

UNIRATIONAL ELLIPTIC SURFACES IN CHARACTERISTIC p

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0. Introduction. Let X be a non-singular projective surface defined over an algebraically closed field k of characteristic p . X is called unirational if there exists a generically surjective rational mapping φ from the projective space P^2 to X . When $p = 0$, it is well-known that a unirational surface is nothing but a rational surface. When $p \neq 0$, however, Zariski found, in 1958, an example of unirational surfaces which are not rational (cf. Zariski [20]). Since then, many such surfaces were found and investigated by Shioda (cf. [13], [14], [15] and [17]), Artin (cf. [1]), Miyanishi (cf. [6] and [7]), Rudakov and Shafarevich (cf. [11]), and others. One of the purposes of this note is to give other examples of unirational surfaces. In fact, we determine irrational unirational elliptic surfaces of base change type with sections in characteristic $p \geq 3$ (for the definition, see Section 1). Another purpose is to investigate the following two classes of elliptic surfaces in characteristic $p \geq 5$:

$$(A) \quad y^2 = 4x^3 - t^3(t-1)^3(t-\alpha)^3x,$$

$$(B) \quad y^2 = 4x^3 - t^4(t-1)^5(t-\alpha)^5,$$

where α is any element of k . We give necessary and sufficient conditions for these elliptic surfaces to be unirational. The main results are summarized in the next section.

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1. Preliminaries and results. Let C be a non-singular complete

curve defined over k , and let $\pi: S \rightarrow C$ be an elliptic surface defined over k . We assume that this elliptic surface is relatively minimal.

DEFINITION 1.1. Let $\pi: S \rightarrow C$ be a unirational elliptic surface. S is said to be of base change type if there exist a curve C' and a morphism f from C' to C such that the fiber product $S \times_C C'$ is rational.

DEFINITION 1.2. X is called a unirational surface of purely inseparable type if there exists a generically surjective purely inseparable rational mapping φ from the projective plane P^2 to X . In particular, X is called a Zariski surface if there exists a rational mapping φ as above with $\deg \varphi = p$.

If an elliptic surface is unirational, then the base curve is isomorphic to the rational curve P^1 . We denote by t the local coordinate of an affine line in P^1 . We consider the following two classes of minimal Weierstrass normal models of elliptic surfaces in characteristic $p \geq 5$:

- Class (I) (i) $y^2 = 4x^3 - t^3(t-1)^3x$,
 (ii) $y^2 = 4x^3 - t^3(t-1)^3(t-\alpha)^3x$,
 Class (II) (i) $y^2 = 4x^3 - t^4(t-1)^4$,
 (ii) $y^2 = 4x^3 - t^4(t-1)^5$,
 (iii) $y^2 = 4x^3 - t^5(t-1)^5$,
 (iv) $y^2 = 4x^3 - t^4(t-1)^5(t-\alpha)^5$,
 (v) $y^2 = 4x^3 - t^5(t-1)^5(t-\alpha)^5$,
 (vi) $y^2 = 4x^3 - t^4(t-1)^5(t-\alpha)^5(t-\beta)^5$,
 (vii) $y^2 = 4x^3 - t^5(t-1)^5(t-\alpha)^5(t-\beta)^5(t-\gamma)^5$,

where α, β and γ are elements of the field k which are neither 0 nor 1, and are mutually distinct.

In this note, we prove the following theorems.

THEOREM I. Suppose that k is of characteristic $p \geq 5$. Then, minimal Weierstrass models of irrational unirational elliptic surfaces of base change type with sections can be listed as follows:

- (1) No such surfaces exist if $p \equiv 1 \pmod{12}$.
- (2) Surfaces in Class (II) if $p \equiv 5 \pmod{12}$.
- (3) Surfaces in Class (I) if $p \equiv 7 \pmod{12}$.
- (4) Surfaces in Classes (I) and (II) if $p \equiv 11 \pmod{12}$.

THEOREM II. Suppose that k is of characteristic 3. Then, minimal Weierstrass models of irrational unirational elliptic surfaces of base change type with sections can be listed as follows:

- (i) $y^2 = 4x^3 + 2t^5(t-1)^3x + t^4(t-1)^4(a+bt+ct^3+dt^4)$, where $(a, b) \neq (0, 0)$.

(ii) $y^2 = 4x^3 + 2t^3(t-1)^3(t+1)^2x + t^3(t-1)^3\{a + bt^2 + (a-b)t^3 + ct^4 + (c-b)t^5 + dt^6\}$.

(iii) $y^2 = 4x^3 + 2t^3(t+1)^3(t-1)^3(t-\alpha)^3x + t^4(t-1)^4(t+1)^4(t-\alpha)^4(a+bt+ct^2) + dt^8(t-1)^3(t+1)^3(t-\alpha)^3$, where $\alpha \neq -1, 0, 1$.

In this table, a, b, c, d and α are elements of the field k .

We denote by $\chi(O_S)$ the Euler Poincaré characteristic of the structure sheaf O_S of the surface S . Then, we have the following:

COROLLARY III. Suppose that k is of characteristic $p \geq 3$. Then, a unirational elliptic surface S of base change type with sections is a Zariski surface. And $\chi(O_S)$ is less than or equal to 5.

COROLLARY IV. Suppose that k is of characteristic $p \geq 3$. Then, the generic fiber of any of the irrational unirational elliptic surfaces of base change type with sections is a supersingular elliptic curve.

REMARK 1.3. In Classes (I) and (II), the values of $\chi(O_S)$ and the types of singular fibers are listed as follows:

Class (I) (i) $\chi(O_S) = 2$, types: C4, C7, C7,

(ii) $\chi(O_S) = 3$, types: C7, C7, C7, C7,

Class (II) (i) $\chi(O_S) = 2$, types: C6, C6, C6,

(ii) $\chi(O_S) = 2$, types: C4, C6, C8,

(iii) $\chi(O_S) = 2$, types: C3, C8, C8,

(iv) $\chi(O_S) = 3$, types: C6, C6, C8, C8,

(v) $\chi(O_S) = 3$, types: C4, C8, C8, C8,

(vi) $\chi(O_S) = 4$, types: C6, C8, C8, C8, C8,

(vii) $\chi(O_S) = 5$, types: C8, C8, C8, C8, C8, C8,

where we use the symbol of Néron (cf. [9]) for the types of singular fibers.

REMARK 1.4. Surfaces in Class (I) (i), Class (II) (i) (ii) (iii) of Theorem I, and in (i) (ii) of Theorem II are examples of Zariski $K3$ surfaces. In the case where $p = 2, 3$, such examples can be found in Shioda [13] and [15]. Miyanishi investigates irrational unirational quasi-elliptic surfaces (cf. [6] and [7]), and gives examples of Zariski $K3$ surfaces in characteristics 2 and 3. Moreover, Rudakov and Shafarevich proves that any unirational $K3$ surface in characteristic 2 is a Zariski surface (cf. [11]).

DEFINITION 1.5. Let X be a non-singular projective surface. X is called supersingular if the second Betti number $B_2(X)$ is equal to the Picard number $\rho(X)$.

Now, we consider two classes of elliptic surfaces which are contained in Classes (I) and (II):

$$(A) \quad y^2 = 4x^3 - t^3(t-1)^3(t-\alpha)^3x,$$

$$(B) \quad y^2 = 4x^3 - t^4(t-1)^5(t-\alpha)^5,$$

where α is any element of k . In the following Theorem V, we denote by S the relatively minimal non-singular model of one of the elliptic surfaces in Classes (A) and (B). We can also regard (A) and (B) as elliptic curves over the rational function field $k(t)$ in one variable. In this case, we write them symbolically by E_t . Then, we have the following:

THEOREM V. *Under the above notations, the following six conditions are equivalent:*

- (i) S is a unirational elliptic surface of base change type.
- (ii) $p \equiv 3 \pmod{4}$ in Case (A) and $p \equiv 2 \pmod{3}$ in Case (B).
- (iii) E_t is a supersingular elliptic curve over $k(t)$.
- (iv) S is a supersingular surface.
- (v) S is a Zariski surface.
- (vi) S is a unirational surface.

The proof of the following is immediate:

LEMMA 1.6. *Let P^1 be the rational curve over k . Then, a purely inseparable morphism f from P^1 to P^1 is always of the form:*

$$(1.1) \quad f^*(t) = s^{p^n}$$

with suitable inhomogenous coordinates s, t of the former and the latter P^1 respectively.

LEMMA 1.7. *Suppose that S is a unirational elliptic surface of base change type. Then S is of purely inseparable type, and in fact is transformed into a rational surface by a purely inseparable base change.*

PROOF. By assumption, there exists a curve $C' (\cong P^1)$ and a morphism f from C' to C such that the fiber product $S \times_{C'} C'$ is birationally isomorphic to the rational surface P^2 . We consider the separable closure K of the function field $k(C)$ in $k(C')$. Then there exists a curve C'' whose function field is isomorphic to K . C'' is also isomorphic to the rational curve P^1 . Choosing suitable coordinates u and v of C' and C'' , respectively, we can describe the morphism from C' to C'' by $v = u^{p^n}$ with a suitable integer n . Denoting by g the morphism from C'' to C , we have

$$(1.2) \quad t = f(u) = g(u^{p^n}).$$

We can regard $g(v)$ as a rational function on P^1 . If we denote by $g^{(1/p^n)}(v)$ the rational function whose coefficients are the p^n -th roots of those of $g(v)$, then we have

$$(1.3) \quad t = [g^{(1/p^n)}(u)]^{p^n},$$

that is, we have the following diagram:

$$(1.4) \quad \begin{array}{ccc} C' & \xrightarrow{u^{p^n}=v} & C'' \\ g^{(1/p^n)} \downarrow & & \downarrow g \\ P^1 & \xrightarrow{s^{p^n}=v} & C, \end{array}$$

where s is a suitable local coordinate of P^1 . Then we have the following diagram:

$$(1.5) \quad \begin{array}{ccccc} S \times_c C' & \xrightarrow{h} & S \times_c P^1 & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ C' & \xrightarrow{g^{(1/p^n)}} & P^1 & \xrightarrow{s^{p^n}=v} & C, \end{array}$$

Since g is a separable morphism, so is $g^{(1/p^n)}$. Therefore, h is also separable. Since $S \times_c C'$ is rational, so is $S \times_c P^1$ by Enriques-Castelnuovo's criterion of rationality (see, for instance, Zariski [20]). Since $v = s^{p^n}$ is purely inseparable, S is of purely inseparable type. q.e.d.

Now, we need the following well-known lemma.

LEMMA 1.8. *Let $\pi: S \rightarrow C$ be an elliptic surface with a section. Suppose that it is relatively minimal, i.e. any fiber of $\pi: S \rightarrow C$ does not contain exceptional curves of the first kind. Then we have*

$$(1.6) \quad C_2(S) = \sum_{P \in C} \text{ord}_P \Delta_P,$$

where $C_2(S)$ is the second Chern number of S and Δ_P is the discriminant of the minimal Weierstrass model of $\pi: S \rightarrow C$ at $P \in C$. Moreover, S is rational if and only if $C_2(S)$ is equal to 12.

PROOF. The first part is well-known (cf. Ogg [10]). The second part follows from the canonical bundle formula for elliptic surfaces, and Enriques-Castelnuovo's criterion of rationality (see, for instance, Zariski [20]). q.e.d.

Let C and C' be algebraic curves defined over k . Let $f: C' \rightarrow C$ be a purely inseparable morphism of degree p , and let P be a point on C' . In the following sections, we denote also by P the point $f(P)$ on C for

the sake of simplicity.

2. The case of char. $k = p \geq 5$. In this section, we assume char. $k = p \geq 5$. Then, an elliptic surface $\pi: S \rightarrow \mathbf{P}^1$ with a section is birationally isomorphic to a Weierstrass normal form

$$(2.1) \quad \begin{aligned} y^2 &= 4x^3 - g_2(t)x - g_3(t), \\ \Delta(t) &= g_2(t)^3 - 27g_3(t)^2, \\ j_s(t) &= 1728g_2(t)^3/\Delta(t), \end{aligned}$$

where $g_2(t)$ and $g_3(t)$ are polynomials of t .

LEMMA 2.1. *By the base change by a purely inseparable morphism of degree p , the types of singular fibers are transformed as follows:*

	original type		transformed type			
	type	ord Δ	$p \equiv 1 \pmod{12}$	$p \equiv 5 \pmod{12}$	$p \equiv 7 \pmod{12}$	$p \equiv 11 \pmod{12}$
ord $j \geq 0$	C1	2	C1	C8	C1	C8
	C2	3	C2	C2	C7	C7
	C3	4	C3	C6	C3	C6
	C4	6	C4	C4	C4	C4
	C6	8	C6	C3	C6	C3
	C7	9	C7	C7	C2	C2
	C8	10	C8	C1	C8	C1
ord $j < 0$	b_m	m	b_{pm}			
	$C5_m$	$m+6$	$C5_{pm}$			

PROOF. We assume that the singular fiber exists at the point $t=0$, and we denote ord_t simply by ord . To calculate this table, we have only to consider the following cases in view of Ogg [10]:

Case (I) ord $j(t) < 0$.

(a) ord $g_2(t) = 0$, type b_m .

(b) ord $g_2(t) = 2$, type $C5_m$.

Case (II) ord $j(t) \geq 0$.

(a) ord $g_2(t) \geq 1$, ord $g_3(t) = 1$, type C1.

(b) ord $g_2(t) = 1$, ord $g_3(t) \geq 2$, type C2.

(c) ord $g_2(t) \geq 2$, ord $g_3(t) = 2$, type C3.

(d₁) ord $g_2(t) = 2$, ord $g_3(t) \geq 3$, type C4.

(d₂) ord $g_2(t) \geq 3$, ord $g_3(t) = 3$, type C4.

(e) ord $g_2(t) \geq 3$, ord $g_3(t) = 4$, type C6.

(f) ord $g_2(t) = 3$, ord $g_3(t) \geq 5$, type C7.

(g) ord $g_2(t) \geq 4$, ord $g_3(t) = 5$, type C8.

Using this fact, we can obtain the table by some direct calculations.
q.e.d.

LEMMA 2.2. *Let f be a purely inseparable morphism from $\tilde{C} = P^1$ to $C = P^1$. Let $\tilde{\pi}: \tilde{S} \rightarrow \tilde{C}$ be a relatively minimal elliptic surface obtained from the fiber product $S \times_C \tilde{C} \rightarrow \tilde{C}$. Then, $\tilde{\pi}^{-1}(P)$ ($P \in C$) is a singular fiber if and only if so is $\pi^{-1}(f(P))$.*

PROOF. This follows from Lemma 2.1. q.e.d.

LEMMA 2.3. Any irrational unirational elliptic surface of base change type with sections can have singular fibers neither of type b_m nor of type $C5_m$.

PROOF. By Lemma 1.7, we have the following diagram:

$$(2.2) \quad \begin{array}{ccc} \tilde{P}^2 & \xrightarrow{\tilde{f}} & S \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ P^1 & \xrightarrow{f} & P^1 \end{array},$$

where f is the purely inseparable morphism of degree p^α , \tilde{f} is a rational mapping and \tilde{P}^2 is a rational surface with the structure of a relatively minimal elliptic surface $\tilde{\pi}: \tilde{P}^2 \rightarrow P^1$ which is birationally isomorphic to the fiber product $S \times_{P^1} P^1$ by f and π .

Suppose that $\pi: S \rightarrow P^1$ has a singular fiber of type $C5_m$. Then, $\tilde{\pi}: \tilde{P}^2 \rightarrow P^1$ has a singular fiber of type $C5_{p^\alpha m}$ by Lemma 2.1. Since \tilde{P}^2 is rational, we have

$$(2.3) \quad C_2(\tilde{P}^2) = \sum_P \text{ord}_P(\Delta_P) = p^\alpha m + 6 + \sum'_P \text{ord}(\Delta_P) = 12,$$

where \sum'_P means the sum over all points on P^1 except for one point over which a singular fiber of type $C5_{p^\alpha m}$ lies. Since we assume $p \geq 5$ and we have no singular fiber with $\text{ord}_P(\Delta_P) = 1$, this is impossible.

Next, suppose that $\pi: S \rightarrow P^1$ has a singular fiber of type b_m . Then $\tilde{\pi}: \tilde{P}^2 \rightarrow P^1$ has a singular fiber of type $b_{p^\alpha m}$. Since \tilde{P}^2 is rational, we have

$$(2.4) \quad C_2(\tilde{P}^2) = p^\alpha m + \sum'_P (\Delta_P) = 12,$$

where \sum'_P means the sum over all points on P^1 except for one point over which a singular fiber of type $b_{p^\alpha m}$ lies. Therefore, we have only

the following possibilities for the singular fibers of $\tilde{\pi}: \tilde{P}^2 \rightarrow P^1$:

- (a) $p = 11, m = 1, \alpha = 1, \sum'_P \text{ord}_P(\Delta_P) = 1,$
- (b) $p = 7, m = 1, \alpha = 1, \sum'_P \text{ord}_P(\Delta_P) = 5,$
- (c) $p = 5, m = 2, \alpha = 1, \sum'_P \text{ord}_P(\Delta_P) = 2,$
- (d) $p = 5, m = 1, \alpha = 1, \sum'_P \text{ord}_P(\Delta_P) = 7.$

Now we show that these cases do not occur.

Case (a). Since there exists no singular fiber with $\text{ord}_P(\Delta_P) = 1$, this cannot occur.

Case (b). Since we have $\sum'_P \text{ord}_P(\Delta_P) = 5$ and $p = 7$, we have singular fibers neither of type b_n nor of type $C5_n$ other than the given one of type $b_{p^{\alpha_m}}$. By Condition (b) and Lemma 2.1, we have only three singular fibers of types b_7 , $C2$ and $C1$ on \tilde{P}^2 . Therefore, by Lemma 2.1, we have only three singular fibers of types b_1 , $C1$ and $C7$ on S . Hence, we have also $C_2(S) = 12$, which contradicts the irrationality assumption by Lemma 1.8.

Case (c). This case cannot occur for the same reason as in Case (b).

Case (d). By assumption, the possibilities for singular fibers on \tilde{P}^2 are as follows:

$$(d_1) \quad b_5, b_5, C1, \quad (d_2) \quad b_5, C3, C2, \quad (d_3) \quad b_5, C2, C1, C1.$$

Therefore, by Lemma 2.1, the types of singular fibers on S are as follows:

$$(d'_1) \quad b_1, b_1, C8, \quad (d'_2) \quad b_1, C6, C2, \quad (d'_3) \quad b_1, C2, C8, C8.$$

By Lemma 1.8, (d'_1) and (d'_2) contradict the irrationality assumption of S . As for (d'_3) , we consider the j -invariant of the elliptic surface $\pi: S \rightarrow P^1$. Then, $j_S(t)$ has only one pole of order one at the point of P^1 over which the singular fiber of type b_1 lies. On the other hand, it has a zero of order at least two at the point of P^1 over which the singular fiber of type $C8$ lies (cf. Néron [9]), a contradiction. q.e.d.

LEMMA 2.4. *Under the assumption of Theorem I, S is a Zariski surface. Moreover, S is transformed into a rational surface by the base change by a purely inseparable morphism of degree p .*

PROOF. By Lemma 2.3, $\pi: S \rightarrow P^1$ has singular fibers neither of type b_m nor of type $C5_m$. Therefore, by Lemma 2.1, types of singular fibers are stable under the base change by a purely inseparable morphism of degree p^2 . On the other hand, the second Chern number of an elliptic surface with a section is determined only by the types of singular fibers. Therefore, by Lemma 1.8, S is a Zariski surface. q.e.d.

Now we prove Theorem I. We have only to consider the following case:

$$(2.5) \quad \begin{array}{ccc} \tilde{P}^2 & \xrightarrow{\tilde{f}} & S \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ P^1 & \xrightarrow{f} & P^1 \end{array},$$

where $\pi: S \rightarrow P^1$ and $\tilde{\pi}: \tilde{P}^2 \rightarrow P^1$ are relatively minimal elliptic surfaces with sections, f is the purely inseparable morphism of degree p , \tilde{f} is a rational mapping, \tilde{P}^2 is a rational surface and is birationally isomorphic to the fiber product $S \times_{P^1} P^1$ of f and π . Since \tilde{P}^2 is rational, \tilde{P}^2 is birationally isomorphic to a model with the following normal form:

$$(2.6) \quad \begin{aligned} y^2 &= 4x^3 - G_2(s)x - G_3(s), \\ \tilde{A}(s) &= G_2(s)^3 - 27G_3(s)^2, \\ \tilde{j}(s) &= 1728G_2(s)^3/\tilde{A}(s), \end{aligned}$$

where $G_2(s)$ and $G_3(s)$ are polynomials of degrees four and six, respectively. Moreover, we can assume that this is a minimal Weierstrass model over every point of the affine line A^1 and has a regular fiber over the point at infinity. $\pi: S \rightarrow P^1$, hence $\tilde{\pi}: \tilde{P}^2 \rightarrow P^1$, have singular fibers neither of type b_m nor of type $C5_m$. Therefore, $\tilde{j}(s)$ has no pole on P^1 . Hence, $\tilde{j}(s)$ is equal to a constant c . We have the following possibilities:

$$(A) \quad G_2(s) \not\equiv 0, \quad (B) \quad G_2(s) \equiv 0.$$

In Case (A), we have the equality

$$(2.7) \quad (1728 - c)G_2(s)^3 = -27cG_3(s)^2.$$

Suppose $c \neq 1728$. Then for all values of s , we have

$$(2.8) \quad 3 \text{ ord } G_2(s) = 2 \text{ ord } G_3(s).$$

By our assumption of minimality and Néron's table (cf. Néron [9] or Ogg [10]), only the singular fiber of type $C4$ satisfies this equality. Therefore, \tilde{P}^2 has only two singular fibers of type $C4$. Hence, so does S by Lemma 2.1, a contradiction to the irrationality assumption. Therefore, we have $c = 1728$. Hence we have

$$(2.9) \quad G_3(s) \equiv 0.$$

By Néron's table, only the types $C2$, $C4$ and $C7$ can occur as singular

fibers of $\tilde{\pi}: \tilde{P}^2 \rightarrow P^1$. By Lemma 2.1 and the table of Néron, the possibilities for singular fibers of $\tilde{\pi}: \tilde{P}^2 \rightarrow P^1$ are as follows:

- (a) C7, C2, (b) C4, C4, (c) C4, C2, C2, (d) C2, C2, C2, C2.

In cases $p \equiv 7 \pmod{12}$ and $p \equiv 11 \pmod{12}$, we have the following possibilities for singular fibers of $\pi: S \rightarrow P^1$:

- (a') C2, C7, (b') C4, C4, (c') C4, C7, C7, (d') C7, C7, C7, C7.

By Lemma 1.8 and the irrationality assumption, we can exclude the cases (a') and (b'). By (2.9) we have also $g_s(t) \equiv 0$. Since we can transform any three points on P^1 to $(0, 1, \infty)$, we can normalize them as in Case (I). In cases $p \equiv 1 \pmod{12}$ and $p \equiv 5 \pmod{12}$, S is always rational. Therefore, we can exclude these cases.

In Case (B), by Néron's table, only the types C1, C3, C4, C6 and C8 can occur as the types of singular fibers of $\tilde{\pi}: \tilde{P}^2 \rightarrow P^1$. By Lemma 1.8, the possibilities for singular fibers of $\tilde{\pi}: \tilde{P}^2 \rightarrow P^1$ are as follows:

- (a) C8, C1, (b) C6, C3, (c) C6, C1, C1, (d) C4, C4,
 (e) C4, C3, C1, (f) C4, C1, C1, C1, (g) C3, C3, C3,
 (h) C3, C3, C1, C1, (i) C3, C1, C1, C1, C1,
 (j) C1, C1, C1, C1, C1, C1.

In cases $p \equiv 5 \pmod{12}$ and $p \equiv 11 \pmod{12}$, we have the following possibilities for singular fibers of $\pi: S \rightarrow P^1$ by Lemma 2.1:

- (a') C1, C8, (b') C3, C6, (c') C3, C8, C8, (d') C4, C4,
 (e') C4, C6, C8, (f') C4, C8, C8, C8, (g') C6, C6, C6,
 (h') C6, C6, C8, C8, (i') C6, C8, C8, C8, C8,
 (j') C8, C8, C8, C8, C8, C8.

Since S is rational in Cases (a'), (b') and (d'), we exclude these three cases. Since we can transform any three points on P^1 to $(0, 1, \infty)$, we can normalize them as in Case (II).

In cases $p \equiv 1 \pmod{12}$ and $p \equiv 7 \pmod{12}$, S is always rational. Therefore, we can exclude these cases.

Conversely, by our construction, it is clear that the surfaces listed in Theorem I have the required properties. q.e.d.

COROLLARY 2.5. *The generic fiber of any of the elliptic surfaces listed in Theorem I is a supersingular elliptic curve.*

PROOF. This follows from the fact that for the elliptic surfaces in Class (I) we have $j_s(t) \equiv 1728$, and that for the elliptic surfaces in Class (II) we have $j_s(t) \equiv 0$ (cf. Deuring [2]).

3. Some lemmas for char. $k = 3$. In this section, we assume that k is of characteristic $p = 3$. Then, an elliptic surface $\pi: S \rightarrow P^1$ with sections is birationally isomorphic to a model with the Weierstrass normal form:

$$(3.1) \quad \begin{aligned} y^2 &= 4x^3 + \beta_2(t)x^2 + 2\beta_4(t)x + \beta_6(t), \\ \Delta(t) &= \beta_2(t)^2\beta_4(t)^2 - \beta_2(t)^3\beta_6(t) + \beta_4(t)^3, \\ j(t) &= \beta_2(t)^6/\Delta(t), \end{aligned}$$

where $\beta_2(t)$, $\beta_4(t)$ and $\beta_6(t)$ are polynomials of t .

For the following lemma, we regard (3.1) as an elliptic curve E_t over the ring $k[[t]]$ of formal power series. We assume (3.1) is a minimal Weierstrass model over $k[[t]]$. Let F be the Frobenius morphism defined by $t = s^3$. Then the minimal Weierstrass model of the pull-back of (3.1) by F is expressed in the following form:

$$(3.2) \quad y^2 = 4x^3 + \tilde{\beta}_2(s)x^2 + 2\tilde{\beta}_4(s)x + \tilde{\beta}_6(s),$$

where $\tilde{\beta}_2(s)$, $\tilde{\beta}_4(s)$ and $\tilde{\beta}_6(s)$ are polynomials of s .

Using these notations, we have the following:

LEMMA 3.1. *By the base change by F , the types of singular fibers are transformed as follows:*

Case (I) (a) *type C1, ord $\Delta = 2 + \delta$.*

(a1) *ord $\beta_2 < \text{ord } \beta_4$, ord β_2 : odd.*

transformed type: type C8, ord $\tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4 - 1$, ord $\tilde{\beta}_4$: even.

(a2) *ord $\beta_2 < \text{ord } \beta_4$, ord β_2 : even.*

transformed type: type C3, ord $\tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4$, ord $\tilde{\beta}_4$: odd.

(a3) *ord $\beta_2 \geq \text{ord } \beta_4$, ord β_4 : odd.*

transformed type: type C6, ord $\tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4$, ord $\tilde{\beta}_4$: odd.

(a4) *ord $\beta_2 \geq \text{ord } \beta_4$, ord β_4 : even.*

transformed type: type C1, ord $\tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4$, ord $\tilde{\beta}_4$: even.

(b) *type C3, ord $\Delta = 4 + \delta$.*

(b1) *ord $\beta_2 < \text{ord } \beta_4$, ord β_2 : odd.*

transformed type: type C1, ord $\tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4$, ord $\tilde{\beta}_4$: odd.

(b2) *ord $\beta_2 < \text{ord } \beta_4$, ord β_2 : even.*

transformed type: type C6, ord $\tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4$, ord $\tilde{\beta}_4$: even.

(b3) *ord $\beta_2 \geq \text{ord } \beta_4$, ord β_4 : odd.*

transformed type: type C8, ord $\tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4$, ord $\tilde{\beta}_4$: odd.

(b4) *ord $\beta_2 \geq \text{ord } \beta_4$, ord β_4 : even.*

transformed type: type C3, ord $\tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4$, ord $\tilde{\beta}_4$: even.

(c) *type C6, ord $\Delta = 8 + \delta$.*

- (c1) $\text{ord } \beta_2 + 1 < \text{ord } \beta_4, \text{ord } \beta_2: \text{ odd.}$
transformed type: type C8, $\text{ord } \tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4, \text{ord } \beta_4: \text{ odd.}$
- (c2) $\text{ord } \beta_2 + 1 < \text{ord } \beta_4, \text{ord } \beta_2: \text{ even.}$
transformed type: type C3, $\text{ord } \tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4, \text{ord } \beta_4: \text{ even.}$
- (c3) $\text{ord } \beta_2 + 1 \geq \text{ord } \beta_4, \text{ord } \beta_4: \text{ odd.}$
transformed type: type C1, $\text{ord } \tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4, \text{ord } \tilde{\beta}_4: \text{ odd.}$
- (c4) $\text{ord } \beta_2 + 1 \geq \text{ord } \beta_4, \text{ord } \beta_4: \text{ even.}$
transformed type: type C6, $\text{ord } \tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4, \text{ord } \tilde{\beta}_4: \text{ even.}$
- (d) *type C8, $\text{ord } \Delta = 10 + \delta.$*
- (d1) $\text{ord } \beta_2 + 1 < \text{ord } \beta_4, \text{ord } \beta_2: \text{ odd.}$
transformed type: type C1, $\text{ord } \tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4, \text{ord } \tilde{\beta}_4: \text{ even.}$
- (d2) $\text{ord } \beta_2 + 1 < \text{ord } \beta_4, \text{ord } \beta_2: \text{ even.}$
transformed type: type C6, $\text{ord } \tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4 - 1, \text{ord } \tilde{\beta}_4: \text{ odd.}$
- (c3) $\text{ord } \beta_2 + 1 \geq \text{ord } \beta_4, \text{ord } \beta_4: \text{ odd.}$
transformed type: type C3, $\text{ord } \tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4, \text{ord } \tilde{\beta}_4: \text{ odd.}$
- (d4) $\text{ord } \beta_2 + 1 \geq \text{ord } \beta_4, \text{ord } \beta_4: \text{ even.}$
transformed type: type C8, $\text{ord } \tilde{\beta}_2 \geq \text{ord } \tilde{\beta}_4, \text{ord } \tilde{\beta}_4: \text{ even.}$
- Case (II) (i) (a) *type C2, $\text{ord } \Delta = 3.$*
transformed type: type C7.
- (b) *type C4, $\text{ord } \Delta = 6.$*
transformed type: type C4.
- (c) *type C7, $\text{ord } \Delta = 9.$*
transformed type: type C2.
- (ii) (a) *type b_m , $\text{ord } \Delta = m.$*
transformed type: type $b_{3m}.$
- (b) *type $C5_m$, $\text{ord } \Delta = m + 6.$*
transformed type: type $C5_{3m}.$

In this table, δ denotes an integer which describes the wildness of ramification (cf. Serre [12]).

PROOF. We obtain this table, using the results in Ogg [10]. In (3.1), we put

$$\begin{aligned}
 \beta_2(t) &= a_0 + a_1 t + \cdots + a_i t^i + \cdots, \\
 \beta_4(t) &= b_0 + b_1 t + \cdots + b_i t^i + \cdots, \\
 \beta_6(t) &= c_0 + c_1 t + \cdots + c_i t^i + \cdots,
 \end{aligned}
 \tag{3.3}$$

where a_i, b_i and c_i are elements of the field k . Setting $t = s^3$, we have

$$\begin{aligned}
 \beta'_2(s) &= \beta_2(s^3) = a_0 + a_1 s^3 + a_2 s^6 + \cdots + a_i s^{3i} + \cdots, \\
 \beta'_4(s) &= \beta_4(s^3) = b_0 + b_1 s^3 + b_2 s^6 + \cdots + b_i s^{3i} + \cdots, \\
 \beta'_6(s) &= \beta_6(s^3) = c_0 + c_1 s^3 + c_2 s^6 + \cdots + c_i s^{3i} + \cdots.
 \end{aligned}
 \tag{3.4}$$

By the base change by F , the elliptic surface (3.1) is transformed into the following surface:

$$(3.5) \quad Y^2 = 4X^3 + \beta'_2(s)X^2 + 2\beta'_4(s)X + \beta'_6(s).$$

We need to calculate the minimal Weierstrass model (3.2) of this surface.

We regard the elliptic surface (3.1) as an elliptic curve E_t defined over the ring $k[[t]]$ of formal power series. We denote by E_2 the reduced group of points of order 2 of E_t over $k[[t]]$. By Ogg [10], we have only to consider the following cases.

Case (I) $j(t)$: integral, $E_2 = 0$, $\text{ord } \beta_i \geq 1$ ($i = 2, 4, 6$).

- (a) $\text{ord } \beta_6 = 1$, type C1, $\text{ord } \Delta = 2 + \delta$,
- (b) $\text{ord } \beta_6 = 2$, type C3, $\text{ord } \Delta = 4 + \delta$,
- (c) $\text{ord } \beta_6 = 4$, type C6, $\text{ord } \Delta = 8 + \delta$,
- (d) $\text{ord } \beta_6 = 5$, type C8, $\text{ord } \Delta = 10 + \delta$.

Case (II) all the other cases ($\delta = 0$).

(i) $j(t)$: integral, $E_2 \neq 0$, $\text{ord } \beta_2 \geq 1$, $\text{ord } \beta_4 \geq 1$ and $\beta_6 = 0$.

- (a) $\text{ord } \beta_4 = 1$, $\text{ord } \Delta = 3$, type C2,
- (b) $\text{ord } \beta_2 \geq 1$, $\text{ord } \Delta = 6$, type C4,
- (c) $\text{ord } \beta_2 \geq 2$, $\text{ord } \beta_4 = 3$, $\text{ord } \Delta = 9$, type C7.

(ii) $\text{ord } j(t) < 0$.

- (a) β_2 : unit, $\beta_4 = 0$, type b_m , $\text{ord } \Delta = -\text{ord } j(t) = m$,
- (b) $\text{ord } \beta_2 = 1$, $\beta_4 = 0$, $\text{ord } \beta_6 \geq 3$, type C5_m, $\text{ord } \Delta = m + 6$.

In the following, we put

$$(3.6) \quad \text{ord } \beta_2 = k, \text{ord } \beta_4 = l, \text{ord } \beta_6 = n.$$

Let $\gamma(s)$ be an element of $k[[s]]$. Putting

$$(3.7) \quad X = X' - \gamma(s),$$

we have

$$(3.8) \quad Y^2 = 4X'^3 + \beta'_2 X'^2 + (\beta'_2 \gamma + 2\beta'_4)X' + (\beta'_2 \gamma^2 + \beta'_4 \gamma).$$

Let $\rho(s)$ be another element of $k[[s]]$. Putting

$$(3.9) \quad Y = \rho(s)^3 Y'', \quad X' = \rho(s)^2 X'',$$

we have the final form (3.2) with

$$(3.10) \quad \begin{aligned} \tilde{\beta}_2(s) &= \beta'_2 / \rho(s)^2, \\ \tilde{\beta}_4(s) &= (\beta'_2 \gamma + 2\beta'_4) / \rho(s)^4, \\ \tilde{\beta}_6(s) &= (\beta'_2 \gamma^2 + \beta'_4 \gamma) / \rho(s)^6. \end{aligned}$$

Now, we make case by case calculation.

Case (I) (a) $c_0 = 0$, $c_1 \neq 0$, $k \geq 1$ and $l \geq 1$. type C1. Setting

$$(3.11) \quad \gamma(s) = c_1^{1/3}s + c_2^{1/3}s^2 + \dots,$$

we have

$$(3.12) \quad \beta'_2\gamma^2 + \beta'_4\gamma = a_k c_1^{2/3}s^{3k+2} + b_l c_1^{1/3}s^{3l+1} + \dots$$

We have the following four cases.

(a1) $k < l$, k : odd. Setting $k = 2\alpha + 1$ and $\rho(s) = s^\alpha$, we have

$$(3.13) \quad \text{ord } \tilde{\beta}_2 = 2k + 1, \text{ ord } \tilde{\beta}_4 = k + 3 \text{ and } \text{ord } \tilde{\beta}_6 = 5.$$

We are in type C8 by the above list.

(a2) $k < l$, k : even. Setting $k = 2\alpha$ and $\rho(s) = s^\alpha$, we have

$$(3.14) \quad \text{ord } \tilde{\beta}_2 = 2k, \text{ ord } \tilde{\beta}_4 = k + 1 \text{ and } \text{ord } \tilde{\beta}_6 = 2.$$

We are in type C3.

(a3) $k \geq l$, l : odd. Setting $l = 2\alpha + 1$ and $\rho(s) = s^\alpha$, we have

$$(3.15) \quad \text{ord } \tilde{\beta}_2 = 3k + 1 - l, \text{ ord } \tilde{\beta}_4 = l + 2 \text{ and } \text{ord } \tilde{\beta}_6 = 4.$$

We are in type C6.

(a4) $k \geq l$, l : even. Setting $k = 2\alpha$ and $\rho(s) = s^\alpha$, we have

$$(3.16) \quad \text{ord } \tilde{\beta}_2 = 2k, \text{ ord } \tilde{\beta}_4 = l \text{ and } \text{ord } \tilde{\beta}_6 = 1.$$

We are in type C1.

We can deal with all the other cases in the same way. So we list only the results.

(b) $c_0 = c_1 = 0$, $c_2 \neq 0$, $k \geq 1$ and $l \geq 2$. type C3.

$$\gamma(s) = c_2^{1/3}s^2 + c_3^{1/3}s^3 + \dots$$

(b1) $k < l$ and $k = 2\alpha + 1$. $\rho(s) = s^{\alpha+1}$, $\text{ord } \tilde{\beta}_2 = 2k - 1$, $\text{ord } \tilde{\beta}_4 = k$ and $\text{ord } \tilde{\beta}_6 = 1$. We are in type C1.

(b2) $k < l$ and $k = 2\alpha$. $\rho(s) = s^\alpha$, $\text{ord } \tilde{\beta}_2 = 2k$, $\text{ord } \tilde{\beta}_4 = k + 2$ and $\text{ord } \tilde{\beta}_6 = 4$. We are in type C6.

(b3) $k \geq l$ and $l = 2\alpha + 1$. $\rho(s) = s^\alpha$, $\text{ord } \tilde{\beta}_2 = 3k - l + 1$, $\text{ord } \tilde{\beta}_4 = l + 2$ and $\text{ord } \tilde{\beta}_6 = 5$. We are in type C8.

(b4) $k \geq l$ and $l = 2\alpha$. $\rho(s) = s^\alpha$, $\text{ord } \tilde{\beta}_2 = 2k - l$, $\text{ord } \tilde{\beta}_4 = l$ and $\text{ord } \tilde{\beta}_6 = 2$. We are in type C3.

(c) $c_0 = c_1 = c_2 = c_3 = 0$, $c_4 \neq 0$, $k \geq 1$ and $l \geq 3$. type C6.

$$\gamma(s) = c_4^{1/3}s^4 + c_5^{1/3}s^5 + \dots$$

(c1) $k + 1 < l$ and $k = 2\alpha - 1$. $\rho(s) = s^\alpha$, $\text{ord } \tilde{\beta}_2 = 2k + 3$, $\text{ord } \tilde{\beta}_4 = k + 2$ and $\text{ord } \tilde{\beta}_6 = 5$. We are in type C8.

(c2) $k + 1 < l$ and $k = 2\alpha$. $\rho(s) = s^{\alpha+1}$, $\text{ord } \tilde{\beta}_2 = 2k - 2$, $\text{ord } \tilde{\beta}_4 = k$ and $\text{ord } \tilde{\beta}_6 = 2$. We are in type C3.

(c3) $k + 1 \geq l$ and $l = 2\alpha + 1$. $\rho(s) = s^{\alpha+1}$, $\text{ord } \tilde{\beta}_2 = 3k - l - 1$, $\text{ord } \tilde{\beta}_4 = l - 1$ and $\text{ord } \tilde{\beta}_6 = 1$. We are in type C1.

(c4) $k + 1 \geq l$ and $l = 2\alpha$. $\rho(s) = s^\alpha$, $\text{ord } \tilde{\beta}_2 = 3k - l$, $\text{ord } \tilde{\beta}_4 = l$ and $\text{ord } \tilde{\beta}_6 = 4$. We are in type C6.

(d) $c_0 = c_1 = c_2 = c_3 = c_4 = 0$, $c_5 \neq 0$, $k \geq 1$ and $l \geq 3$. type C8.

$$\gamma(s) = c_5^{1/3}s^5 + c_6^{1/3}s^6 + \dots$$

(d1) $k + 1 < l$ and $k = 2\alpha - 1$ ($k \geq 3$, since $j(t)$ is integral). $\rho(s) = s^{\alpha+1}$, $\text{ord } \tilde{\beta}_2 = 2k - 3$, $\text{ord } \tilde{\beta}_4 = k - 1$ and $\text{ord } \tilde{\beta}_6 = 1$. We are in type C1.

(d2) $k + 1 < l$ and $k = 2\alpha$. $\rho(s) = s^{\alpha+1}$, $\text{ord } \tilde{\beta}_2 = 2k - 2$, $\text{ord } \tilde{\beta}_4 = k + 1$ and $\text{ord } \tilde{\beta}_6 = 4$. We are in type C6.

(d3) $k + 1 \geq l$ and $l = 2\alpha + 1$ ($l \geq 3$, since $\text{ord } \beta_4 > (\text{ord } \beta_2)/2$). $\rho(s) = s^{\alpha+1}$, $\text{ord } \tilde{\beta}_2 = 3k - l - 1$, $\text{ord } \tilde{\beta}_4 = l - 2$ and $\text{ord } \tilde{\beta}_6 = 2$. We are in type C3.

(d4) $k + 1 \geq l$ and $l = 2\alpha$. $\rho(s) = s^\alpha$, $\text{ord } \tilde{\beta}_2 = 3k - l$, $\text{ord } \tilde{\beta}_4 = l$ and $\text{ord } \tilde{\beta}_6 = 5$. We are in type C8.

Case (II) (i) (a) $k \geq l$, $b_0 = 0$, $b_1 \neq 0$, $\beta_6 = 0$ and $\text{ord } \Delta = 3$. type C2.

In this case, (3.5) is itself a minimal Weierstrass model. Therefore, we have $\text{ord } \tilde{\beta}_2 = 3k$, $\text{ord } \tilde{\beta}_4 = 3$ and $\tilde{\beta}_6 = 0$. We are in type C7.

(b) $k \geq 1$, $\beta_6 = 0$ and $\text{ord } \Delta = 6$. type C4.

Setting $\gamma(s) = 0$ and $\rho(s) = s$, we have $\text{ord } \tilde{\beta}_2 = 3k - 2$, $\text{ord } \tilde{\beta}_4 = 2$, $\tilde{\beta}_6 = 0$ and $\text{ord } \Delta = 6$. We are in type C4.

(c) $k \geq 2$, $b_0 = b_1 = b_2 = b_3 = 0$, $b_4 \neq 0$, $\beta_6 = 0$ and $\text{ord } \Delta = 9$. type C7.

Setting $\gamma(s) = 0$ and $\rho(s) = s^2$, we have $\text{ord } \tilde{\beta}_2 = 3k - 4$, $\text{ord } \tilde{\beta}_4 = 1$, $\tilde{\beta}_6 = 0$ and $\text{ord } \Delta = 3$. We are in type C2.

(ii) (a) $c_0 = c_1 = 0$ and $\beta_4 = 0$. type b_m .

In this case, (3.5) is itself a minimal Weierstrass model. Therefore, we are in type b_{3m} .

(b) $c_0 = 0$, $c_1 \neq 0$, $\beta_4 = 0$ and $n \geq 3$. type $C5_m$.

Setting $\gamma(s) = 0$ and $\rho(s) = s$, we have $\text{ord } \tilde{\beta}_2 = 1$, $\tilde{\beta}_4 = 0$ and $\text{ord } \tilde{\beta}_6 = 3l + 3$. We are in type $C5_{3m}$.

Thus, we have completed our calculation. q.e.d.

LEMMA 3.2. *Let f be a purely inseparable morphism from $\tilde{C} = \mathbf{P}^1$ to $C = \mathbf{P}^1$. Let $\tilde{\pi}: \tilde{S} \rightarrow \tilde{C}$ be a relatively minimal elliptic surface obtained from the fiber product $S \times_C \tilde{C} \rightarrow \tilde{C}$. Then, $\tilde{\pi}^{-1}(P)$ ($P \in \tilde{C}$) is a singular fiber if and only if so is $\pi^{-1}(f(P))$.*

PROOF. This follows from Lemma 3.1. q.e.d.

As for the integer δ , we have the following:

LEMMA 3.3. δ does not change under the base change by any purely

inseparable morphism.

PROOF. We have only to check this lemma in the case of the base change by a purely inseparable morphism of degree 3 for the singular fiber of types C1, C3, C6 and C8. According to Ogg [10], we can choose the Weierstrass normal form (3.1) with $\text{ord } \beta_8 = 1, 2, 4$ or 5 in these cases. Then, we have

$$(3.17) \quad \delta = \text{ord } \Delta - 2 \text{ord } \beta_8.$$

Using this formula and the calculation in Lemma 3.1, we can show Lemma 3.3 case by case. q.e.d.

LEMMA 3.4. *Suppose that $6 \text{ord}_P \beta_2 = \text{ord}_P \Delta_P$ at P over which a singular fiber lies. Then, the singular fiber is of type C4. Moreover, if the singular fiber is expressed in a minimal Weierstrass model at P , we have $\text{ord}_P \beta_2(t) = 1$.*

PROOF. Since $j(t) = \beta_2^3/\Delta$ is invariant under the allowable transformation of this model, we can assume that the singular fiber is already expressed in a minimal Weierstrass model at P . Suppose that the singular fiber is of type C4. Then, we have $\text{ord}_P \beta_2 \geq 1$ and $\text{ord}_P \Delta_P = 6$. Therefore, by assumption, we have $\text{ord}_P \beta_2 = 1$, and this case can occur.

Neither type C2 nor type C7 can occur as the type of the singular fiber, since $j(t)$ has a zero of order three or more in these cases. Neither type b_m nor type $C5_m$ can occur, for $j(t)$ has a pole in these cases. Hence, from now on, we may assume that it is of type C1, C3, C6 or C8. In these cases, we may assume $\text{ord}_P \beta_8 = 1, 2, 4$ or 5, respectively (cf. Ogg [10]). We analyze the terms of Δ .

Suppose that $\text{ord}_P \beta_2^6 \leq \text{ord}_P \beta_2^3 \beta_8$. Then, we have

$$3 \text{ord}_P \beta_2 \leq \text{ord}_P \beta_8.$$

By the assumption on $\text{ord}_P \beta_8$, we have only the following cases:

(a₁) $\text{ord}_P \beta_2 = 1$, $\text{ord}_P \beta_8 = 4$ and type C6,

(a₂) $\text{ord}_P \beta_2 = 1$, $\text{ord}_P \beta_8 = 5$ and type C8.

By our assumption and $\text{ord}_P \beta_2 = 1$, we have $\text{ord}_P \Delta_P = 6$, which is impossible for the types C6 and C8. Hence, we have

$$(3.18) \quad \text{ord}_P \beta_2^6 > \text{ord}_P \beta_2^3 \beta_8.$$

Therefore, in order for $6 \text{ord}_P \beta_2 = \text{ord}_P \Delta_P$ to hold, one of the following three cases must occur:

(b₁) $\text{ord}_P \beta_2^3 \beta_8 = \text{ord}_P \beta_4^3$,

(b₂) $\text{ord}_P \beta_2^3 \beta_8 = \text{ord}_P \beta_2^2 \beta_4^2 < \text{ord}_P \beta_4^3$,

(b₃) $\text{ord}_P \beta_4^3 = \text{ord}_P \beta_2^2 \beta_4^2 < \text{ord}_P \beta_2^3 \beta_8$.

None, however, can occur. Indeed, Case (b₁) is impossible by the assumption on $\text{ord}_P \beta_6$. In Cases (b₂) and (b₃), we have $3 \text{ord}_P \beta_2 < \text{ord}_P \beta_6$, which contradicts (3.18). q.e.d.

By Lemma 3.4, the j -invariant satisfies the following property.

LEMMA 3.5. *Suppose $j(t)$ is not constant on P^1 . Then, $j(t)$ has a zero at any point of P^1 over which the singular fiber of type C1, C2, C3, C6, C7 or C8 lies.*

Now, we can prove the following lemma as in the case of $\text{char. } k = p \geq 5$.

LEMMA 3.6. *When $\text{char. } k = 3$, any irrational unirational elliptic surface S of base change type with sections can have singular fibers neither of type b_m nor of type $C5_m$.*

PROOF. By Lemma 1.7, we have the following diagram by base change:

$$(3.19) \quad \begin{array}{ccccc} \tilde{P}^2 & \xrightarrow{\quad\quad\quad} & \tilde{S} & \xrightarrow{\quad\quad\quad} & S \\ \downarrow \varphi & & \downarrow \tilde{\pi} & & \downarrow \pi \\ P^1 & \xrightarrow{\quad F \quad} & P^1 & \xrightarrow{\quad f_1 \quad} & P^1 \end{array},$$

where f_1 is a purely inseparable morphism, the dotted arrows are rational mappings and all elliptic surfaces are relatively minimal. Moreover, we can assume that \tilde{S} is not rational. If π has a singular fiber of type b_m or of $C5_m$, then $\tilde{\pi}$ also has a singular fiber of type b_m or $C5_m$. Therefore, it suffices to prove that $\tilde{\pi}$ has no such singular fibers. In the following, we denote by Δ the discriminant of the elliptic surface $\tilde{\pi}: \tilde{S} \rightarrow P^1$, and by $\tilde{\Delta}$ the discriminant of the elliptic surface $\varphi: \tilde{P}^2 \rightarrow P^1$.

Suppose that $\tilde{\pi}$ has a singular fiber of type $C5_m$. Then, by Lemma 3.1, we have on \tilde{P}^2 a singular fiber of type $C5_{3m}$. Then, we have

$$(3.20) \quad 3m + 6 \leq 12.$$

Therefore, we have $m = 1$ or 2 . If $m = 2$, then we have no other singular fibers on \tilde{P}^2 . Hence, $\tilde{\pi}$ has only one singular fiber of type $C5_2$. Then, by Lemma 3.1, we have $C_2(\tilde{S}) = 8$, which is impossible. If $m = 1$, we have on \tilde{P}^2

$$(3.21) \quad C_2(\tilde{P}^2) = \sum_P \text{ord}_P(\tilde{\Delta}_P) = (3 + 6) + \sum'_P \text{ord}_P(\tilde{\Delta}_P) = 12,$$

where \sum'_P means the sum over all points on P^1 except for one point

over which a singular fiber of type $C5_3$ lies. The possibilities for the singular fibers on \tilde{P}^2 except that of type $C5_3$ are as follows:

(a) $C2$ ($\delta_P = 0$), (b) $C1$ ($\delta_P = 1$).

Therefore, by Lemmas 3.1 and 3.3, the possibilities for the types of singular fibers on \tilde{S} except that of type $C5_1$ are as follows:

(a') $C7$; $\sum_P \text{ord}_P A_P = 16$,

(b₁') $C1$ ($\delta_P = 1$); $\sum_P \text{ord}_P A_P = 10$,

(b₂') $C3$ ($\delta_P = 1$); $\sum_P \text{ord}_P A_P = 12$,

(b₃') $C6$ ($\delta_P = 1$); $\sum_P \text{ord}_P A_P = 16$,

(b₄') $C8$ ($\delta_P = 1$); $\sum_P \text{ord}_P A_P = 18$.

On the other hand, by the assumption of irrationality of \tilde{S} , we have

$$(3.22) \quad C_2(\tilde{S}) = \sum_P \text{ord}_P A_P \geq 24.$$

Therefore, any of the above cases cannot occur. Hence, $\tilde{\pi}$ has no singular fiber of type $C5_m$.

Now, suppose that $\tilde{\pi}$ has singular fibers of type b_m , that is, $\{b_{m_i}\}$ ($i = 1, \dots, r$). Then, by Lemma 3.1, φ has singular fibers of types b_{m_i} ($i = 1, \dots, r$). We note that the j -invariants of \tilde{P}^2 and \tilde{S} are not constant. Since $C_2(\tilde{P}^2) = 12$, we have

$$(3.23) \quad 3m_1 + 3m_2 + \dots + 3m_r \leq 12 \text{ and } m_i \geq 1 \ (i = 1, \dots, r).$$

Suppose that $3m_1 + 3m_2 + \dots + 3m_r = 12$. Then, all the singular fibers on \tilde{S} are of type b_{m_i} and we have

$$(3.24) \quad C_2(\tilde{S}) = \sum_{i=1}^r m_i = 4,$$

which is impossible.

Suppose that $3m_1 + 3m_2 + \dots + 3m_r = 9$. Then only the Cases (a) and (b) above are the possibilities for the singular fibers on \tilde{P}^2 other than those of types b_{3m_i} . Therefore, the types of singular fibers on \tilde{S} except those of type b_{m_i} are (a'), (b₁'), (b₂'), (b₃') and (b₄'), which is impossible, for we have $C_2(\tilde{S}) < 24$ in all these cases.

Suppose that $3m_1 + 3m_2 + \dots + 3m_r = 6$. Then, the possibilities for the singular fibers on \tilde{P}^2 except those of type b_m is as follows:

(a) $C1$ ($\delta_P = 4$),

(b) $C3$ ($\delta_P = 2$),

(c) $C4$,

(d) $C2, C2$,

(e) $C1$ ($\delta_P = 1$), $C2$,

(f) $C1$ ($\delta_P = 0$), $C3$ ($\delta_{P'} = 0$),

(g) $C1$ ($\delta_P = 1$), $C1$ ($\delta_{P'} = 1$), (h) $C1$ ($\delta_P = 0$), $C1$ ($\delta_{P'} = 0$), $C1$ ($\delta_{P''} = 0$).

Therefore, using Lemmas 3.1 and 3.3, we have $C_2(\tilde{S}) < 24$ for (a), (b), (c), (d), (e) and (f), which is impossible. In Case (g), since we must

have $C_2(\tilde{S}) \geq 24$, we have by Lemmas 3.1 and 3.3 the following possibilities for the types of singular fibers on \tilde{S} :

$$(g') \quad b_1, b_1, C8 \ (\delta_P = 1), \ C8 \ (\delta_{P'} = 1) .$$

On the other hand, the elliptic surface $\tilde{\pi}: \tilde{S} \rightarrow P^1$ is birationally isomorphic to a model with the Weierstrass normal form:

$$(3.25) \quad y^2 = 4x^3 + \tilde{\beta}_2(t)x^2 + 2\tilde{\beta}_4(t)x + \tilde{\beta}_6(t) .$$

By the base change by the purely inseparable morphism of degree 3, (g') must be transformed into (g) . Therefore, by Lemma 3.1, we have

$$(3.26) \quad \text{both } \text{ord}_P \tilde{\beta}_2(t) \text{ and } \text{ord}_{P'} \tilde{\beta}_2(t) \text{ are odd} .$$

On the other hand, the j -invariant $j_{\tilde{S}}(s)$ has only two poles of order 1 at the points of P^1 over which the singular fibers of type b_1 lies. Therefore, the sum of the orders of zeros of $j_{\tilde{S}}(s)$ must be equal to 2. Since we have $\text{ord}_P \tilde{A}_P = 11$, $\text{ord}_{P'} \tilde{A}_{P'} = 11$ and $j_{\tilde{S}}(t) = \tilde{\beta}_2(t)^6 / \tilde{A}(t)$, we have $\text{ord}_P \tilde{\beta}_2(t) = 2$ and $\text{ord}_{P'} \tilde{\beta}_2(t) = 2$, that is, both $\text{ord}_P \tilde{\beta}_2(t)$ and $\text{ord}_{P'} \tilde{\beta}_2(t)$ are even, a contradiction to (3.26). In Case (h), $j_{\tilde{P}^2}(s)$ has only two poles of order 3 at the points of P^1 over which the singular fibers of type b_3 lie. On the other hand, since we have $\text{ord}_Q \mathcal{A}_{\tilde{P}^2, Q} = 2$ ($Q = P, P', P''$), by the form of $j_{\tilde{P}^2}(s)$ we have $\text{ord}_Q j_{\tilde{P}^2}(s) \geq 4$ ($Q = P, P', P''$), which contradicts $\sum_{R \in P^1} \text{ord}_R j_{\tilde{P}^2}(s) = 0$.

Suppose that $3m_1 + 3m_2 + \cdots + 3m_r = 3$. Then, we have

$$(3.27) \quad r = 1 \text{ and } m_1 = 1 .$$

Suppose, moreover, that φ has only one singular fiber E other than the singular fiber of type b_3 . Let E lie over the point P on P^1 . Then we have $\text{ord}_P \mathcal{A}_P = 9$. Therefore, the possibilities for the singular fibers over P are as follows:

- (a) $C1$ ($\delta_P = 7$), (b) $C3$ ($\delta_P = 5$),
- (c) $C6$ ($\delta_P = 1$), (d) $C7$.

Hence, by Lemma 3.1, we have $C_2(\tilde{S}) \leq 1 + (10 + 7) = 18$, which contradicts $C_2(\tilde{S}) \geq 24$. Now, suppose that φ has two or more singular fibers, say E_1, E_2, \dots , other than the singular fiber of type b_3 . Suppose that E_1 is of type $C4$. Then, the order of $\mathcal{A}_{\tilde{P}^2}$ at the point P of P^1 over which E_1 lies must be three. Therefore, the type of E_2 is $C1$ with $\delta_P = 1$, or $C2$. Then, by Lemmas 3.1 and 3.3, we have $C_2(\tilde{S}) \leq 1 + 6 + (10 + 1) = 18$ in both cases, a contradiction to $C_2(\tilde{S}) \geq 24$. Hence, φ cannot have singular fibers of type $C4$. Thus, by Lemma 3.5, $j_{\tilde{P}^2}(s)$ has two zeros at the points of P^1 over which E_1 and E_2 lie. Since F

is a purely inseparable morphism of degree 3, the orders of these zeros are both at least three. However, $j_{\tilde{\pi}^2}(s)$ has only one pole of order 3 at the point over which the singular fiber of type b_3 lies, a contradiction to the fact that $\sum_{P \in P^1} \text{ord}_P j_{\tilde{\pi}^2}(s) = 0$. Hence, $\tilde{\pi}$ has no singular fibers of type b_m . Thus, π has no singular fibers of type b_m , either. q.e.d.

COROLLARY 3.7. *Under the same assumption as in Lemma 3.6, $j_s(t)$ is a constant.*

PROOF. By Lemma 3.6, $j_s(t)$ has no pole on P^1 . Therefore, $j_s(t)$ is a constant. q.e.d.

LEMMA 3.8. *Under the same assumption as in Lemma 3.6, $\beta_2(t)$ is identically equal to zero.*

PROOF. By Lemma 3.7, we have a constant c such that

$$(3.29) \quad j_s(t) = \beta_2(t)^6 / \Delta(t) \equiv c.$$

Suppose $\beta_2(t) \not\equiv 0$. Then, we have $c \neq 0$, and $6 \text{ord}_P \beta_2(t) = \text{ord}_P \Delta(t)$ for any point P on P^1 . Therefore, by Lemma 3.4, all the singular fibers are of type $C4$. However, in this case, S cannot be an irrational unirational elliptic surface of base change type by Lemma 3.1. q.e.d.

LEMMA 3.9. *Under the same assumption as in Lemma 3.6, the elliptic surface $\pi: S \rightarrow P^1$ is a Zariski surface. Moreover, $\pi: S \rightarrow P^1$ is transformed into a rational surface by the base change by the purely inseparable morphism of degree 3.*

PROOF. Suppose $\beta_4(t) \equiv 0$. Then, we have $\Delta(t) \equiv 0$ by Lemma 3.8, which is impossible. Therefore, we have $\beta_4(t) \not\equiv 0$. Since we have $\text{ord}_P \beta_2(t) = \infty$, by Lemma 3.1 the types of the singular fibers on S are stable under the base change by the purely inseparable morphism of degree p^2 . Since S is of base change type, S must be a Zariski surface. q.e.d.

4. Conclusion in char. $k = 3$. In this section, we again assume that k is of characteristic $p = 3$. Let $\pi: S \rightarrow P^1$ be an irrational unirational elliptic surface of base change type with sections. Then, by Lemma 3.8, S is given in a Weierstrass normal form:

$$(4.1) \quad y^2 = 4x^3 + 2\beta_4(t)x + \beta_6(t), \quad \Delta(t) = \beta_4(t)^3.$$

By Lemma 3.9, we have the following diagram:

$$(4.2) \quad \begin{array}{ccc} \tilde{P}^2 & \xrightarrow{\quad\quad\quad} & S \\ \downarrow \varphi & & \downarrow \pi \\ P^1 & \xrightarrow{\quad F \quad} & P^1 \end{array},$$

where F is the purely inseparable morphism of degree 3 given by $t=s^3$, and $\varphi: \tilde{P}^2 \rightarrow P^1$ is the relatively minimal elliptic surface obtained by the pull-back of $\pi: S \rightarrow P^1$ by F . We can write the Weierstrass normal form of $\varphi: \tilde{P}^2 \rightarrow P^1$ as

$$(4.3) \quad y^2 = 4x^3 + 2\tilde{\beta}_4(s)x + \tilde{\beta}_0(s), \quad \tilde{A}(s) = \tilde{\beta}_4(s)^3.$$

By a suitable transformation of P^1 , we can assume that φ has a regular fiber at the point $s = \infty$. Since we have

$$(4.4) \quad 12 = C_2(\tilde{P}^2) = \sum_{P \in P^1} \text{ord}_P \tilde{A}(s) = 3 \sum_{P \in P^1 \setminus \{\infty\}} \text{ord}_P \tilde{\beta}_4(s),$$

we have

$$(4.5) \quad \deg \tilde{\beta}_4(s) = 4.$$

Since φ has a regular fiber over the point $s = \infty$, we have

$$(4.6) \quad \deg \tilde{\beta}_0(s) \leq 6.$$

Up to isomorphism, we need to consider only the following five cases:

$$(4.7) \quad \begin{array}{ll} (1) & \tilde{\beta}_4(s) = s^4, \\ (2) & \tilde{\beta}_4(s) = s^3(s-1), \\ (3) & \tilde{\beta}_4(s) = s^2(s-1)^2, \\ (4) & \tilde{\beta}_4(s) = s(s-1)(s+1)^2, \\ (5) & \tilde{\beta}_4(s) = s(s-1)(s+1)(s-\alpha), \text{ where } \alpha \neq -1, 0, 1. \end{array}$$

By a suitable transformation of x , we may assume that $\tilde{\beta}_0(s)$ is in the following form:

$$(4.8) \quad \tilde{\beta}_0(s) = c_1s + c_2s^2 + c_4s^4 + c_5s^5.$$

We denote by E_t an elliptic curve defined by the equation (4.1) over the ring $k[[t]]$ of formal power series. We can also regard E_t as an elliptic curve over the field of fractions $k((t))$ of $k[[t]]$. Then, we have the following:

LEMMA 4.1. *E_t has some rational points of order 2 over $k((t))$ if and only if there exists an element $\gamma(t)$ of $k[[t]]$ such that*

$$(4.9) \quad \text{ord}_t [4\gamma(t)^3 + 2\beta_4(t)\gamma(t) + \beta_6(t)] \geq 2 \text{ord}_t \beta_4(t).$$

PROOF. The x -coordinates of the points of order 2 are the solutions of the equation:

$$(4.10) \quad 4x^3 + 2\beta_4(t)x + \beta_6(t) = 0.$$

Suppose that (4.10) has a root $\gamma(t)$ over $k((t))$. Then, it is clear that it must be contained in $k[[t]]$. Therefore, (4.9) is satisfied. Conversely, suppose that there exists an element $\gamma(t)$ such that (4.9) is satisfied. Then, in (4.10), setting:

$$(4.11) \quad x = \beta_4(t)X + \gamma(t),$$

we have

$$(4.12) \quad X = 4\beta_4(t)X^3 + \beta(t),$$

with $\beta(t) = [4\gamma(t)^3 + 2\beta_4(t)\gamma(t) + \beta_6(t)]/\beta_4(t)^2$. By Condition (4.9), $\beta(t)$ is contained in $k[[t]]$. It is clear that the equation (4.12) has a root in $k[[t]]$. Therefore, the equation (4.10) also has a root in $k[[t]]$. q.e.d.

In order to formulate Lemma 4.3, we introduce the following function.

DEFINITION 4.2. $\Phi(E_{t-a}) = \text{Max ord}_t \{4\gamma(t)^3 + 2\beta_4(t+a)\gamma(t) + \beta_6(t+a)\}$, where $\gamma(t)$ runs over all elements of $k[[t]]$.

Using this definition, we have the following:

LEMMA 4.3. *When char. $k = 3$, irrational unirational elliptic surfaces of base change type with sections are in one of the following forms:*

$$(A) \quad y^2 = 4x^3 + 2t^5(t-1)^3x + \beta(t),$$

with $\Phi(E_t) = 4$ or 5 , $\Phi(E_{t-1}) \geq 4$ and $\deg \beta(t) \leq 12$.

$$(B) \quad y^2 = 4x^3 + 2t^3(t-1)^3(t+1)^2x + \beta(t),$$

with $\Phi(E_t) \geq 4$, $\Phi(E_{t-1}) \geq 4$ and $\deg \beta(t) \leq 12$.

$$(C) \quad y^2 = 4x^3 + 2t^3(t-1)^3(t+1)^3(t-\alpha)^3x + \beta(t),$$

with $\Phi(E_t) \geq 4$, $\Phi(E_{t-1}) \geq 4$, $\Phi(E_{t+1}) \geq 4$, $\Phi(E_{t-a}) \geq 4$, $\deg \beta(t) \leq 18$ and $\alpha \neq -1, 0, 1$.

PROOF. We are in the situations (4.1), (4.2) and (4.3). Moreover, we may assume (4.6) and (4.7). We consider separately the five cases in (4.6).

Case (1) Suppose that E_s in (4.3) has a rational point of order 2

over $k((s))$. Then, by Lemma 4.1, we have a formal power series:

$$(4.13) \quad \gamma(s) = \gamma_0 + \gamma_1 s + \gamma_2 s^2 + \cdots + \gamma_n s^n + \cdots$$

with $\gamma_i \in k$, such that $\gamma(s)$ satisfies (4.9). Therefore, we have

$$(4.14) \quad \gamma_0^3 = 0, \quad c_1 = 0, \quad c_2 = 0, \quad \gamma_1^3 = 0, \quad -\gamma_0 + c_4 = 0, \quad -\gamma_1 + c_5 = 0.$$

Hence, we have $\tilde{\beta}_6(s) = 0$. Setting

$$(4.15) \quad x = s^2 X, \quad y = s^3 Y,$$

we can transform (4.3) into the following form:

$$(4.16) \quad Y^2 = 4X^3 + 2X,$$

that is, the surface defined by (4.3) is a ruled surface over the elliptic curve (4.16). Therefore, this surface is not rational, a contradiction. Hence, E_s has no rational points of order 2 over $k((s))$.

We put $i = \text{ord}_s \tilde{\beta}_3(s)$. Then, the types of the singular fibers of φ are as follows (cf. Ogg [10, Proposition 2]):

$$(4.17) \quad C1 \text{ for } i = 1, \quad C3 \text{ for } i = 2, \quad C6 \text{ for } i = 4, \quad C7 \text{ for } i = 5.$$

On the other hand, since we have $\text{ord}_t \beta_2(t) = \infty > \text{ord}_t \beta_4(t)$ on S , and $\text{ord}_s \tilde{\beta}_4(s)$ is even, the types of the singular fibers are stable under the base change by the purely inseparable morphism of degree 3. Using Lemmas 1.8 and 3.3, we conclude that S is a rational surface, a contradiction.

Case (2) Suppose that E_s has a rational point of order 2 over $k((s))$. Then the singular fiber over the point $s = 0$ is of type $C7$. Therefore, the singular fiber over the point $s = 1$ must be of type $C1$ with $\delta = 1$, or of type $C2$. Hence, by Lemmas 1.8, 3.1 and 3.3, the surface S is also rational, a contradiction. Therefore, we may assume that E_s has no rational points of order 2 over $k((s))$. Then, by Lemma 4.1, we have $(c_1, c_2, c_4) \neq (0, 0, 0)$. We put $s - 1 = u$. Then, the equation (4.3) is transformed into the following form:

$$(4.18) \quad E_u: y^2 = 4x^3 + 2(u+1)^3 ux + c_1(u+1) + c_2(u+1)^2 \\ + c_4(u+1)^4 + c_5(u+1)^5.$$

By Lemma 1.8, the possibilities for singular fibers of φ are as follows:

(a) The case where E_u has a rational point of order 2 over $k((u))$.

The singular fibers of φ are of

- (a₁) type $C1$ ($\delta = 7$) and type $C2$ if $c_1 \neq 0$,
- (a₂) type $C3$ ($\delta = 5$) and type $C2$ if $c_1 = 0$ and $c_2 \neq 0$,
- (a₃) type $C6$ ($\delta = 1$) and type $C2$ if $c_1 = c_2 = 0$ and $c_4 \neq 0$.

(b) The case where E_u has no rational points of order 2 over $k((u))$. The singular fibers of φ are of

(b₁) type C1 ($\delta = 7$) and type C1 ($\delta = 1$) if $c_1 \neq 0$,

(b₂) type C3 ($\delta = 5$) and type C1 ($\delta = 1$) if $c_1 = 0$ and $c_2 \neq 0$,

(b₃) type C6 ($\delta = 1$) and type C1 ($\delta = 1$) if $c_1 = c_2 = 0$ and $c_4 \neq 0$.

Therefore, using Lemma 3.1, the possibilities for the singular fibers of π are as follows:

(a₁) type C6 ($\delta=7$), type C7, (b₁) type C6 ($\delta=7$), type C6 ($\delta=1$),

(a₂) type C8 ($\delta=5$), type C7, (b₂) type C8 ($\delta=5$), type C6 ($\delta=1$),

(a₃) type C1 ($\delta=1$), type C7, (b₃) type C1 ($\delta=1$), type C6 ($\delta=1$).

The cases (a₃) and (b₃) are excluded, for S is rational in these cases by the calculation of $\text{ord } \Delta$. By Lemma 4.1 and Ogg [10, Proposition 2], the other cases are summarized as in Case (A).

Case (3) In this case, we have $\text{ord}_p \Delta_p = 6$ at each singular fiber of φ . The possibilities for the singular fibers with $\text{ord}_p \Delta_p = 6$ are as follows:

(a) C4, (b) C1 ($\delta = 4$), (c) C3 ($\delta = 2$).

Therefore, φ has two singular fibers whose types are (a), (b) or (c). By Lemma 3.1, the original surface S has singular fibers of the same type as \tilde{P}^2 in any of these cases. Therefore, S is rational by Lemmas 1.8, 3.1 and 3.3, a contradiction.

Case (4) Over the points $s = 0, 1, -1$, the possibilities for the singular fibers are respectively as follows:

(a) $s = -1$; C4, C1 ($\delta = 4$) or C3 ($\delta = 2$),

(b) $s = 0$ or 1 ; C2 or C1 ($\delta = 1$).

Hence, by Lemmas 3.1 and 4.1 the possibilities for the singular fibers on S are summarized as in Case (B).

Case (5) We have $\text{ord}_p \Delta_p = 2$ at any point over which a singular fiber lies. Therefore, the possibilities for the singular fibers are as follows:

(a) C2, (b) C1 ($\delta = 1$).

Hence, by Lemmas 3.1 and 4.1, the possibilities for the singular fibers are summarized as in Case (C). q.e.d.

Finally, to finish the proof of Theorem II, we need to calculate the normal forms of the equations in Lemma 4.3.

Case (A) Since $\Phi(E_{t-1}) \geq 4$, by a suitable transformation of x , we may assume

$$(4.19) \quad \beta(t) = (t - 1)^4 \alpha(t),$$

where $\alpha(t)$ is a polynomial of degree less than or equal to 8.

LEMMA 4.4. *Let $h(t)$ be a polynomial of degree less than n . Then, we have a unique expression for $h(t)$ in terms of a homogeneous polynomial of degree n in t and $t - 1$:*

$$(4.20) \quad h(t) = a_0(t-1)^n + a_1(t-1)^{n-1}t + \cdots + a_{n-1}(t-1)t^{n-1} + a_nt^n$$

with elements a_i ($i = 0, 1, \dots, n$) of k .

PROOF. Clear.

Let us set

$$(4.21) \quad \begin{aligned} \alpha(t) &= \alpha_0(t-1)^8 + \alpha_1(t-1)^7t + \alpha_2(t-1)^6t^2 + \cdots + \alpha_8t^8. \\ \gamma(t) &= \gamma_0(t-1)^4 + \gamma_1(t-1)^3t + \gamma_2(t-1)^2t^2 + \gamma_3(t-1)t^3 \\ &\quad + \gamma_4t^4 + \gamma_5t^5 + \gamma_6t^6 + \cdots. \end{aligned}$$

Then we have

$$(4.22) \quad \begin{aligned} 4\gamma(t)^3 + 2t^5(t-1)^3\gamma(t) + (t-1)^4\alpha(t) \\ = (\alpha_0 + \gamma_0^3)(t-1)^{12} + \alpha_1(t-1)^{11}t + \alpha_2(t-1)^{10}t^2 \\ + (\alpha_3 + \gamma_1^3)(t-1)^9t^3 + \alpha_4(t-1)^8t^4 + (\alpha_5 - \gamma_0)(t-1)^7t^5 \\ + (\text{higher order terms in } t). \end{aligned}$$

Since we have $\Phi(E_t) = 4$ or 5 , we must have

$$(4.23) \quad \alpha_0 + \gamma_0^3 = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 + \gamma_1^3 = 0,$$

and

$$(4.24) \quad \alpha_4 \neq 0 \text{ or } \alpha_5 - \gamma_0 \neq 0.$$

Setting

$$(4.25) \quad \gamma_0 = -\alpha_0^{1/3}, \gamma_1 = -\alpha_3^{1/3}, \gamma_2 = -(\alpha_3^{1/3} + \alpha_6)^{1/3}, \gamma_i = 0 \text{ for } i \geq 3,$$

$$a = \alpha_4, b = \gamma_0 - \alpha_4 - \alpha_5, c = \gamma_2 - \alpha_4 - \alpha_7,$$

$$d = -\gamma_0 - \gamma_2 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8, \text{ and}$$

$$(4.26) \quad x = X + \gamma(t),$$

We have the following normal form for (4.1) in Case (A).

$$(4.27) \quad y^2 = 4X^3 + 2t^5(t-1)^3X + t^4(t-1)^4(a + bt + ct^3 + dt^4),$$

where $(a, b) \neq (0, 0)$ by (4.24).

Case (B) $\Phi(E_t) \geq 4$ and $\Phi(E_{t-1}) \geq 4$ imply $\text{ord}_t \beta(t) \geq 3$ and $\text{ord}_{t-1} \beta(t) \geq 3$. We set

$$(4.28) \quad \begin{aligned} \beta(t) &= t^3(t-1)^3\alpha(t), \\ \alpha(t) &= \alpha_0(t-1)^6 + \alpha_1t(t-1)^5 + \alpha_2t^2(t-1)^4 + \cdots + \alpha_6t^6, \end{aligned}$$

$$\gamma(t) = \gamma_0(t-1)^4 + \gamma_1 t(t-1)^3 + \cdots + \gamma_4 t^4 + \gamma_5 t^5 + \cdots, \\ x = X + \gamma(t).$$

Since we have $(t+1)^2 = (t-1)^2 - (t-1)t + t^2$, in the same method as in Case (A) we have

$$(4.29) \quad \begin{aligned} \gamma_0 &= 0, \gamma_1 = \alpha_1, \gamma_2 - \gamma_1 + \gamma_2 - \gamma_3 + \alpha_3 = 0, \\ \gamma_3 &= \alpha_3, \gamma_4 = 0, \text{ and } \gamma_i = 0 \text{ for } i \geq 5. \end{aligned}$$

Moreover, we set

$$(4.30) \quad \begin{aligned} a &= \gamma_1^3 + \alpha_0, \quad b = \gamma_1 - \gamma_2 + \alpha_2, \quad c = \gamma_3 - \gamma_2 + \alpha_4, \\ d &= \gamma_1^3 + \gamma_1 + \gamma_2 + \gamma_3^3 + \gamma_3 + \alpha_0 + \alpha_2 + \alpha_4 + \alpha_6. \end{aligned}$$

Then, we have the following normal form for (4.1) in Case (B).

$$(4.31) \quad \begin{aligned} y^2 &= 4X^3 + 2t^3(t-1)^3(t+1)^2X \\ &+ t^3(t-1)^3\{a + bt^2 + (a-b)t^3 + ct^4 + (c-b)t^5 + dt^6\}. \end{aligned}$$

Case (C) In this case, $\Phi(E_i) \geq 4$, $\Phi(E_{i-1}) \geq 4$, $\Phi(E_{i+1}) \geq 4$ and $\Phi(E_{i-\alpha}) \geq 4$ imply $\text{ord}_i \beta(t) \geq 3$, $\text{ord}_{i-1} \beta(t) \geq 3$, $\text{ord}_{i+1} \beta(t) \geq 3$, and $\text{ord}_{i-\alpha} \beta(t) \geq 3$. We put

$$(4.32) \quad \begin{aligned} \beta(t) &= t^3(t-1)^3(t+1)^3(t-\alpha)^3\alpha(t), \\ \deg \alpha(t) &\leq 6. \end{aligned}$$

Then, $\alpha(t)$ can be uniquely expressed as

$$(4.33) \quad \begin{aligned} \alpha(t) &= \alpha_0(t+1)^3(t-\alpha)^3 + \alpha_1(t+1)^2(t-\alpha)^2t(t-1) \\ &+ \alpha_2(t+1)(t-\alpha)t^2(t-1)^2 + \alpha_3t^3(t-1)^3 \\ &+ \alpha_4(t+1)(t-\alpha)t^3(t-1) + \alpha_5t^5 + \alpha_6t^6. \end{aligned}$$

By a suitable transformation of x , we may assume $\alpha_0 = 0$. We set

$$(4.34) \quad \begin{aligned} \gamma(t) &= t(t-1)(t+1)(t-\alpha)\{\gamma_0(t+1)(t-\alpha) + \gamma_1t(t-1) + \gamma_2t^2\}, \\ x &= X + \gamma(t). \end{aligned}$$

Moreover, we set

$$(4.35) \quad \begin{aligned} \gamma_0 &= 0, \gamma_1^3 + \alpha_3 = 0, \gamma_2^3 + \alpha_6 = 0, \\ a &= -\alpha\alpha_1, \quad b = \gamma_1 + (-\alpha + 1)\alpha_1 - \alpha_2, \\ c &= -\gamma_1 - \gamma_2 + \alpha_1 + \alpha_2 + \alpha_4, \quad d = \alpha_5. \end{aligned}$$

Then, we have the following normal form for (4.1) in Case (C).

$$(4.36) \quad \begin{aligned} y^2 &= 4X^3 + 2t^3(t+1)^3(t-1)^3(t-\alpha)^3X \\ &+ t^4(t-1)^4(t+1)^4(t-\alpha)^4(a + bt + ct^2) \\ &+ dt^3(t-1)^3(t+1)^3(t-\alpha)^3. \end{aligned}$$

Hence, Theorem II is proved.

COROLLARY 4.5. *The generic fiber of any of the elliptic surfaces listed in Theorem II is a supersingular elliptic curve.*

PROOF. This follows from the fact that $j_s(t) \equiv 0$ and $\text{char. } k = 3$ (cf. Deuring [2]).

Now, Corollary III follows from Lemmas 2.4 and 3.9, while Corollary IV follows from Corollaries 2.5 and 4.5.

5. Unirationality of some elliptic surfaces. In this section, we assume that k is of characteristic $p \geq 5$. We prove Theorem V in Section 1. For this purpose, we consider the following three generalized Kummer surfaces (cf. Ueno [19] and Shioda-Inose [16]).

(1) Let E_1 and E_2 be two elliptic curves defined as follows:

$$(5.1) \quad \begin{cases} E_1: y_1^2 = x_1^4 - 1, \\ E_2: y_2^2 = x_2^4 - 1. \end{cases}$$

We put

$$(5.2) \quad A = E_1 \times E_2.$$

Let G be the cyclic group of order four generated by an element g defined by

$$(5.3) \quad \begin{cases} x_1 \mapsto ix_1, & y_1 \mapsto y_1, \\ x_2 \mapsto -ix_2, & y_2 \mapsto y_2, \end{cases}$$

where i is a primitive fourth root of unity. Then, the minimal non-singular model of A/G is a generalized Kummer $K3$ surface.

(2) Let E_1 and E_2 be two elliptic curves defined as follows:

$$(5.4) \quad \begin{cases} E_1: y_1^2 = x_1^3 - 1, \\ E_2: y_2^2 = x_2^3 - 1. \end{cases}$$

We put

$$(5.5) \quad A = E_1 \times E_2.$$

Let G be the cyclic group of order three generated by an element g defined by

$$(5.6) \quad \begin{cases} x_1 \mapsto \omega x_1, & y_1 \mapsto y_1, \\ x_2 \mapsto \omega^2 x_2, & y_2 \mapsto y_2, \end{cases}$$

where ω is a primitive cube root of unity. Then, the minimal non-singular model of A/G is a generalized Kummer $K3$ surface.

(3) Let A be the same one as in (2). Let G be the cyclic group of order six generated by an element defined by

$$(5.7) \quad \begin{cases} x_1 \mapsto \omega x_1, & y_1 \mapsto -y_1, \\ x_2 \mapsto \omega^2 x_2, & y_2 \mapsto -y_2, \end{cases}$$

where ω is a primitive cube root of unity. Then, the minimal non-singular model of A/G is a generalized Kummer $K3$ surface.

Using these surfaces, we investigate the structures of some elliptic surfaces in Classes (I) and (II).

PROPOSITION 5.1. *The minimal non-singular models of the elliptic surfaces of Classes (I) (i), (II) (i) and (II) (ii) are respectively isomorphic to the generalized Kummer surfaces given in (1), (2) and (3).*

PROOF. By the existence of the minimal models in these cases, we have only to prove that these surfaces are birationally isomorphic to each other in each case. For the surface of Class (I) (i), we consider the finite algebraic extension $k(x, y, t, s)$ of the function field $k(x, y, t)$ of the surface of Class (I) (i) given by

$$(5.8) \quad s^4 = t(t - 1).$$

The Galois group $H = \mathbb{Z}/4\mathbb{Z}$ of the extension $k(x, y, t, s)/k(x, y, t)$ is generated by an element σ defined by

$$(5.9) \quad \begin{cases} x \mapsto x, & y \mapsto y, \\ s \mapsto is, & t \mapsto t. \end{cases}$$

We put

$$(5.10) \quad \begin{cases} x = s^9 X^2 / 2(Y + i), & y = s^9 X / \zeta(Y + i), \\ s = \zeta U / 2, & t = (iV + 1) / 2, \end{cases}$$

where ζ is a primitive eighth root of unity such that $\zeta^2 = i$. Then, the function field $k(X, Y, U, V)$ has the relations

$$(5.11) \quad \begin{cases} Y^2 = X^4 - 1, \\ V^2 = U^4 - 1. \end{cases}$$

Then, σ acts on this field $k(X, Y, U, V)$ by

$$(5.12) \quad \begin{cases} X \mapsto iX, & Y \mapsto Y, \\ U \mapsto -iU, & V \mapsto V. \end{cases}$$

Therefore, the surface of Class (I) (i) is isomorphic to the generalized Kummer surface given in (1). In the other cases, we can show the birationality by the same method. q.e.d.

Shioda proved that a Kummer surface is unirational if and only if the original abelian surface is supersingular (cf. Shioda [15]). For the surfaces given in (1), (2) and (3), we have a similar proposition as follows.

PROPOSITION 5.2. *For the generalized Kummer surface S given in (1) (resp. in (2) and (3)), the following four conditions are equivalent:*

- (i) S is a unirational surface.
- (ii) S is a Zariski surface.
- (iii) S is supersingular.
- (iv) $p \equiv 3 \pmod{4}$ (resp. $p \equiv 2 \pmod{3}$).

PROOF. (iv) implies (ii) by Theorem I, Corollary III and Proposition 5.1. It is trivial that (ii) implies (i). By Shioda [13], (i) implies (iii).

Finally, we must prove that (iii) implies (iv). We prove it for the surface given in (1). For the other surfaces, we can prove it by the same method. Suppose that Condition (iv) does not hold. Let $l (\geq 5)$ be a prime number different from the characteristic p . We denote by $H^i(X, Q_l)$ the i -th l -adic étale cohomology group of the algebraic variety X . We put $H^i(X, \bar{Q}_l) = H^i(X, Q_l) \otimes_{Q_l} \bar{Q}_l$, where \bar{Q}_l denotes an algebraic closure of the l -adic number field Q_l . We have the following canonical isomorphism (Künneth formula):

$$(5.13) \quad H^1(A, \bar{Q}_l) \cong H^1(E_1, \bar{Q}_l) \oplus H^1(E_2, \bar{Q}_l).$$

We fix a finite field F_q with $q = p^e$, over which the elliptic curves and the actions in (1) are defined. Let F be the Frobenius morphism of the elliptic curves E_1 and E_2 relative to F_q . Then, $F_A = F \times F$ is the Frobenius morphism of the abelian surface $A = E_1 \times E_2$ relative to F_q . Under our assumption to the effect that Condition (iv) does not hold, E_1 (resp. E_2) is not supersingular (cf. Deuring [2]). Therefore, any powers of eigenvalues of the linear mapping F^* on $H^1(E_i, \bar{Q}_l)$ ($i = 1, 2$) are not powers of p (cf. Mumford [8]). On the other hand, the automorphism g acts on the cohomology group $H^1(A, \bar{Q}_l)$. We denote it by g^* . Using the Lefschetz fixed-point formula (cf. Milne [5]) and (5.13), we can easily calculate the eigenvalues of g^* . Since we have

$$(5.14) \quad F^{*} \circ g^{*} = g^{*} \circ F^{*},$$

we have the following:

$$(5.15) \quad \begin{aligned} F_A^* e_1 &= \alpha_1 e_1, \quad F_A^* e_2 = \alpha_2 e_2, \quad F_A^* e'_1 = \alpha_1 e'_1, \quad F_A^* e'_2 = \alpha_2 e'_2, \\ g^* e_1 &= i e_1, \quad g^* e_2 = -i e_2, \quad g^* e'_1 = -i e'_1, \quad g^* e'_2 = i e'_2, \end{aligned}$$

where α_1 and α_2 are suitable elements of \bar{Q}_l with $\alpha_1 \neq \alpha_2$. Since A is

an abelian surface, we have an isomorphism:

$$(5.16) \quad H^2(A, \bar{Q}_l) \cong \bigwedge^2 H^1(A, \bar{Q}_l).$$

By Harder-Narashimhan [3], we also have an isomorphism:

$$(5.17) \quad H^2(A/G, \bar{Q}_l) \cong H^2(A, \bar{Q}_l)^G,$$

where $H^2(A, \bar{Q}_l)^G$ means G -invariants in $H^2(A, \bar{Q}_l)$. Using, (5.15), (5.16) and (5.17), we have an element $e_1 \wedge e'_1$ of $H^2(A/G, \bar{Q}_l)$ such that

$$(5.18) \quad F_{A/G}^*(e_1 \wedge e'_1) = \alpha_1^2 e_1 \wedge e'_1,$$

where $F_{A/G}$ is the Frobenius morphism of A/G .

Let $\pi: S \rightarrow A/G$ be the minimal resolution of A/G . Let W be the set of singular points of A/G , and let E be the exceptional divisor of π on S . Since we have an isomorphism:

$$(5.19) \quad \pi|_{S \setminus E}: S \setminus E \rightarrow A/G \setminus W,$$

we have an isomorphism:

$$(5.20) \quad H_c^i(A/G \setminus W, \bar{Q}_l) \cong H_c^i(S \setminus E, \bar{Q}_l).$$

There exists a long exact sequence

$$(5.21) \quad \begin{array}{ccccccc} \rightarrow & H_c^{i-1}(W, \bar{Q}_l) & \rightarrow & H_c^i(A/G \setminus W, \bar{Q}_l) & \rightarrow & H_c^i(A/G, \bar{Q}_l) & \rightarrow & H_c^i(W, \bar{Q}_l) \rightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \rightarrow & H_c^{i-1}(E, \bar{Q}_l) & \rightarrow & H_c^i(S \setminus E, \bar{Q}_l) & \rightarrow & H_c^i(S, \bar{Q}_l) & \rightarrow & H_c^i(E, \bar{Q}_l) \rightarrow, \end{array}$$

where $H_c^i(X, \bar{Q}_l)$ means the i -th l -adic cohomology with compact support (cf. Milne [5]), and we put $H_c^i(X, \bar{Q}_l) = H_c^i(X, \bar{Q}_l) \otimes_{\bar{Q}_l} \bar{Q}_l$. Since the singularities of A/G in our case are rational singularities of type A_n by direct calculation, it is easy to see

$$(5.22) \quad H_c^1(E, \bar{Q}_l) = 0.$$

Since we have

$$(5.23) \quad \begin{aligned} H_c^1(W, \bar{Q}_l) &= H_c^2(W, \bar{Q}_l) = 0, \quad H_c^2(A/G, \bar{Q}_l) \cong H^2(A/G, \bar{Q}_l), \\ H_c^2(S, \bar{Q}_l) &\cong H^2(S, \bar{Q}_l) \end{aligned}$$

(cf. Milne [5]), we have an injection:

$$(5.24) \quad \pi^*: H^2(A/G, \bar{Q}_l) \hookrightarrow H^2(S, \bar{Q}_l).$$

By this injection π^* , we can regard $e_1 \wedge e'_1$ as a non-zero element of $H^2(S, \bar{Q}_l)$. By our assumption, any power of α_1^2 is not a power of p . Therefore, $e_1 \wedge e'_1$ cannot come from an algebraic cycle. Hence, S is not supersingular. Thus, Condition (iii) implies Condition (iv). q.e.d.

REMARK 5.3. Professor T. Shioda commented to the author that he has another proof of Proposition 5.2, using the theory in [4] and [15].

Now, we prove Theorem V. (ii) implies (i) by Theorem I. The equivalence of (ii) and (iii) follows from Deuring [2]. (ii) implies (v) by Theorem I and Corollary III. It is clear that (i) or (v) implies (vi). (vi) implies (iv) by Shioda [13].

Finally, we have to show that Condition (iv) implies Condition (ii). Suppose that Condition (ii) does not hold. If $\alpha = 1$, then the surface (A) is isomorphic to the surface of Class (I) (i) in Theorem I, and the surface (B) is isomorphic to the surface of Class (II) (i). If $\alpha = 0$, then the surface (A) is also isomorphic to the surface of Class (I) (i), and the surface (B) is isomorphic to the surface of Class (II) (ii). Therefore, by Propositions 5.1 and 5.2, we conclude that (iv) implies (ii), when $\alpha = 0$ or 1.

Now, we assume $\alpha \neq 0, 1$.

We first consider the surface (A). We consider the following morphism f from the projective line P^1 with inhomogeneous coordinate s to P^1 with inhomogeneous coordinate t :

$$(5.25) \quad t = (s - \alpha) / [as^2 + (1 - a)s - \alpha],$$

where a is a solution of the equation:

$$(5.26) \quad (1 - a)^2 + 4a\alpha = 0.$$

By the base change by f , the surface of Class (I) (i) in Theorem I is transformed into the surface (A) with $\alpha \neq 0, 1$. Since the surface of Class (I) (i) is not supersingular under our assumption, by Propositions 5.1 and 5.2, we conclude that the surface (A) with $\alpha \neq 0, 1$ is not supersingular (cf. Shioda [13]). Therefore, (iv) implies (ii).

Now, we consider the surface (B). We consider the following morphism f from the projective line P^1 with inhomogeneous coordinate s to P^1 with inhomogeneous coordinate t :

$$(5.27) \quad t = s / [a(s - 1)(s - \alpha) + s],$$

where a is a solution of the equation:

$$(5.28) \quad a^2(1 - \alpha)^2 - 2a(1 + \alpha) + 1 = 0.$$

Then, by the base change by f , the surface of Class (II) (ii) in Theorem I is transformed into the surface (B). Since the surface of Class (II) (ii) is not supersingular under our assumption, by Propositions 5.1 and 5.2, we conclude that the surface (B) with $\alpha \neq 0, 1$ is not supersingular. Therefore, (iv) implies (ii). q.e.d.

REMARK 5.4. For the elliptic surface of Class (II) (iii) in Theorem I, we can also prove the equivalence similar to that in Theorem V with Condition (ii) replaced by (ii') " $p \equiv 3 \pmod{4}$ ".

In this case, to prove the fact that (iv) implies (ii), we have only to consider the morphism f from P^1 with inhomogeneous coordinate s to P^1 with inhomogeneous coordinate t defined by

$$(5.29) \quad t = 1/(2s - 1)^2.$$

By the base change by f , the surface of Class (II) (ii) is transformed into the surface of Class (II) (iii).

Thus, by this fact and Theorem V, we know the necessary and sufficient conditions for the surfaces of Classes (I) and (II) (i), (ii), (iii), (iv) to be unirational. For the surfaces of Class (II), we have a more general conjecture.

CONJECTURE. *The elliptic surface*

$$(5.30) \quad y^2 = 4x^3 - t^5(t - 1)^5(t - \alpha)^5(t - \beta)^5(t - \gamma)^5,$$

where α, β and γ are any elements of k , is unirational if and only if $p \equiv 3 \pmod{4}$.

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