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Unit Distances and Diameters in Euclidean Spaces

Konrad J. Swanepoel

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Abstract We show that the maximum number of unit distances or of diameters in a set of n points in d-dimensional Euclidean space is attained only by specific types of Lenz constructions, for all $d \geq 4$ and n sufficiently large depending on d. As a corollary, we determine the exact maximum number of unit distances for all even $d \geq 6$ and the exact maximum number of diameters for all $d \geq 4$ and all n sufficiently large depending on d.

Keywords Erdős problem · Combinatorial geometry · Unit distance problem · Lenz construction · Diameters · Erdős–Simonovits stability theorem · Erdős–Stone theorem

1 Introduction

1.1 Unit Distances

For a finite subset S of Euclidean d-space $\mathbb{R}^d$, let $u(S)$ denote the number of pairs of points in S at distance 1. Define

$$u_d(n) = \max\{u(S) : S \subset \mathbb{R}^d,\ |S| = n\}.$$

Erdős initiated the study of $u_2(n)$ in [4] and of $u_d(n)$ in the higher-dimensional case $d \geq 3$ in [5]. The cases $d = 2$ and $d = 3$ are the most difficult. Erdős [4] obtained

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K.J. Swanepoel (✉)
Fakultät für Mathematik, Technische Universität Chemnitz, 09107 Chemnitz, Germany
e-mail: konrad.swanepoel@gmail.com

K.J. Swanepoel
Department of Mathematical Sciences, University of South Africa, PO Box 392, Pretoria 0003, South Africa

the superlinear lower bound

$$u_2(n) \geq n^{1+\frac{c}{\log\log n}},$$

which he conjectured to be tight [9–12]. The best known upper bound is

$$u_2(n) \leq cn^{4/3},$$

due to Spencer, Szemerédi, and Trotter [27]. See Székely [32] for a particularly simple proof.

For $d = 3$, the known lower (Erdős [5]) and upper bounds (Clarkson et al. [3]) are the following:

$$cn^{4/3} \log\log n \leq u_3(n) \leq cn^{3/2}\beta(n),$$

where $\beta(n)$ is an extremely slowly growing function related to the inverse Ackerman function.

For $d \geq 4$ (the subject of this paper), the situation changes drastically. Lenz, as reported in [5], observed that if we take $p := \lfloor d/2 \rfloor$ circles in pairwise orthogonal 2-dimensional subspaces, each with center at the origin and radius $1/\sqrt{2}$, then any two points on different circles are at unit distance. Therefore, if n points are chosen by taking $n/p + O(1)$ points on each circle, $\frac{p-1}{2p}n^2 - O(1)$ unit distances are obtained. Erdős [5] showed that since $K_{p+1}(3)$, the complete $(p+1)$-partite graph with three vertices in each class, does not occur as a unit-distance graph in $\mathbb{R}^d$, the Erdős–Stone theorem [17] gives:

$$u_d(n) = \frac{p-1}{2p}n^2 + o\big(n^2\big) \quad \text{for all } d \geq 4.$$

Using an extremal graph theory result of Erdős [6] and Simonovits [26], Erdős [7] determined the exact value of $u_d(n)$ when $d \geq 4$ is even and n is a sufficiently large (depending on d) multiple of $2d = 4p$. The n/p points on each circle are then the vertices of $n/(4p)$ squares. This determines $u_d(n)$ asymptotically for all sufficiently large n up to an $O(1)$ term (still for d even). Brass [1] (together with a number theoretical result of van Wamelen [33]) determined $u_4(n)$ completely. For $n \geq 5$,

$$u_4(n) = \begin{cases} \lfloor n^2/4 \rfloor + n & \text{if } n \text{ is divisible by 8 or 10,} \\ \lfloor n^2/4 \rfloor + n - 1 & \text{otherwise.} \end{cases}$$

For odd $d \geq 5$, Erdős and Pach [14] showed that

$$u_d(n) = \frac{p-1}{2p}n^2 + \Theta\big(n^{4/3}\big).$$

For the lower bound, they observed that the Lenz construction can be improved when d is odd by replacing one of the circles by a 2-sphere of radius $1/\sqrt{2}$ in a 3-dimensional space orthogonal to the other 2-dimensional subspaces and by placing the points on the sphere such that the unit distance occurs at least $cn^{4/3}$ times (a construction of Erdős, Hickerson, and Pach [13]). For the upper bound, they used a stability result in extremal graph theory [2, Chap. 5, Remark 4.5(ii)] together with the fact that the maximum number of unit distances among n points on a 2-sphere is $O(n^{4/3})$ [3].

1.2 Diameters

For a finite subset S of $\mathbb{R}^d$, we call a pair of points in S a *diameter* if their distance equals the diameter of S. Let $M(S)$ denote the number of diameters in S. Define

$$M_d(n) = \max\{M(S) : S \subset \mathbb{R}^d,\ |S| = n\}.$$

Erdős [4] showed that $M_2(n) = n$ for $n \geq 3$. Vázsonyi conjectured, as reported in [4], that $M_3(n) = 2n - 2$ for $n \geq 4$. This was independently proved by Grünbaum [17], Heppes [18], and Straszewicz [28]. For a new proof, see [30].

As in the case of unit distances, the situation is completely different when $d \geq 4$. Erdős [5] showed that for $d \geq 4$, $M_d(n) = \frac{p-1}{2p}n^2 + o(n^2)$, the same asymptotics as $u_d(n)$. For other work on this problem by Hadwiger, Lenz, and Yugai, see the survey of Martini and Soltan [22].

2 New Results

If $d \geq 4$ is even, let $p = d/2$, and consider any orthogonal decomposition $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ with all V_i 2-dimensional. In each V_i, let C_i be the circle with center at the origin o and radius r_i such that $r_i^2 + r_j^2 = 1$ for all distinct i and j. When $d \geq 6$, this implies that each $r_i = 1/\sqrt{2}$. For the purposes of Lemma 8 below, we call the p circles $C_1, \dots, C_p$ an *even-dimensional Lenz system*. We define an *even-dimensional Lenz configuration* to be any translate of a finite subset of $\bigcup_{i=1}^p C_i$.

If $d \geq 5$ is odd, let $p = \lfloor d/2 \rfloor$, and consider any orthogonal decomposition $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ with V_1 3-dimensional and all other V_i $(i = 2, \dots, p)$ 2-dimensional. Let Σ_1 be the sphere in V_1 with center o and radius r_1, and for each $i = 2, \dots, p$, let C_i be the circle with center o and radius r_i such that $r_i^2 + r_j^2 = 1$ for all distinct i, j. When $d \geq 7$, necessarily each $r_i = 1/\sqrt{2}$. Again, as needed for Lemma 8, we call the 2-sphere and $p - 1$ circles $\Sigma_1, C_2, \dots, C_p$ an *odd-dimensional Lenz system*. We define an *odd-dimensional Lenz configuration* to be any translate of a finite subset of $\Sigma_1 \cup \bigcup_{i=2}^p C_i$. (Later we distinguish between *weak* and *strong* Lenz configurations as a technical notion inside the proofs. The definition here coincides with a strong Lenz construction in the sequel.)

We call a set S of n points in $\mathbb{R}^d$ an *extremal set* with respect to unit distances [diameters] if $u(S) = u_d(n)$ [$M(S) = M_d(n)$].

Theorem 1 *For each $d \geq 4$, there exists $N(d)$ such that all extremal sets of $n \geq N(d)$ points (with respect to unit distances or diameters) are Lenz configurations.*

The proof uses a typical technique in extremal graph and hypergraph theory [16, 20, 23, 26]: First prove a stability result for sets that are close to extremal, thus giving approximate structural information, and then use extremality to deduce more exact structural information.

For even $d \geq 6$, it is then possible to determine $u_d(n)$ exactly. On the other hand, for odd $d \geq 5$, the main obstacle to determine $u_d(n)$ is our lack of knowledge of the function $f(m)$ which gives the exact maximum number of unit distances between m points on a 2-sphere of radius $1/\sqrt{2}$ (for odd $d \geq 7$) and the function $g(m)$ which gives the exact maximum number of unit distances between m points on a 2-sphere of arbitrary radius [13, 31] (for $d = 5$).

Let $t_p(n)$ denote the number of edges of the *Turán p-partite graph* on n vertices. This is the complete p-partite graph with $\lfloor n/p \rfloor$ or $\lceil n/p \rceil$ vertices in each class [2, Chapter VI]. We do not need the exact value of $t_p(n)$, only that

$$t_p(n) = \frac{p-1}{2p}n^2 - O(1).$$

Corollary 2 *Let $d \geq 6$ be even. For all sufficiently large n (depending on d),*

$$u_d(n) = \begin{cases} t_p(n) + n - r & \text{if } 0 \leq r \leq p-1, \\ t_p(n) + n - p & \text{if } p \leq r \leq 3p-1, \\ t_p(n) + n - 2d + r & \text{if } 3p \leq r \leq 4p-1, \end{cases}$$

where $p = d/2$, and r is the remainder when dividing n by $4p = 2d$.

For all $d \geq 4$, it is possible to determine $M_d(n)$ exactly if n is large. The most complicated case is $d = 5$, where it is necessary to know the maximum number of diameters in a set of n points on a 2-sphere in $\mathbb{R}^3$. For each $n \geq 6$, we construct a set of n points in $\mathbb{R}^3$ with $2n - 2$ diameters, all lying on a sphere (see Lemma 7(e) below).

Corollary 3 *For all sufficiently large n (depending on d),*

$$M_4(n) = \begin{cases} t_2(n) + \lceil n/2 \rceil + 1 & \text{if } n \not\equiv 3 \pmod 4, \\ t_2(n) + \lceil n/2 \rceil & \text{if } n \equiv 3 \pmod 4; \end{cases}$$

$$M_5(n) = t_2(n) + n;$$

$$M_d(n) = t_p(n) + p \quad \text{for even } d \geq 6, \text{ where } p = d/2;$$

$$M_d(n) = t_p(n) + \lceil n/p \rceil + p - 1 \quad \text{for odd } d \geq 7, \text{ where } p = \lfloor d/2 \rfloor.$$

We use two stability theorems to prove Theorem 1, one for even dimensions and one for odd dimensions.

Theorem 4 *For all $\varepsilon > 0$ and even $d \geq 4$, there exist $\delta > 0$ and N such that any set of $n \geq N$ points in $\mathbb{R}^d$ with at least $(\frac{p-1}{2p} - \delta)n^2$ unit distance pairs can be partitioned into $S_0, S_1, \dots, S_p$ such that $|S_0| < \varepsilon n$ and for each $i = 1, \dots, p$,*

$$\frac{n}{p} - \varepsilon n < |S_i| < \frac{n}{p} + \varepsilon n$$

and S_i is on a circle C_i such that the circles $C_1, \ldots, C_p$ have the same center and are mutually orthogonal.

Theorem 5 *For all $\varepsilon > 0$ and odd $d \geq 5$, there exist $\delta > 0$ and N such that any set S of $n \geq N$ points in $\mathbb{R}^d$ with at least $(\frac{p-1}{2p} - \delta)n^2$ unit distance pairs can be partitioned into $S_0, S_1, \ldots, S_p$ such that $|S_0| < \varepsilon n$ and for each $i = 1, \ldots, p$,*

$$\frac{n}{p} - \varepsilon n < |S_i| < \frac{n}{p} + \varepsilon n,$$

S_1 is on a 2-sphere Σ_1, S_i is on a circle C_i, $i = 2, \ldots, p$, and $\Sigma_1, C_2, \ldots, C_p$ have the same center and are mutually orthogonal.

Corollary 6 *Let $d \geq 4$. If a set S of n points in $\mathbb{R}^d$ has at least $(\frac{p-1}{2p} - o(1))n^2$ unit distance pairs, then S is a Lenz configuration except for $o(n)$ points.*

3 Overview of the Paper

In Sect. 4 we prove some geometrical lemmas.

In Sect. 5 we determine the maximum number of unit distances and diameters in even-dimensional Lenz configurations, introduce the notions of weak and strong Lenz configuration in odd dimensions, show that the weak Lenz configurations with the largest number of unit distances or diameters are strong Lenz configurations, and determine the maximum number of diameters in strong Lenz configurations. Corollaries 2 and 3 then follow, given that extremal sets are Lenz configurations (weak Lenz configurations when $d \geq 5$ is odd).

In Sect. 6 we use the Erdős–Simonovits stability theorem from extremal graph theory to prove Theorems 4 and 5, from which Corollary 6 is immediate.

Finally, in Sect. 7 we use the stability theorems (Theorems 4 and 5) to show that sets of points that are extremal with respect to unit distances or diameters are (weak) Lenz configurations, thereby finishing the proof of Theorem 1.

4 Geometric Preliminaries

We denote the distance between points a and b in $\mathbb{R}^d$ by $|ab|$. The *unit distance graph* of a set S of n points in $\mathbb{R}^d$ is defined by joining any two points at distance 1. Let $u(S)$ denote the number of (unordered) unit distance pairs in S. Two points in S at distance 1 are *neighbors*. For any point x and finite set S, let $u(x, S)$ denote the number of points in S that are at distance 1 to x. Similarly, for any finite sets A and B, let $u(A, B)$ denote the number of (ordered) unit distance pairs (a, b) with $a \in A$ and $b \in B$.

Whenever we work with diameters, we assume that the diameter of S is 1 and then we use the notation $u(S)$, $u(x, S)$, and $u(A, B)$ as before. In this case we call the unit distance graph of S the *diameter graph* of S.

We continually use the following two basic lemmas in the sequel. The first deals with unit distances and diameters on circles and 2-spheres, and the second one with unit distances in dimensions higher than 3.

Lemma 7 *Let S be a set of n points in $\mathbb{R}^3$.*

(a) *If S lies on a circle of radius $1/\sqrt{2}$, then*

$$u(S) \le \begin{cases} n & \text{if } n \text{ is divisible by } 4, \\ n-1 & \text{otherwise.} \end{cases}$$

Equality is possible for all n by letting S be the union of the vertices of $\lfloor n/4 \rfloor$ inscribed squares and $n - 4\lfloor n/4 \rfloor$ vertices of an additional square.

(b) *If S has diameter 1 and lies on a circle, then*

$$u(S) \le \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$$

Equality is possible for all $n \ge 2$, for a circle of suitable radius depending on n.

(c) *If S has diameter 1 and lies on a circle of radius $> 1/\sqrt{3}$, then $u(S) = 1$.*

(d) *If S lies on a 2-sphere, then $u(S) = O(n^{4/3})$. When the radius equals $1/\sqrt{2}$, there exists a set S with $u(S) = \Omega(n^{4/3})$.*

(e) *If S has diameter 1 and lies on a 2-sphere, then $u(S) \le 2n - 2$. Equality is possible for each $n \ge 4$, $n \ne 5$, for a 2-sphere of suitable radius depending on n.*

(f) *If S has diameter 1 and lies on a 2-sphere of radius $\ge 1/\sqrt{2}$, then $u(S) \le n$. Equality is possible for all $n \ge 3$ and all radii $\ge 1/\sqrt{2}$.*

Proof Statements (a), (b), and (c) are straightforward, except perhaps that $u(S) \le n-1$ for an even number of concyclic points of diameter 1. This follows essentially from the easily seen observation that if the diameter graph of points of some concyclic points contains a cycle, then it consists only of this cycle, together with the well-known fact that all cycles in diameter graphs in the plane are odd [19, 29].

The upper bound in (d) is due to Clarkson et al. [3]. The simplest known proof of it is by adapting Székely's proof [32] for the planar case. The lower bound in (d) is due to Erdős, Hickerson, and Pach [13].

Statement (f) can be found in Kupitz, Martini, and Wegner [21]. It follows as in the planar case [25, Theorem 13.13] from the observation that any two diameters, when drawn as short great circular arcs on the 2-sphere, must intersect. Examples of n points with n diameters are easily found for all radii larger than $1/\sqrt{2}$; they have essentially the same structure as in the plane; see [21] for details.

The upper bound of $2n - 2$ in (e) is the Grünbaum–Heppes–Straczewicz upper bound for diameters in $\mathbb{R}^3$ [25, Theorem 13.14]. (For a new proof, see [30].) The following is a short proof for points on a 2-sphere. For a point x on the sphere, denote its opposite point by x'. Colour the n given points blue and their opposite points red. For any diameter xy, join the blue point x and the red point y' by a short arc of the great circle passing through them and do the same with x' and y. This defines a

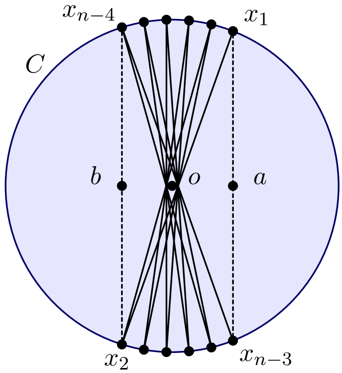

Fig. 1 Circle C with points x_1 to x_{n-3}

bipartite geometric graph on the sphere, with all the arcs of the same length r, say. It is easily seen that this graph is planar: if the arcs ab' and cd' intersect, then by the triangle inequality the arc ad' or the arc $b'c$ will be shorter than r. Then either $|ad|$ or $|bc|$ will be larger than the diameter, a contradiction. This graph has $2n$ vertices. By Euler's formula, a bipartite planar graph on $2n$ vertices has at most $4n-4$ edges. Since this is twice the number of diameters, the upper bound follows.

The only statement that remains to be proved is that $2n-2$ diameters can be attained on a 2-sphere for each $n \geq 4$, $n \neq 5$. For even $n \geq 4$, the following construction is described in Neaderhouser and Purdy [24]. Consider the vertex set of a regular $(n-1)$-gon of diameter 1 and choose another point on the axis of symmetry of the polygon at distance 1 to the $n-1$ vertices. This gives n points with $2n-2$ diameters.

For odd $n \geq 7$, a more involved construction is needed. Place $n-3$ points $x_1, \dots, x_{n-3}$ on the circle C of radius r and center o in the xy-plane such that the diameter 1 occurs between consecutive x_i's (Fig. 1).

Note that r and n determine everything up to isometry. We fix r later in the proof. Let x_{n-2} be the point on the positive z-axis at distance 1 to each point of C. Then x_{n-2} and C are on a unique sphere Σ with center o' and radius s, say. Note that o' is on the positive z-axis.

We now want to find points x_{n-1} and x_n on Σ such that

$$|x_1x_{n-1}| = |x_{n-3}x_{n-1}| = |x_2x_n| = |x_{n-4}x_n| = |x_{n-1}x_n| = 1$$

and

$$|x_{n-2}x_{n-1}| \leq 1, \qquad |x_{n-2}x_n| \leq 1,$$

as in Fig. 2.

This will give $2n-2$ diameters in the set $S := \{x_1, \dots, x_n\}$. For any value of r, there will clearly be unique points $x_{n-1}, x_n \in \Sigma \setminus \{x_{n-2}\}$ that satisfy

$$|x_{n-3}x_{n-1}| = |x_1x_{n-1}| = |x_2x_n| = |x_{n-4}x_n| = 1.$$

It remains to find an appropriate value of r so that

$$|x_{n-1}x_n| = 1, \qquad |x_{n-2}x_{n-1}| \leq 1, \qquad |x_{n-2}x_n| \leq 1.$$

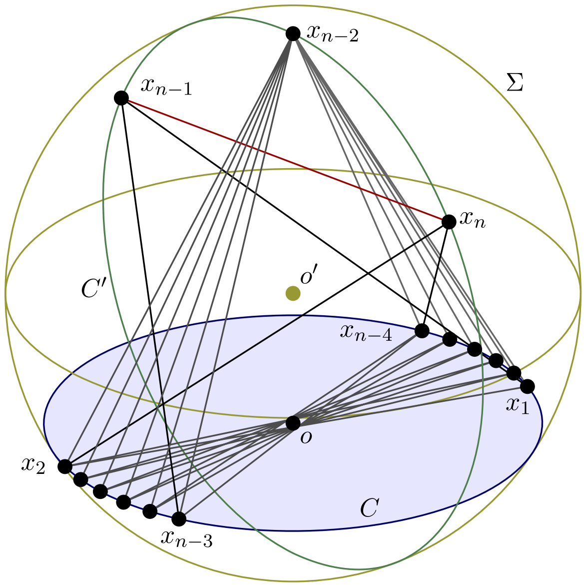

Fig. 2 15 points with 28 diameters on a sphere

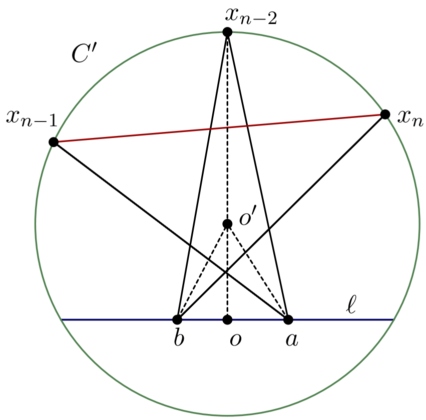

Fig. 3 Circle C'

We reduce this to a two-dimensional problem (Fig. 3). Let a and b be the mid-points of x_1x_{n-3} and x_2x_{n-4}, respectively. Consider the intersection of Σ with the plane $oabx_{n-2}$. This is a circle C' with center o' and radius s. By symmetry, x_{n-1} and x_n lie on C', and $|ax_{n-2}| = |ax_{n-1}|$ and $|bx_{n-2}| = |bx_n|$. Therefore, ao' bisects $\sphericalangle x_{n-2}ax_{n-1}$, and bo' bisects $\sphericalangle x_{n-2}bx_n$. Clearly, $|oa| > |ob|$, and both $|oa|$ and $|ob|$ are strictly monotone functions of r.

We now consider r to be a variable ranging in the interval $(1/2, r_0)$, where

$$r_0 := \left(2\cos\frac{\pi}{2(n-4)}\right)^{-1}.$$

On the one hand, $r > \frac{1}{2}$, and in the limit, as $r \to \frac{1}{2}$, the diameters $x_i x_{i+1}$ all coincide, and $\lim_{r\to 1/2}|oa| = \lim_{r\to 1/2}|ob| = 0$. It follows that

$$\lim_{r\to 1/2}|x_{n-2}x_{n-1}| = \lim_{r\to 1/2}|x_{n-2}x_n| = 0,$$

hence

$$\lim_{r\to 1/2}|x_{n-1}x_n| = 0. \tag{1}$$

On the other hand, $r < r_0$, where in the limit, as $r \to r_0$, x_1 and x_{n-3} coincide, and the points form the vertex set of a regular $(n-4)$-gon. Thus

$$\lim_{r\to r_0}|oa| = r_0,$$

$$\lim_{r\to r_0}|ob| \to 2r_0 \sin\frac{\pi}{n-4},$$

and

$$\lim_{r\to r_0}|x_{n-2}a| = 1.$$

Since $2r_0 > 1$, $\lim_{r\to r_0} x_{n-1}$ is a point below the chord ℓ of C' through a and b. (Note that ℓ is a diameter of C.) Also,

$$\lim_{r\to r_0}|x_2 a| = \lim_{r\to r_0}|x_{n-4}a| = 1,$$

hence $\lim_{r\to r_0} x_n = a$. Since x_{n-1} is lower than x_n (because $|oa| > |ob|$), x_{n-1} reaches ℓ before x_n. Since $|x_n b| = |x_{n-2}b|$, the chord $x_n b$ is below o'. Since at this stage (with $x_{n-1} \in \ell$) the chord bx_{n-1} is below o', the chord $x_{n-1}x_n$ is also below o'. Thus, before x_{n-1} reaches ℓ, there is a stage where $x_{n-1}x_n$ passes through o' with both x_{n-1} and x_n still above ℓ, and therefore their distances to x_{n-2} are at most 1. From $s > r > 1$ it follows that $|x_{n-1}x_n| > 1$. By (1), at some earlier stage, $|x_{n-1}x_n| < 1$. Therefore, at some inbetween stage, $|x_{n-1}x_n| = 1$. This finishes the construction for odd $n \geq 7$. □

We remark that the exception $n \neq 5$ in Lemma 7(f) is genuine. Suppose that there exist 5 points on a 2-sphere with 8 diameters. Then one of the points must be incident to 4 diameters. The other 4 points are then concyclic, and among them there can be at most 3 diameters (Lemma 7(b)), a contradiction.

Lemma 8 *Let $d \geq 4$, and set $p := \lfloor d/2 \rfloor$.*

(a) *Suppose that $A_1, \dots, A_p \subseteq \mathbb{R}^d$ are given such that $|A_i| \geq 3$ for each i, and $|ab| = 1$ for all $a \in A_i$ and $b \in A_j$ with distinct i and j. Then the A_i are contained in some translate of a Lenz system, i.e., there exists a Lenz system and $v \in \mathbb{R}^d$ such that after some relabeling of the indices, $A_i - v \subseteq C_i$ for $i \geq 2$, and $A_1 - v \subseteq C_1$ or $A_1 - v \subseteq \Sigma_1$ depending on whether d is even or odd.*

(b) *There does not exist* $p+1$ *sets* $A_1, \dots, A_{p+1} \subseteq \mathbb{R}^d$ *each containing at least* 3 *points such that any two points in different sets are at distance* 1.
(c) *Suppose that the point* $x \in \mathbb{R}^d$ *is at distance* 1 *to at least three points of each of the circles* C_i, $i \geq 2$, *in some Lenz system. Then* $x \in C_1$ *if* d *is even, or* $x \in \Sigma_1$ *if* d *is odd.*
(d) *Let* $d \geq 7$ *be odd. Let* C *be a circle on* Σ_1 *in some Lenz system with* $v \in V_1$ *perpendicular to* C. *Suppose that the point* $x \in \mathbb{R}^d$ *is at distance* 1 *to at least three points of each of the circles* C *and* C_i, $i \geq 3$. *Then* x *is on the sphere with center* o *and radius* $1/\sqrt{2}$ *in the* 3*-dimensional space spanned by* V_2 *and* v.

Proof In this proof we denote the affine hull of a set A by $\operatorname{aff}(A)$. Statements (c) and (d) are easy to prove. Note that if a point is at the same distance to 3 points on a circle, then its orthogonal projection on the affine hull of the circle is the center of the circle.

Statement (b) was originally observed by Erdős [5]. In fact, both (b) and (a) follow from the easily seen observation that if $A, B \subseteq \mathbb{R}^d$ are such that $|A|, |B| \geq 3$ and $|ab| = 1$ for all $a \in A$, $b \in B$, then A and B lie on unique spheres in $\operatorname{aff}(A)$ and $\operatorname{aff}(B)$, respectively, and these affine spaces are orthogonal. However, $\operatorname{aff}(A)$ and $\operatorname{aff}(B)$ need not intersect, so the only difficulty is perhaps to prove the existence of v in (a).

Consider, for example, the case of odd d in (a). The pairwise orthogonality of all $\operatorname{aff}(A_i)$ implies that $p-1$ of them have dimension 2, and one has dimension 3. Without loss of generality, let $\dim \operatorname{aff}(A_1) = 3$.

We show that any two $\operatorname{aff}(A_i)$ intersect. Suppose that $\operatorname{aff}(A_1) \cap \operatorname{aff}(A_2) = \emptyset$. Then $\dim \operatorname{aff}(A_1 \cup A_2) = 6$, and since $\operatorname{aff}(A_1 \cup A_2)$ and $\operatorname{aff}(A_i)$, $i \geq 3$, are pairwise orthogonal, we obtain a contradiction with the dimension. It follows similarly that any other pair of affine hulls has nonempty intersection.

Thus $\operatorname{aff}(A_1) \cap \operatorname{aff}(A_2) = \{v\}$, say. It remains to show that all other $\operatorname{aff}(A_i)$ also contain v. Suppose, for example, that $v \notin \operatorname{aff}(A_3)$. Then $\operatorname{aff}(A_1) \cap \operatorname{aff}(A_3) = \{v_1\}$ and $\operatorname{aff}(A_2) \cap \operatorname{aff}(A_3) = \{v_2\}$ with $v_1, v_2 \neq v$. Furthermore, $v_1 \neq v_2$, otherwise $\operatorname{aff}(A_1) \cap \operatorname{aff}(A_2)$ would contain more than one point. Thus $\operatorname{aff}(A_1 \cup A_2) \cap \operatorname{aff}(A_3)$ contains a line. This contradicts the orthogonality of $\operatorname{aff}(A_1 \cup A_2)$ and $\operatorname{aff}(A_3)$.

The case where d is even is similar. □

5 Optimized Lenz Configurations

5.1 Even Dimensions $d \geq 6$

We have already defined a Lenz configuration in the introduction. For any Lenz configuration S on n points lying on $p = d/2$ mutually orthogonal circles C_i with center o and radius $1/\sqrt{2}$, we define $S_i := S \cap C_i$ and $n_i := |S_i|$.

5.1.1 Unit Distances

When $d \geq 6$ is even, define

$$u_d^L(n) = \max\{u(S) : S \text{ is a Lenz configuration of } n \text{ points in } \mathbb{R}^d\}.$$

We call any Lenz configuration S of n points in $\mathbb{R}^d$ for which $u(S) = u_d^L(n)$ an *optimized Lenz configuration* (for unit distances).

Proposition 9 *Let $d \geq 6$ be even, $n \geq 1$, $p = d/2$, and $n \equiv r (\mathrm{mod}\, 2d)$, $0 \leq r \leq 2d-1$. Then*

$$u_d^L(n) = \begin{cases} t_p(n) + n - r & \text{if } 0 \leq r \leq p-1, \\ t_p(n) + n - p & \text{if } p \leq r \leq 3p-1, \\ t_p(n) + n - 2d + r & \text{if } 3p \leq r \leq 4p-1. \end{cases}$$

Proof Consider an optimized Lenz configuration S on p pairwise orthogonal circles $C_1, \dots, C_p$. We may rearrange the points on each circle without changing the number of unit distances between circles. By Lemma 7(a) and maximality, each $u(S_i) = n_i$ if $n_i \equiv 0 (\mathrm{mod}\, 4)$ and $u(S_i) = n_i - 1$ otherwise. The problem is now that of maximizing the function

$$u(n_1, \dots, n_p) := \left(\sum_{1 \leq i < j \leq p} n_i n_j \right) + n - p + k(n_1, \dots, n_p)$$

over all nonnegative $n_1, \dots, n_p$ that sum to n, where $k(n_1, \dots, n_p)$ equals the number of n_i divisible by 4. This easy but tedious exercise finishes the proof. □

5.1.2 Diameters

Still for even $d \geq 6$, define

$$M_d^L(n) = \max\{M(S) : S \text{ is a Lenz configuration of } n \text{ points in } \mathbb{R}^d\}.$$

We call any diameter 1 Lenz configuration S of n points in $\mathbb{R}^d$ for which $M(S) = u(S) = M_d^L(n)$ an *optimized Lenz configuration* (for diameters).

Proposition 10 *Let $d \geq 6$ be even, $n \geq d$, and $p = d/2$. Then*

$$M_d^L = t_p(n) + p.$$

Proof Consider an optimized Lenz configuration S of diameter 1 on p pairwise orthogonal circles $C_1, \dots, C_p$. By Lemma 7(c), each $u(S_i) \leq 1$. Therefore, $u(S) \leq t_p(n) + p$. Equality is clearly possible if $n \geq d$ by dividing the n points as equally as possible between the p circles and ensuring that a diameter occurs within each S_i. □

5.2 The Dimension $d = 4$

For any Lenz configuration S on n points lying on orthogonal circles C_1 and C_2 with common center o and radii r_1 and r_2 such that $r_1^2 + r_2^2 = 1$, define $S_i := S \cap C_i$ and $n_i := |S_i|$.

5.2.1 Unit Distances

This section is included for the sake of completeness. As for even $d \geq 6$, define

$$u_4^L(n) = \max\{u(S) : \ S \text{ is a Lenz configuration of } n \text{ points in } \mathbb{R}^4\}.$$

As shown by Brass [1] and van Wamelen [33]:

Proposition 11 *Let $n \geq 5$. Then*

$$u_4^L(n) = \begin{cases} t_2(n) + n & \text{if } n \text{ is divisible by } 8 \text{ or } 10, \\ t_2(n) + n - 1 & \text{otherwise.} \end{cases}$$

5.2.2 Diameters

As for even $d \geq 6$, define

$$M_4^L(n) = \max\{u(S) : S \text{ is a Lenz configuration of } n \text{ points in } \mathbb{R}^4\}.$$

We call any diameter 1 Lenz configuration S of n points in $\mathbb{R}^4$ for which $M(S) = u(S) = M_4^L(n)$ an *optimized Lenz configuration* (for diameters).

Proposition 12 *Let $n \geq 6$. Then*

$$M_4^L(n) = \begin{cases} t_2(n) + \lceil n/2 \rceil + 1 & \text{if } n \not\equiv 3 \pmod 4, \\ t_2(n) + \lceil n/2 \rceil & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof Consider an optimized Lenz configuration S of diameter 1 on pairwise orthogonal circles C_1 and C_2. Without loss of generality, $r_1 \leq r_2$. We now apply Lemma 7(b), (c). If $u(S_2) > 1$, then $r_2 \leq 1/\sqrt{3}$, hence $r_1 \geq \sqrt{2/3} > r_2$, a contradiction. Therefore, $u(S_2) \leq 1$. Also, $u(S_1) \leq n_1$, and if n_1 is even, $u(S_1) \leq n_1 - 1$. It follows that

$$u(S) \leq \begin{cases} n_1 n_2 + n_1 + 1 & \text{if } n_1 \text{ is odd}, \\ n_1 n_2 + n_1 & \text{if } n_1 \text{ is even}. \end{cases}$$

By considering the four cases of n modulo 4, it is easily checked that the maximum over all nonnegative n_i with $n_1 + n_2 = n$ is as in the statement of the theorem. For $n \geq 6$, it is also easy to see that there are configurations that attain this maximum. □

5.3 Odd Dimensions $d \geq 7$

We introduce the notion of a weak Lenz configuration. Let $d \geq 7$ be odd, $p = (d-1)/2$, and consider any orthogonal decomposition $\mathbb{R}^d = V_0 \oplus V_1 \oplus \cdots \oplus V_p$ with $\dim V_0 = 1$ and $\dim V_i = 2$ $(i = 1, \ldots, p)$. For each $i = 1, \ldots, p$, let Σ_i be the sphere in $V_0 \oplus V_i$ with center o and radius $1/\sqrt{2}$, and let C_i be the circle $V_i \cap \Sigma_i$. Let p^+ and p^- be the two points in V_0 at distance $1/\sqrt{2}$ from o. Thus p^+ and p^- are the north and south poles of each Σ_i when C_i is considered to be its equator.

A *strong Lenz configuration* of n points in $\mathbb{R}^d$ is a translate of a finite subset of $C_1 \cup \cdots \cup C_{p-1} \cup \Sigma_p$ for some orthogonal decomposition. (This is merely the odd-dimensional “Lenz configuration” of Sect. 2.) A *weak Lenz configuration* of n points in $\mathbb{R}^d$ is a translate of a finite subset of a $\Sigma_1 \cup \cdots \cup \Sigma_p$ for some orthogonal decomposition. Strong Lenz configurations are clearly also weak. If S is a weak Lenz configuration, we assume without loss of generality that it is a subset of $\Sigma_1 \cup \cdots \cup \Sigma_p$ and we define $S_i := S \cap \Sigma_i \setminus \{p^+, p^-\}$ $(i = 1, \ldots, p)$, $S_0 := S \cap \{p^+, p^-\}$, $n_i := |S_i|$ $(i = 0, \ldots, p)$, $n := |S|$.

5.3.1 Unit Distances

For odd $d \geq 7$, define

$$u_d^L(n) = \max\bigl\{u(S) : S \text{ is a weak Lenz configuration of } n \text{ points in } \mathbb{R}^d\bigr\}.$$

We call any weak Lenz configuration S of n points in $\mathbb{R}^d$ for which $u(S) = u_d^L(n)$ an *optimized Lenz configuration* (for unit distances). Unlike the even-dimensional case, we cannot give an expression for $u_d^L(n)$ more accurate than the estimate $u_d^L(n) = t_p(n) + \Theta(n^{4/3})$ of Erdős and Pach [14]. However, we next show that an optimized Lenz configuration must be strong for n sufficiently large depending on d. This implies that $u_d^L(n)$ can be determined if the function $f(n)$, which gives the maximum number of unit distances for n points on a 2-sphere of radius $1/\sqrt{2}$, is known.

Lemma 13 *For any distinct i and j, no point of $\Sigma_i \setminus C_i$ can be at unit distance to any point of $\Sigma_j \setminus C_j$.*

Proof Choose $x \in \Sigma_i$ and $y \in \Sigma_j$. Then $x = \lambda_i p^+ + v_i$ and $y = \lambda_j p^+ + v_j$ for unique $\lambda_i, \lambda_j \in \mathbb{R}$ and $v_i \in V_i$, $v_j \in V_j$. Then $\langle x, y\rangle = \lambda_i\lambda_j/2$, hence $|xy|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 = 1 - \lambda_i\lambda_j$. If $x \notin C_i$ and $y \notin C_j$, then $\lambda_i, \lambda_j \neq 0$ and $|xy| \neq 1$. $\square$

Proposition 14 *For each odd $d \geq 7$, there exists $N(d)$ such that all optimized Lenz configurations for unit distances on $n \geq N(d)$ points in $\mathbb{R}^d$ are strong Lenz configurations.*

Proof Let S be an optimized Lenz configuration on n points. Suppose that S is not a strong Lenz configuration. Thus, without loss of generality, $S_i \setminus C_i \neq \emptyset$ for $i = 1, 2$. We aim for a contradiction.

Since $u(S_1 \setminus C_1) = O(|S_1 \setminus C_1|^{4/3})$ (Lemma 7(d)) and $S_1 \setminus C_1 \neq \emptyset$, there exists $x \in S_1 \setminus C_1$ with $u(x, S_1 \setminus C_1) = O(|S_1 \setminus C_1|^{1/3}) = O(n^{1/3})$. Also, since $x \neq p^{\pm}$, $u(x, C_1) \leq 2$. Therefore, $u(x, S_1) = O(n^{1/3})$. Note that for each $i = 2, \ldots, p$, x is at distance 1 to all points in $S_i \cap C_i$ but, by Lemma 13, to none of $S_i \setminus C_i$. If we replace x by a new point on C_1, we lose at most $u(x, S_1)$ unit distances and gain $\sum_{i=2}^{p} |S_i \setminus C_i|$. Since $u(S)$ is the maximum over all weak Lenz configurations,

$$\sum_{i=2}^{p} |S_i \setminus C_i| \leq u(x, S_1) = O\bigl(n^{1/3}\bigr).$$

By instead considering a point $x \in S_2 \setminus C_2$ we similarly obtain that

$$\sum_{\substack{i=1\\ i\neq 2}}^{p} |S_i \setminus C_i| = O\bigl(n^{1/3}\bigr).$$

Therefore, $|S_i \setminus C_i| = O(n^{1/3})$ for each $i = 1, \ldots, p$.

We can now bound $u(S)$ from above. Note that each point of S_0 is at unit distance to each point of C_i, but to none of $\Sigma_i \setminus C_i$, each point of $\Sigma_i \setminus \{p^+, p^-\}$ is at unit distance to at most two points of C_i, $u(S_i \cap C_i) \le |S_i \cap C_i|$ (Lemma 7(a)), and $u(S_i \setminus C_i) = O(|S_i \setminus C_i|^{4/3})$ (Lemma 7(d)). This gives:

$$\begin{aligned} u(S_i) &\le u(S_0 \cup S_i)\\ &= u(S_0, S_i) + u(S_i \cap C_i) + u(S_i \cap C_i, S_i \setminus C_i) + u(S_i \setminus C_i)\\ &\le 2|S_i \cap C_i| + |S_i \cap C_i| + 2|S_i \setminus C_i| + O\bigl(|S_i \setminus C_i|^{4/3}\bigr)\\ &= O(n) + O\bigl(\bigl(n^{1/3}\bigr)^{4/3}\bigr) = O(n). \end{aligned}$$

Therefore, (grouping S_0 and S_1 together)

$$\begin{aligned} u(S) &\le t_p(n) + u(S_0 \cup S_1) + \sum_{i=2}^{p} u(S_i)\\ &= t_p(n) + O(n), \end{aligned}$$

which contradicts $u(S) = u_d^L(n) = t_p(n) + \Theta(n^{4/3})$ for large n. □

5.3.2 Diameters

For odd $d \ge 7$, define

$$M_d^L(n) = \max\bigl\{M(S) : S \text{ is a weak Lenz configuration of } n \text{ points in } \mathbb{R}^d\bigr\}.$$

We call any diameter 1 weak Lenz configuration S of n points in $\mathbb{R}^d$ for which $M(S) = u(S) = M_d^L(n)$ an *optimized Lenz configuration* (for diameters). We show that an optimized Lenz configuration must be strong for large n, and furthermore determine the exact value of $M_d^L(n)$.

Proposition 15 *For each odd $d \ge 7$, there exists $N(d)$ such that all optimized Lenz configurations for diameters on $n \ge N(d)$ points in $\mathbb{R}^d$ are strong Lenz configurations. Furthermore,*

$$M_d^L(n) = t_p(n) + \left\lceil \frac{n}{p} \right\rceil + p - 1 = t_p(n-1) + n - 1 + p.$$

Proof Choose a set S of n points equally distributed between the orthogonal circles $C_1, \ldots, C_{p-1}$ and 2-sphere Σ_p such that the diameter of each $S \cap C_i$ is 1 and also

$S \cap \Sigma_p \setminus C_p = \{p^+\}$ and $|S \cap C_p| = \lceil n/p \rceil - 1$. Then clearly $u(S) = t_p(n) + \lceil n/p \rceil + p - 1$. Therefore, $M_d^L(n) \geq t_p(n) + \lceil n/p \rceil + p - 1$. We need this lower bound in a moment.

Now let S be any optimized Lenz configuration on n points. Let $k_i := |S_i \setminus C_i|$ ($i = 1, \ldots, p$). We have to show that S is a strong Lenz configuration, i.e., that $k_i = 0$ for all $i = 1, \ldots, p$ except at most one.

First consider the case where $S_0 \neq \emptyset$; without loss of generality, $S_0 = \{p^+\}$. Then

$$
\begin{aligned}
u(S) &= u\big(S \setminus \{p^+\}\big) + \sum_{i=1}^{p} u\big(p^+, S_i\big) \\
&= \sum_{1 \leq i < j \leq p} u(S_i, S_j) + \sum_{i=1}^{p} u(S_i) + \sum_{i=1}^{p} u\big(p^+, S_i\big) \\
&= \sum_{1 \leq i < j \leq p} u(S_i, S_j) + \sum_{i=1}^{p} u\big(S_i \cup \{p^+\}\big) \\
&= \sum_{1 \leq i < j \leq p} |S_i||S_j| - \sum_{1 \leq i < j \leq p} k_i k_j + \sum_{i=1}^{p} u\big(S_i \cup \{p^+\}\big) \quad \text{(Lemma 13)} \\
&\leq t_p(n-1) - \sum_{1 \leq i < j \leq p} k_i k_j + \sum_{i=1}^{p} (n_i + 1) \quad \text{(Lemma 7(f))} \\
&= t_p(n-1) - \sum_{1 \leq i < j \leq p} k_i k_j + n - 1 + p \\
&= t_p(n) + \lceil n/p \rceil + p - 1 - \sum_{1 \leq i < j \leq p} k_i k_j.
\end{aligned}
$$

However, $u(S) = M_d^L(n) \geq t_p(n) + \lceil n/p \rceil + p - 1$, hence $\sum_{1 \leq i < j \leq p} k_i k_j = 0$, which implies that $k_i = 0$ for all i except one. This finishes the case $S_0 \neq \emptyset$.

Next consider the case where $S_0 = \emptyset$. Without loss of generality, $S_1 \setminus C_1 \neq \emptyset$, otherwise, by Proposition 10, $u(S) \leq t_p(n) + p$, a contradiction for large n. By Lemma 7(f), $u(S_1) \leq n_1$. If we remove the points in S_1, place p^+ into S_0 and choose $n_1 - 1$ new points of diameter 1 on C_1 to form another set S' of diameter 1, we then lose at most n_1 diameters and gain $n_1 + k_1 \sum_{i=2}^{p} k_i$. By maximality, $\sum_{i=2}^{p} k_i = 0$, i.e., the original S was already a strong Lenz configuration and $u(S) = u(S')$. Since $p^+ \in S'$, this case is reduced to the previous one. The theorem is proved. □

5.4 The Dimension $d = 5$

Consider an orthogonal decomposition $\mathbb{R}^5 = V_0 \oplus V_1 \oplus V_2$ such that $\dim V_0 = 1$ and $\dim V_1 = \dim V_2 = 2$. Choose $r_1 \in (0, 1)$. Let Σ_1 be the 2-sphere in $V_0 \oplus V_1$ with center o and radius r_1. Let C_2 be the circle in V_2 with center o and radius $r_2 := \sqrt{1 - r_1^2}$. Then any point of Σ_1 and any point of C_2 are at unit distance. We call

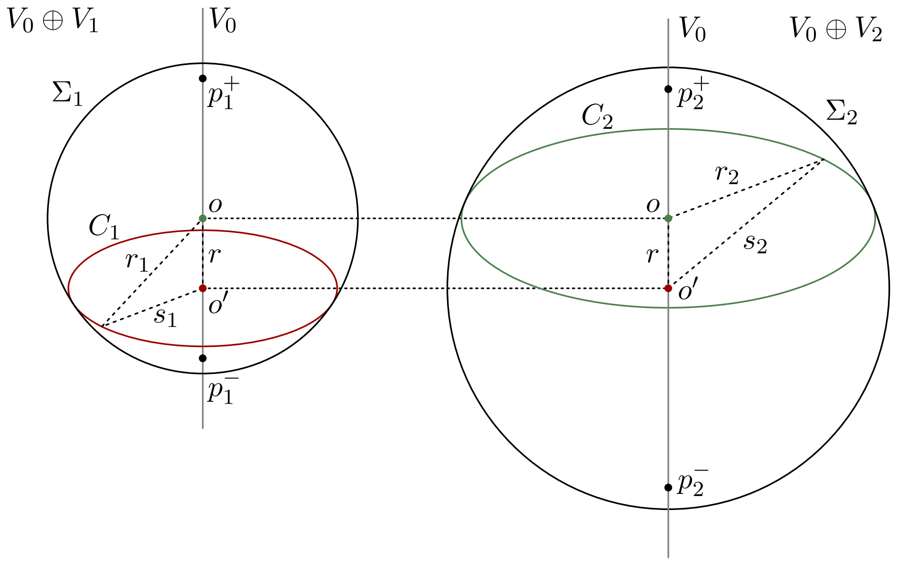

Fig. 4 Spheres Σ_i and circles C_i of a weak Lenz configuration in $\mathbb{R}^5$

a translate of a finite subset of $\Sigma_1 \cup C_2$ a *strong Lenz configuration* (equivalent to the 5-dimensional "Lenz configuration" of Sect. 2).

To define a weak Lenz configuration takes more care than for odd $d \geq 7$. Choose an additional parameter $r \in [0, r_1)$ and a point $o' \in V_0$ at distance r to o. Let C_1 be the circle with center o' and radius $s_1 := \sqrt{r_1^2 - r^2}$ in the plane of $V_0 \oplus V_1$ parallel to V_1 that passes through o'. Let Σ_2 be the 2-sphere in $V_0 \oplus V_2$ with center o' and radius $s_2 := \sqrt{r_2^2 + r^2}$. Then $C_i \subset \Sigma_i$ $(i = 1, 2)$ (Fig. 4). Note that $s_1^2 + s_2^2 = 1$, hence any point of Σ_2 and any point of C_1 are at unit distance. Similarly to Lemma 13, no point of $\Sigma_1 \setminus C_1$ can be at unit distance to a point of $\Sigma_2 \setminus C_2$. We call a translate of a finite subset of $\Sigma_1 \cup \Sigma_2$ a *weak Lenz configuration*. As before, strong Lenz configurations are clearly weak. Assume without loss of generality that the weak Lenz configuration $S \subset \Sigma_1 \cup \Sigma_2$. Each Σ_i has two *poles*: $\{p_1^+, p_1^-\} := V_0 \cap \Sigma_1$ and $\{p_2^+, p_2^-\} := V_0 \cap \Sigma_2$. In general, Σ_1 and Σ_2 may not have a point in common. If they do, the common points will be coinciding poles. Define $S_0 := S \cap V_0$, $S_i := S \cap \Sigma_i \setminus V_0$ $(i = 1, 2)$, $n_i := |S_i|$ $(i = 0, 1, 2)$, and $n := |S|$.

5.4.1 *Unit Distances*

As for odd $d \geq 7$, define

$$u_5^L(n) = \max\{u(S) : S \text{ is a weak Lenz configuration of } n \text{ points in } \mathbb{R}^5\}.$$

We call any weak Lenz configuration S of n points in $\mathbb{R}^5$ satisfying $u(S) = u_5^L(n)$ an *optimized Lenz configuration* (for unit distances). Again the best known estimate is $u_5^L(n) = t_2(n) + \Theta(n^{4/3})$, due to Erdős and Pach [14]. We show that an optimized

Lenz configuration is strong for sufficiently large n. As before, this implies that $u_5^L(n)$ can be determined if the function $g(n)$, which gives the maximum number of unit distances for n points on a 2-sphere of arbitrary radius, is known.

Proposition 16 *For all sufficiently large n, all optimized Lenz configurations for unit distances on n points in $\mathbb{R}^5$ are strong Lenz configurations.*

Proof Let S be an optimized Lenz configuration on n points. Suppose that $S_1 \setminus C_1 \neq \emptyset$ and $S_2 \setminus C_2 \neq \emptyset$. By Lemma 7(d), there exist points $x_i \in S_i \setminus C_i$ with $u(x_i, S_i \setminus C_i) = O(n^{1/3})$ $(i = 1, 2)$. Since $x_i \notin S_0$, $u(x_i, C_i) \leq 2$. Thus $u(x_i, S_i) = O(n^{1/3})$. If we replace each x_i by a new point on C_i, we lose at most $O(n^{1/3})$ unit distances and gain $|S_1 \setminus C_1| + |S_2 \setminus C_2|$. Since S is extremal, $|S_1 \setminus C_1| + |S_2 \setminus C_2| = O(n^{1/3})$. We bound $u(S)$ from above as in the case of odd $d \geq 7$. For each $i = 1, 2$:

$$\begin{aligned} u(S_i) &\leq u(S_i \cup S_0) \\ &= u(S_0) + u(S_0, S_i) + u(S_i \cap C_i) + u(S_i \cap C_i, S_i \setminus C_i) + u(S_i \setminus C_i) \\ &\leq 4 + 4|S_i \cap C_i| + |S_i \cap C_i| + 2|S_i \setminus C_i| + O\big(|S_i \setminus C_i|^{4/3}\big) \\ &= O(n), \end{aligned}$$

hence,

$$\begin{aligned} u(S) &= u(S_1, S_2) + u(S_0 \cup S_1) + u(S_0 \cup S_2) + u(S_0) + u(S_1) + u(S_2) \\ &\leq t_2(n) + O(n), \end{aligned}$$

which contradicts $u(S) = t_2(n) + \Theta(n^{4/3})$.

Therefore, some $S_i \setminus C_i = \emptyset$; without loss of generality, $S_2 \setminus C_2 = \emptyset$. To show that S is a strong Lenz configuration, it remains to show that $S_0 \subset \Sigma_1$. Suppose then without loss of generality that $p_2^+ \in S_0$ and $p_2^+ \notin \Sigma_1$. Then $p_1^\pm \neq p_2^+$. Since p_1^+ is at unit distance to each point of C_2, and p_1^+ and p_2^+ are different points in V_0, it follows that p_2^+ is not at unit distance to any point in S_2. If we replace p_2^+ by a new point on C_2, we lose at most one unit distance (possibly between p_2^+ and p_2^-) and gain $|S \cap \Sigma_1 \setminus C_1|$ unit distances. By extremality, $|S \cap \Sigma_1 \setminus C_1| \leq 1$. Therefore, except for at most 3 points (in addition, $p_2^+ \in S_0$ and possibly $p_2^- \in S_0$), S is on two orthogonal circles, and for this essentially 4-dimensional configuration, we obtain $u(S) \leq t_2(n) + O(n)$ as before, a contradiction.

It follows that S is a strong Lenz configuration. □

5.4.2 Diameters

As for odd $d \geq 7$, define

$$M_5^L(n) = \max\big\{u(S) : S \text{ is a weak Lenz configuration of } n \text{ points in } \mathbb{R}^5\big\}.$$

We call any diameter 1 weak Lenz configuration S of n points in $\mathbb{R}^5$ satisfying $M(S) = u(S) = M_5^L(n)$ an *optimized Lenz configuration* (for diameters). Again an optimized Lenz configuration is strong for large n, and the exact value of $M_5^L(n)$ can be determined. This is more intricate than for odd $d \geq 7$.

Proposition 17 *For all sufficiently large n, all optimized Lenz configurations for diameters on n points in $\mathbb{R}^5$ are strong Lenz configurations. Furthermore, $M_5^L(n) = t_2(n) + n$.*

Proof We first describe two types of strong Lenz configurations on n points with $t_2(n) + n$ diameters.

In the first construction, choose r_1 such that there exists a set S_1 of n_1 points of diameter 1 on Σ_1 with $2n_1 - 2$ diameters. By Lemma 7(e) this is possible if $n_1 \geq 4$, $n_1 \neq 5$. Choose any set S_2 of $n_2 = n - n_1$ points of diameter 1 on C_2. (Note that $r_1 < 1/\sqrt{2}$ by Lemma 7(f), which gives $r_2 > 1/\sqrt{2} > 1/\sqrt{3}$. Then by Lemma 7(c), we can have at most one diameter of length 1 on C_2.) Let $S := S_1 \cup S_2$. Then

$$\begin{aligned} u(S) &= u(S_1, S_2) + u(S_1) + u(S_2) \\ &= n_1 n_2 + 2n_1 - 2 + 1 = n_1(n_2 + 2) - 1 \\ &\leq t_2(n+2) - 1 = t_2(n) + n. \end{aligned}$$

Equality is possible by taking $n_1 = \lfloor n/2 \rfloor + 1$ or $\lceil n/2 \rceil + 1$. Keeping in mind that $n_1 \geq 4$, $n_1 \neq 5$, we obtain $t_2(n) + n$ diameters for all $n \geq 6$, $n \neq 8$.

In the second construction, first choose r_2 such that there exists a set S_2 of n_2 points of diameter 1 on C_2 with n_2 diameters (a regular star polygon). By Lemma 7(b) this is possible if $n_2 \geq 3$ is odd. Then $r_2 \leq 1/\sqrt{3}$ by Lemma 7(c), and $r_1 \geq \sqrt{2/3} > 1/\sqrt{2}$. By Lemma 7(f) we can then choose a set S_1 of $n_1 = n - n_2$ points of diameter 1 on Σ_1 with n_1 diameters if $n_1 \geq 3$. Let $S := S_1 \cup S_2$. Then

$$\begin{aligned} u(S) &= u(S_1, S_2) + u(S_1) + u(S_2) \\ &= n_1 n_2 + n_1 + n_2 = (n_1 + 1)(n_2 + 1) - 1 \\ &\leq t_2(n+2) - 1 = t_2(n) + n. \end{aligned}$$

Equality is possible by taking $n_1 = \lfloor n/2 \rfloor$, $n_2 = \lceil n/2 \rceil$ or $n_1 = \lceil n/2 \rceil$, $n_2 = \lfloor n/2 \rfloor$. Keeping in mind the requirements that $n_2 \geq 3$ must be odd and $n_1 \geq 3$, we obtain $t_2(n) + n$ diameters for all $n \geq 6$, $n \not\equiv 0 \pmod 4$. (It is because this second, simpler construction does not work for all n that we need the construction in Lemma 7(e) of an odd number n_1 of points on a 2-sphere with $2n_1 - 2$ diameters.)

Summarizing, $M_5^L(n) \geq t_2(n) + n$ for all $n \geq 9$. It is easy to see that every strong Lenz configuration with at least $t_2(n) + n$ diameters must be one of the above two constructions for sufficiently large n. We now turn to weak Lenz configurations.

Let S be an optimized Lenz configuration on n points. We distinguish between two cases.

First case: $S \cap \Sigma_1 \cap \Sigma_2 \neq \emptyset$. Any point in $S \cap \Sigma_1 \cap \Sigma_2$ must be a common pole of Σ_1 and Σ_2, say $p_1^+ = p_2^+$. Since this point is at distance 1 to C_1 and C_2, it follows that $|p_1^+ p_1^-|, |p_2^+ p_2^-| > 1$. Therefore, $S \cap \Sigma_1 \cap \Sigma_2$ contains only one point $p := p_1^+ = p_2^+$

at distance 1 to both C_1 and C_2. Let $k_i := |S_i \setminus C_i|$ $(i = 1, 2)$. Then

$$\begin{aligned} t_2(n) + n &\le u(S) \\ &= u(S_1, S_2) + u\big(S_1 \cup \{p\}\big) + u\big(S_2 \cup \{p\}\big) \\ &= n_1 n_2 - k_1 k_2 + u\big(S_1 \cup \{p\}\big) + u\big(S_2 \cup \{p\}\big). \end{aligned} \tag{2}$$

If $u(S_i \cup \{p\}) \le n_i + 1$ for both $i = 1, 2$, then by substituting into (2),

$$\begin{aligned} t_2(n) + n &\le n_1 n_2 - k_1 k_2 + n_1 + 1 + n_2 + 1 \\ &= (n_1 + 1)(n_2 + 1) - k_1 k_2 + 1 \\ &\le t_2(n + 1) - k_1 k_2 + 1 \quad (\text{note } n_1 + n_2 + 1 = n) \\ &= t_2(n) + \lceil n/2 \rceil - k_1 k_2 + 1. \end{aligned}$$

Therefore, $\lfloor n/2 \rfloor + k_1 k_2 \le 1$, a contradiction.

Without loss of generality, we may therefore assume that $u(S_1 \cup \{p\}) > n_1 + 1$. By Lemma 7(f), $r_1 < 1/\sqrt{2}$, which gives $r_2 > 1/\sqrt{2}$ and $u(S_2 \cup \{p\}) \le n_2 + 1$ (again Lemma 7(f)). Also, $u(S_1 \cup \{p\}) \le 2(n_1 + 1) - 2 = 2n_1$ (Lemma 7(e)). Substituting into (2), we have

$$\begin{aligned} t_2(n) + n &\le n_1 n_2 - k_1 k_2 + 2n_1 + n_2 + 1 \\ &= (n_1 + 1)(n_2 + 2) - k_1 k_2 - 1 \\ &\le t_2(n + 2) - k_1 k_2 - 1 \\ &= t_2(n) + n - k_1 k_2. \end{aligned}$$

It follows that $k_1 k_2 = 0$, S is a strong Lenz configuration, and $u(S) = t_2(n) + n$.

Second case: $S \cap \Sigma_1 \cap \Sigma_2 = \emptyset$. Then S may still contain poles, but a pole of Σ_i in S is not at distance 1 to C_i (otherwise it would also be a pole of the other sphere). We now define $T_i = S \cap \Sigma_i$ $(i = 1, 2)$. Then T_1, T_2 partition S (and we forget about the partition S_0, S_1, S_2). Let $m_i := |T_i|$ and $k_i := |T_i \setminus C_i|$ $(i = 1, 2)$. As in the first case,

$$\begin{aligned} t_2(n) + n &\le u(S) \\ &= u(T_1, T_2) + u(T_1) + u(T_2) \\ &= m_1 m_2 - k_1 k_2 + u(T_1) + u(T_2). \end{aligned} \tag{3}$$

If $u(T_i) \le m_i$ for both $i = 1, 2$, then by substituting into (3),

$$\begin{aligned} t_2(n) + n &\le m_1 m_2 - k_1 k_2 + m_1 + m_2 \\ &= (m_1 + 1)(m_2 + 1) - k_1 k_2 - 1 \\ &\le t_2(n + 2) - k_1 k_2 - 1 \\ &= t_2(n) + n - k_1 k_2. \end{aligned}$$

It follows that $k_1k_2=0$, S is a strong Lenz configuration, and $u(S)=t_2(n)+n$.

Otherwise, without loss of generality, $u(T_1)>m_1$. As in the first case,

$$u(T_1)\le 2m_1-2 \tag{4}$$

and

$$u(T_2)\le m_2. \tag{5}$$

Since each point in $T_i\setminus C_i$ is at distance 1 to at most two points of $T_i\cap C_i$ (recall that in this case a pole is not at distance 1 to any point on C_i), we also obtain

$$\begin{aligned}u(T_1)&=u(T_1\cap C_1)+u(T_1\cap C_1,T_1\setminus C_1)+u(T_1\setminus C_1)\\&\le |T_1\cap C_1|+2|T_1\setminus C_1|+2|T_1\setminus C_1|-2\\&=m_1+3k_1-2\end{aligned} \tag{6}$$

and, since $r_2>1/\sqrt{2}$,

$$\begin{aligned}u(T_2)&=u(T_2\cap C_2)+u(T_2\cap C_2,T_2\setminus C_2)+u(T_2\setminus C_2)\\&\le 1+2|T_2\setminus C_2|+|T_2\setminus C_2|\\&=1+3k_2.\end{aligned} \tag{7}$$

Substituting (5) and (6) into (3), we have

$$\begin{aligned}t_2(n)+n&\le m_1m_2-k_1k_2+m_1+3k_1-2+m_2\\&=(m_1+1)(m_2+1)-k_1(k_2-3)-3\\&\le t_2(n+2)-k_1(k_2-3)-3\\&=t_2(n)+n-k_1(k_2-3)-2.\end{aligned}$$

Therefore, $k_1(k_2-3)+2\le 0$, hence $k_2\le 2$.

Substituting (4) and (7) into (3), we have

$$\begin{aligned}t_2(n)+n&\le m_1m_2-k_1k_2+2m_1-2+3k_2+1\\&=m_1(m_2+2)-(k_1-3)k_2-1\\&\le t_2(n+2)-(k_1-3)k_2-1\\&=t_2(n)+n-(k_1-3)k_2.\end{aligned}$$

Therefore, $(k_1-3)k_2\le 0$. If $k_2>0$, then $k_1\le 3$, and substituting (6) and (7) into (3),

$$\begin{aligned}t_2(n)+n&\le m_1m_2-k_1k_2+m_1+3k_1-2+3k_2+1\\&=m_1(m_2+1)+O(1)\\&\le t_2(n+1)+O(1)\\&=t_2(n)+\lceil n/2\rceil+O(1),\end{aligned}$$

a contradiction. It follows that $k_2 = 0$, hence S is a strong Lenz configuration, and $u(S) = t_2(n) + n$. □

6 Stability Theorems

We formulate the stability theorem of Erdős [8] and Simonovits [26] (see also [2, Chap. 5, Theorem 4.2]) in the following convenient way. Let $K_r(t)$ denote the complete r-partite graph with t vertices in each class.

Stability Theorem *For any $p, t \geq 2$ and any $\varepsilon > 0$, there exist N and $\delta > 0$ such that if G is any graph with $n \geq N$ vertices and at least $(\frac{p-1}{2p} - \delta)n^2$ edges which does not contain $K_{p+1}(t)$, then the vertices of G can be partitioned into sets $S_0, S_1, \dots, S_p$ such that $|S_0| < \varepsilon n$,*

$$\frac{n}{p} - \varepsilon n < |S_i| < \frac{n}{p} + \varepsilon n \quad \text{for each } i = 1, \dots, p,$$

and for each $i = 1, \dots, p$, each $x \in S_i$ is nonadjacent to less than εn vertices of $G - S_i$.

We now use the Stability Theorem to prove Theorems 4 and 5.

Proof of Theorem 4 Without loss of generality, $\varepsilon < 1/(3p^2)$. By Lemma 8(b), $K_{p+1}(3)$ does not occur in the unit distance graph of S. Let $S_0, S_1, \dots, S_p$ be the partition coming from the Stability Theorem. Suppose that S_1 does not lie on any circle. Let A_1 be a set of 4 nonconcyclic points of S_1. For each $i = 2, \dots, p$, let A_i consist of 3 points of S_i such that any two vertices in distinct A_i's are adjacent. This is possible, since each $x \in S_i$ is at unit distance to all points in $S \setminus S_i$ except for εn points, and $(4 + 3(p-2))\varepsilon n + 3 < n/p - \varepsilon n$ if $n > 9p^2$. The unit distance graph of $\bigcup_{i=1}^{p} A_i$ contains a complete p-partite graph with 4 vertices in the one class A_1 and 3 vertices in each other class A_i, $i = 2, \dots, p$. By Lemma 8(a), each A_i is concyclic, a contradiction.

Therefore, each S_i $(i = 1, \dots, p)$ is concyclic. To see that these circles are orthogonal, choose 3 points from each S_i as above to form a $K_p(3)$. Again by Lemma 8(a) each class lies on a circle C_i, with $C_1, \dots, C_p$ mutually orthogonal. Since there is a unique circle through any 3 noncollinear points, $S_i \subset C_i$ for each $i = 1, \dots, p$. □

The following is the even-dimensional case of Corollary 6.

Corollary 18 *Fix an even $d \geq 4$. If a set S of n points in $\mathbb{R}^d$ has at least $(\frac{p-1}{2p} - o(1))n^2$ unit distance pairs, then S is a Lenz configuration except for $o(n)$ points.*

Proof of Theorem 5 Without loss of generality, $\varepsilon < 1/(4p^2)$. By Lemma 8(b), $K_{p+1}(3)$ does not occur in the unit distance graph of S. Let $S_0, S_1, \dots, S_p$ be the partition coming from the Stability Theorem using $\varepsilon' = \varepsilon/5$. Suppose that S_1 does not lie on any 2-sphere. Let A_1 be a set of 5 noncospherical points of S_1. For each

$i = 2, \dots, p$, let A_i consist of 3 points of S_i such that any two vertices in distinct A_i's are adjacent. This is possible, since each $x \in S_i$ is at unit distance to all points in $S \setminus S_i$ except for $\varepsilon' n$ points, and $(5 + 3(p-2))\varepsilon' n + 3 < n/p - \varepsilon' n$ if $n > 4p$. The unit distance graph of $\bigcup_{i=1}^{p} A_i$ contains a complete p-partite graph with 5 vertices in one class and 3 vertices in each other class. By Lemma 8(a), each A_i is on a 2-sphere, a contradiction.

Therefore, each S_i $(i = 1, \dots, p)$ is on a 2-sphere. If each S_i lies on a circle, then, as in the even-dimensional case, it follows that these circles are orthogonal, and the proof is finished. Without loss of generality S_1 is then not concyclic. Let Σ_1 denote the unique 2-sphere on which S_1 lies. Let A_1 be a set of 4 noncoplanar points of S_1.

We now slightly modify the partition of S. There are less than $4\varepsilon' n$ points of $\bigcup_{i=2}^{p} S_i$ not at distance 1 to each point of A_1. Remove these points from $\bigcup_{i=2}^{p} S_i$ and add them to S_0. Thus we may assume that each point of A_1 is joined to all of $\bigcup_{i=2}^{p} S_i$, but then $|S_0| < 5\varepsilon' n = \varepsilon n$ for each $i = 1, \dots, p$, $||S_i| - n/p| < \varepsilon n$, and each point of S_i is joined to less than εn points of $S \setminus S_i$. We show that for this modified partition, $S_2, \dots, S_p$ are on circles $C_2, \dots, C_p$, with $\Sigma_1, C_2, \dots, C_p$ mutually orthogonal.

Suppose that some S_i $(i = 2, \dots, p)$ is not concyclic, without loss of generality S_2. Let A_2 be a set of 4 nonconcyclic points from S_2, and, as before, for $i = 3, \dots, p$, let A_i consist of 3 points from S_i such that all vertices in different A_i's are adjacent. By Lemma 8(a), all A_i, $i = 2, \dots, p$, must lie on circles, a contradiction.

Therefore, each S_i $(i = 2, \dots, p)$ is on a circle C_i. As in the proof of Theorem 4, to see that $\Sigma_1, C_2, \dots, C_p$ are mutually orthogonal, choose 4 noncoplanar points from S_1 and 3 points from the other S_i that form a complete p-partite graph, and apply Lemma 8(a). □

The following is the odd-dimensional case of Corollary 6.

Corollary 19 *Fix an odd $d \geq 5$. If a set S of n points in $\mathbb{R}^d$ has at least $(\frac{p-1}{2p} - o(1))n^2$ unit distance pairs, then S is a strong Lenz configuration except for $o(n)$ points.*

7 Extremal Sets Are (Weak) Lenz Configurations

The following three propositions, completing the proof of the main theorem, now follow relatively simply from the stability theorems.

Proposition 20 *For each even $d \geq 4$, there exists $N(d)$ such that all sets of $n \geq N(d)$ points in $\mathbb{R}^d$ extremal with respect to unit distances or diameters are Lenz configurations.*

Proof When considering diameters assume that the diameter is 1. In both cases, an extremal set S on n points has at least $t_p(n) = \frac{p-1}{2p} n^2 - O(1)$ unit distances, so we may apply Theorem 4 (with $\varepsilon = 1/(2p^2)$). Thus, for n sufficiently large depending on d, there is a partition $S_0, S_1, \dots, S_p$ of S with $|S_0| < \varepsilon n$ and for $i = 1, \dots, p$, $||S_i| - n/p| < \varepsilon n$ and the S_i are on orthogonal circles C_i.

We use the extremality of S to show that $S_0 \subset \bigcup_{i=1}^{p} C_i$. Let $x \in S_0$. If $u(x, S_i) \geq 3$ for all $i = 2, \dots, p$, then by Lemma 8(c), $x \in C_1$. Thus, without loss of generality, $u(x, S_i) \leq 2$ for at least two i's, say $i = 1, 2$. Then

$$\begin{aligned} u(x, S) &= \sum_{i=0}^{p} u(x, S_i) \leq |S_0| - 1 + 2 + 2 + \sum_{i=3}^{p} |S_i| \\ &< \varepsilon n - 1 + 4 + (p-2)\left(\frac{n}{p} + \varepsilon n\right) = \left(1 - \frac{2}{p} + \varepsilon(p-1)\right)n + 3. \end{aligned}$$

If we remove x and replace it with a new point $x' \in C_1$, then

$$\begin{aligned} u\big(x', S \setminus \{x\}\big) &\geq u\left(x', \bigcup_{i=2}^{p} S_i\right) = \sum_{i=2}^{p} |S_i| \\ &> (p-1)\left(\frac{n}{p} - \varepsilon n\right) = \left(1 - \frac{1}{p} - (p-1)\varepsilon\right)n. \end{aligned}$$

In the case of diameters, we have to take care that x' does not increase the diameter. This can be done as follows.

Since all points of C_1 are already at unit distance to all points of $\bigcup_{i=2}^{p} C_i$, it is sufficient to choose x' at distance at most 1 to each point of S_0. When $d \geq 6$, C_1 has radius $1/\sqrt{2}$, hence S_1 is contained in a $90°$ arc γ of C_1. The set of points on C_1 at distance larger than 1 from some $y \in S_0$ is a (perhaps empty) subarc of γ. Such a subarc does not contain any point of S_1 and is therefore between some two consecutive points of S_1. Since $|S_1| \geq |S_0| + 1$ for n sufficiently large, there exist two consecutive points of S_1, say a and b, with no subarc between them. Therefore, all points on C_1 between a and b are at distance at most 1 to all points of S_0, and we may choose x' to be any point on C_1 between a and b. When $d = 4$, one of the two circles C_1 and C_2 has radius at least $1/\sqrt{2}$, and the above argument also works for this circle.

Since S is extremal, such a modification cannot increase the number of unit distances. Thus, $u(S) \geq u(S \cup \{x'\} \setminus \{x\})$, hence $u(x, S) \geq u(x', S \setminus \{x\})$, i.e.,

$$\left(1 - \frac{2}{p} + \varepsilon(p-1)\right)n + 3 > \left(1 - \frac{1}{p} - \varepsilon(p-1)\right)n,$$

which is a contradiction if $\varepsilon = 1/(2p^2)$ and $n \geq 3p^2$.

We have shown that $S_0 \subset \bigcup_{i=1}^{p} C_i$ for large n, which implies that S is a Lenz configuration. □

Proposition 21 *For each odd $d \geq 7$, there exists $N(d)$ such that all sets of $n \geq N(d)$ points in $\mathbb{R}^d$ extremal with respect to unit distances or diameters are weak Lenz configurations.*

Proof Again in the case of diameters, assume that the diameter is 1. Again we apply Theorem 5 (with $\varepsilon = 1/(4p^2)$). Thus, for n sufficiently large depending on d, there is a partition $S_0, S_1, \dots, S_p$ of S with $|S_0| < \varepsilon n$ and for $i = 1, \dots, p$, $||S_i| - n/p| < \varepsilon n$, S_1 is on a sphere Σ_1, each S_i ($i = 2, \dots, p$) is on a circle C_i, and $\Sigma_1, C_2, \dots, C_p$ are mutually orthogonal and all have radius $1/\sqrt{2}$.

To show that S is a weak Lenz configuration, it is sufficient to show that each point of S_0 not on Σ_1 lies on the 2-sphere of radius $1/\sqrt{2}$ containing some C_i ($i = 2, \dots, p$) in the subspace generated by C_i and some fixed diameter of Σ_1.

As in the proof of Proposition 20, there exists a point $x' \in C_2$ that does not increase the diameter. Since S is extremal, for any $x \in S$,

$$\begin{aligned} u(x, S) &\geq u\bigl(x', S \setminus \{x\}\bigr) \geq \sum_{\substack{i=1\\ i\neq 2}}^{p} |S_i| \\ &> (p-1)\left(\frac{n}{p} - \varepsilon n\right) = \left(1 - \frac{1}{p} - (p-1)\varepsilon\right)n. \end{aligned} \tag{8}$$

For $i = 2, \dots, p$, define

$$T_i := \bigl\{x \in S_0 : u(x, S_i) \leq 2\bigr\}.$$

For any point $x \in \Sigma_1$, $u(x, S_i) = |S_i| > \frac{n}{p} - \varepsilon n \geq 3$ for $n > 4p$, and therefore $\bigcup_{i=2}^{p} T_i \subseteq S_0 \setminus \Sigma_1$. Conversely, if $x \in S_0$ and $u(x, S_i) \geq 3$ for each $i = 2, \dots, p$, then $x \in \Sigma_1$ (Lemma 8(c)). It follows that $\bigcup_{i=2}^{p} T_i = S_0 \setminus \Sigma_1$. We next show that $T_2, \dots, T_p$ partition $S_0 \setminus \Sigma_1$. If not, there exists $x \in S_0 \setminus \Sigma_1$ with $u(x, S_i) \leq 2$ and $u(x, S_j) \leq 2$ for distinct $i, j \in \{2, \dots, p\}$. Then

$$\begin{aligned} u(x, S) &= u(x, S_0) + u(x, S_1) + \sum_{i=2}^{p} u(x, S_i) \\ &< \varepsilon n + \left(\frac{n}{p} + \varepsilon n\right) + 2 + 2 + (p-3)\left(\frac{n}{p} + \varepsilon n\right) \\ &= \left(1 - \frac{2}{p} + (p-1)\varepsilon\right)n + 4, \end{aligned}$$

which contradicts the lower bound (8) when $n > 8p$.

Note that the neighbors in S_1 of an $x \in S_0 \setminus \Sigma_1$ all lie on a circle C_x, say, on Σ_1. We now show that C_x is the same circle for all $x \in S_0 \setminus \Sigma_1$. First we bound $u(x, S_1)$ from below:

$$\begin{aligned} u(x, S) &= u(x, S_0) + u(x, S_1) + \sum_{i=2}^{p} u(x, S_i) \\ &< \varepsilon n + u(x, S_1) + 2 + (p-2)\left(\frac{n}{p} + \varepsilon n\right) \\ &= u(x, S_1) + \left(1 - \frac{2}{p} + (p-1)\varepsilon\right)n + 2, \end{aligned}$$

which, together with estimate (8), gives

$$u(x, S_1) > \left(\frac{1}{p} - 2(p-1)\varepsilon\right)n - 2. \tag{9}$$

If $C_y \neq C_x$ for some $y \in S_0 \setminus \Sigma_1$, then

$$|S_1| \geq u(x, S_1) + u(y, S_1) - 2 > 2\left(\frac{1}{p} - 2(p-1)\varepsilon\right)n - 6.$$

This contradicts $|S_1| < \frac{n}{p} + \varepsilon n$ when $n > 8p^2$.

Therefore, the neighbors in S_1 of any $x \in S_0 \setminus \Sigma_1$ are on the same circle $C_x =: C$. For each $i = 2, \dots, p$, let Σ_i be the sphere of radius $1/\sqrt{2}$ which has C_i as great circle in the 3-space containing C_i and the diameter of Σ_1 perpendicular to C. Since $T_2, \dots, T_p$ form a partition of $S_0 \setminus \Sigma_1$, each point of T_i is at distance 1 to at least 3 points of each C_j, $j \neq i$. By (9), each point of $S_0 \setminus \Sigma_1$ also has at least 3 neighbors on C if $n \geq 10p$. By Lemma 8(d) it follows that $T_i \subset \Sigma_i$. Since also $S_i \subset \Sigma_i$, we obtain that S is a weak Lenz configuration for large n. □

Proposition 22 *For all sufficiently large n, all sets of n points in $\mathbb{R}^5$ extremal with respect to unit distances or diameters are weak Lenz configurations.*

Proof An extremal set S of n points has at least $n^2/4$ unit distances, so by Theorem 5 with $\varepsilon = 1/11$ we obtain that for sufficiently large n, S can be partitioned into S_0, S_1, S_2 such that $|S_0| < \varepsilon n$, $||S_i| - n/2| < \varepsilon n$ $(i = 1, 2)$, S_1 is on a sphere Σ_1 of radius r_1, S_2 is on a circle C_2 of radius r_2, with Σ_1 and C_2 orthogonal, and $r_1^2 + r_2^2 = 1$.

As in the proof for odd $d \geq 7$, if $r_2 \geq 1/\sqrt{2}$, we can find a point $x' \in C_2$ that does not increase the diameter. Otherwise, $r_1 \geq 1/\sqrt{2}$, and we consider the intersection of Σ_1 and all balls in the 3-space of Σ_1 of radius 1 centered at points in $S \cap \Sigma_1$. This gives a spherically convex set on Σ_1 containing $S \cap \Sigma_1$. Any new point x' in this set is at distance at most 1 to all points of S. As before, replacing any point $x \in S$ by x' gives $u(x, S) > (\frac{1}{2} - \varepsilon)n$. Note that if $u(x, S_2) \geq 3$ for some $x \in S_0$, then $x \in \Sigma_1$. Therefore, $u(x, S_2) \leq 2$ for all $x \in S_0 \setminus \Sigma_1$. Next we bound $u(x, S_1)$ from below for all $x \in S_0 \setminus \Sigma_1$:

$$\begin{aligned}\left(\frac{1}{2} - \varepsilon\right)n &< u(x, S) = u(x, S_0) + u(x, S_1) + u(x, S_2)\\ &< \varepsilon n + u(x, S_1) + 2,\end{aligned}$$

hence

$$u(x, S_1) > \left(\frac{1}{2} - 2\varepsilon\right)n - 2.$$

The neighbors in S_1 of an $x \in S_0 \setminus \Sigma_1$ lie on a circle C_1, say, of Σ_1. If the neighbors of some other $y \in S_0 \setminus \Sigma_1$ lie on another circle of Σ_1, then

$$\frac{n}{2} + \varepsilon n > |S_1| > u(x, S_1) + u(y, S_1) - 2$$
$$> (1 - 4\varepsilon)n - 6.$$

Therefore, $5\varepsilon n > n/2 - 6$, a contradiction for n sufficiently large. It follows that all neighbors in S_1 of any $y \in S_0 \setminus \Sigma_1$ are on C_1. Let the radius of C_1 be s_1. By Lemma 8(c), each $x \in S_0 \setminus \Sigma_1$ lies on the complementary sphere Σ_2 of radius s_2, where $s_1^2 + s_2^2 = 1$, and $C_2 \subset \Sigma_2$. We have shown that S is a weak Lenz configuration for large n. □

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