UNIT ROOTS IN PERIODIC AUTOREGRESSIONS

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First version received March 1995

Abstract. This paper analyzes the presence and consequences of a unit root in periodic autoregressive models for univariate quarterly time series. First, we consider various representations of such models, including a new parametrization which facilitates imposing a unit root restriction. Next, we propose a class of likelihood ratio tests for a unit root, and we derive their asymptotic null distributions. Likelihood ratio tests for periodic parameter variation are also proposed. Finally, we analyze the impact on unit root inference of misspecifying a periodic process by a constant-parameter model.

Keywords. Periodic integration; periodic models; seasonality; unit roots.

1. INTRODUCTION

The defining property of a periodic autoregressive (PAR) model of a univariate seasonally observed time series is that its parameters vary with the seasons. The early literature on univariate periodic time series models includes Gladyshev (1961), Jones and Brelsford (1967), Pagano (1978), Troutman (1979) and Tiao and Grupe (1980), *inter alia*. Recently, and accompanied by a revived interest in the analysis of seasonality in general, periodic models have received growing attention in the econometric analysis of seasonal time series; see, for example, Osborn (1988, 1991), Birchenhall *et al.* (1989), Ghysels *et al.* (1994), Franses (1994) and Boswijk and Franses (1995a, b). Given that many economic time series display patterns which may be best described by stochastic trends, the possibility of unit roots in periodic autoregressive models merits attention. Osborn (1988) and Franses (1994) discuss theoretical arguments for, and consequences of, the presence of such roots, and Osborn *et al.* (1988) define the concept of periodic integration. This notion refers to the situation where the original series contains a stochastic trend, which is eliminated by particular linear combinations of successive quarterly observations (called *periodic differences*).

In this paper we extend the literature on PAR models in three different directions. First, we propose a new likelihood ratio (LR) test for a periodic unit root in (possibly higher-order) periodic autoregressions. An earlier test for this purpose was developed by Franses (1994), who applied Johansen's (1988) LR

0143-9782/96/03 221–245 JOURNAL OF TIME SERIES ANALYSIS Vol. 17, No. 3

test to a vector autoregressive (VAR) representation of the annual vector process of quarterly observations. However, the over-parametrization of this (unrestricted) VAR model may easily lead to a relatively low power of unit root tests. Therefore, this paper proposes a new representation which facilitates imposing the unit root restriction directly in the original (univariate) model specification. This leads to a new class of LR tests for a single unit root, which may be seen as a periodic generalization of the well-known augmented Dickey–Fuller test.

Second, we compare our analysis to a related testing problem which has recently been advanced by Ghysels *et al.* (1994). They derive Wald tests for non-periodic (either seasonal or non-seasonal) unit root restrictions in unrestricted periodic autoregressions. In the present paper we show that these non-periodic unit root hypotheses entail simple parameter restrictions on a periodically integrated autoregression, which may be tested using LR statistics with an asymptotically χ^2 null distribution. This is a specific application of an LR test for periodic parameter variation, which we consider in detail.

Third, we extend Tiao and Grupe's (1980) analysis of misspecified non-periodic models for periodic processes to the case of periodic integration. We show that such misspecification in general leads to an overstatement of the number of unit roots and an increase in the lag length. This provides an explanation of the trade-off between lag length and periodic parameter variation, which is often found in empirical practice.

The outline of the paper is as follows. In Section 2, we consider various representations of periodic autoregressions, and we define the concept of periodic integration. Throughout the paper, we focus on quarterly time series, but all results can be extended to monthly and other seasonally observed processes. In Section 3, we propose and analyze a class of LR tests for a single unit root. Furthermore, we analyze LR tests for parameter constancy over the seasons, which may be used to test for a non-periodic unit root. In Section 4, we consider the presence of a unit root in Tiao and Grupe's (1980) misspecified homogeneous model, and study the use of Dickey *et al.*'s (1984) seasonal unit root test in periodic autoregressions. In Section 5 we discuss the results.

2. REPRESENTATION

Consider the periodic autoregressive model of order p (PAR(p)) for a quarterly observed univariate time series $\{y_t, t = 1, \ldots, n\}$:

$$y_t = \varphi_{1s} y_{t-1} + \cdots + \varphi_{ps} y_{t-p} + \varepsilon_t \qquad t = 1, \ldots, n \tag{1}$$

where φ_{is}, $i = 1, \ldots, p$, are periodically varying parameters, i.e. the coefficient of y_{t-i} equals φ_{is} if time t corresponds to season s. Defining $\{D_{st}, t = 1, \ldots, n, s = 1, \ldots, 4\}$ as a set of seasonal dummy variables, so that

$D_{st} = 1$ in season s and 0 elsewhere, the model may be expressed more explicitly as

$$y_t = \sum_{s=1}^{4} \varphi_{1s} D_{st} y_{t-1} + \cdots + \sum_{s=1}^{4} \varphi_{ps} D_{st} y_{t-p} + \varepsilon_t. \tag{2}$$

In the sequel, we shall use (1) as a short-hand notation for (2); moreover, for any set $\{a_1, \ldots, a_4\}$ we shall use the convention $a_{s-4k} = a_s$, $k \in \mathbb{N}$ (i.e. the index s satisfies arithmetic modulo 4).

We assume that the starting values $\{y_{1-p}, \ldots, y_0\}$ are fixed, and that $\{\varepsilon_t\}$ is an independent $N(0, \sigma^2)$ sequence. As usual, the normality assumption is required only for the construction of LR statistics, the asymptotic properties of which may be evaluated under wider distributional assumptions. One of the extensions of the model (1) discussed below is to allow for seasonal heteroskedasticity, i.e. periodic variation of the error variance. Another extension is to include deterministic variables such as seasonal dummies or trends. Note that by restricting some appropriately chosen φ_{is} parameters to zero, the lag length may also be varied over the seasons; thus p represents the maximum lag length.

An important aspect of (1) is that the autocorrelation function and hence the spectral density varies throughout the year. Therefore, the process $\{y_t\}$ is non-stationary, which implies that (1) is not a useful representation for the analysis of stationarity, unit roots and stochastic trends, see also Osborn (1991). These issues are analyzed most conveniently in the multivariate representation of (1) that originates from stacking the observations of $\{y_t\}$ in the annual sequence of (4×1) vectors $Y_T = (Y_{1T}, \ldots, Y_{4T})'$, where $Y_{sT} = y_{4(T-1)+s}$ is the observation in season s of year T, with $T = 1, \ldots, N$ and $N = n/4$. This idea of stacking y_t into Y_T was first proposed by Gladyshev (1961); we shall call $\{Y_T\}$ the vector of quarters (VQ) process of $\{y_t\}$. The univariate model (1) of $\{y_t\}$ can be expressed as the following multivariate model of $\{Y_T\}$:

$$\Phi_0 Y_T = \Phi_1 Y_{T-1} + \cdots + \Phi_P Y_{T-P} + E_T \qquad T = 1, \ldots, N. \tag{3}$$

Here Φ_i, $i = 0, \ldots, P$, are 4×4 parameter matrices; Φ_0 is lower triangular with unit elements on the diagonal, and $\Phi_{0,sj} = -\varphi_{s-j,s}$ for $j < s$; further, $\Phi_{i,sj} = \varphi_{(4i+s-j),s}$ for $i = 1, \ldots, P$. The maximum lag length P equals $[(p-1)/4] + 1$, where $[x]$ denotes the integer part of x. The (4×1) vector $\{E_T\}$ is the VQ process of $\{\varepsilon_t\}$, an independent $N(0, \sigma^2 I_4)$ sequence. We shall refer to (3) as the VQ representation of (1). Notice that, in contrast with (1), this representation has constant parameters. We emphasize that (3) is merely a useful representation to analyze the presence of unit roots and stochastic trends; the statistical analysis in the next section will be based on the original model (1).

Let L denote the lag operator, which may operate both on quarterly and on annual data, so that $L^k x_t = x_{t-k}$ and $L^k X_T = X_{T-k}$, $k \in \mathbb{N}$. Define the matrix lag polynomial

$$\Phi(L) = \Phi_0 - \Phi_1 L - \cdots - \Phi_P L^P \tag{4}$$

so that the VQ representation may be concisely expressed as $\Phi(L)Y_T = E_T$. The vector process $\{Y_T\}$ in (3) is stationary if the roots of the characteristic equation

$$\varphi(z) = |\Phi(z)| = 0 \tag{5}$$

are all outside the unit circle. The process is integrated (and possibly cointegrated) at the zero frequency if the characteristic equation has some roots equal to one and all other roots outside the unit circle. These zero-frequency unit roots in the VQ process may correspond to unit roots in the original process $\{y_t\}$ at the zero frequency, but also at seasonal frequencies. An intuitive explanation of this is that seasonal cycles in $\{y_t\}$ cannot show up as cycles in the annual process, because their frequency is too high. For example, the VQ representation of a first-order periodic autoregression ($p = 1$) has as its characteristic equation $\varphi(z) = |1 - \varphi_{11}\varphi_{12}\varphi_{13}\varphi_{14}z| = 0$ (see, for example, Franses, 1994). Defining $\rho = \varphi_{11}\varphi_{12}\varphi_{13}\varphi_{14}$, we thus find that $\{Y_T\}$ is stationary if $|\rho| < 1$, and has a single unit root if $\rho = 1$. The latter includes the non-periodic non-seasonal unit root case $\varphi_{1s} = 1$, $\forall s$, but also the non-periodic seasonal unit root case $\varphi_{1s} = -1$, $\forall s$, corresponding to a half-year cycle (cf. Hylleberg *et al.*, 1990).

DEFINITION 1. Let $\{y_t\}$ have the VQ representation (3), with characteristic equation (5).

(i) If (5) has all roots outside the unit circle, then $\{y_t\}$ is said to be periodically integrated of order 0, denoted by $y_t \sim \mathrm{PI}(0)$.

(ii) If (5) has a single root equal to 1 and all other roots outside the unit circle, and $Y_{sT} \sim I(1)$, $\forall s$, then $\{y_t\}$ is said to be periodically integrated of order 1, denoted by $y_t \sim \mathrm{PI}(1)$.

We shall also refer to PI(0) processes as periodically stationary. Until recently, the literature on periodic time series has concentrated on such PI(0) processes; see, for example, Tiao and Grupe (1980).

The implications of Definition 1 are analyzed most easily in the error correction form of (3):

$$\Delta Y_T = \Pi Y_{T-1} + \Gamma_1 \Delta_1 Y_{T-1} + \cdots + \Gamma_{P-1}\Delta_1 Y_{T-P+1} + E_T^* \tag{6}$$

where $\Delta_1 = (1 - L)$, the first-difference operator, and where $\Pi = -\Phi_0^{-1}\Phi(1)$, $\Gamma_i = \Phi_0^{-1}(\Phi_{i+1} + \cdots + \Phi_P)$, and $E_T^* = \Phi_0^{-1}E_T$. Periodic stationarity implies that $\Phi(1)$, and hence Π, is non-singular. Periodic integration, on the other hand, implies that $\operatorname{rank}\Pi = 3$, so that $\{Y_T\}$ is cointegrated of order (1, 1), with cointegrating rank 3, see Engle and Granger (1987). Let B denote the 4×3 matrix of cointegrating vectors. As is well known, B is not unique; what is identified is the space spanned by the columns of B, the so-called cointegration space (see Johansen, 1991). From the assumption that all components of Y_T are I(1), it follows that B may be represented as

$$B' = \begin{bmatrix} -\varphi_2 & 1 & 0 & 0 \\ 0 & -\varphi_3 & 1 & 0 \\ 0 & 0 & -\varphi_4 & 1 \end{bmatrix} \tag{7}$$

for some $\varphi_s \neq 0$, $s = 2, 3, 4$. Defining $\varphi_1 = 1/(\varphi_2\varphi_3\varphi_4)$, so that $\varphi_1\varphi_2\varphi_3\varphi_4 = 1$, we have

$$\begin{aligned} Y_{1,T} - \varphi_1 Y_{4,T-1} &= -\varphi_1\left(Y_{4,T-1} - \frac{1}{\varphi_1} Y_{1,T}\right) \\ &= -\varphi_1(Y_{4,T} - \varphi_4\varphi_3\varphi_2 Y_{1,T}) + \varphi_1 \Delta_1 Y_{4,T} \end{aligned} \tag{8}$$

and since cointegration of order (1, 1) with cointegrating matrix B implies that both terms on the right-hand side of Equation (8) are stationary, it follows that $y_t - \varphi_s y_{t-1}$ has a stationary VQ representation, and thus is periodically stationary. In fact, Osborn *et al.* (1988) take this as the defining property of periodic integration, obtained as a specific case of Granger's (1986) time-varying parameter integration. They call a time series y_t periodically integrated if the periodic differencing filter $\delta_s y_t$ is required to render it stationary, where $\delta_s \equiv (1 - \varphi_s L)$ with $\varphi_1\varphi_2\varphi_3\varphi_4 = 1$.

This leads to the following representation, which forms the basis of the LR tests to be derived in the next section. For any set of constants φ_s whose product equals one, rewrite (1) as

$$y_t - \varphi_s y_{t-1} = \sum_{i=1}^{p-1} \psi_{is}(y_{t-i} - \varphi_{s-i} y_{t-i-1}) + \pi_s y_{t-1} + \varepsilon_t \tag{9}$$

(recall that $\varphi_{s-4k} = \varphi_s$, $k \in \mathbb{N}$), where the parameters ψ_{is} and π_s are defined from the backward recursion (starting from $\psi_{ps} = 0$, $\forall s$)

$$\begin{aligned} \psi_{is} &= \frac{\psi_{i+1,s} - \varphi_{i+1,s}}{\varphi_{s-i}} \qquad i = p-1, \ldots, 1 \\ \pi_s &= \varphi_{1s} - \psi_{1s} - \varphi_s. \end{aligned} \tag{10}$$

If $y_t \sim \mathrm{PI}(1)$, then we may choose φ_s such that all terms in (9), except $\pi_s y_{t-1}$, are periodically stationary, which implies the restriction $\pi_s = 0$, $\forall s$. Conversely, this restriction implies that the VQ representation of (9) has at least one unit root. Let U_T denote the VQ process of $u_t = (y_t - \varphi_s y_{t-1})$, so that $U_T = \bar{\Phi}_0 Y_T - \bar{\Phi}_1 Y_{T-1}$, where

$$\bar{\Phi}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\varphi_2 & 1 & 0 & 0 \\ 0 & -\varphi_3 & 1 & 0 \\ 0 & 0 & -\varphi_4 & 1 \end{bmatrix} \qquad \bar{\Phi}_1 = \begin{bmatrix} 0 & 0 & 0 & \varphi_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

If the VAR representation of U_T is given by $\Psi(L)U_T = E_T$, then

$$\Phi(L)Y_T = \Psi(L)(\bar{\Phi}_0 - \bar{\Phi}_1 L)Y_T = E_T \tag{12}$$

which has the characteristic equation

$$|\Psi(z)|\,|\bar{\Phi}_0 - \bar{\Phi}_1 z| = |\Psi(z)|(1 - \varphi_1\varphi_2\varphi_3\varphi_4 z) = 0. \tag{13}$$

Since $\varphi_1\varphi_2\varphi_3\varphi_4 = 1$, this characteristic equation has at least one unit root; periodic integration requires that all roots of $|\Psi(z)| = 0$ are outside the unit circle, so that $U_T = \Psi(L)^{-1}E_T$ is stationary.

Observe that (9) may be expressed, using the periodic differencing filter δ_s, as

$$\delta_s y_t = \sum_{i=1}^{p-1} \psi_{is}\delta_{s-i}y_{t-i} + \pi_s y_{t-1} + \varepsilon_t. \tag{14}$$

If all periodic parameter variation would be excluded, then $\delta_s = \Delta_1$, and (14) reduces to the well-known Dickey–Fuller regression. An intermediate case is considered by Ghysels *et al.* (1994), who focus on non-periodic differences but allow for other parameter variation; they propose Wald tests for $\pi_s = 0$, $\forall s$ in (14), but with δ_s restricted to either $(1 - L)$ (a non-seasonal unit root) or $(1 + L)$ (a seasonal unit root). If the only requirement on φ_s is that their product equals 1, then the representation (9) has three more parameters than (1), so that it is not identified. From (10) it is clear that we may choose φ_s such that $\pi_s = 0$, $s = 2, 3, 4$; the unit root restriction may then be formulated as $\pi_1 = 0$. Note that under this restriction, the periodic differences will have to be estimated simultaneously with the other parameters. Therefore, the likelihood ratio statistic for the hypothesis of periodic integration, derived in the next section, will require non-linear least-squares estimation under the unit root restriction.

Since $u_t = (y_t - \varphi_s y_{t-1})$ is periodically stationary with $\varphi_1\varphi_2\varphi_3\varphi_4 = 1$, we find, by repeated substitution,

$$\begin{aligned} y_t &= \varphi_s y_{t-1} + u_t \\ &\;\;\vdots \\ &= \rho y_{t-4} + u_t + \varphi_s u_{t-1} + \varphi_s\varphi_{s-1}u_{t-2} + \varphi_s\varphi_{s-1}\varphi_{s-2}u_{t-3} \end{aligned} \tag{15}$$

where $\rho = \varphi_1\varphi_2\varphi_3\varphi_4 = 1$, so that $\Delta_4 y_t$ is a periodic moving average of order 3 in u_t (here $\Delta_4 = (1 - L^4)$, the annual difference operator). Hence $\Delta_4 y_t$ is periodically stationary, which suggests that annual differencing a PI(1) process may remove its non-stationarity. However, this would lead to overdifferencing, which can be seen as follows. The VQ process of $\Delta_4 y_t$ is $\Delta_1 Y_T$, which has the following vector moving-average (VMA) representation:

$$\Delta_1 Y_T = (\Theta_0 + \Theta_1 L)U_T = (\Theta_0 + \Theta_1 L)\Psi(L)^{-1}E_T \tag{16}$$

where

$$\Theta_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \varphi_2 & 1 & 0 & 0 \\ \varphi_2\varphi_3 & \varphi_3 & 1 & 0 \\ \varphi_2\varphi_3\varphi_4 & \varphi_3\varphi_4 & \varphi_4 & 1 \end{bmatrix}$$

$$\Theta_1 = \begin{bmatrix} 0 & \varphi_3\varphi_4\varphi_1 & \varphi_4\varphi_1 & \varphi_1 \\ 0 & 0 & \varphi_4\varphi_1\varphi_2 & \varphi_1\varphi_2 \\ 0 & 0 & 0 & \varphi_1\varphi_2\varphi_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

Defining $C(L) = (\Theta_0 + \Theta_1 L)\Psi(L)^{-1}$, the characteristic equation of this VMA matrix polynomial is

$$|C(z)| = |\Theta_0 + \Theta_1 z|\,|\Psi(z)^{-1}| = (1 - z)^3|\Psi(z)^{-1}| = 0 \tag{18}$$

so the VMA model is not invertible, and, in fact, has three unit roots. Thus three of the four unit roots of the $I_4(1 - L)$ filter applied to Y_T cancel, leaving only one unit root as required.

The moving-average representation (16) leads to the so-called common trends representation. Consider the power series decomposition $C(L) = C(1) + C^*(L)(1 - L)$ (see, for example, Banerjee *et al.*, 1993, p. 257), where

$$C(1) = (\Theta_0 + \Theta_1)\Psi(1)^{-1} = ab'\Psi(1)^{-1} \tag{19}$$

with

$$a = \begin{bmatrix} 1 \\ \varphi_2 \\ \varphi_2\varphi_3 \\ \varphi_2\varphi_3\varphi_4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ \varphi_3\varphi_4\varphi_1 \\ \varphi_4\varphi_1 \\ \varphi_1 \end{bmatrix}. \tag{20}$$

Substitution of (19) in (16) and integrating both sides yields

$$Y_T = Y_0 + ab'\Psi(1)^{-1}\sum_{j=1}^{T} E_j + C^*(L)E_T. \tag{21}$$

This shows that the four components of Y_T have a single common stochastic trend, given by the partial sum of $b'\Psi(1)^{-1}E_T$. This is in contrast with seasonally integrated models, the simplest example of which is $\Delta_4 y_t = \varepsilon_t$. The VQ representation for this model is $\Delta_1 Y_T = E_T$, so that in comparison with (6), the cointegrating relationships have disappeared. Thus, in contrast with (21), each of the quarters of a seasonally integrated process have their own stochastic trend, in this specific case given by the partial sum of E_{sT} in season s. As a consequence, the observations of seasonally integrated processes in successive quarters may drift apart unboundedly (see Osborn, 1993).

All representations considered in this section may be extended to allow for periodic intercepts and linear trends:

$$y_t = \varphi_{1s}y_{t-1} + \cdots + \varphi_{ps}y_{t-p} + \alpha_s + \beta_s T_t + \varepsilon_t \tag{22}$$

where T_t denotes the year at time t. The corresponding VQ representation is

$$\Phi_0 Y_T = \Phi_1 Y_{T-1} + \cdots + \Phi_P Y_{T-P} + \alpha + \beta T + E_T \tag{23}$$

where $\alpha = (\alpha_1, \ldots, \alpha_4)'$ and $\beta = (\beta_1, \ldots, \beta_4)'$. The common trends representation is obtained simply by replacing E_T in (21) by $(\alpha + \beta T + E_T)$, which yields after some rewriting:

$$\begin{aligned} Y_T &= \mu + \tau T + \kappa T^2 + X_T \\ X_T &= ab'\Psi(1)^{-1}\sum_{j=1}^{T} E_j + C^*(L)E_T \end{aligned} \tag{24}$$

where μ, τ and κ are functions of α, β, $C(L)$ and the starting value Y_0. In particular, if $b'\Psi(1)^{-1}\beta = 0$, then $\kappa = 0$ (no quadratic trend), and if $\beta = 0$ and $b'\Psi(1)^{-1}\alpha = 0$, then $\tau = \kappa = 0$ (no linear trend). Note that X_T in (24) represents the deviation of Y_T from its mean (conditional upon the starting value).

3. TESTING FOR A UNIT ROOT AND PARAMETER VARIATION

3.1. *Testing for a periodic unit root*

Consider again the PAR(p) model (1). The null hypothesis to be tested is

$$\mathrm{H}_0 : |\Phi(1)| = 0 \tag{25}$$

i.e. the characteristic equation $|\Phi(z)| = 0$ of the VQ representation of (1) has a single unit root. As we have shown in the previous section, under the null hypothesis the model can be represented as (9) with $\varphi_1\varphi_2\varphi_3\varphi_4 = 1$ and $\pi_s = 0$, $\forall s$, i.e.

$$y_t = \sum_{s=1}^{4} \varphi_s D_{st} y_{t-1} + \sum_{i=1}^{p-1}\sum_{s=1}^{4} \psi_{is} D_{st}(y_{t-i} - \varphi_{s-i} y_{t-i-1}) + \varepsilon_t \qquad t = 1, \ldots, n. \tag{26}$$

Recall from (14) that (26) may be seen as a periodic autoregression in periodic differences $\delta_s y_t$; replacing these by ordinary differences $\Delta_1 y_t$ entails a further restriction, namely $\varphi_s = 1$, $\forall s$.

Under the assumption that $\{\varepsilon_t\}$ is an independent $\mathrm{N}(0, \sigma^2)$ sequence and the starting values are fixed, the unrestricted maximum likelihood (ML) estimators of the parameters of (2) are obtained by linear least-squares, and the unrestricted maximized log-likelihood is given by

$$\mathscr{L}(\hat{\theta}) = c - \frac{n}{2}\ln\hat{\sigma}^2 = c - \frac{n}{2}\ln\left(\frac{1}{n}\sum_{t-1}^{n}\hat{\varepsilon}_t^2\right) \tag{27}$$

where θ is the full parameter vector, $\hat{\theta}$ is the unrestricted maximum likelihood estimator, c is a constant and $\hat{\varepsilon}_t$ are the least-squares residuals from (2).

Similarly, the restricted maximized log-likelihood is obtained from non-linear least-squares estimation of (26), yielding $\tilde{\sigma}^2$ and hence $\mathcal{L}(\tilde{\theta}) = c - (n/2)\ln\tilde{\sigma}^2$. Thus, the LR statistic is given by

$$\mathrm{LR} = -2\{\mathcal{L}(\tilde{\theta}) - \mathcal{L}(\hat{\theta})\} = n(\ln\tilde{\sigma}^2 - \ln\hat{\sigma}^2). \tag{28}$$

In order to allow for a non-zero (periodic) mean under the alternative, a set of seasonal dummies may be added to the regressors, leading to the unrestricted and restricted estimators $\hat{\sigma}^2_\mu$ and $\tilde{\sigma}^2_\mu$, respectively, and the LR statistic $\mathrm{LR}_\mu = n(\ln\tilde{\sigma}^2_\mu - \ln\hat{\sigma}^2_\mu)$. Similarly, if a set of periodic intercepts and linear trends is added (see (22)), to test against trend-stationarity, then we obtain $\mathrm{LR}_\tau = n(\ln\tilde{\sigma}^2_\tau - \ln\hat{\sigma}^2_\tau)$.

In order to derive the asymptotic null distribution of these statistics, we make the following assumption.

ASSUMPTION 1. $\{y_t, t = 1, \ldots, n\}$ is generated by (22), with VQ representation (23), where

(i) $\{\varepsilon_t, t = 1, \ldots, n\}$ are independent and identically distributed (i.i.d.) with mean zero and variance σ^2;

(ii) the characteristic equation $|\Phi(z)| = 0$ has at most a single unit root, and all other roots outside the unit circle.

Note that we do not require normality of the disturbances for the asymptotic analysis. In fact, the i.i.d. assumption may be weakened even further to allow for martingale differences with constant conditional variance σ^2 (and an additional higher moment condition). The second part of Assumption 1 excludes the possibility that the VQ process is either I(2) or I(1) with cointegrating rank smaller than 3; thus we limit ourselves here to the choice between PI(1) and PI(0), see Definition 1. An extension of the present analysis to multiple unit roots is the subject of current research by the authors.

LEMMA 1. *Consider X_T in (24). Under Assumption 1, we have as $N \to \infty$,*

$$\frac{1}{\sqrt{N}} X_{[rN]} \xrightarrow{\mathrm{d}} B(r) = \omega a W(r) \qquad r \in [0, 1] \tag{29}$$

where $[rN]$ denotes the integer part of rN, where $B(r)$ is a 4×1 vector Brownian motion process with variance matrix $\Omega = \omega^2 aa'$, $W(r)$ is a standard (scalar) Brownian motion process and

$$\omega = \sigma\{b'\Psi(1)^{-1}\Psi(1)'^{-1}b\}^{1/2}. \tag{30}$$

Proofs are given in the Appendix. Note that $\{\Delta X_T\}$ has a singular long-run variance matrix Ω, which is due to the unit roots in the moving-average (MA) polynomial. The fact that the four VQ series $\{X_T\}$ have a single common stochastic trend is reflected in the four Brownian motions $B(r)$ being defined from a single standard Brownian motion $W(r)$.

For any deterministic function $k(r)$ on the unit interval, define

$$W_k(r) = W(r) - \int_0^1 W(t)k(t)' \, dt \left\{ \int_0^1 k(t)k(t)' \, dt \right\}^{-1} k(r) \tag{31}$$

i.e. the projection in L^2 of $W(r)$ on the orthogonal complement of $k(r)$, see Park and Phillips (1988). With this notation, we can state the main result:

THEOREM 1. *Under* H_0 *and Assumption 1, we have, as* $n \to \infty$,

$$\mathrm{LR}, \mathrm{LR}_\mu, \mathrm{LR}_\tau \xrightarrow{d} \left\{ \int_0^1 F(r)^2 \, dr \right\}^{-1} \left\{ \int_0^1 F(r) \, dW(r) \right\}^2 \tag{32}$$

where $W(r)$ *is a standard Brownian motion process, and*

(i) *for* LR, $F(r) = W(r)$;
(ii) *for* LR_μ, $F(r) = W_k(r)$ *with* $k(r) = 1$, *provided* $\tau = 0$ *in (24)*;
(iii) *for* LR_τ, $F(r) = W_k(r)$ *with* $k(r) = (1, r)'$, *provided* $\kappa = 0$ *in (24)*.

Theorem 1 implies that the LR statistics for a unit root in periodic autoregressions have the same asymptotic null distributions as the square of Fuller's (1976) $\hat{\tau}$, $\hat{\tau}_\mu$ and $\hat{\tau}_\tau$ statistics. The same distributions appear as special cases in Johansen's (1988, 1991) cointegration analysis, and as such asymptotic critical values for LR and LR_μ are given in Osterwald-Lenum (1992), Tables 0 and 1.1, respectively (with $p - r = 1$). Since the distribution of Fuller's $\hat{\tau}_\tau$ hardly has any mass at the positive part of the line, critical values for LR_τ may be obtained simply by taking the square of the corresponding critical values of $\hat{\tau}_\tau$.

If $p = 1$, then a one-sided test may be based on the studentized statistic sign $(\hat{\varphi}_1\hat{\varphi}_2\hat{\varphi}_3\hat{\varphi}_4 - 1)\sqrt{\mathrm{LR}}$, where $\hat{\varphi}_s$ are the unrestricted least-squares estimators; this statistic has the same asymptotic null distribution as Fuller's τ statistic (this is proved in a previous version of the paper, available from the authors upon request). Boswijk and Franses (1995a) provide a Monte Carlo study of the size and power properties of this studentized LR test, and show that it outperforms a Wald-type t test, as well as a 'coefficient test' (analogous to Fuller's (1976) $n(\hat{\rho} - 1)$). If $p > 1$, however, such one-sided tests are not feasible, because the φ_s parameters are not identified without the unit root restriction. Therefore, if the null hypothesis is rejected using the LR statistic, we still need to check whether all roots of the characteristic equation are outside the unit circle.

Notice that the null distributions of LR_μ and LR_τ are evaluated under the additional restrictions $\tau = 0$ (no linear trend in the level of y_t) and $\kappa = 0$ (no quadratic trend), respectively, but these restrictions are not imposed in the calculation of $\tilde{\sigma}^2$. If, alternatively, we impose such restrictions under the null, then the resulting test statistics, say LR_μ^* and LR_τ^*, are similar to Dickey and Fuller's (1981) Φ_1 and Φ_3 statistics, respectively. Without proof we state that LR_μ^* and LR_τ^* in fact have the same asymptotic null distributions as Φ_1 and

Φ_3, multiplied by 2 (since these are F-type statistics), so that critical values from Dickey and Fuller (1981, Tables IV and VI, multiplied by 2) can be used.

One may wish to extend the tests given above to allow for periodic variation in the variance of $\{\varepsilon_t\}$ or periodic heteroskedasticity, i.e. replace σ^2 by σ_s^2. The representational issues discussed in Section 2 do not change with this possibility, but of course the format of the LR statistics does. Maximum likelihood is now equivalent to weighted (non-linear) least-squares estimation. It is easily shown that without the unit root restriction, this is identical to ordinary least-squares estimation. Letting $\hat{\sigma}_s^2$ and $\tilde{\sigma}_s^2$ denote the unrestricted and restricted ML variance estimators, the LR statistic becomes

$$\mathrm{LR} = n \sum_{s=1}^{4} (\ln \tilde{\sigma}_s^2 - \ln \hat{\sigma}_s^2). \tag{33}$$

It can be shown that the LR statistic in (33) has the same asymptotic null distribution as the original LR statistic (28) has under the restriction of periodic homoskedasticity; to save space we do not provide this proof here. An LR test for periodic heteroskedasticity is easily constructed from $\hat{\sigma}_s^2$ and $\hat{\sigma}^2$, or from the corresponding restricted estimates. Alternatively, a Lagrange multiplier-type diagnostic is given by nR^2, where R^2 is the coefficient of determination in the auxiliary regression

$$\hat{\varepsilon}_t^2 = h_0 + \sum_{s=1}^{3} h_s D_{st} + e_t. \tag{34}$$

3.2. *Testing for periodic parameter variation*

As discussed in Section 2, periodically integrated models are capable of capturing non-stationary behaviour that is not explained by non-periodic, possibly seasonally integrated models. However, this is in general at the cost of a larger number of parameters. Therefore, it is worthwhile to test whether the periodic variation in some or all of the parameters is significant, and if not, to specify more parsimonious (partially) constant-parameter models. We shall discuss two tests for periodic variation. First, we consider the LR test for the null hypothesis

$$\mathrm{H}_0^{\mathrm{NP}} : \varphi_{is} = \varphi_{i1} \qquad s = 2, 3, 4;\ i = 1, \ldots, p \tag{35}$$

to be tested in the general unrestricted model (1) (denoted by H_1), possibly extended by constants and trends. Secondly, we analyze, within the periodically integrated model (denoted by H_0), the null hypotheses

$$\begin{aligned} \mathrm{H}_0^{\mathrm{NS}} &: \varphi_s = 1 \qquad \forall s \\ \mathrm{H}_0^{\mathrm{S}} &: \varphi_s = -1 \qquad \forall s. \end{aligned} \tag{36}$$

Notice that this implies three restrictions in addition to the unit root restriction (25). The motivation for this hypothesis is that it reduces the periodic difference

$\delta_s = (1 - \varphi_s L)$, required to obtain stationarity, to the ordinary difference filter $\Delta_1 = (1 - L)$, or the filter $(1 + L)$, corresponding to a seasonal unit root -1. We denote a LR statistic for a hypothesis H_0 within a more general model H_1 by $LR(H_0|H_1)$. The asymptotic properties of such tests are given in the next theorem.

THEOREM 2. *Under Assumption 1, and as $n \to \infty$:*

(i) under H_0^{NP}, $LR(H_0^{NP}|H_1) \to_d \chi^2(3p)$, *whether* $y_t \sim PI(0)$ *or* $y_t \sim PI(1)$;
(ii) under H_0^{NS}, $LR(H_0^{NS}|H_0) \to_d \chi^2(3)$; *under* H_0^S, $LR(H_0^S|H_0) \to_d \chi^2(3)$.

The first part of Theorem 2 implies that we can always use conventional critical values from the χ^2 distribution to test periodic variation, even if it is not known whether or not the series contains a unit root. This suggests to use this test (or an F-version thereof) as a starting point in empirical modelling. If the null hypothesis is not rejected, we may proceed with Hylleberg *et al.*'s (1990) analysis of seasonal unit roots; if significant periodicity is found, the periodic unit root tests proposed above become relevant.

Theorem 2 suggests two alternative ways to test the null hypothesis of a (non-periodic) random walk in a first-order periodic autoregression. The first approach is to test the hypothesis of a non-periodic AR(1) first, using a $\chi^2(3)$ test, and then test for a unit root in this AR model using the Dickey–Fuller statistic. Alternatively, we may first test for a periodic unit root using our LR statistic, and subsequently test whether $\varphi_s = 1$, $\forall s$, using a $\chi^2(3)$ test. As a third possibility, Ghysels *et al.* (1994) propose to take the two steps together and test (*inter alia*) H_0^{NS} against H_1, i.e. in the unrestricted PAR(1) (or a higher-order generalization). The results above indicate that the null distribution of such a test should be the sum of a $\chi^2(3)$ distribution and the distribution in (32). Indeed, Ghysels *et al.* (1994) find for their Wald test for H_0^{NS}, denoted W_4^R,

$$W_4^R \xrightarrow{d} \sum_{s=1}^{4} \left\{ \int_0^1 G_1(r)^2 \, dr \right\}^{-1} \left\{ \int_0^1 G_1(r) \, dB_{4s}(r) \right\}^2 \tag{37}$$

where $B_4(r) = (B_{41}(r), \ldots, B_{44}(r))'$ is a standard vector Brownian motion process and $G_1(r) = b' B_4(r)$, where in this case $b = (1, 1, 1, 1)'$. Let S denote an orthogonal matrix, the first column of which is $0.5b$, and let $W(r) = S' B_4(r)$, a vector Brownian motion process with variance matrix $S' I_4 S = I_4$. Partition $W(r)$ as $(W_1(r), \ldots, W_4(r))'$, so that $W_s(r)$, $s = 1, \ldots, 4$, are independent Brownian motions, with $W_1(r) = 0.5b' B_4(r) = 0.5 G_1(r)$. It is easily checked that $G_1(r)$ and $B_4(r)$ in (37) may be replaced by $W_1(r)$ and $W(r)$, respectively, so that

$$W_4^R \xrightarrow{d} \sum_{s=1}^{4} \left\{ \int_0^1 W_1(r)^2 \, dr \right\}^{-1} . \left\{ \int_0^1 W_1(r) \, dW_s(r) \right\}^2 = \tau^2 + \chi^2 \tag{38}$$

where τ is a random variable with the asymptotic distribution of the Dickey–Fuller $\hat{\tau}$ statistic, and χ^2 is a random variable with a $\chi^2(3)$ distribution. The latter distribution arises because $W_s(r)$, $s = 2, 3, 4$, are independent of $W_1(r)$, so that the signed square root of the terms in (38) for $s = 2, 3, 4$, are standard normal variates (this holds conditionally on $W_1(r)$, and since the conditional distribution is independent of $W_1(r)$, it holds unconditionally as well). This shows that the W_4^R test can be decomposed into a unit root test and a χ^2 test. Critical values of this distribution, as well as extensions to deterministic components and multiple unit roots, are tabulated in Ghysels *et al.* (1994); note that the two constituents τ^2 and χ^2 have well-known (and tabulated) distributions. Furthermore, observe that if the W_4^R stastistic rejects the null hypothesis of a non-periodic unit root, then the series may be either PI(0) or PI(1) with $\varphi_s \neq 1$. In other words, the alternative hypothesis for this test includes processes with stochastic trending behaviour in addition to mean-reverting processes.

4. UNIT ROOTS IN THE MISSPECIFIED HOMOGENEOUS MODEL

4.1. *Representation*

The misspecified homogeneous model (MHM) analyzed by Tiao and Grupe (1980; see also Osborn, 1991) is a time-invariant approximation to the periodic model. The basis of this approximation is that although the autocovariance function (and hence the spectrum) of a periodically stationary process is time varying, it averages out to a constant autocovariance function, which corresponds to some constant stationary autoregressive moving-average (ARMA) model. Let γ_{ks} denote the kth order autocovariance of a stationary periodic process $\{y_t\}$ in season s, and define its annual average

$$\bar{\gamma}_k = \frac{1}{4}\sum_{s=1}^{4}\gamma_{ks} \qquad k = 0, 1, \ldots. \tag{39}$$

Then the MHM of $\{y_t\}$ is the ARMA model $\varphi(L)y_t = \theta(L)\varepsilon_t$, which has $\{\bar{\gamma}_k, k = 0, 1, \ldots\}$ as its autocovariance function.

In this section we shall derive the MHM of a periodically integrated process. For notational ease, we shall concentrate on the first-order PAR model without any deterministic components. Analogous results can be obtained for extensions of this model. Because such a process is not (periodically) stationary, we cannot define its autocovariances, and we have to analyze its fourth difference $\Delta_4 y_t$ to obtain the MHM. Its covariance functions can be readily obtained from (16)–(17) with $U_T = E_T$. For example,

$$\gamma_{11} = E[\Delta_4 y_t \Delta_4 y_{t-1} | s = 1] = E[\Delta_1 Y_{1T} \Delta_1 Y_{4,T-1}] = \sigma^2(\varphi_1 + \varphi_1\varphi_4^2 + \varphi_1\varphi_4^2\varphi_3^2). \tag{40}$$

In general, we have (recall the notational convention $\varphi_0 = \varphi_4$ and $\varphi_{-1} = \varphi_3$)

$$\begin{aligned} \gamma_{0s} &= \sigma^2[1 + \varphi_s^2\{1 + \varphi_{s-1}^2(1 + \varphi_{s-2}^2)\}] \\ \gamma_{1s} &= \sigma^2\varphi_s\{1 + \varphi_{s-1}^2(1 + \varphi_{s-2}^2)\} \\ \gamma_{2s} &= \sigma^2\varphi_s\varphi_{s-1}(1 + \varphi_{s-2}^2) \\ \gamma_{3s} &= \sigma^2\varphi_s\varphi_{s-1}\varphi_{s-2} \end{aligned} \tag{41}$$

and $\gamma_{ks} = 0$, $k > 3$ $(s = 1, \ldots, 4)$. Thus $\{\Delta_4 y_t\}$ is a periodic MA(3) process, the MHM of which is an MA(3) model. The parameters of this MA(3) model may be derived from the autocovariances $\bar{\gamma}_k$, which in turn are obtained by substituting (41) in (39). From (15) we observe that without the unit root restriction $\rho = 1$, $(y_t - \rho y_{t-4})$ is also a periodic MA(3) process with the same parameters. Thus the MHM of a PAR(1) process, whether or not periodically integrated, is the ARMA(4, 3) model (with intermediate autoregressive parameters equal to zero)

$$y_t = \rho y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \theta_3\varepsilon_{t-3}. \tag{42}$$

We shall derive the values of the MA parameters θ_j for two polar cases explicitly. First, suppose $\varphi_s = 1$, $\forall s$, so that y_t is a random walk. This means that $\Delta_4 y_t$ is overdifferenced; indeed, the autocovariances are given by $\bar{\gamma}_0 = 4\sigma^2$, $\bar{\gamma}_1 = 3\sigma^2$, $\bar{\gamma}_2 = 2\sigma^2$ and $\bar{\gamma}_3 = \sigma^2$, which corresponds to $\theta_1 = \theta_2 = \theta_3 = 1$. This defines an MA polynomial with three roots on the unit circle (-1, i and $-i$). Thus the MHM of y_t becomes

$$(1 - L^4)y_t = (1 + L + L^2 + L^3)\varepsilon_t \tag{43}$$

and since $(1 - L^4) = (1 - L)(1 + L + L^2 + L^3)$, three of the four roots on the unit circle cancel, and the resulting model is, as expected, $(1 - L)y_t = \varepsilon_t$. Thus in this example the MHM has exactly the same number of unit roots as the original model, simply because the two models coincide.

Suppose, as a second example, that $\varphi_1 = \varphi_2 = -1$, $\varphi_3 = \varphi_4 = 1$. Then the autocovariances $(\gamma_{0s}, \gamma_{1s}, \gamma_{2s}, \gamma_{3s})$ are given by $\sigma^2(4, -3, -2, -1)$ for $s = 1$, $\sigma^2(4, -3, 2, 1)$ for $s = 2$, $\sigma^2(4, 3, -2, 1)$ for $s = 3$ and $\sigma^2(4, 3, 2, -1)$ for $s = 4$. This implies that $\bar{\gamma}_0 = 4\sigma^2$ and $\bar{\gamma}_k = 0$, $k > 0$, so that the MHM of $\{\Delta_4 y_t\}$ is simply white noise. Therefore, the misspecified model now has four roots on the unit circle, whereas the original (VQ) process has only one unit root.

In the general case, the relationship between $\{\theta_i\}$ and $\{\bar{\gamma}_i\}$ can be derived from the autocovariance-generating function. It can be shown that almost everywhere in the parameter space, the MHM of $\{\Delta_4 y_t\}$ is an invertible MA(3) model, so that the conclusion of four roots on the unit circle will hold in general. Thus, misspecifying a periodically integrated process with a constant-parameter AR model will lead to an overstatement of the number of unit roots, and to the erroneous conclusion that the Δ_4 filter is required to eliminate the stochastic trend in the process. Since seasonal adjustment filters often have a factor equal to $(1 + L + L^2 + L^3)$ (see, for example, Ericsson *et al.*, 1994), we may expect that the MHM of a seasonally adjusted PIAR(1, 1) process will have a single, non-seasonal unit root.

A notable effect of the misspecification, which can be clearly observed from (42), is that the first-order AR structure is lost. Since the MHM is an ARMA process (albeit with most of its AR parameters equal to zero), approximating this by an AR model, as is often done in the econometric analysis of unit roots, may require quite a large lag length and hence a large number of parameters. This provides a theoretical explanation of the phenomenon often found in empirical practice, that there appears to be a trade-off between lag length and periodic parameter variation. That is, whereas a non-periodic model may require quite a large number of lags to obtain white noise errors, a periodic model often requires a smaller lag length. Thus, the loss in parsimony from allowing parameters to vary over the seasons may be offset by a gain from this lag length reduction.

4.2. *Testing*

The number of unit roots in a process critically determines the relevant asymptotic null distribution of a unit root statistic. For example, Dickey and Fuller's (1979) $\hat{\tau}$ statistic for a single unit root in a non-periodic autoregression has a distribution that depends upon only one Brownian motion process, and is given by the signed square root of (32). On the other hand, Dickey *et al.*'s (1984, henceforth DHF) $\hat{\tau}_4$ statistic for $\rho = 1$ in

$$y_t = \rho y_{t-4} + \varepsilon_t \qquad t = 1, \ldots, n \tag{44}$$

satisfies, under the null hypothesis,

$$\hat{\tau}_4 \xrightarrow{\mathrm{d}} \left\{ \sum_{s=1}^{4} \int_0^1 W_s(r)^2 \,\mathrm{d}r \right\}^{-1/2} \sum_{s=1}^{4} \int_0^1 W_s(r)\,\mathrm{d}W_s(r) \tag{45}$$

where $\{W_s(r), r \in [0, 1], s = 1, \ldots, 4\}$ are four independent standard Brownian motion processes. Because the MHM of the first-order periodic autoregression is quite similar (and in some cases identical) to (44), we analyze the properties of the DHF statistic in a PIAR(1, 1) model in the next theorem.

THEOREM 3. *Let $\{y_t\}$ be generated by a first-order periodically integrated autoregression. As $n \to \infty$,*

$$\hat{\tau}_4 \xrightarrow{\mathrm{d}} 2\left\{ \int_0^1 W(r)^2 \,\mathrm{d}r \right\}^{-1/2} \int_0^1 W(r)\,\mathrm{d}W(r) \tag{46}$$

where $W(r)$ is a standard Brownian motion process.

Since $W(r)$ is the same Brownian motion as in Theorem 1, representing the single common trend in the process, it follows from this theorem that the squared DHF statistic, divided by 4, is asymptotically equivalent to the LR statistic if $\{y_t\}$ is a PIAR(1, 1) process. Thus, if the $\hat{\tau}_4$ statistic is used for the hypothesis of a periodic unit root, it should be compared with the critical values tabulated in Fuller (1976), multiplied by 2, rather than those given in DHF. The asymptotic distribution of $\hat{\tau}_4$ is the same as given in (45), but with $W_s(r) = W(r)$ $(s = 1, \ldots, 4)$. The fact that this distribution only depends upon one Brownian motion instead of four is yet another reflection of the fact that H_0 entails only a single unit root, and hence a single common stochastic trend in the four VQ series. Observe that the equivalence of $(\hat{\tau}_4^2/4)$ and LR only holds under the null hypothesis; because LR is designed to discriminate between periodic integration and periodic stationarity, it can be expected to be more powerful than $\hat{\tau}_4$, which is based on a misspecified model.[1]

Theorem 3 again indicates that specifying a non-periodic model for a periodic process leads to an overstatement of the number of unit roots: under the hypothesis of periodic integration, the test statistic $\hat{\tau}_4$ does not diverge, but has a non-degenerate asymptotic distribution. If the critical values from DHF are 'naively' used, then the null hypothesis will seldom be rejected. Furthermore, the second example above indicates that it is possible that the residuals from (44) do not display any serial correlation, so that the misspecification may not be detected. A diagnostic for *periodic* serial correlation will then be more informative to indicate the misspecification of the non-periodic model.

5. CONCLUDING REMARKS

In this paper we have discussed inference on unit roots in periodic autoregressions. We have shown that the analysis of a non-periodic unit root can be easily embedded within periodically integrated autoregressions. Moreover, we have analyzed the consequences of specifying a non-periodic model for a periodically integrated process. In particular, it has been shown that such misspecification may lead to an overstatement of the number of (seasonal) unit roots and a spuriously high lag length.

The main point of the paper is that, for valid inference on stochastic trends in seasonal time series, the possibility of *periodic differencing* should be entertained as an intermediate case between first-differencing and annual differencing to obtain (periodic) stationarity. For time series with the same stochastic trend in each quarter, the annual filter leads to *over*differencing, since it neglects the cointegrating relationships between the quarters. On the other hand, if this common stochastic trend has a (slightly) different scale factor in each quarter, then the first-differencing filter will not eliminate the stochastic trend, and thus can be said to *under*difference the time series. Applying the tests developed in the present paper to quarterly UK macroeconomic time series, Franses and Paap (1994) show that this periodic differencing filter may be relevant in practice: for most series a periodic unit root is found, whereas the restriction corresponding to a first-differencing filter is rejected.

In empirical practice, the use of seasonal time series that are adjusted for seasonality using, for example, the Census X-11 filter is still widespread. One of the underlying assumptions to justify seasonal adjustment techniques is that a univariate time series can be linearly decomposed into four independent components, namely a trend, a cyclical component, a seasonal component and an irregular (noise) process. This assumption is invalid if a time series is periodically integrated, in which case the stochastic seasonal fluctuations cannot be separated from the stochastic trend. Moreover, since seasonal adjustment filters treat the observations in each of the seasons the same, they are non-periodic. Therefore, such filters cannot entirely remove the periodicity in a process, even though in practice the parameter variation will be substantially decreased. As discussed in Section 4, we may expect that the periodic difference filter for a PI(1) series will be drawn towards the non-periodic first-difference filter. Thus a PI(1) series may appear to be I(1) after linear seasonal adjustment.

Throughout the paper we have concentrated on periodic models with a purely autoregressive structure of a known order. Analogously to non-seasonal unit root tests, this could in principle be extended to periodic ARMA models. The evidence from the non-seasonal unit root literature, see, for example, Schwert (1989), shows that unit root tests can have very poor size and power properties in the presence of moving-average components with a near-unit root. Moreover, Schwert's results suggest that unit root tests can best be performed in so-called long autoregressions, where the possibly infinite-order autoregressive representation is approximated by an AR model of an order which is finite but growing with the sample size. As for the selection of the lag order in practice, the analysis of Hall (1994) indicates that a general-to-specific testing strategy leads to the most reliable inference on the presence of a unit root if the true process is an autoregression of unknown (but finite) order; Ng and Perron (1995) have found the same result when the data are generated by an ARMA process.

In this paper we have only considered the possibility of a single unit root. Multiple unit roots may imply either that the VQ process is integrated of order $d > 1$, or that there are less than three cointegration relationships (or both). The latter possibility is considered by Franses (1994), using Johansen's (1988) procedure in the VAR model of the VQ process. Within this framework, the number of cointegrating relationships can be determined via a sequence of LR tests. As noted in the introduction, a disadvantage of this approach is that the structure of the original model is lost in the unrestricted VAR model, which leads to over-parametrization. This provides a motivation to extend the present approach to LR tests for multiple unit roots; this extension is currently under investigation by the authors. Of specific interest is the hypothesis of four unit roots, since that corresponds to a seasonally integrated model, i.e. a (periodically) stationary autoregression in the annual differences. A test for this hypothesis is considered by Ghysels *et al.* (1994), who provide a periodic generalization of Hylleberg *et al.*'s (1990) tests for seasonal unit roots. For the intermediate cases (i.e. with less than four roots on the unit circle), their tests statistics may again be decomposed into a test for multiple periodic unit roots, and a test for periodic parameter variation, analogously to (38).

Finally, a multivariate extension of the notion of periodic integration has recently been put forward by Boswijk and Franses (1995b), who define and analyze periodic error correction and cointegration. A motivation for such models is given in Osborn (1993), *inter alia*. In order to reduce the potential over-parametrization of such models, Boswijk and Franses (1995b) consider single-equation error correction models with periodic parameter variation only in the error correction terms. Their analysis comprises a test for cointegration, an estimator of the cointegrating vectors and a test for the weak exogeneity assumption implicit in their model.

APPENDIX

PROOF OF LEMMA 1. Because $\{E_T\}$ is an i.i.d. sequence with mean zero and variance matrix $\sigma^2 I_4$, it satisfies the conditions of a multivariate invariance principle (see, for example, Phillips and Durlauf, 1986):

$$\frac{1}{\sqrt{N}}\sum_{j=1}^{[rN]} E_j \xrightarrow{\mathrm{d}} E(r)$$

where $E(r)$ is a 4×1 vector Brownian motion process with variance matrix $\sigma^2 I_4$. Since $U_T = \Psi(L)^{-1}E_T$ is a stationary vector autoregression, it can be expressed as $U_T = \Psi(1)^{-1}E_T + D(L)\Delta_1 E_T$, with $D(L)$ some matrix power series with exponentially decreasing weights, so that

$$\frac{1}{\sqrt{N}}\sum_{j=1}^{[rN]} U_j = \frac{1}{\sqrt{N}}\sum_{j=1}^{[rN]} \Psi(1)^{-1}E_j + \mathrm{o_p}(1) \xrightarrow{\mathrm{d}} U(r) = \Psi(1)^{-1}E(r)$$

where $U(r)$ is a 4×1 vector Brownian motion process with variance matrix $\sigma^2\Psi(1)^{-1}\Psi(1)'^{-1}$. Finally,

$$\frac{1}{\sqrt{N}}X_{[rN]} = \frac{1}{\sqrt{N}}\left(\Theta_0\sum_{j=1}^{[rN]} U_j + \Theta_1\sum_{j=0}^{[rN]-1} U_j\right) = (\Theta_0 + \Theta_1)\frac{1}{\sqrt{N}}\sum_{j=1}^{[rN]} U_j + \mathrm{o_p}(1)$$

which converges in distribution to $(\Theta_0 + \Theta_1)U(r) = ab'U(r)$, cf. (19). Letting $W(r) = \omega^{-1}b'U(r)$, the required result obtains.

PROOF OF THEOREM 1. We shall first prove (32) for LR, and then discuss the extension to LR_μ and LR_τ. Consider the original model (2), and denote the full parameter vector by $\varphi = (\varphi_{1.}', \ldots, \varphi_{p.}')'$, with $\varphi_{i.}' = (\varphi_{i1}, \ldots, \varphi_{i4})$. Next, consider the reparametrization (9), with $\varphi_1\varphi_2\varphi_3\varphi_4 = 1$ and $\pi_2 = \pi_3 = \pi_4 = 0$:

$$y_t = \sum_{s=1}^{4}\varphi_s D_{st}y_{t-1} + \sum_{i=1}^{p-1}\sum_{s=1}^{4}\psi_{is}D_{st}(y_{t-i} - \varphi_{s-i}y_{t-i-1}) + \pi_1 D_{1t}y_{t-1} + \varepsilon_t.$$

Recall that the zero restrictions on π_s, $s > 1$, are without loss of generality; without these restrictions, the model is not identified. Under the null hypothesis, $\pi_1 = 0$ and the remaining parameters are identified. Let $\theta = (\theta_1, \theta_2', \theta_3')'$ denote the full parameter vector, where $\theta_1 = \pi_1$, $\theta_2 = (\varphi_2, \varphi_3, \varphi_4)'$ and $\theta_3 = \psi = (\psi_{1.}', \ldots, \psi_{p-1,.}')'$, with $\psi_{i.}' = (\psi_{i1}, \ldots, \psi_{i4})$. Note that θ_1 represents the unit root parameter; θ_2 contains the cointegration parameters (with φ_1 not included, since it is simply defined by $1/\varphi_2\varphi_3\varphi_4$); and θ_3 contains the coefficients of (periodically) stationary regressors. Under the null hypothesis, the relationship between φ and θ is one-to-one; thus the Jacobian matrix of the inverse transformation $\varphi(\theta)$, denoted by $J = \partial\varphi/\partial\theta'$, is non-singular.

Let $x_t' = (x_{1t}', \ldots, x_{pt}')'$ with $x_{it}' = (D_{1t}, \ldots, D_{4t})y_{t-i}$, so that the model is expressed concisely as $y_t = \varphi' x_t + \varepsilon_t$, and the unrestricted ML estimator of φ is

$$\hat{\varphi} = \left(\sum_{t=1}^{n} x_t x_t'\right)^{-1} \sum_{t=1}^{n} x_t y_t = \varphi + \left(\sum_{t=1}^{n} x_t x_t'\right)^{-1} \sum_{t=1}^{n} x_t \varepsilon_t.$$

Below we shall prove that $\hat{\varphi}$ is consistent; if the restricted estimator $\tilde{\varphi}$ is consistent as well (which we shall assume henceforth), then the usual quadratic expansion of the LR statistic will yield

$$\begin{aligned}
\text{LR} &= (\hat{\varphi} - \tilde{\varphi})' Q_\varphi (\hat{\varphi} - \tilde{\varphi}) + o_p(1) \\
&= (\hat{\varphi} - \tilde{\varphi})' J^{-1} Y_N (Y_N^{-1} J' Q_\varphi J Y_N^{-1}) Y_N J'^{-1} (\hat{\varphi} - \tilde{\varphi}) + o_p(1) \\
&= (\hat{\theta} - \tilde{\theta})' Y_N (Y_N^{-1} Q_\theta Y_N^{-1}) Y_N (\hat{\theta} - \tilde{\varphi}) + o_p(1)
\end{aligned}$$

where $Y_N = \text{diag}(N \cdot I_4, \sqrt{N} \cdot I_{4(p-1)})$, and where Q_φ and Q_θ denote minus the Hessian matrices of the log-likelihood formulated in terms of φ and θ, respectively:

$$Q_\varphi = \frac{1}{\sigma^2} \sum_{t=1}^{n} x_t x_t' \qquad Q_\theta = J' Q_\varphi J = \frac{1}{\sigma^2} \sum_{t=1}^{n} z_t z_t'$$

with $z_t = J' x_t$. Note that because of the usual block-diagonality of the information matrix with respect to σ^2 and the remaining parameters, we have excluded σ^2 from the parameter vectors φ and θ. Because $\tilde{\theta}_1 = 0$, and because θ_2 and θ_3 are unrestricted under the null hypothesis, we find

$$\text{LR} = \frac{(N\hat{\theta}_1)^2}{(Y_N^{-1} Q_\theta Y_N^{-1})^{11}} + o_p(1)$$

where A^{11} is the first diagonal element of A^{-1}. From

$$\begin{aligned}
Y_N J^{-1} (\hat{\varphi} - \varphi) &= \left(\frac{1}{\sigma^2} Y_N^{-1} \sum_{t=1}^{n} z_t z_t' Y_N^{-1}\right)^{-1} \frac{1}{\sigma^2} Y_N^{-1} \sum_{t=1}^{n} z_t \varepsilon_t \\
&= (Y_N^{-1} Q_\theta Y_N^{-1})^{-1} Y_N^{-1} q_\theta
\end{aligned}$$

where q_θ, the score vector evaluated in the true value, is implicitly defined, it is clear that all results will follow from the limiting behaviour of the normalized Hessian matrix and score vector.

Partition $z_t = (z_{1t}, z_{2t}', z_{3t}')'$ conformably with θ. Since z_t can be expressed as $\partial\varphi' x_t / \partial\theta$, we find (recall that $\varphi_1 = 1/\varphi_2\varphi_3\varphi_4$)

$$z_{1t} = D_{1t} y_{t-1} \qquad z_{2t} = \begin{bmatrix} v_{2t} \\ v_{3t} \\ v_{4t} \end{bmatrix} - \begin{bmatrix} \varphi_1/\varphi_2 \\ \varphi_1/\varphi_3 \\ \varphi_1/\varphi_4 \end{bmatrix} v_{1t} \qquad z_{3t} = \begin{bmatrix} w_{1t} \\ \vdots \\ w_{p-1,t} \end{bmatrix}$$

where

$$v_{st} = D_{st}y_{t-1} - \sum_{i=1}^{p-1} \psi_{i,\,s+i} D_{s+i,\,t} y_{t-i-1} \qquad s = 1, \ldots, 4$$

and $w_{it}' = (D_{1t}, \ldots, D_{4t})u_{t-i}$. Note that z_{1t} and z_{2t} contain stochastic trends, whereas z_{3t} is periodically stationary.

For the I(1) regressors, we first note that from Lemma 1 and the continuous mapping theorem,

$$\frac{1}{N^2}\sum_{T=1}^{N} Y_{s-1,\,T} Y_{q-1,\,T} \xrightarrow{\mathrm{d}} \omega^2 a_{s-1} a_{q-1} \int_0^1 W(r)^2\, dr$$

$$\frac{1}{N}\sum_{T=1}^{N} Y_{s-1,\,T} E_T \xrightarrow{\mathrm{d}} \omega a_{s-1} \int_0^1 W(r)\, dE(r)$$

for s, $q = 1, \ldots, 4$. This implies, first, that

$$\frac{1}{\sigma^2 N^2}\sum_{t=1}^{n} z_{1t}^2 = \frac{1}{\sigma^2 N^2}\sum_{T=1}^{N} Y_{4,\,T-1}^2 \xrightarrow{\mathrm{d}} \frac{\omega^2}{\sigma^2} a_4^2 \int_0^1 W(r)^2\, dr$$

$$\frac{1}{\sigma^2 N}\sum_{t=1}^{n} z_{1t}\varepsilon_t = \frac{1}{\sigma^2 N}\sum_{T=1}^{N} Y_{4,\,T-1} E_{1T} \xrightarrow{\mathrm{d}} \frac{\omega}{\sigma^2} a_4 \int_0^1 W(r)\, dE_1(r).$$

Next, let $v_t = (v_{1t}, \ldots, v_{4t})'$, and let H denote the 4×3 matrix such that $z_{2t} = H'v_t$. Let V_{sT} denote the VQ process of v_{st}; note that this is a different 4×1 vector process for each s. Partition $\Psi(L)$ as $[\psi_1(L), \ldots, \psi_4(L)]$, where $\psi_s(L)$ are 4×1 lag polynomials, and let $\Psi = \Psi(1)$ and $\psi_s = \psi_s(1)$. Using the fact that v_{st} collects all terms with coefficient φ_s, it can be derived from (12) that

$$V_{sT} = \psi_s(L) Y_{s-1,\,T} = \psi_s Y_{s-1,\,T} + \psi_s^*(L)\Delta_1 Y_{s-1,\,T}$$

where $\psi_s^*(L)$ are 4×1 lag polynomials; the second equality is a consequence of the well-known power series decomposition. Because $\Delta_1 Y_{s-1,\,T}$ is I(0) and hence of lower order than $\psi_s Y_{s-1,\,T}$, we have

$$\frac{1}{N^2}\sum_{t=1}^{n} v_{st} v_{qt} = \frac{1}{N^2}\sum_{T=1}^{N} V_{sT}' V_{qT} = \psi_s'\psi_q \frac{1}{N^2}\sum_{T=1}^{N} Y_{s-1,\,T} Y_{q-1,\,T} + \mathrm{o_p}(1)$$

which implies

$$\frac{1}{\sigma^2 N^2}\sum_{t=1}^{n} z_{2t} z_{2t}' \xrightarrow{\mathrm{d}} \frac{\omega^2}{\sigma^2} H'A\Psi'\Psi A H \int_0^1 W(r)^2\, dr$$

where $A = \operatorname{diag}(a_4, a_1, a_2, a_3)$, with a_s defined in (20). Similarly, we find

$$\frac{1}{\sigma^2 N}\sum_{t=1}^{n} z_{2t}\varepsilon_t \xrightarrow{\mathrm{d}} \frac{\omega}{\sigma^2} H'A\Psi' \int_0^1 W(r)\, dE(r)$$

and

$$\frac{1}{\sigma^2 N^2}\sum_{t=1}^{n} z_{2t} z_{1t} \xrightarrow{\mathrm{d}} \frac{\omega^2}{\sigma^2} H'A\Psi' A_1 \int_0^1 W(r)^2\, dr$$

where $A_1 = (a_4, 0, 0, 0)'$, the first column of A.

From the general results of Park and Phillips (1989), we find for the stationary regressors

$$\frac{1}{\sigma^2 N}\sum_{t=1}^{n} z_{3t}z_{3t}' = \sum_{s=1}^{4}\frac{1}{\sigma^2 N}\sum_{T=1}^{N}\bar{U}_{s-1,T}\bar{U}_{s-1,T}' \xrightarrow{d} V_3$$

$$\frac{1}{\sigma^2\sqrt{N}}\sum_{t=1}^{n} z_{3t}\varepsilon_t = \sum_{s=1}^{4}\frac{1}{\sigma^2\sqrt{N}}\sum_{T=1}^{N}\bar{U}_{s-1,T}E_{sT} \xrightarrow{d} \mathrm{N}(0, V_3)$$

where $\bar{U}_{s-1,T} = (U_{s-1,T}, \ldots, U_{s-p,T})'$, and where V_3 is a fixed positive definite matrix. Moreover,

$$\frac{1}{\sigma^2 N}\sum_{t=1}^{n} z_{3t}(z_{1t}, z_{2t}') = \mathrm{O_p}(1).$$

In order to summarize the results, define the matrix $K = (\omega/\sigma)[A_1{:}\Psi AH]$. Then

$$Y_N^{-1}Q_\theta Y_N^{-1} \xrightarrow{d} \begin{bmatrix} K'K\int_0^1 W(r)^2\,dr & 0 \\ 0 & V_3 \end{bmatrix} \qquad Y_N^{-1}q_\theta \xrightarrow{d} \begin{bmatrix} \sigma^{-1}K'\int_0^1 W(r)\,dE(r) \\ N(0, V_3) \end{bmatrix}.$$

and hence

$Y_N(\hat{\theta} - \theta)$

$$= Y_N J^{-1}(\hat{\varphi} - \varphi) + \mathrm{o_p}(1) \xrightarrow{d} \begin{bmatrix} \left\{\int_0^1 W(r)^2\,\mathrm{d}r\right\}^{-1}\int_0^1 W(r)\,\mathrm{d}\sigma^{-1}(K'K)^{-1}K'E(r) \\ N(0, V_3^{-1}) \end{bmatrix}.$$

Note that this implies that $\hat{\varphi}$ is consistent.

Define $S(r) = \sigma^{-1}(K'K)^{-1}K'E(r)$, a 4×1 vector Brownian motion with variance matrix $(K'K)^{-1}$. Letting $S_1(r)$ denote the first component of $S(r)$, the above results imply

$$N\hat{\theta}_1 \xrightarrow{d} \left\{\int_0^1 W(r)^2\,dr\right\}^{-1}\int_0^1 W(r)\,dS_1(r).$$

Partition $K = [K_1{:}K_2]$ conformably with $S(r)$. It is well known from partitioned regression theory that $S_1(r)$ may be expressed as

$$S_1(r) = \sigma^{-1}(K_1'M_2K_1)^{-1}K_1'M_2E(r)$$

where $M_2 = I - K_2(K_2'K_2)^{-1}K_2'$. Thus the variance of S_1 equals $(K_1'M_2K_1)^{-1}$. The covariance between $S_1(r)$ and $\sigma^{-1}K_2'E(r)$ is equal to $(K_1'M_2K_1)^{-1}K_1'M_2K_2 = 0$. However, the same is true for the covariance between $W(r) = \omega^{-1}b'E(r)$ and $\sigma^{-1}K_2'E(r)$, which is

$$\frac{1}{\omega\sigma}b'\Psi^{-1}\Psi AH = \frac{1}{\omega\sigma}b'AH = 0$$

where the final equality may be checked simply from the definitions of A, b and H. Thus $S_1(r)$ and $W(r)$ are independent of the same 3×1 vector Brownian motion $\sigma^{-1}K_2'E(r)$; and since these are all defined from the same 4×1 vector Brownian motion $E(r)$, it follows that $W(r)$ and $S_1(r)$ must be the same up to a scale factor, i.e. $S_1(r) = (K_1'M_2K_1)^{-1/2}W(r)$. Since

$$(Y_N^{-1}Q_\theta Y_N^{-1})^{11} \xrightarrow{d} (K'K)^{11}\left\{\int_0^1 W(r)^2\,dr\right\}^{-1}$$

and $(K'K)^{11} = (K_1'M_2K_1)^{-1}$, we find that

$$\mathrm{LR} \xrightarrow{\mathrm{d}} (K_1^* M_2 K_1)\left\{\int_0^1 W(r)^2\,dr\right\}^{-1}\left\{\int_0^1 W(r)\,dS_1(r)\right\}^2$$

which, using the relation between $S_1(r)$ and $W(r)$, reduces to the required expression (32).

For LR_μ, we note that since the periodic intercepts enter unrestrictedly, their implication is that all regressors should be replaced by the residual of a regression on four dummies. This in turn implies that in all relevant formulae, Y_{sT} should be replaced by $(Y_{sT} - \bar{Y}_s)$, where $\bar{Y}_s$ is the average over N years. If $\tau = 0$ in (24), then $(Y_{sT} - \bar{Y}_s) = (X_{sT} - \bar{X}_s)$. The general results from Park and Phillips (1988) now imply that $W(r)$ should be replaced by $W_k(r)$ with $k = 1$. The analysis for LR_τ (periodic intercepts and linear trends) is entirely analogous. Here we need $\kappa = 0$ so that Y_T does not contain a quadratic trend, and so the residual of a regression of Y_{sT} on a constant and trend is equal to the residual of the same regression with X_{sT} as the dependent variable.

Proof of theorem 2. Consider first the periodicity test in the unrestricted periodic autoregression. If $y_t \sim \mathrm{PI}(0)$, then it follows from the general results on stationary periodic autoregressions that the LR statistic has a $\chi^2(3p)$ distribution; cf. Pagano (1978, Theorem 4).

Now suppose $y_t \sim \mathrm{PI}(1)$. It will be convenient to consider a minor variation on the parametrization used in the proof of Theorem 1, where π_1 is restricted to zero, but the restriction $\varphi_1 = 1/\varphi_2\varphi_3\varphi_4$ is no longer imposed. Thus, we replace $\theta_1 = \pi_1$ by $\theta_1^* = \varphi_1$, and $\theta = (\theta_1, \theta_2', \theta_3')'$ by $\theta^* = (\theta_1^*, \theta_2', \theta_3')'$. Define the 4×3 matrix $R = ([I_3{:}0] - [0{:}I_3])'$; note that $R'(1, 1, 1, 1)' = 0$. Moreover, let $\bar{R} = (I_p \otimes R)$. The parameter constancy restriction may expressed in terms of θ^* as

$$\bar{R}'\theta^* = \begin{bmatrix} R'\varphi^* \\ R'\psi_{1.} \\ \vdots \\ R'\psi_{p-1,.} \end{bmatrix} = 0$$

where $\varphi^* = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)' = (\theta_1^*, \theta_2')'$.

A quadratic expansion of the LR statistic yields

$$\begin{aligned}\mathrm{LR} &= (\bar{R}'\hat{\theta}^*)'[\bar{R}'Q_\theta^{-1}\bar{R}]^{-1}\bar{R}'\hat{\theta}^* + o_p(1)\\ &= (\hat{\theta}^* - \theta^*)'Y_N\bar{R}[\bar{R}'(Y_N^{-1}Q_\theta \cdot Y_N^{-1})^{-1}\bar{R}]^{-1}\bar{R}'Y_N(\hat{\theta}^* - \theta^*)' + o_p(1)\end{aligned}$$

where the second equality follows from the block-diagonality of $\bar{R}$. Repeating the above analysis for the new parametrization yields the same results as in the proof of Theorem 1, but with K replaced by $K^* = (\omega/\sigma)\Psi A$. This implies that the LR statistic can be decomposed (asymptotically) as $\mathrm{LR} = \mathrm{LR}_\varphi + \mathrm{LR}_\psi$, where LR_φ is the LR statistic for $R'\varphi^* = 0$, and LR_ψ is the LR statistic for $(I_{p-1} \otimes R')\psi = 0$. Moreover, it is easily seen that $\mathrm{LR}_\psi \to_{\mathrm{d}} \chi^2\{3(p-1)\}$, independently of LR_φ. For the latter statistic, we find

$$\mathrm{LR}_\varphi \xrightarrow{\mathrm{d}} \int_0^1 W(r)\,dV(r)'\left\{\int_0^1 W(r)^2\,dr\right\}^{-1}\int_0^1 W(r)\,dV(r)$$

where

$$V(r) = \sigma^{-1}[R'(A\Psi'\Psi A)^{-1}R]^{-1/2}R'(A\Psi'\Psi A)^{-1}A\Psi'E(r)$$

a 3×1 standard vector Brownian motion process. The covariance between $W(r)$ and $V(r)$ is given by

$$\frac{1}{\sigma\omega}[R'(A\Psi'\Psi A)^{-1}R]^{-1/2}R'(\Psi'\Psi)^{-1}A^{-1}b.$$

Under the null hypothesis, we either have $\varphi_s = 1, \forall s$, or $\varphi_s = -1, \forall s$. In the former case, $a = b = (1, 1, 1, 1)'$ and $A = I_4$, and in the latter case $a = b = (1, -1, 1, -1)'$ and $A = \operatorname{diag}(-1, 1, -1, 1)$; thus $A^{-1}b = \pm(1, 1, 1, 1)'$. Moreover, under the null hypothesis the matrix Ψ has a particular structure, satisfying $\Psi_{s+k,\, q+k} = \Psi_{sq}$, where modulo 4 addition is used both for s and q. It can be checked that this structure implies $R'(\Psi'\Psi)^{-1}A^{-1}b = 0$. Thus $V(r)$ and $W(r)$ are independent, which implies

$$\left\{\int_0^1 W(r)^2\,dr\right\}^{-1/2}\int_0^1 W(r)\,dV(r) \sim N(0, I_3)$$

and hence $\mathrm{LR}_\varphi \to_{\mathrm{d}} \chi^2(3)$. Because of the limiting $\chi^2\{3(p-1)\}$ distribution of LR_ψ, independent of LR_φ, we conclude that $\mathrm{LR} \to_{\mathrm{d}} \chi^2(3p)$. The proof in the case of fitted intercepts and trends is entirely analogous; simply replace $W(r)$ by a demeaned or detrended standard Brownian motion.

Consider now part (ii). We first need to derive the properties of $\tilde{\theta}$, the restricted ML estimator. The score vector and Hessian matrix for the restricted model, denoted by $\tilde{q}_\theta$ and $\tilde{Q}_\theta$, are simply obtained by deleting the first element (corresponding to π_1) of q_θ and the first row and column of Q_θ. Their properties are immediately found from the proof of Theorem 1, with K replaced by K_2. Define $\tilde{V}(r) = \sigma^{-1}(K_2')^{-1/2}K_2'E(r)$, a 3×1 standard vector Brownian motion process. We have already shown that $K_2'E(r)$, and hence $\tilde{V}(r)$, is independent of $W(r)$. Assuming (as before) consistency of $\tilde{\theta}$, as a Taylor series expansion of the LR statistic for $\theta_2 = (1, 1, 1)'$ or $\theta_2 = -(1, 1, 1)'$ yields

$$\mathrm{LR} = N(\tilde{\theta}_2 - \theta_2)'(N^{-2}\tilde{Q}_{\theta_2})N(\tilde{\theta}_2 - \theta_2) + o_p(1)$$

$$\xrightarrow{\mathrm{d}} \int_0^1 W(r)\,d\tilde{V}(r)'\left\{\int_0^1 W(r)^2\,dr\right\}^{-1}\int_0^1 W(r)\,d\tilde{V}(r)$$

and by the same argument as before, the distribution of the last expression is $\chi^2(3)$.

PROOF OF THEOREM 3. The least-squares estimate $\hat{\rho}$ from (44) can be expressed as

$$N(\hat{\rho} - 1) = \left(\frac{1}{N^2}\sum_{t=1}^{n} y_{t-4}^2\right)^{-1}\frac{1}{N}\sum_{t=1}^{n} y_{t-4}\Delta_4 y_t$$

so that the DHF statistic satisfies

$$\hat{\tau}_4 = \frac{1}{\hat{\sigma}}\left(\frac{1}{N^2}\sum_{t=1}^{n} y_{t-4}^2\right)^{-1/2}\frac{1}{N}\sum_{t=1}^{n} y_{t-4}\Delta_4 y_t$$

with $\hat{\sigma}$ the residual standard error from (44). Using Lemma 1 and the continuous mapping theorem, we find

$$\frac{1}{N^2}\sum_{t=1}^{n} y_{t-4}^2 = \frac{1}{N^2}\sum_{s=1}^{4}\sum_{T=1}^{N} Y_{s,\,T-1}^2 \xrightarrow{\mathrm{d}} \sum_{s=1}^{4}\int_0^1 B_s(r)^2\,dr$$

and

$$\frac{1}{N}\sum_{t=1}^{n} y_{t-4}\Delta_4 y_t = \frac{1}{N}\sum_{s=1}^{4}\sum_{T=1}^{N} Y_{s,\,T-1}\Delta_1 Y_{sT} \xrightarrow{\mathrm{d}} \sum_{s=1}^{4}\int_0^1 B_s(r)\,dB_s(r).$$

Notice that although the joint vector process $\{\Delta_1 Y_T\}$ is not serially uncorrelated, the individual components $\{\Delta_1 Y_{sT}\}$, $s = 1, \ldots, 4$, are white noise, because $\Delta_4 y_t$ is uncorrelated with $\Delta_4 y_{t-4}$. Therefore, there is no 'bias parameter' added to the stochastic integral, cf. Park and Phillips (1988). Together these results imply that $(\hat{\rho} - 1) = O_p(N^{-1})$, so that

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{t=1}^{n} \Delta_4 y_t^2 + o_p(1) = \frac{1}{4}\sum_{s=1}^{4}\frac{1}{N}\sum_{T=1}^{n} \Delta_1 Y_{s,T}^2 + o_p(1) \xrightarrow{p} \bar{\gamma}_0 = \frac{1}{4}\sum_{s=1}^{4}\gamma_{0s}.$$

Thus we have

$$\hat{\tau}_4 \xrightarrow{d} \left\{\bar{\gamma}_0 \sum_{s=1}^{4}\int_0^1 B_s(r)^2\, dr\right\}^{-1/2} \sum_{s=1}^{4}\int_0^1 B_s(r)\, dB_s(r)$$

$$= \left\{\bar{\gamma}_0 \omega^2(a'a)\int_0^1 W(r)^2\, dr\right\}^{-1/2} \omega^2(a'a)\int_0^1 W(r)\, dW(r).$$

Finally, because from (17) the diagonal elements of $\Theta_0\Theta_1'$ are equal to zero, we have

$$\bar{\gamma}_0 = \frac{\sigma^2}{4}\,\mathrm{tr}\,(\Theta_0\Theta_0' + \Theta_1\Theta_1') = \frac{\sigma^2}{4}\,\mathrm{tr}\,([\Theta_0 + \Theta_1][\Theta_0 + \Theta_1]')$$

$$= \frac{\sigma^2}{4}\,\mathrm{tr}\,(ab'ba') = \frac{\omega^2}{4}(a'a)$$

because $\omega^2 = \sigma^2(b'b)$. Substitution leads to the required result.

NOTE

1. A Monte Carlo experiment which compares the finite sample size and power performance of the DHF statistic to that of the LR statistic in PAR models is available from the authors upon request.

ACKNOWLEDGEMENTS

The authors would like to thank Eric Ghysels, Svend Hylleberg, Koichi Maekawa, Marius Ooms, Denise Osborn, Richard Paap, Murray Smith and an anonymous referee for helpful comments. Both authors gratefully acknowledge financial support from the Royal Netherlands Academy of Arts and Sciences and the Tinbergen Institute.

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