}{e^{(n+1)\gamma} - 1} \quad \text{and} \quad g = -\frac{a(e^{n\gamma} - 1)}{e^{(n+1)\gamma} - 1}. \tag{11}$$

If e^{γ} is a constant, then it follows from (11) that f is also a constant. This is a contradiction. If e^{γ} is non-constant, then we have from (11) that

$$T(r, f) = nT(r, e^{\gamma}) + S(r, f), \quad \bar{N}\left(r, \frac{1}{f - \iota}\right) = nT(r, e^{\gamma}) + S(r, f), \quad \iota \neq 0, -a.$$

It follows that $\delta(\iota, f) = 0$. This contradicts $\delta(\iota, f) > 0$.

Now we assume that $\bar{E}_X(\{0\}, f) = \bar{E}_X(\{0\}, g)$ holds for a domain X in $\boldsymbol{C}$. Similarly to the above discussion, we see that f and g share 0 and ∞ CM in the domain X, and therefore $f \equiv g$ unless there is a non-constant meromorphic function h in $\boldsymbol{C}$ which is analytic and zero-free in X such that f and g are given by

$$f = -\frac{ah(h^n - 1)}{h^{n+1} - 1} = hg \quad \text{and} \quad g = -\frac{a(h^n - 1)}{h^{n+1} - 1}.$$

Further if f is an entire function, its denominator $h^n + h^{n-1} + \cdots + h + 1$ should also be zero-free in $\boldsymbol{C}$, which is however impossible for $n \geq 3$. When $n = 2$, h does not attain two primitive cubic roots of unity, c, c^2, and therefore there is a non-constant entire function H such that $h = c(e^H - c)/(e^H - 1)$. Then we obtain the desired expressions by substituting this into the above expressions for f and g, respectively.

This completes the proof of Lemma 6.

Note that by using the above process of the proof, we can also obtain the result of Lemma 6 in the case where $\delta(0, f) > 1/n$.

LEMMA 7. *Let S_j $(j = 1, 2, 3)$ be defined as in Theorem* 1, *and let F and G be defined as* (8). *Assume that $\bar{E}_X(S_1, f) = \bar{E}_X(S_1, g)$ and $E_X(S_j, f) = E_X(S_j, g)$ $(j = 2, 3)$*

for $X = X(\alpha, \beta)$ *with* $0 \le \alpha < \beta \le 2\pi$. *If* $F \not\equiv G$, *then*

$$\bar{C}\left(r, \frac{1}{f}\right) = \bar{C}\left(r, \frac{1}{g}\right) = Q(r, f) + Q(r, g)\,. \tag{12}$$

PROOF. Set

$$H_1 := \frac{F'}{F-1} - \frac{G'}{G-1}\,. \tag{13}$$

Since $E_X(S_j, f) = E_X(S_j, g)$ for $j = 2, 3$, we see that F and G share $1, \infty$ CM in X. Hence, (13) implies that $C(r, H_1) \equiv 0$, so that by Lemma 3 we have

$$\boldsymbol{S}(r, H_1) = Q(r, f) + Q(r, g)\,. \tag{14}$$

We discuss the following two cases.

Case 1. Suppose that $H_1 \equiv 0$. By integration, we have from (13)

$$F - 1 = A(G - 1)\,, \tag{15}$$

where A is a non-zero constant. Since $F \not\equiv G$, we have $A \ne 1$. This together with the assumption gives $\bar{E}_X(S_1, f) = \bar{E}_X(S_1, g) = \phi$. Thus, (12) holds.

Case 2. Suppose that $H_1 \not\equiv 0$. Assume that z_0 is a zero of f and g of multiplicities p and q, respectively. Then z_0 is a multiple zero of F and G of multiplicities np and nq, respectively, so that we have $H_1(z_0) = 0$. By the assumption $\bar{E}_X(S_1, f) = \bar{E}_X(S_1, g)$ and (14), we have

$$\bar{C}\left(r, \frac{1}{f}\right) = \bar{C}\left(r, \frac{1}{g}\right) \le C\left(r, \frac{1}{H_1}\right) = Q(r, f) + Q(r, g)\,,$$

which proves Lemma 7. □

LEMMA 8. *Under the conditions of Lemma* 7, *we have*

$$C(r, f) = C(r, g) \le \frac{1}{n}\{\boldsymbol{S}(r, f) + \boldsymbol{S}(r, g)\} + Q(r, f) + Q(r, g)\,. \tag{16}$$

PROOF. Set

$$H_2 := \left(\frac{F'}{F-1} - \frac{G'}{G-1}\right) - \left(\frac{F'}{F} - \frac{G'}{G}\right). \tag{17}$$

Then

$$H_2 = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}\,.$$

It follows that

$$C(r, H_2) \le \bar{C}\left(r, \frac{1}{f}\right) + \bar{C}\left(r, \frac{1}{f+a}\right) + \bar{C}\left(r, \frac{1}{g+a}\right). \tag{18}$$

Therefore, by a lemma on the logarithmic derivative, (12) and (18), we obtain that

$$\boldsymbol{S}(r, H_2) \le \boldsymbol{S}(r, f) + \boldsymbol{S}(r, g) + Q(r, f) + Q(r, g)\,. \tag{19}$$

We discuss the following two cases.

Case 1. Suppose that $H_2 \equiv 0$. By integration, we have from (17)

$$\frac{F-1}{F} = B\frac{G-1}{G}\,, \tag{20}$$

where B is a non-zero constant. Since $F \not\equiv G$, we have $B \neq 1$. Again by (20), we deduce that $E_X(S_2, f) = E_X(S_2, g) = \phi$. Therefore, (16) holds.

Case 2. Suppose that $H_2 \not\equiv 0$. Assume that z_1 is a pole of f with multiplicity p; then an elementary calculation gives that z_1 is a zero of H_2 with multiplicity at least $(n+1)p-1 \geq np$. From this and (19), we obtain

$$nC(r, f) \leq C\left(r, \frac{1}{H_2}\right) \leq \boldsymbol{S}(r, f) + \boldsymbol{S}(r, g) + Q(r, f) + Q(r, g)\,. \tag{21}$$

We obtain from (21) that (16) holds. This completes the proof of Lemma 8.

Moreover, we need the following important lemmas concerning Pólya peaks (see [3, 20]).

LEMMA 9. *Let $f(z)$ be a transcendental meromorphic function of finite lower order μ and order λ $(0 < \lambda \leq \infty)$ in $\boldsymbol{C}$. Then for an arbitrary positive number σ satisfying $\mu \leq \sigma \leq \lambda$, there exist a set E of finite linear measure and a sequence of positive numbers $\{r_n\}$ such that*:

(i) $r_n \notin E$, $\lim_{n\to\infty}(r_n/n) = \infty$;
(ii) $\liminf_{n\to\infty}(\log T(r_n, f))/\log r_n \geq \sigma$;
(iii) $T(t, f) < (1+o(1))(t/r_n)^{\sigma}T(r_n, f)$, $t \in [r_n/n, nr_n]$.

A sequence $\{r_n\}$ in Lemma 9 is called a sequence of Pólya peaks of order σ outside E, which was proved in [20].

Given a positive function $\Lambda = \Lambda(r)$ on $(0, \infty)$ with $\Lambda \to 0$ as $r \to \infty$, we define for $r > 0$ and $a \in \boldsymbol{C}$

$$D_\Lambda(r, a) := \left\{\theta \in [-\pi, \pi) \,\middle|\, \log^+ \frac{1}{|f(re^{i\theta}) - a|} > \Lambda(r)T(r, f)\right\}$$

and

$$D_\Lambda(r, \infty) := \{\theta \in [-\pi, \pi) \mid \log^+ |f(re^{i\theta})| > \Lambda(r)T(r, f)\}\,.$$

The following lemma was proved by Baernstein [1].

LEMMA 10. *Let $f(z)$ be a transcendental meromorphic function of finite lower order μ and order λ $(0 < \lambda \leq \infty)$ in $\boldsymbol{C}$. Suppose that $\delta := \delta(a, f) > 0$ for some $a \in \hat{\boldsymbol{C}}$; then for arbitrary Pólya peaks $\{r_n\}$ of positive and finite order σ $(\mu \leq \sigma \leq \lambda)$ and an arbitrary positive function $\Lambda = \Lambda(r)$ with $\Lambda \to 0$ as $r \to \infty$, we have*

$$\liminf_{n\to\infty} \operatorname{meas} D_\Lambda(r_n, a) \geq \min\left\{2\pi, \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}}\right\}.$$

Further, we need one more important lemma given in [18, Theorem VII.3]. We first introduce some notation.

Let $f(z)$ be a meromorphic function in a domain containing an angular domain $\Delta := \{z \mid |\arg z| < \alpha\}$ and its closure with respect to $\boldsymbol{C}$. Let $\Delta(r)$ be the part of Δ which is contained in $|z| \le r$. We put

$$S_f^*(r, \Delta) = \frac{1}{\pi} \iint_{\Delta(r)} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 t\,dt\,d\theta\,, \quad z = te^{i\theta},$$

$$T_f^*(r, \Delta) = \int_0^r \frac{S_f^*(t, \Delta)}{t}\, dt,$$

which are called the Ahlfors-Shimizu characteristic functions of f on Δ. We denote just simply by $S^*(r, f)$, $T^*(r, f)$ the above characteristic functions of $f(z)$ in the whole plane, $S_f^*(r, \boldsymbol{C})$, $T_f^*(r, \boldsymbol{C})$.

Let $\bar{n}_f(r, \Delta, a) = \bar{n}(r, \Delta, f = a)$ $(a \in \hat{\boldsymbol{C}})$ be the number of distinct zeros of $f_a(z)$ contained in $\Delta(r)$. We put

$$\bar{N}_f(r, \Delta, a) := \bar{N}(r, \Delta, f = a) = \int_1^r \frac{\bar{n}_f(t, \Delta, a)}{t}\, dt\,.$$

Then we shall give the following analogue of the second fundamental theorem.

LEMMA 11. *Let $f(z)$ be a meromorphic function in the plane. Then, for any three distinct points a_1, a_2, a_3 in $\hat{\boldsymbol{C}}$, we have*

$$S_f^*(r, \Delta) \le 3 \sum_{i=1}^{3} \bar{n}_f(2r, \Delta, a_i) + O(\log r)$$

and

$$T_f^*(r, \Delta) \le 3 \sum_{i=1}^{3} \bar{N}_f(2r, \Delta, a_i) + O((\log r)^2)\,.$$

Finally, from [11, Theorem 1.4], we have the following lemmas.

LEMMA 12. *Let $f(z)$ be a meromorphic function in the plane. Then*

$$|T(r, f) - T^*(r, f)| \le \log^+ |f(0)| + (1/2)\log 2\,.$$

LEMMA 13 ([12, Lemma 1.1.1]). *Let $g : (0, +\infty) \to \boldsymbol{R}$, $h : (0, +\infty) \to \boldsymbol{R}$ be monotone increasing functions such that $g(r) \le h(r)$ outside of an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

3. Proof of theorems. We shall prove Theorem 1 by the method whose idea comes from Zheng [30].

3.1. Proof of Theorem 1. First, we define F and G as in (8); then F and G share 1 and ∞ CM in X.

Suppose that $F \not\equiv G$. Lemma 7 implies that

$$\bar{C}(r, 1/f) = \bar{C}(r, 1/g) = Q(r, f) + Q(r, g)\,. \tag{22}$$

Therefore, by the definition of F and G and (22), we have

$$\begin{aligned}(23)\qquad & C_2\left(r, \frac{1}{F}\right) + C_2\left(r, \frac{1}{G}\right) + 2\bar{C}(r, F) \\ & \le C\left(r, \frac{1}{f+a}\right) + C\left(r, \frac{1}{g+a}\right) + 2\bar{C}(r, f) + Q(r, f) + Q(r, g)\,.\end{aligned}$$

Set $\boldsymbol{S}_1(r) := \max\{\boldsymbol{S}(r, f), \boldsymbol{S}(r, g)\}$. Then we have from (8)

$$(24)\qquad \boldsymbol{S}(r) = (n+1)\boldsymbol{S}_1(r) + O(1)\,,$$

where $\boldsymbol{S}(r) = \max\{\boldsymbol{S}(r, F), \boldsymbol{S}(r, G)\}$. By the estimation (16) obtained in Lemma 8 and (23), we deduce that

$$(25)\qquad C_2\left(r, \frac{1}{F}\right) + C_2\left(r, \frac{1}{G}\right) + 2\bar{C}(r, F) \le \left(2 + \frac{4}{n}\right)\boldsymbol{S}_1(r) + Q(r, f) + Q(r, g)\,.$$

Suppose also that $FG \not\equiv 1$. By Lemma 5 and noting that $n \ge 3$, we have from (24) and (25) that, $\boldsymbol{S}_1(r) \le Q(r, f) + Q(r, g)$. Therefore, from (22), we have

$$(26)\qquad \boldsymbol{S}(r, f) = O(\log(rT(r, f)T(r, g)))\,, \quad r \notin E_2\,,$$

for some set $E_2 \subset [0, \infty)$ of finite linear measure.

By (1), we choose a real number $\varepsilon \in (0, (\beta - \alpha)/4)$ such that

$$(27)\qquad 2\pi + \alpha - \beta + 4\varepsilon < \frac{4}{\sigma} \arcsin\sqrt{\frac{\delta}{2}}\,.$$

Recalling that $\mu(f) < \infty$, $\lambda(f) > 0$ and $\mu(f) \le \sigma \le \lambda(f)$, we can apply Lemma 9 to $f(z)$ in order to confirm the existence of the Pólya peaks $\{r_n\}$ of order σ of f outside the set $E := E_1 \cup E_2$, where E_1 and E_2 are sets of finite linear measure appearing in the assumption $(*)$ and (26), respectively. Furthermore, applying Lemma 10 to the Pólya peaks $\{r_n\}$, we have either

$$(28)\qquad \operatorname{meas} D_\Lambda(r_n, \iota) > \frac{4}{\sigma} \arcsin\sqrt{\frac{\delta}{2}} - \varepsilon$$

or

$$(29)\qquad \operatorname{meas} D_\Lambda(r_n, \iota) > 2\pi - \varepsilon\,,$$

for each sufficiently large n, say $n \ge n_0$. Thus, it follows from (28) that

$$\begin{aligned}& \operatorname{meas}\{D_\Lambda(r_n, \iota) \cap (\alpha + \varepsilon, \beta - \varepsilon)\} \\ & \quad \ge \operatorname{meas} D_\Lambda(r_n, \iota) - \operatorname{meas}\{[-\pi,\ \pi] \setminus (\alpha + \varepsilon, \beta - \varepsilon)\} \\ & \quad = \operatorname{meas} D_\Lambda(r_n, \iota) - \operatorname{meas}\{[-\pi,\ \alpha + \varepsilon] \cup [\beta - \varepsilon, \pi]\} \\ & \quad = \operatorname{meas} D_\Lambda(r_n, \iota) - (2\pi + \alpha - \beta + 2\varepsilon) > \varepsilon > 0\,.\end{aligned}$$

It also follows from (29) that

$$\operatorname{meas}\{D_\Lambda(r_n, \iota) \cap (\alpha + \varepsilon, \beta - \varepsilon)\} > \varepsilon > 0\,.$$

Therefore, by the definition of $D_\Lambda(r_n, \iota)$ and setting $\Lambda(r) = 1/\log r$, we have

$$\begin{aligned}
(30)\qquad \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \log^+\left|\frac{1}{f(r_n e^{i\theta}) - \iota}\right| d\theta &\geq \int_{D_\Lambda(r_n,\iota)\cap(\alpha+\varepsilon,\beta-\varepsilon)} \log^+\left|\frac{1}{f(r_n e^{i\theta}) - \iota}\right| d\theta \\
&\geq \varepsilon \frac{T(r_n, f)}{\log r_n}, \quad n \geq n_0 .
\end{aligned}$$

Furthermore, we have by the definition of $B_{\alpha,\beta}(r_n, 1/(f-\iota))$ and (26) that

$$\begin{aligned}
(31)\qquad \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \log^+\left|\frac{1}{f(r_n e^{i\theta}) - \iota}\right| d\theta &\leq \frac{\pi r_n^\omega}{2\omega \sin(\varepsilon\omega)} B_{\alpha,\beta}\left(r_n, \frac{1}{f-\iota}\right) \\
&= \frac{\pi r_n^\omega}{2\omega \sin(\varepsilon\omega)} O(\log r_n T(r_n, f) T(r_n, g)), \quad n \geq n_0 .
\end{aligned}$$

From (30) and (31), we obtain

$$(32)\qquad T(r_n, f) \leq \frac{\pi {r_n}^\omega \log r_n}{2\omega\varepsilon \sin(\varepsilon\omega)} O(\log r_n T(r_n, f) T(r_n, g)) .$$

Hence, from the assumptions (1) and $(*)$, (32) implies that

$$\sigma \leq \liminf_{n\to\infty} \frac{\log T(r_n, f)}{\log r_n} \leq \omega < \sigma ,$$

which is impossible.

Therefore, we have $FG \equiv 1$. From (8) we obtain

$$f^n(f+a) g^n(g+a) \equiv b^2 ,$$

which implies that f does not take 0, $-a$ and ∞ in X. By using Lemma 4, we obtain

$$\boldsymbol{S}(r, f) = O(\log r T(r, f)) , \quad r \notin E ,$$

for some set $E \subset [0, \infty)$ of finite linear measure.

By a similar argument as above that (26) results in a contradiction, we can also deduce a contradiction. We remark here that $FG \equiv 1$ does not hold, even when $E_X(S_j, f) = E_X(S_j, g)$ $(j = 1, 2, 3)$ and f attains one of the three values 0, ∞ and $-a$ at least once in X. Hence, we obtain $F \equiv G$. By Lemma 6, we have $f \equiv g$.

This completes the proof of Theorem 1.

3.2. Proof of Theorem 2. Proceeding as in the proof of Theorem 1 with $\iota = \infty$, we have (23) and (24), and hence

$$C_2\left(r, \frac{1}{F}\right) + C_2\left(r, \frac{1}{G}\right) + 2\bar{C}(r, F) \leq 2\boldsymbol{S}_1(r) + Q(r, f) + Q(r, g) .$$

By Lemma 5 and noting that $n \geq 2$, we deduce that

$$\boldsymbol{S}_1(r) \leq Q(r, f) + Q(r, g) .$$

Therefore, we also obtain the conclusion of Theorem 2. $\square$

3.3. Proofs of Theorem 3 and Theorem 4. Suppose that Theorem 3 does not hold. Then, for any $\alpha \in [0, 2\pi)$, we have a constant $\varepsilon_\alpha \in (0, \pi/2)$ and a meromorphic function $g := g^{[\alpha]}$ in $\boldsymbol{C}$ such that $E(S_1, f) = E(S_1, g^{[\alpha]})$ and $E_{X(\alpha)}(S_j, f) = E_{X(\alpha)}(S_j, g^{[\alpha]})$ for $j = 2, 3$, but $f \not\equiv g^{[\alpha]}$, where $X(\alpha) := \{z \mid |\arg z - \alpha| < \varepsilon_\alpha\}$.

We define F and G as in (8); then F and G share 1 CM in $X(\alpha)$. By Lemma 6, we deduce that $F \not\equiv G$. Proceeding as in the proof of Theorem 1, we have (26) or $FG \equiv 1$. Suppose that $FG \equiv 1$. Then we can deduce a contradiction as proved there. Therefore, by Lemma 1 and (26), for any value $a \in \hat{\boldsymbol{C}}$, we have

$$C_{X(\alpha)}(r, a) := C_{\alpha-\varepsilon_\alpha, \alpha+\varepsilon_\alpha}(r, a) = O(\log rT(r, f)T(r, g^{[\alpha]}))\,, \quad r \notin E_\alpha\,, \tag{33}$$

where E_α is a set of r of finite linear measure depending on α and possibly on f and g.

On the other hand, we define $X_1(\alpha) = \{z \mid |\arg z - \alpha| < \varepsilon_\alpha/2\}$. Note that if an a-point $b_m = |b_m|e^{i\theta_m}$ of f in $X(\alpha)$ is in $X_1(\alpha)$,

$$\sin\omega(\theta_m - (\alpha - \varepsilon_\alpha)) \ge \sin\left(\frac{\omega\varepsilon_\alpha}{2}\right) = \frac{1}{\sqrt{2}}$$

since $\omega = \omega_\alpha := \pi/(2\varepsilon_\alpha)(> 1)$. Then we have

$$\begin{aligned}
C_{X(\alpha)}(2r, a) &= 2\sum_{\substack{1<|b_m|<2r\\ |\theta_m-\alpha|<\varepsilon_\alpha}} \left\{\frac{1}{|b_m|^\omega} - \frac{|b_m|^\omega}{(2r)^{2\omega}}\right\}\sin\omega(\theta_m - (\alpha - \varepsilon_\alpha))\\
&\ge \sqrt{2}\sum_{\substack{1<|b_m|<2r\\ |\theta_m-\alpha|<\varepsilon_\alpha/2}} \left\{\frac{1}{|b_m|^\omega} - \frac{|b_m|^\omega}{(2r)^{2\omega}}\right\}\\
&= \sqrt{2}\left\{\frac{n_f(2r, X_1(\alpha), a)}{(2r)^\omega} + \omega\int_1^{2r}\frac{n_f(t, X_1(\alpha), a)}{t^{\omega+1}}\,dt\right.\\
&\quad\left. - \frac{1}{(2r)^\omega}n_f(2r, X_1(\alpha), a) + \frac{\omega}{(2r)^{2\omega}}\int_1^{2r} n_f(t, X_1(\alpha), a)t^{\omega-1}\,dt\right\} + O(1)\\
&\ge \sqrt{2}\left\{\frac{n_f(r, X_1(\alpha), a)}{(2r)^{\omega+1}}(2r - r) + n_f(r, X_1(\alpha), a)r^{\omega-1}(2r - r)\right\} + O(1)\\
&\ge \sqrt{3}\, n_f(r, X_1(\alpha), a)\left\{\frac{1}{2^{\omega+1}}r^{-\omega} + \frac{2^\omega - 1}{2^{2\omega}}r^{-\omega}\right\} + O(1)\,.
\end{aligned}$$

It follows from (33) that

$$n_f(r, X_1(\alpha), a) = O(r^\omega \log rT(2r, f)T(2r, g^{[\alpha]}))\,, \quad r \notin E_\alpha\,. \tag{34}$$

If we identify the interval $[0, 2\pi)$ with the unit circle and $(\alpha - \varepsilon_\alpha/4, \alpha + \varepsilon_\alpha/4)$ with the corresponding open arc on the unit circle then, since the unit circle is compact and

$$[0,\ 2\pi) \subseteq \bigcup_{\alpha\in[0,2\pi)}\left(\alpha - \frac{\varepsilon_\alpha}{4},\ \alpha + \frac{\varepsilon_\alpha}{4}\right),$$

we can choose finitely many coverings $(\alpha_1 - \varepsilon_{\alpha_1}/4,\ \alpha_1 + \varepsilon_{\alpha_1}/4)$, $(\alpha_2 - \varepsilon_{\alpha_2}/4,\ \alpha_2 + \varepsilon_{\alpha_2}/4), \dots,$ $(\alpha_k - \varepsilon_{\alpha_k}/4,\ \alpha_k + \varepsilon_{\alpha_k}/4)$ of the interval $[0, 2\pi)$. Therefore, using Lemma 11 and (34), for any three distinct complex numbers a_j, $j = 1, 2, 3$, and recalling the assumption $(**)$, we have

$$
\begin{aligned}
S^*(r, f) &\le \sum_{i=1}^{k} S_f^*(r, \Delta_i) \\
&\le \sum_{i=1}^{k} \left\{ 3 \sum_{j=1}^{3} \bar{n}_f(2r, \Delta_i, a_j) \right\} + O(\log r) \qquad (35)\\
&= O(r^{\Omega} \log r T(4r, f)), \quad r \notin E_0,
\end{aligned}
$$

where $\Delta_i = \Delta_{(\alpha_i - \varepsilon_{\alpha_i}/4, \alpha_i + \varepsilon_{\alpha_i}/4)}$ $(i = 1, \dots, k)$, $\Omega := \max\{\omega(\alpha_1), \dots, \omega(\alpha_k), \tau\}$ and $E_0 := E_1 \cup \cdots \cup E_k \cup E$. Thus,

$$
T^*(r, f) = O(r^{\Omega+1} \log r T(4r, f)), \quad r \notin E_0. \qquad (36)
$$

By Lemma 12, it follows from (36) that

$$
T(r, f) = O(r^{\Omega+1} \log r T(4r, f)), \quad r \notin E_0.
$$

Here we use Lemma 13 in order to remove the exceptional set E_0.

Now we have by Lemma 13 that $T(r, f) \le M(r^{\Omega+1} \log r T(\kappa r, f))$ without any exceptional set of r, for some $M > 0$ and $\kappa > 4$. Therefore, by the assumption, we have $\limsup_{r\to\infty} \log T(r, f)/\log r < \infty$. Since $\lambda(f) = \infty$, this is a contradiction. Therefore, we have Theorem 3. The proof of Theorem 4 is similar to that of Theorem 3, so we omit it. □

4. Concluding remarks. In fact, we can obtain the following result from Section 3.

REMARK 5. The assumption '$E(S_1, f) = E(S_1, g)$' in Theorem 1 can be replaced by '$\bar{E}(S_1, f) = \bar{E}(S_1, g)$'. Similarly, in Theorem 2, Theorem 3 and Theorem 4, the assumption '$E(S_1, f) = E(S_1, g)$' can be replaced by '$\bar{E}(S_1, f) = \bar{E}(S_1, g)$'.

The above theorems in Section 1 indeed show the existence of an angular domain X such that if $E(S_1, f) = E(S_1, g)$ and $E_X(S_j, f) = E_X(S_j, g)$ for $j = 2, 3$ for certain meromorphic or entire functions f and g, then $f \equiv g$.

Finally, we make one remark concerning the conditions $(*)$ and $(**)$, which look really redundant. However, it does not seem straightforward to remove them in our discussion given above.

In fact, Lemmas 11 and 12, and the assumptions $\bar{E}_X(S_j, f) = \bar{E}_X(S_j, g)$ $(j = 1, 2, 3)$ imply that

$$
\begin{aligned}
T_f^*(r, X) &\le \frac{1}{n+1} T_F^*(r, X) + O(1) \\
&\le \frac{3}{n+1} \{\bar{N}(r, X, G = 0) + \bar{N}(r, X, G = \infty) + \bar{N}(r, X, G = 1)\} + O((\log r)^2) \\
&\le O(T_G^*(r, X)) + O((\log r)^2) \le O(T_g^*(r, X)) + O((\log r)^2).
\end{aligned}
$$

If we hope to remove the condition $(*)$ in Theorems 1 and 2 by using this estimate, we need however to have taken $X = \boldsymbol{C}$, which is inconsistent with our viewpoint mentioned in Section 1. Regarding the condition $(**)$ in Theorems 3 and 4, the $g^{[\alpha_i]}$ in the proof of those theorems must have been all independent of the angle α_i $(1 \leq i \leq k)$. However, we now should be careful enough to doubt the truth of this matter in general.

Under the condition $Q(r, f) = O(\log r \boldsymbol{S}_{\alpha,\beta}(r, f))$, Zheng in [31] obtained uniqueness of the meromorphic functions which have five shared values in a precise subset of $\boldsymbol{C}$ and in [32] considered the case of dealing with four shared values. Following his proof, the condition $(*)$ would imply the same conclusion as his result for meromorphic functions in $\boldsymbol{C}$.

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