*Uniqueness Under Nonlinear Boundary Conditions for Elliptic Problems**

SIMON LEVIN

Communicated by J. B. DIAZ

1. Introduction. The purpose of this paper is to develop a method for generalizing Green's identities in order to study questions of uniqueness for quasilinear partial differential equations of second order under nonlinear boundary conditions. In particular, the general theory for elliptic equations is expounded in detail (§3); and in §4, a number of examples illustrative of the technique are given

Given a plane region S bounded by a simple closed curve C, it is well-known that two regular solutions ϕ_1, ϕ_2 of Neumann's problem

$$\Delta\phi = \phi_{xx} + \phi_{yy} = 0 \text{ in } S, \qquad \frac{\partial\phi}{\partial n} = h(s) \text{ on } C \tag{1.1}$$

for Laplace's equation differ by at most a constant. Here s denotes arc length along C. The usual proof [5] rests upon Green's integral identity

$$\int_C \phi \frac{\partial\phi}{\partial n}\, ds = \int_S [\phi_x^2 + \phi_y^2 + \phi\Delta\phi]\, dS.$$

One sets $\phi = \phi_1 - \phi_2$ to obtain the identity

$$\int_C (\phi_1 - \phi_2)\left(\frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n}\right) ds$$
$$= \int_S [(p_1 - p_2)^2 + (q_1 - q_2)^2 + (\phi_1 - \phi_2)(\Delta\phi_1 - \Delta\phi_2)]\, dS,$$

where, following Monge, we write

$$p_i = \frac{\partial\phi_i}{\partial x}, \quad q_i = \frac{\partial\phi_i}{\partial y}, \quad r_i = \frac{\partial^2\phi_i}{\partial x^2}, \quad s_i = \frac{\partial^2\phi_i}{\partial x\, \partial y}, \quad t_i = \frac{\partial^2\phi_i}{\partial y^2}, \quad (i = 1, 2).$$

If ϕ_1 and ϕ_2 are two solutions of the boundary problem (1.1), the integral over C

* This research was partially supported by a National Science Foundation Fellowship, and by Grants ARO(D)-31-124-G563 and (NSF)-GP-5882.

and hence the integral over S in (1.2) vanish, from which $p_1 = p_2$, $q_1 = q_2$ readily follows to imply the stated result.

In place of Laplace's equation, we consider the quasilinear equation

$$L[\phi] = Ar + 2Bs + Ct = D, \tag{1.3}$$

where A, B, C, D are functions of x, y, ϕ, p, q; and instead of (1.1), pose the boundary problem

$$L[\phi] = D \quad \text{in} \quad S, \qquad \frac{\partial \phi}{\partial n} = f(x, y, \phi, p, q) \quad \text{on} \quad C. \tag{1.4}$$

We seek those functions f for which a suitable generalization of the Green identity (1.2) can be found which will yield information on the uniqueness of the solution.

We begin by replacing (1.2) by the generalized identity

$$\begin{aligned} \int_C \tau\left(f_2 \frac{\partial \phi_1}{\partial n} - f_1 \frac{\partial \phi_2}{\partial n}\right) ds \\ = \int_S [u_{\phi_i} p_i + v_{\phi_i} q_i + u_{p_i} r_i + (u_{q_i} + v_{p_i}) s_i + v_{q_i} t_i + u_x + v_y]\, dS, \end{aligned} \tag{1.5}$$

obtained through Gauss's theorem. Here

$$f_1 = f(x, y, \phi_1, p_1, q_1), \qquad f_2 = f(x, y, \phi_2, p_2, q_2), \tag{1.6}$$

and

$$\tau = \tau(x, y, \phi_i, p_i, q_i),$$

with

$$u = \tau[f_2 p_1 - f_1 p_2], \qquad v = \tau[f_2 q_1 - f_1 q_2];$$

and we employ the summation convention for repeated indices. The region S is permitted to be multiply connected, with its boundary C composed of a finite number of smooth, simple closed curves. For simplicity, the functions f_1, f_2, τ are generally assumed single-valued and real analytic in their arguments. The functions ϕ_1, ϕ_2, which may be multi-valued, are of class C' in $S + C$ and C'' in S.

If for suitable multipliers λ_1, λ_2 we impose the conditions

$$\begin{aligned} u_{p_1} r_1 + (u_{q_1} + v_{p_1}) s_1 + v_{q_1} t_1 = \lambda_1 L[\phi_1], \\ u_{p_2} r_2 + (u_{q_2} + v_{p_2}) s_2 + v_{q_2} t_2 = \lambda_2 L[\phi_2] \end{aligned} \tag{1.7}$$

on τ and f, the identity (1.5) becomes

$$\int_C \tau \cdot \left(f_2 \frac{\partial \phi_1}{\partial n} - f_1 \frac{\partial \phi_2}{\partial n}\right) ds = \int_S (u_{\phi_i} p_i + v_{\phi_i} q_i + \lambda_i L[\phi_i] + u_x + v_y)\, dS. \tag{1.8}$$

Hence, if ϕ_1 and ϕ_2 are two solutions of the boundary problem (1.4), then (1.8) reduces to

$$\int_S (u_{\phi_i} p_i + v_{\phi_i} q_i + \lambda_i D_i + u_x + v_y)\, dS = 0, \tag{1.9}$$

where $D_i = D(x, y, \phi_i, p_i, q_i)$; and our problem is to avail ourselves of whatever arbitrariness remains in the choice of τ and f so that (1.9) will enable us to determine the conditions under which the solution to (1.4) is unique.

The question of existence of solutions to (1.4) is, of course, of great interest. Our investigation is limited to the study of uniqueness in case a solution to a given problem exists. We leave untouched the question of whether a solution actually exists.

2. Overdetermined systems. In solving (1.7), we are led to seek solutions of the system ([1])

$$\frac{u_{p_1}}{A_1} = \frac{u_{q_1} + v_{p_1}}{2B_1} = \frac{v_{q_1}}{C_1} = \lambda_1, \qquad \frac{u_{p_2}}{A_2} = \frac{u_{q_2} + v_{p_2}}{2B_2} = \frac{v_{q_2}}{C_2} = \lambda_2;$$

or equivalently,

$$\begin{aligned} u_{p_1} &= \frac{A_1}{C_1} v_{q_1}, & u_{p_2} &= \frac{A_2}{C_2} v_{q_2}, \\ u_{q_1} &= -v_{p_1} + \frac{2B_1}{C_1} v_{q_1}, & u_{q_2} &= -v_{p_2} + \frac{2B_2}{C_2} v_{q_2}, \end{aligned} \tag{2.1}$$

provided $C_1 \neq 0$, $C_2 \neq 0$. Setting $\sigma = \tau f_1 f_2$ in (1.6), we see that u, v must have the special form

$$u = \sigma\left(\frac{p_1}{f_1} - \frac{p_2}{f_2}\right), \qquad v = \sigma\left(\frac{q_1}{f_1} - \frac{q_2}{f_2}\right). \tag{2.2}$$

The system (2.1) is a special case of the general *overdetermined system*,

$$\begin{aligned} u_{x_1} &= a_1 v_{x_1} + b_1 v_{y_1}, & u_{x_2} &= a_2 v_{x_2} + b_2 v_{y_2}, \\ u_{y_1} &= c_1 v_{x_1} + d_1 v_{y_1}, & u_{y_2} &= c_2 v_{x_2} + d_2 v_{y_2}, \end{aligned} \tag{2.3}$$

of four partial differential equations for two unknown functions u, v of four independent variables x_1, y_1, x_2, y_2. Here, the coefficients a_1, b_1, c_1, d_1 in the *first component system* are functions of x_1, y_1 only, and are assumed to be single-valued and real analytic in some region R_1 of the (x_1, y_1)-plane; similarly, the coefficients a_2, b_2, c_2, d_2 in the *second component system* are functions of x_2, y_2 only, and are single-valued and real analytic in some region R_2 of the (x_2, y_2)-plane. Finally, u and v are single-valued and real analytic functions of x_1, y_1, x_2, y_2 in $R_1 \times R_2$.

A detailed study of such systems may be found in [6] and [7], with particular emphasis being placed, in view of (2.2), on solutions of the form

$$u = \sigma\cdot[P_1 - P_2], \qquad v = \sigma\cdot[Q_1 - Q_2], \tag{2.4}$$

in which P_1, Q_1 depend only on x_1, y_1; P_2, Q_2 only on x_2, y_2; and σ depends on all four variables. For the remainder of this paper, we shall be concerned only with some particular systems (2.1) arising in the study of the uniqueness of solutions for certain partial differential equations. We shall prove only those properties of such systems relevant to our analysis.

It is clear that every overdetermined system (2.3) will possess solutions

$$u = u_1 - u_2, \qquad v = v_1 - v_2, \tag{2.5}$$

where u_1, v_1 depend only on x_1, y_1; and u_2, v_2, only on x_2, y_2. Such solutions, we term *separable*; and moreover, functions such as u and v in (2.5) are also called separable. More generally, if a solution may be written in the form (2.4), it is termed *quasi-separable.*

3. Elliptic systems. The system

$$u_x = av_x + bv_y$$
$$u_y = cv_x + dv_y$$

is termed elliptic if $(a - d)^2 + 4bc < 0$. It may be shown ([6], [7]) that if one of the component systems in the overdetermined system (2.3) is elliptic, then every solution will be separable unless the other component system is also elliptic and the two systems share a pair of constant, complex conjugate eigenvalues $r \pm is$. In this case, the solution of (1.3) is

$$u = U + \frac{r}{s} V, \qquad v = \frac{V}{s},$$

where $U + iV$ is an arbitrary analytic function of $\zeta_1 = \xi_1 + i\eta_1$, $\zeta_2 = \xi_2 + i\eta_2$, and the functions

$$\xi_j(x_j, y_j), \qquad \eta_j(x_j, y_j)$$

are solutions of the Beltrami equations ([4])

$$s\xi_{x_j} = (a_j - r)\eta_{x_j}, \qquad s\xi_{y_j} = c_j\eta_{x_j} + (d_j - r)\eta_{y_j}, \qquad j = 1, 2.$$

This reduces the search for quasi-separable solutions in the elliptic case to the search for quasi-separable solutions of the Cauchy–Riemann equations

$$\begin{aligned} U_{\xi_1} &= V_{\eta_1}, \qquad & U_{\xi_2} &= V_{\eta_2}, \\ U_{\eta_1} &= -V_{\xi_1}, \qquad & U_{\eta_2} &= -V_{\xi_2}; \end{aligned} \tag{3.1}$$

i.e., to the determination of those analytic functions $F = U + iV$ of two complex variables ζ_1, ζ_2 for which

$$U = \sigma P, \qquad V = \sigma Q, \tag{3.2}$$

with

$$P = P_1(\xi_1, \eta_1) - P_2(\xi_2, \eta_2), \qquad Q = Q_1(\xi_1, \eta_1) - Q_2(\xi_2, \eta_2),$$
$$\sigma = \sigma(\xi_1, \eta_1, \xi_2, \eta_2).$$

Such functions will in general be termed *quasi-separable* analytic functions of ζ_1, ζ_2; and in particular, those for which

$$U = U_1(\xi_1, \eta_1) - U_2(\xi_2, \eta_2), \qquad V = V_1(\xi_1, \eta_1) - V_2(\xi_2, \eta_2),$$

so that

$$F = f_1(\zeta_1) - f_2(\zeta_2),$$

are called *separable*. The question of quasi-separable analytic functions of ζ_1, ζ_2 is completely answered by the following theorem ([6], [7]):

Theorem 3.1. *The only quasi-separable analytic functions F of two complex variables $\zeta_1 = \xi_1 + i\eta_1$, $\zeta_2 = \xi_2 + i\eta_2$ are those obtained from separable analytic functions H through the operations H, H^{-1}, and e^H. Conversely, if H is separable analytic, H, H^{-1}, and e^H are quasi-separable analytic.*

If σ is constant in (3.2), F is separable and the conclusion of the first statement of the theorem is satisfied. In completing the proof of the first statement, we therefore without loss of generality assume σ is not constant. By definition, if F is quasi-separable, it has the form (3.2). Thus, since $H = \log F$ is an analytic function of ζ_1, ζ_2 in a cut domain, its real and imaginary parts

$$R = \log (U^2 + V^2)^{\frac{1}{2}}, \qquad \Theta = \arctan \frac{V}{U}$$

satisfy the Cauchy–Riemann equations

$$(3.4) \qquad \begin{aligned} R_{\xi_1} &= \Theta_{\eta_1}, & R_{\xi_2} &= \Theta_{\eta_2} \\ R_{\eta_1} &= -\Theta_{\xi_1}, & R_{\eta_2} &= -\Theta_{\xi_2}. \end{aligned}$$

Consequently, $\Theta = \arctan (Q_1 - Q_2)/(P_1 - P_2)$ must satisfy the integrability conditions

$$(3.5) \qquad \begin{aligned} \Theta_{\xi_1\xi_1} + \Theta_{\eta_1\eta_1} &= 0, & \Theta_{\xi_2\xi_2} + \Theta_{\eta_2\eta_2} &= 0, \\ \Theta_{\xi_1\eta_2} - \Theta_{\eta_1\xi_2} &= 0, & \Theta_{\xi_1\xi_2} + \Theta_{\eta_1\eta_2} &= 0. \end{aligned}$$

If Θ is separable (see 2.5), then (3.4) insures that R is also separable; and again the claim of the first statement is satisfied, since $F = e^H$ for a separable H.

If Θ is not separable, then it is immediate that for each $i = 1, 2$, the functions P_i and Q_i are independent. For suppose, without loss of generality, that P_1, Q_1 are functionally dependent. Then

$$P_{\xi_1}Q_{\eta_1} - P_{\eta_1}Q_{\xi_1} = 0,$$

and from this we deduce that

$$(3.6) \qquad P_{\xi_1}\Theta_{\eta_1} - P_{\eta_1}\Theta_{\xi_1} = 0, \qquad Q_{\xi_1}\Theta_{\eta_1} - Q_{\eta_1}\Theta_{\xi_1} = 0.$$

Differentiation of (3.6) yields the system

$$(3.7) \qquad \begin{aligned} P_{\xi_1}\Theta_{\eta_1\xi_2} - P_{\eta_1}\Theta_{\xi_1\xi_2} &= 0, & Q_{\xi_1}\Theta_{\eta_1\xi_2} - Q_{\eta_1}\Theta_{\xi_1\xi_2} &= 0, \\ P_{\xi_1}\Theta_{\eta_1\eta_2} - P_{\eta_1}\Theta_{\xi_1\eta_2} &= 0, & Q_{\xi_1}\Theta_{\eta_1\eta_2} - Q_{\eta_1}\Theta_{\xi_1\eta_2} &= 0; \end{aligned}$$

and discarding the case $P_{\xi_1} = P_{\eta_1} = 0$, $Q_{\xi_1} = Q_{\eta_1} = 0$ as leading to a separable Θ, we get

$$\Theta_{\xi_1\xi_2}\Theta_{\eta_1\eta_2} - \Theta_{\xi_1\eta_2}\Theta_{\eta_1\xi_2} = 0.$$

From this and (3.5), we would immediately arrive at a contradiction to the hypothesis: Θ not separable. Thus functional independence is established, and the first equation in (3.5) yields:

$$\begin{aligned}(3.8)\qquad & P(P^2 + Q^2)(Q_{\xi_1\xi_1} + Q_{\eta_1\eta_1}) - Q(P^2 + Q^2)(P_{\xi_1\xi_1} + P_{\eta_1\eta_1}) \\ & - 2(P^2 - Q^2)(Q_{\xi_1}P_{\xi_1} + Q_{\eta_1}P_{\eta_1}) + 2PQ(P_{\xi_1}^2 + P_{\eta_1}^2 - Q_{\xi_1}^2 - Q_{\eta_1}^2) = 0.\end{aligned}$$

This identity in the variables ξ_1, η_1, ξ_2, η_2 continues to hold if the variables ξ_1, η_1 are held constant, and ξ_2, η_2 allowed to vary. Since P_2, Q_2 are functionally independent, the variables P, Q may be regarded as independent variables, and the coefficients of the cubic (3.8) in P, Q, as constants with respect to ξ_2, η_2. It follows that the coefficients in (3.8) must vanish identically, and we have

$$(3.9)\qquad \begin{aligned} P_{\xi_1\xi_1} + P_{\eta_1\eta_1} = 0, \qquad & P_{\xi_1}Q_{\xi_1} + P_{\eta_1}Q_{\eta_1} = 0, \\ Q_{\xi_1\xi_1} + Q_{\eta_1\eta_1} = 0, \qquad & P_{\xi_1}^2 + P_{\eta_1}^2 - Q_{\xi_1}^2 - Q_{\eta_1}^2 = 0.\end{aligned}$$

These equations are satisfied if and only if $P + iQ$ or $P - iQ$ is an analytic function of ζ_1; and by analogy, we obtain the identical result for ζ_2.

Finally, we shall conclude that $P - iQ$ is in fact an analytic function of ζ_1, ζ_2. Since we already know that $U + iV = \sigma(P + iQ)$ is also analytic, it follows that the product $\sigma(P^2 + Q^2)$ of these two analytic functions, being real, must be a constant. Hence $F = H^{-1}$ for a separable H, and the proof of the first statement of the theorem is completed.

The proof that $P - iQ$ is analytic is by contradiction. For let us suppose, without loss of generality, that it is not an analytic function of ζ_1. Then, as already observed, it follows from (3.9) that $P + iQ$ is an analytic function of ζ_1, as is $U + iV = \sigma(P + iQ)$, so that σ is independent of ξ_1, η_1. Thus

$$F(\zeta_1, \zeta_2) = \sigma(\zeta_2)[P + iQ],$$

where $P + iQ$ is analytic in ζ_1; and F, in ζ_1, ζ_2. It further is immediate that

$$0 = F_{\zeta_1\bar{\zeta}_2} = \sigma_{\bar{\zeta}_2}[P + iQ]_{\zeta_1} = \sigma_{\bar{\zeta}_2}\overline{[P - iQ]_{\bar{\zeta}_1}}.$$

Since by assumption, $[P - iQ]_{\bar{\zeta}_1} \neq 0$, it follows that σ must be complex analytic in ζ_2, and hence constant, a contradiction. Thus $P - iQ$ is in fact analytic.

Conversely, if H is a separable analytic function of ζ_1, ζ_2, it is by definition quasi-separable; and it is moreover easy to see that H^{-1} is also quasi-separable analytic, for

$$H^{-1} = \frac{1}{|H|^2}\bar{H},$$

and $\bar{H}$ is separable. To complete the proof of the theorem, all that remains to be shown is that e^H is quasi-separable. If we write $H = R + i\Theta$, with

$$R = R_1(\xi_1, \eta_1) - R_2(\xi_2, \eta_2), \qquad \Theta = \theta_1(\xi_1, \eta_1) - \theta_2(\xi_2, \eta_2),$$

we find

$$e^H = e^{R_1 - R_2} e^{i(\theta_1 - \theta_2)} = e^{R_1 - R_2} \frac{e^{i2\theta_1} + e^{-i2\theta_2}}{e^{i(\theta_1+\theta_2)} + e^{-i(\theta_1+\theta_2)}}.$$

But then

$$e^H = \sigma(e^{i2\theta_1} + e^{-i2\theta_2}),$$

where

$$\sigma = \tfrac{1}{2}e^R \sec(\theta_1 + \theta_2)$$

is real, so that e^H is quasi-separable, as claimed.

The representations utilized in the preceding section are essentially unique up to arbitrary constants. and this fact will be demonstrated in the ensuing corollaries. We shall confine our interest to *non-trivial* analytic functions of two complex variables, *i.e.* ones which actually depend on both variables.

Corollary 3.1. *If F is a non-trivial separable analytic function of ζ_1, ζ_2, and if we write $F = U + iV$, where U, V has the form* (3.2), *then σ is a constant.*

It is clear that the proof of Theorem 3.1 also suffices to establish that if $F = U + iV$ is any quasi-separable analytic function, even a separable one, and if U, V has the form (3.2) with σ not constant, then $F = H^{-1}$ or e^H for some separable function H. But since F itself is also separable analytic, this cannot happen for non-trivial F, and thus Corollary 3.1 is proven.

Corollary 3.2. *If $F = H^{-1}$, where H is a non-trivial separable analytic function of ζ_1, ζ_2, and if we write $F = U + iV$, where U, V has the form* (3.2), *then $\sigma = k/|H|^2$ for some real constant k.*

By hypothesis, $\bar{F}/|F|^2$ is a separable analytic function of ζ_1, ζ_2, and hence $F/|F|^2$ is a separable analytic function of $\bar{\zeta}_1, \bar{\zeta}_2$. Corollary 3.2 is thus an immediate consequence of Corollary 3.1.

Corollary 3.3. *If $F = e^H$, where*

$$H = R + i\Theta = R_1(\xi_1, \eta_1) - R_2(\xi_2, \eta_2) + i\theta_1(\xi_1, \eta_1) - i\theta_2(\xi_2, \eta_2)$$

is a non-trivial separable analytic function of $\zeta_1 = \xi_1 + i\eta_1$, $\zeta_2 = \xi_2 + i\eta_2$, and if we write $F = U + iV$, where U, V has the form (3.2), *then $\sigma = k\, e^R \sec(\theta_1 + \theta_2 - \theta_0)$ for some real constants k, θ_0.*

From (3.2), it is clear that

$$\sigma P = e^R \cos\Theta, \qquad \sigma Q = e^R \sin\Theta. \tag{3.10}$$

Since R_1, θ_1 are independent functions of ξ_1, η_1; and R_2, θ_2, of ξ_2, η_2, it follows that P, Q will be separable functions of ξ_1, η_1 and ξ_2, η_2, *i.e.* of the form (2.5), if and only if they are separable functions of R_1, θ_1 and R_2, θ_2, respectively, *i.e.*, if and only if

$$P_{R_1R_2} = Q_{R_1R_2} = 0, \qquad P_{R_1\theta_2} = Q_{R_1\theta_2} = 0, \qquad (3.11)$$
$$P_{\theta_1R_2} = Q_{\theta_1R_2} = 0, \qquad P_{\theta_1\theta_2} = Q_{\theta_1\theta_2} = 0.$$

If we set $G = e^R/\sigma$ in (3.10), the second pair of equations in (3.11) implies $G_{R_1} = 0$; and the third pair, $G_{R_2} = 0$. Thus $G = G(\theta_1, \theta_2)$, and the corollary follows from (3.11) and elementary trigonometric identities.

4. Nonlinear boundary problems for elliptic equations. The application of the results on elliptic overdetermined systems just obtained to uniqueness questions is most clearly illustrated by the example of the equation

$$\Delta\phi = F(x, y, \phi, p, q). \qquad (4.1)$$

We make the assumptions that S is a simply-connected region with boundary C, and that ϕ is a C^2 solution of (4.1) in S, continuous in the closure; and attempt to pose boundary problems whose solution is unique. In the light of (2.1) and (2.2), we seek to determine those functions $f(\phi, p, q)$ for which there exist real valued functions $\sigma(x, y, \phi_1, \phi_2, p_1, p_2, q_1, q_2)$ such that

$$U + iV = \sigma\left[\frac{z_1}{f(\phi_1, p_1, q_1)} - \frac{z_2}{f(\phi_2, p_2, q_2)}\right] \qquad (4.2)$$

is an analytic function of $z_1 = p_1 + iq_1$, $z_2 = p_2 + iq_2$. By Theorem 1.2, if this is possible, then necessarily $U + iV = H, H^{-1}$, or e^H, where H is a separable analytic function of z_1, z_2. We shall consider these three cases in the ensuing sections.

4.1. $U + iV$ separable. If $U + iV$ is separable analytic (and non-trivial), then Corollary 3.1 tells us that

$$\sigma = \sigma(x, y, \phi_1, \phi_2), \qquad f = f(\phi). \qquad (4.3)$$

Under these restrictions, (1.5) becomes

$$\int_C \sigma(x, y, \phi_1, \phi_2)\left[\frac{\frac{\partial\phi_1}{\partial n}}{f(\phi_1)} - \frac{\frac{\partial\phi_2}{\partial n}}{f(\phi_2)}\right] ds \qquad (4.4)$$
$$= \int_S \left\{U_{\phi_i}p_i + V_{\phi_i}q_i + \sigma\left(\frac{F_1}{f_1} - \frac{F_2}{f_2}\right) + \sigma_x\left(\frac{p_1}{f_1} - \frac{p_2}{f_2}\right) + \sigma_y\left(\frac{q_1}{f_1} - \frac{q_2}{f_2}\right)\right\} dS,$$

where $f_i = f(\phi_i)$ and $F_i = F(x, y, \phi_i, p_i, q_i)$, $i = 1, 2$.

When $\sigma = \sigma(\phi_1, \phi_2)$, this identity has been studied in great detail by M. H. Martin ([8], [9], [10], [11]); and in its simplest forms reduces to the classical identities of potential theory ([5]). Its most familiar application occurs in the special case $f \equiv 1$, $\sigma = \phi_1 - \phi_2$, for which it reduces to the identity (1.2). From this it follows that any two solutions of the boundary problem

$$\Delta\phi = F(x, y, \phi) \text{ in } S, \; \phi_n = -G(s, \phi) \text{ on } C \tag{4.5}$$

differ by a constant if F, G are continuous in their arguments and non-decreasing in ϕ. This result, for $F \equiv 0$, is due to Carleman [3]. Actually, one may clearly extend the result to yield a uniqueness theorem, where $\phi_n = -G(s, \phi)$ holds only on a portion of the boundary, and where ϕ is specified as some function of arc length on the remainder.

Although the identity (1.2) discussed in the preceding paragraph is well-known, the following identity, valid for any two solutions ϕ_1, ϕ_2 of the equation

$$\Delta\phi + [\text{grad } \alpha]\cdot[\text{grad } \phi] + k\,|\text{grad } \phi|^2 = g(\phi, x, y),$$

where the constant k and the function $\alpha(x, y)$ are given, is new, and is a direct application of our methods:

$$\int_C e^{\alpha+2k\phi_1} w(\phi_1 - \phi_2)\left(\frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n}\right) ds$$
$$= \int_S e^{\alpha+2k\phi_1}\{[w' + kw][(p_1 - p_2)^2 + (q_1 - q_2)^2 + w[g(\phi_1, x, y) - g(\phi_2, x, y)]\}\, dS.$$

Here w is an arbitrary function of $\phi_1 - \phi_2$. If we take $w = (\phi_1 - \phi_2)e^{-k(\phi_1-\phi_2)}$, then both w and $w' + kw$ are positive, and the following theorem emerges:

Theorem 4.1. *If $g(x, y, \phi)$ and $G(s, \phi)$ are continuous functions, non-decreasing in ϕ, then any two solutions of the boundary problem*

$$\Delta\phi + [\text{grad } \alpha]\cdot[\text{grad } \phi] + k\,|\text{grad } \phi|^2 = g(x, y, \phi) \quad in \quad S,$$
$$\phi = h(s) \quad on \quad C_1, \qquad \phi_n = -G(s, \phi) \quad on \quad C_2$$

differ by at most a constant. Here the functions $\alpha = \alpha(x, y)$, $h(s)$, and the constant k are also given, and $C = C_1 + C_2$.

4.2. Two integral identities and Stekloff's problem. For harmonic functions, the Stekloff boundary condition that $\phi_n = \phi h(s)$ has long been a source of study, usually with some restriction on the sign of $h(s)$. In seeking to avoid such restrictions, Martin [8] made use of the following identity, obtainable from (4.4) by setting $\sigma = \phi_1^2$:

$$\int_C \lambda\left(\phi_2 \frac{\partial\phi_1}{\partial n} - \phi_1 \frac{\partial\phi_2}{\partial n}\right) ds = \int_S [\phi_2^2(\lambda_x^2 + \lambda_y^2) + \phi_1(\Delta\phi_1 - \lambda\Delta\phi_2)]\, dS. \tag{4.5}$$

Here $\lambda = \phi_1/\phi_2$ must be assumed C^1 in S and continuous in the closure.

In order to study a wide class of problems, of which Stekloff's problem is a special case, we consider the boundary condition $\phi_n = f(\phi)h(\phi, s)$; and in place of λ, introduce the quantity $\mu = f(\phi_1)/f(\phi_2) = f_1/f_2$, which we take to be C^1 in S and continuous in $S + C$. (Here and henceforth it will be understood that all remarks remain valid provided that μ is more generally defined as the unique solution with the required continuity properties for the equation $f(\phi_1) = \mu f(\phi_2)$. It is important to note that this uniqueness implies that $f^2(\phi_2) > 0$ on a dense subset of S.) We obtain at once the identity:

$$(4.6) \qquad \int_C \mu\left(f_2 \frac{\partial \phi_1}{\partial n} - f_1 \frac{\partial \phi_2}{\partial n}\right) ds = \int_S [f_1'(p_1^2 + q_1^2) - 2\mu f_1'(p_1 p_2 + q_1 q_2) + \mu^2 f_2'(p_2^2 + q_2^2) + \mu(f_2 \, \Delta\phi_1 - f_1 \, \Delta\phi_2)] \, dS.$$

The uniqueness theorems obtainable from this identity may be derived from the following basic theorem. Applications to the most familiar cases will be given as corollaries.

Theorem 4.2. *Given $G(\phi, x, y)$ continuous and $f(\phi)$ continuously differentiable in $S + C$, and $h(\phi, s)$ continuous on C, if ϕ_2 is a solution to the boundary problem*

$$(4.8) \qquad \Delta\phi = f(\phi)G(\phi, x, y) \quad in \quad S, \qquad \phi_n = f(\phi)h(\phi, s) \quad on \quad C,$$

then any other solution ϕ_1 for which the ratio $\mu = f(\phi_1)/f(\phi_2)$ is $C'(S)$ and continuous in $S + C$, for which the inequalities $G(\phi_1, x, y) \geqq G(\phi_2, x, y)$ and $0 < f'(\phi_1) \leqq f'(\phi_2)$ hold on a dense subset of S, and for which the inequality $h(\phi_1, S) \leqq h(\phi_2, S)$ holds on C, must be linearly dependent on ϕ_2. In fact, μ and $\phi_1 - \mu\phi_2$ must both be constant.

(If the condition $0 < f'(\phi_1) \leqq f'(\phi_2)$ on a dense subset of S is replaced by the conditions $0 \leqq f'(\phi_1) \leqq f'(\phi_2)$ and $f'(\phi_2) > 0$ on a dense subset W of S, it is still possible to conclude that μ is constant. The integrand in (4.9) is non-positive at every point, and hence identically zero. Moreover, at each point of W such that $f'(\phi_1) > 0$, we have already seen that $(\mu_x^2 + \mu_y^2)f_2^2 = 0$. At those points where $f'(\phi_1) = 0$, we may conclude from (4.9) that $\mu^2(p_2^2 + q_2^2) = 0$. Yet it is still true from the definition of μ that

$$(\mu_x^2 + \mu_y^2)f_2^2 = (f_1'p_1 - \mu f_2'p_2)^2 + (f_1'q_1 - \mu f_2'q_2)^2.$$

In this case, the facts that $f_1' = 0$ and $\mu^2(p_2^2 + q_2^2) = 0$ lead at once to the result $(\mu_x^2 + \mu_y^2)f_2^2 = 0$. Thus $(\mu_x^2 + \mu_y^2)f_2^2 \equiv 0$ on W; and by continuity, on S. Since $f_2^2 > 0$ on a dense subset of S, it follows that $\mu_x = \mu_y \equiv 0$ in S, and so μ is constant.)

It follows from (4.6) and the conditions on μ, G, h that

$$(4.9) \qquad \int_S [f_1'(p_1^2 + q_1^2) - 2\mu f_1'(p_1 p_2 + q_1 q_2) + \mu^2 f_2'(p_2^2 + q_2^2)] \, dS \leq 0.$$

Moreover, since $f_1' \leqq f_2'$, it is immediate from (4.9) that

$$\int_S f_1'[(p_1 - \mu p_2)^2 + (q_1 - \mu q_2)^2] \, dS \leq 0;$$

and hence, because $f_1' > 0$ on a dense subset of S, that

$$p_1 = \mu p_2 , \qquad q_1 = \mu q_2 . \tag{4.10}$$

With (4.10) established, to prove the theorem it is clearly necessary and sufficient to prove that μ is constant; and to this end, we examine the quantity $T = \mu_x p_2 + \mu_y q_2$. On the one hand, T is simply $\Delta\phi_1 - \mu\, \Delta\phi_2$, and so it follows from (4.8) and the condition $G(\phi_1 , x, y) \geqq G(\phi_2 , x, y)$ that T has everywhere the same sign as $f(\phi_1)$. However, from (4.10) and the definition of μ, we have that

$$f_1 T = \mu^2(f_1' - f_2')(p_2^2 + q_2^2).$$

Thus, since $f_1' \leqq f_2'$, the product $f_1 T$ is not positive; and therefore must, by the above, vanish identically. Hence T is identically zero, and it is an immediate consequence of (4.10) and the definition of μ that $(\mu_x^2 + \mu_y^2)f_2^2 \equiv 0$. Finally, since μ is $C^1(S)$ and $f_2^2 > 0$ on a dense subset of S, it follows that $\mu_x \equiv \mu_y \equiv 0$ in S; and so μ is constant.

Corollary 4.1. (***Stekloff Problem***). *If ϕ_2 is a solution to the boundary problem*

$$\Delta\phi = \phi G(\phi, x, y) \quad in \quad S, \qquad \phi_n = \phi h(\phi, s) \quad on \quad C,$$

then any other solution ϕ_1 for which the ratio $\lambda = \phi_1/\phi_2$ is $C^1(S)$ and continuous on $S + C$, for which $G(\phi_1 , x, y) \geqq G(\phi_2 , x, y)$ in S, and for which $h(\phi_1 , s) \leqq h(\phi_2 , s)$ on C, must be a multiple of ϕ_2 .

The corollary is an immediate consequence of Theorem 4.2; and with G and h independent of ϕ, it is proved by Martin in [8]. As he points out ([9]), the requirement that ϕ_1/ϕ_2 be $C^1(S)$ cannot be completely eliminated. For example, given any positive integer k, the real and imaginary parts of the analytic function z^k are both of course harmonic, and they satisfy the boundary condition $\phi_n = k\phi$ on $|z| = 1$. They are certainly not linearly dependent.

The methods and results pertaining to the Stekloff problem can be generalized, and one obtains the following important corollary.

Corollary 4.2. *Let the continuous functions $G(\phi, x, y)$ and $h(\phi, s)$ be given on $S + C$ and on C respectively, and let the positive integer k also be given. Further, let ϕ_1 , ϕ_2 be two distinct solutions to the boundary problem*

$$\Delta\phi = \phi^{1+k}G(\phi, x, y) \quad in \quad S, \qquad \phi_n = \phi^{1+k}h(\phi, s) \quad on \quad C,$$

subject to the following conditions:

(i) *$\mu = \phi_1^{1+k}/\phi_2^{1+k}$ is $C^1(S)$ and continuous on $S + C$, and $|\mu| \leqq 1$,*
(ii) *if k is odd, then ϕ_1 and ϕ_2 are both non-negative,*
(iii) *$G(\phi_1 , x, y) \geqq G(\phi_2 , x, y)$ in S,*
(iv) *$h(\phi_1 , s) \leqq h(\phi_2 , s)$ on C.*

Then either ϕ_1 is constant or $\phi_1 \equiv -\phi_2$. In fact, for k odd, condition (ii) *insures that ϕ_1 is constant.*

Simply define $f(\phi) = \phi^{1+k}$. All of the hypotheses of Theorem 4.2 are satisfied, except possibly the condition that $0 < f_1'(\phi_1) \leqq f_2'(\phi_2)$ on a dense subset of S. But in light of the remark following Theorem 4.2, we may still conclude that ϕ_1 must be a constant multiple of ϕ_2, say $\phi_1 = m\phi_2$. If m is zero, Corollary 4.2 is established. If m is different from zero, the condition $f_2^2 > 0$ on a dense subset of S implies the same condition for f_1^2, and hence, in this case, for $f'(\phi_1)$. Thus, in fact, all of the hypotheses of Theorem 4.2 are satisfied (except in the trivial case), and we may conclude that $\phi_1 - (\phi_1^{1+k}/\phi_2^{1+k})\phi_2$ is constant. Making the substitution $\phi_1 = m\phi_2$, we obtain that $\phi_1(1 - m^k)$ is constant. Thus, either ϕ_1 itself is constant, or $m^k = 1$. Since ϕ_1 and ϕ_2 are distinct, the stated conclusion is immediate.

When G and h do not depend explicitly on ϕ, or more generally, when equality holds in (iii) and (iv), one may prove the same result with the word "positive" substituted for "negative" in (ii). One simply replaces the functions ϕ_1, ϕ_2, G, and h by their negatives. Using a somewhat different technique, Martin ([11]) essentially proved the corollary, extended in the manner just indicated, for the case when G and h are independent of ϕ. His method, however, is valid only for $|\lambda| < 1$, whereas the method of proof here permits relaxation of the strict inequality.

4.3. $(U+iV)^{-1}$ separable. In the case $U+iV=H^{-1}$, where H is a separable analytic (non-trivial) function of z_1, z_2, it follows readily from (4.2) that $f = (p^2 + q^2)K(\phi, x, y)$. For by Corollary 3.2, it is immediate that $\bar{z}_1/f_1 - \bar{z}_2/f_2$ is an analytic function of z_1 and z_2, hence $\bar{z}_i/f_i$ is an analytic function of z_i for each $i = 1, 2$. Moreover, $(p_i^2 + q_i^2)/f_i$, thus an analytic function of z_i, is at the same time real-valued; and therefore constant with respect to p_i, q_i.

We look now only at the simplest such case, $K(\phi, x, y) \equiv 1$, and are led to the identity

$$(4.6)\qquad \int_C \frac{w(\phi_1, \phi_2)}{|\mathbf{v}_1 - \mathbf{v}_2|^2}\left(|\mathbf{v}_2|^2 \frac{\partial \phi_1}{\partial n} - |\mathbf{v}_1|^2 \frac{\partial \phi_2}{\partial n}\right) ds = \int_S \frac{(w_{\phi_1}\mathbf{v}_1 + w_{\phi_2}\mathbf{v}_2)\cdot(|\mathbf{v}_2|^2\,\mathbf{v}_1 - |\mathbf{v}_1|^2\,\mathbf{v}_2)}{|\mathbf{v}_1 - \mathbf{v}_2|^2}\, dS,$$

valid, when the integrands are continuous, for any two harmonic functions ϕ_1, ϕ_2. Here $\mathbf{v}_i = \operatorname{grad} \phi_i$, $i = 1, 2$. Verification is immediate once it is established that the vector

$$\frac{|\mathbf{v}_2|^2\,\mathbf{v}_1 - |\mathbf{v}_1|^2\,\mathbf{v}_2}{|\mathbf{v}_1 - \mathbf{v}_2|^2}$$

is divergenceless. The possible singularities of the integrands in (4.6), albeit isolated, present difficulties when one attempts to obtain uniqueness results, and the following is offered only as a weak example of the results possible.

Theorem 4.3. *If the gradient vector of one non-constant solution of the boundary problem*

$$\Delta\phi = 0 \quad in \quad S, \qquad \phi_n = (p^2 + q^2)h(s) \quad on \quad C$$

never exceeds in length the gradient vector of a second such solution, then the two gradient vectors must at some point be identical. Thus the gradients of any two non-constant solutions must at some point have the same length.

The first statement is the only one which needs proving, and we prove it by contradiction. Assume that $|\mathbf{v}_1 - \mathbf{v}_2|^2 > 0$ at every point for two solutions ϕ_1, ϕ_2 for which, say, $|\mathbf{v}_2| \geqq |\mathbf{v}_1|$. Placing $w = \phi_1$ in (4.6), and introducing the angle θ between $\mathbf{v}_1$ and $\mathbf{v}_2$, we find

$$\int_S \frac{|\mathbf{v}_1|^2\, |\mathbf{v}_2|\, (|\mathbf{v}_2| - |\mathbf{v}_1| \cos\theta)}{|\mathbf{v}_1 - \mathbf{v}_2|^2}\, dS = 0.$$

From $|\mathbf{v}_2| \geqq |\mathbf{v}_1|$, it follows that the integrand is non-negative; and its vanishing implies $\mathbf{v}_1 \equiv \mathbf{v}_2$, a contradiction which establishes the theorem.

4.4. Log $(U + iV)$ separable. When $U + iV = e^H$, where H is separable analytic (and non-trivial) as a function of z_1, z_2, we invoke Corollary 3.3, writing

$$H = R + i\Theta = R_1(p_1, q_1, \phi_1, \phi_2, x, y) - R_2(p_2, q_2, \phi_1, \phi_2, x, y)$$
$$+ i\theta_1(p_1, q_1, \phi_1, \phi_2, x, y) - i\theta_2(p_2, q_2, \phi_1, \phi_2, x, y).$$

We obtain at once that $\sigma = \frac{1}{2}K(\phi_1, \phi_2, x, y)e^R \sec(\theta_1 + \theta_2 - \theta_0)$, where θ_0 may also depend on ϕ_1, ϕ_2, x, y. It follows from (4.2) that

$$\frac{z_1}{f_1} - \frac{z_2}{f_2} = \frac{1}{K}\left(e^{i(2\theta_1 - \theta_0)} + e^{i(\theta_0 - 2\theta_2)}\right);$$

or to rephrase this,

$$\frac{z_1}{f_1} - \frac{z_2}{f_2} = \frac{1}{K}\left(e^{i\psi_1} - e^{i\psi_2}\right),$$

where ψ_1, ψ_2 are harmonic in p_1, q_1 and in p_2, q_2 respectively. Regarding for the moment everything as held fixed except z_1 and z_2, we have the system of relations

$$z_j = \frac{f_j}{K}\left(e^{i\psi_j} + m + in\right), \qquad (j = 1, 2), \tag{4.7}$$

for some constant $m + in$. For given j, this is a system of two equations, and elimination of f_j/K between them yields the identity

$$\frac{p_j}{q_j} = \frac{\cos\psi_j + m}{\sin\psi_j + n}. \tag{4.8}$$

But clearly the only solutions ψ_j of (4.8) that can be harmonic in p_j, q_j are linear combinations of the arc tangent of p_j/q_j, and in fact it is just trigonometric manipulation to show that the coefficient of the arc tangent must be -1 or -2.

In the remainder of this section, we shall develop an integral identity when the coefficient is in fact -1 as an example of the results obtainable. The technique for developing an identity for the coefficient -2 is analogous.

When $\psi_j = -\arctan p_j/q_j + b$, where b is constant, we obtain easily upon differentiation of (4.8) with respect to p_j/q_j that $m = n = 0$; and this result, together with (4.7), yields at once the result that

$$f_j = \pm K\,|z_j|, \qquad j = 1, 2. \tag{4.9}$$

If we now reintroduce dependence on the variables ϕ_1, ϕ_2, x, y, we see that K, since it must satisfy two equations (4.9), can depend only on x, y. Since K is otherwise an arbitrary function, we choose the plus sign in (4.9), so that $f_j = K(x, y)\,|z_j|$.

We now have from (4.2) that the function

$$U + iV = \frac{\sigma}{K}\left(\frac{z_1}{|z_1|} - \frac{z_2}{|z_2|}\right)$$

must be analytic. Multiplication by $z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}}$ yields (in a cut domain) another analytic function

$$\frac{\sigma}{K}\left(\frac{z_1}{|z_1|} - \frac{z_2}{|z_2|}\right) z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} .$$

But it is easily verified that this function is pure imaginary, and hence it must be independent of z_1, z_2. Since σ is real, we may write, for some arbitrary function $w(\phi_1, \phi_2, x, y)$,

$$\sigma = w(\phi_1, \phi_2, x, y)\left|\frac{z_1^{\frac{1}{2}} z_2^{\frac{1}{2}}}{\dfrac{z_1}{|z_1|} - \dfrac{z_2}{|z_2|}}\right|.$$

Choosing $w \equiv \phi_1 - \phi_2$, we obtain from (1.5) the identity

$$\int_C \frac{(\phi_1 - \phi_2)\left(|\mathbf{v}_2| \dfrac{\partial \phi_1}{\partial n} - |\mathbf{v}_1| \dfrac{\partial \phi_2}{\partial n}\right)}{(|\mathbf{v}_1|\,|\mathbf{v}_2| - \mathbf{v}_1 \cdot \mathbf{v}_2)^{\frac{1}{2}}}\, ds = \int_S (|\mathbf{v}_1| + |\mathbf{v}_2|)(|\mathbf{v}_1|\,|\mathbf{v}_2| - \mathbf{v}_1 \cdot \mathbf{v}_2)^{\frac{1}{2}}\, dS, \tag{4.10}$$

valid for any two harmonic functions ϕ_1, ϕ_2 provided $|\mathbf{v}_1|\,|\mathbf{v}_2| \neq \mathbf{v}_1 \cdot \mathbf{v}_2$, or equivalently, $J = p_1 q_2 - p_2 q_1 \neq 0$. Thus we obtain the following theorem.

Theorem 4.4. *If ϕ_1, ϕ_2 are any two non-constant solutions of the boundary problem*

$$\Delta\phi = 0 \quad \text{in} \quad S, \qquad \phi = g(s) \quad \text{on} \quad C_1, \qquad \phi_n = h(s, \phi)(p^2 + q^2)^{\frac{1}{2}} \quad \text{on} \quad C_2,$$

where h is non-increasing in ϕ on the boundary $C = C_1 + C_2$, then at some point the Jacobian of ϕ_1, ϕ_2 with respect to x, y must vanish.

Again, singularities of the integrand over C in (4.10) make the theorem difficult to extend, but this line of investigation appears nevertheless potentially fruitful in studying problems such as these.

BIBLIOGRAPHY

[1] L. BERS, Univalent solutions of linear elliptic systems, *Comm. Pure Appl. Math.*, 6 (1953) 513–526.

[2] ———, *Theory of Pseudo-analytic Functions*, Lecture Notes, New York University, New York, 1953.

[3] T. CARLEMAN, Über eine nichtlineare Randwertaufgabe bei der Gleichung $\Delta u = 0$, *Math. Z.*, 9 (1921) 35–43.

[4] R. COURANT & D. HILBERT, *Methods of Mathematical Physics II: Partial Differential Equations*, Interscience Publishers (John Wiley & Sons), New York, 1962.

[5] O. D. KELLOGG, *Foundations of Potential Theory*, Frederick Ungar Publishing Company, New York, 1929.

[6] S. A. LEVIN, *Uniqueness and nonlinearity*, technical report AD 602-033, Army Research Office (Durham).

[7] ——— & M. H. MARTIN, *Quasi-separable solutions of systems of partial differential equations. I. Elliptic case*, Atti del Simposio Internazionale sulle Applicazioni dell'Analisi alla Fisica Matematica, Edizioni Cremonese, Roma, 1965, 84–96.

[8] M. H. MARTIN, Linear and nonlinear boundary problems for harmonic functions, *Proc. Amer. Math. Soc.*, 10 (1959) 258–266.

[9] ———, Nonlinear boundary problems for harmonic functions, *Rendiconti di Matematica*, 20 (1961) 373–384.

[10] ———, Some aspects of uniqueness for solutions to boundary problems, *Proc. Edinburgh Soc.*, 13 (1962) 25–35.

[11] ———, On the uniqueness of harmonic functions under boundary conditions, *J. Math. and Physics*, 17 (1963) 1–13.

[12] M. W. STEKLOFF, Sur les problèmes fondamentaux de la physique mathématique, *Ann. Sci. Ecole Norm. Sup.*, 19 (1902) 218–259, 455–490.

Cornell University
Date Communicated: APRIL 18, 1967