Eric Boeckx; Lieven Vanhecke

Unit tangent sphere bundles with constant scalar curvature

Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 3, 523–544

Persistent URL: http://dml.cz/dmlcz/127667

Czechoslovak Mathematical Journal, 51 (126) (2001), 523–544

UNIT TANGENT SPHERE BUNDLES WITH CONSTANT SCALAR CURVATURE

E. Boeckx *and L. Vanhecke, Leuven

(Received December 9, 1997)

Abstract. As a first step in the search for curvature homogeneous unit tangent sphere bundles we derive necessary and sufficient conditions for a manifold to have a unit tangent sphere bundle with constant scalar curvature. We give complete classifications for low dimensions and for conformally flat manifolds. Further, we determine when the unit tangent sphere bundle is Einstein or Ricci-parallel.

Keywords: unit tangent sphere bundles, constant scalar curvature, Einstein and Ricci-parallel metrics

MSC 2000: 53C25

1. Introduction

A Riemannian manifold (M,g) is *curvature homogeneous* ([23]) if and only if, for each pair of points p and q in M, there exists a linear isometry $F\colon T_pM \to T_qM$ such that $F^*R_q = R_p$, where R is the Riemann curvature tensor of (M,g). This means, intuitively speaking, that the curvature is the "same" at all points. Clearly, (locally) homogeneous Riemannian manifolds are curvature homogeneous, but the converse is not true. We refer to [7, Chapter 12] for an extensive and up-to-date survey about curvature homogeneous manifolds and for many non-trivial examples, i.e., curvature homogeneous spaces which are not locally homogeneous.

Most, if not all, of these examples, were found or constructed by ad hoc methods. In the search for new ones, we propose to take a different road. Starting from a given Riemannian manifold (M,g), we consider spaces naturally associated to it and investigate under which conditions these are curvature homogeneous. This plan was

* Postdoctoral Researcher of the Fund for Scientific Research-Flanders (FWO-Vlaanderen)

first adopted in [27]. We intend to deal with the tangent bundle and the unit tangent sphere bundle, equipped with the natural Sasaki metrics.

The tangent bundle (TM, Tg) turns out to be uninteresting. Indeed, in [21], E. Musso and F. Tricerri show that (TM, Tg) has constant scalar curvature if and only if the base manifold (M, g) is flat. Hence, (TM, Tg) is curvature homogeneous (even locally homogeneous) only in this specific case.

The unit tangent sphere bundle looks more interesting. In this article, we study as a first step under which conditions (T_1M, g_S) has constant scalar curvature. In contrast to the case of the tangent bundle, we find many non-trivial examples. We give a complete classification for low dimensions ($\dim M = 2$ or 3) and for conformally flat spaces. These results serve as the basis for the upcoming paper [9] devoted to curvature homogeneous unit tangent sphere bundles as such.

In the last two sections, we study some stronger restrictions on the curvature and determine explicitly which unit tangent sphere bundles (T_1M, g) are Einstein or Ricci-parallel. As a consequence, we find a new proof of Blair's theorem about locally symmetric unit tangent sphere bundles ([5]).

2. Curvature expressions

We recall the conventions and notations of [8] and collect the formulas we need. We refer to that paper for a more elaborate exposition.

Let (M, g) be a smooth, n-dimensional ($n \geqslant 2$), connected Riemannian manifold and ∇ its Levi Civita connection. The Riemann curvature tensor R is defined by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ for all vector fields X, Y and Z on M. The tangent bundle of (M, g), denoted by TM, consists of pairs (x, u) where x is a point in M and u a tangent vector to M at x. The mapping $\pi\colon TM \to M\colon (x, u) \mapsto x$ is the natural projection from TM onto M.

It is well-known that the tangent space to TM at (x, u) splits into the direct sum of the vertical subspace $VTM_{(x,u)} = \ker \pi_{*|(x,u)}$ and the horizontal subspace $HTM_{(x,u)}$ with respect to ∇:

$$T_{(x,u)}TM = VTM_{(x,u)} \oplus HTM_{(x,u)}.$$

For $X \in T_xM$, there exists a unique vector X^h at the point $(x, u) \in TM$ such that $X^h \in HTM_{(x,u)}$ and $\pi_*(X^h) = X$. X^h is called the *horizontal lift* of X to (x, u). There is also a unique vector X^v at the point (x, u) such that $X^v \in VTM_{(x,u)}$ and $X^v(df) = Xf$ for all functions f on M. X^v is called the *vertical lift* of X to (x, u). The map $X \mapsto X^h$, respectively $X \mapsto X^v$, is an isomorphism between T_xM and $HTM_{(x,u)}$, respectively T_xM and $VTM_{(x,u)}$. Similarly, one lifts vector fields on M to horizontal or vertical vector fields on TM. The expressions in local coordinates for these lifts are given in [8].

The tangent bundle TM of a Riemannian manifold (M, g) can be endowed in a natural way with a Riemannian metric Tg, the so-called *Sasaki metric,* depending only on the Riemannian structure g of the base manifold M. It is uniquely determined by

$$Tg(X^h, Y^h) = Tg(X^v, Y^v) = g(X, Y) \circ \pi, \qquad Tg(X^h, Y^v) = 0$$

for all vector fields X and Y on M.

In this paper, we consider the hypersurface T_1M, the *unit tangent sphere bundle,* consisting of the unit tangent vectors to (M, g). T_1M is given implicitly by the equation $g_x(u, u) = 1$. A unit normal vector N to T_1M at $(x, u) \in T_1M$ is given by the vertical lift of u to (x, u): $N_{|(x,u)} = u^v$.

As the vertical lift of a vector (field) is not tangent to T_1M in general, we define the *tangential lift* of $X \in T_xM$ to $(x, u) \in T_1M$ by

$$X^t_{(x,u)} = (X - g(X, u)\, u)^v_{(x,u)}.$$

The tangent space to T_1M at (x, u) is spanned by vectors of the form X^h and X^t where $X \in T_xM$. When working at a fixed point $(x, u) \in T_1M$, we will use $\overline{X}$ for $X - g(X, u)u$ in order to keep the notation as simple as possible.

We endow T_1M with the Riemannian metric g_S induced from the Sasaki metric Tg on TM. It is given explicitly by

$$\begin{aligned} g_{S|(x,u)}(X^t, Y^t) &= g_x(\overline{X}, \overline{Y}), \\ g_{S|(x,u)}(X^t, Y^h) &= 0, \\ g_{S|(x,u)}(X^h, Y^h) &= g_x(X, Y). \end{aligned}$$

The Levi Civita connection $\overline{\nabla}$ associated to this metric is given at (x, u) by

$$\begin{aligned} \overline{\nabla}_{X^t} Y^t &= -g(Y, u)\, X^t, \\ \overline{\nabla}_{X^t} Y^h &= \frac{1}{2}\,(R(u, X)Y)^h, \\ \overline{\nabla}_{X^h} Y^t &= (\nabla_X Y)^t + \frac{1}{2}\,(R(u, Y)X)^h, \\ \overline{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2}\,(R(X, Y)u)^t \end{aligned} \tag{1}$$

for all vector fields X and Y on M. (See [8].)

Also the Riemann curvature tensor $\overline{R}$ associated to g_S was calculated in [8] (see also [28]). In its $(0,4)$-tensor form, $\overline{R}$ is given by

$$\begin{aligned}
\overline{R}_{|(x,u)}(X^t,Y^t,Z^t,V^t) &= -\,g_x(\overline{X},\overline{Z})g_x(\overline{Y},\overline{V})+g_x(\overline{Y},\overline{Z})g_x(\overline{X},\overline{V}),\\
\overline{R}_{|(x,u)}(X^t,Y^t,Z^t,V^h) &= 0,\\
\overline{R}_{|(x,u)}(X^h,Y^t,Z^t,V^h) &= -\,\frac{1}{2}g_x(R(\overline{Y},\overline{Z})X,V)\\
&\quad+\frac{1}{4}g_x(R(u,Z)X,R(u,Y)V),\\
\overline{R}_{|(x,u)}(X^h,Y^t,Z^h,V^h) &= \frac{1}{2}g_x((\nabla_X R)(u,Y)Z,V),\\
\overline{R}_{|(x,u)}(X^h,Y^h,Z^t,V^t) &= g_x(R(X,Y)\overline{Z},\overline{V})\\
&\quad+\frac{1}{4}g_x(R(u,Z)X,R(u,V)Y)\\
&\quad-\frac{1}{4}g_x(R(u,V)X,R(u,Z)Y),\\
\overline{R}_{|(x,u)}(X^h,Y^h,Z^h,V^h) &= g_x(R(X,Y)Z,V)\\
&\quad+\frac{1}{2}g_x(R(X,Y)u,R(Z,V)u)\\
&\quad+\frac{1}{4}g_x(R(X,Z)u,R(Y,V)u)\\
&\quad-\frac{1}{4}g_x(R(X,V)u,R(Y,Z)u).
\end{aligned}$$

Next, we determine the Ricci tensor $\bar{\varrho}$ of (T_1M,g_S) at the point $(x,u)\in T_1M$. For this purpose, let $E_1,\dots,E_{n-1},E_n=u$ be an orthonormal basis of T_xM. Then $E_1^t,\dots,E_{n-1}^t,E_1^h,\dots,E_n^h$ is an orthonormal basis of $T_{(x,u)}T_1M$ and $\bar{\varrho}$ is defined by

$$\bar{\varrho}(A,B)=\sum_{i=1}^{n-1}\overline{R}(E_i^t,A,B,E_i^t)+\sum_{i=1}^{n}\overline{R}(E_i^h,A,B,E_i^h).$$

We obtain

$$\begin{aligned}
\bar{\varrho}_{|(x,u)}(X^t,Y^t) &= (n-2)g_x(\overline{X},\overline{Y})+\frac{1}{4}\sum_{i=1}^{n}g_x(R(u,X)E_i,R(u,Y)E_i),\\
\bar{\varrho}_{|(x,u)}(X^t,Y^h) &= \frac{1}{2}\Big((\nabla_u\varrho)_x(X,Y)-(\nabla_X\varrho)_x(u,Y)\Big),\\
\bar{\varrho}_{|(x,u)}(X^h,Y^h) &= \varrho_x(X,Y)-\frac{1}{2}\sum_{i=1}^{n}g_x(R(u,E_i)X,R(u,E_i)Y).
\end{aligned}\tag{2}$$

Another metric contraction gives the scalar curvature $\bar{\tau}$:

$$\begin{aligned}\bar{\tau}_{|(x,u)} &= \sum_{i=1}^{n-1} \bar{\varrho}_{|(x,u)}(E_i^t, E_i^t) + \sum_{i=1}^{n} \bar{\varrho}_{|(x,u)}(E_i^h, E_i^h)\\ &= \tau_x + (n-1)(n-2) - \xi_x(u,u)/4\end{aligned}$$

where, as in [2] and [11], $\xi(u,v) = \sum_{i,j=1}^{n} g(R(u,E_i)E_j, R(v,E_i)E_j)$.

We note that the natural projection $\pi_1\colon (T_1M, g_S) \to (M,g)\colon (x,u) \mapsto x$ is a Riemannian submersion with totally geodesic fibres. Hence, one can also use the standard formulas in [3, pp. 243–244] to obtain the above expressions for the curvature. O'Neill's integrability tensor A is given in this case by

$$\begin{aligned}A_{X^h} Y^t &= \frac{1}{2}\,(R(u,Y)X)^h,\\ A_{X^h} Y^h &= -\frac{1}{2}\,(R(X,Y)u)^t\end{aligned}$$

as follows from the expressions (1) for the covariant derivative, while the shape tensor T is zero.

Finally, we also need the first covariant derivative of the Ricci tensor. A straightforward calculation, using the expressions (2) above and the formulas (1) for the covariant derivative $\overline{\nabla}$, gives

$$\begin{aligned}
(3)\qquad (\overline{\nabla}_{Z^t}\bar{\varrho})_{|(x,u)}(X^t,Y^t) &= \frac{1}{4}\sum_{i=1}^{n}\Big(g_x(R(\overline{Z},\overline{X})E_i, R(u,\overline{Y})E_i)\\
&\qquad + g_x(R(u,\overline{X})E_i, R(\overline{Z},\overline{Y})E_i)\Big),\\
(\overline{\nabla}_{Z^t}\bar{\varrho})_{|(x,u)}(X^t,Y^h) &= \frac{1}{2}\Big((\nabla_{\overline{Z}}\varrho)_x(\overline{X},Y) - (\nabla_{\overline{X}}\varrho)_x(\overline{Z},Y)\Big)\\
&\quad - \frac{1}{4}\Big((\nabla_u\varrho)_x(\overline{X}, R(u,\overline{Z})Y)\\
&\qquad - (\nabla_{\overline{X}}\varrho)_x(u, R(u,\overline{Z})Y)\Big),\\
(\overline{\nabla}_{Z^t}\bar{\varrho})_{|(x,u)}(X^h,Y^h) &= -\frac{1}{2}\Big(\varrho_x(X, R(u,Z)Y) + \varrho_x(R(u,Z)X,Y)\Big)\\
&\quad - \frac{1}{2}\sum_{i=1}^{n}\Big(g_x(R(\overline{Z},E_i)X, R(u,E_i)Y)\\
&\qquad + g_x(R(u,E_i)X, R(\overline{Z},E_i)Y)\Big)\\
&\quad + \frac{1}{4}\sum_{i=1}^{n}\Big(g_x(R(u,E_i)R(u,Z)X, R(u,E_i)Y)\\
&\qquad + g_x(R(u,E_i)X, R(u,E_i)R(u,Z)Y)\Big),
\end{aligned}$$

$$
\begin{aligned}
(\overline{\nabla}_{Z^h}\bar{\varrho})_{|(x,u)}(X^t,Y^t) = \frac{1}{4}\sum_{i=1}^{n}&\Big(g_x((\nabla_Z R)(u,\overline{X})E_i, R(u,\overline{Y})E_i) \\
&\quad + g_x(R(u,\overline{X})E_i,(\nabla_Z R)(u,\overline{Y})E_i)\Big) \\
&- \frac{1}{4}\Big((\nabla_u\varrho)_x(R(u,\overline{X})Z,\overline{Y}) \\
&\quad + (\nabla_u\varrho)_x(\overline{X},R(u,\overline{Y})Z) \\
&\quad - (\nabla_{\overline{X}}\varrho)_x(R(u,\overline{Y})Z,u) \\
&\quad - (\nabla_{\overline{Y}}\varrho)_x(R(u,\overline{X})Z,u)\Big), \\
(\overline{\nabla}_{Z^h}\bar{\varrho})_{|(x,u)}(X^t,Y^h) = \frac{1}{2}&\Big((\nabla^2_{Zu}\varrho)_x(X,Y) - (\nabla^2_{ZX}\varrho)_x(u,Y)\Big) \\
&+ \frac{1}{2}(n-2)g_x(X,R(Z,Y)u) - \frac{1}{2}\varrho_x(R(u,X)Z,Y) \\
&+ \frac{1}{4}\sum_{i=1}^{n} g_x(R(u,E_i)R(u,X)Z,R(u,E_i)Y) \\
&+ \frac{1}{8}\sum_{i=1}^{n} g_x(R(u,X)E_i,R(u,R(Z,Y)u)E_i), \\
(\overline{\nabla}_{Z^h}\bar{\varrho})_{|(x,u)}(X^h,Y^h) = (\nabla_Z&\varrho)_x(X,Y) \\
&- \frac{1}{2}\sum_{i=1}^{n}\Big(g_x((\nabla_Z R)(u,E_i)X,R(u,E_i)Y) \\
&\qquad + g_x(R(u,E_i)X,(\nabla_Z R)(u,E_i)Y)\Big) \\
&+ \frac{1}{4}\Big((\nabla_u\varrho)_x(R(Z,X)u,Y) \\
&\quad + (\nabla_u\varrho)_x(X,R(Z,Y)u) \\
&\quad - (\nabla_{R(Z,X)u}\varrho)_x(u,Y) \\
&\quad - (\nabla_{R(Z,Y)u}\varrho)_x(u,X)\Big).
\end{aligned}
$$

In the rest of this paper, especially in calculations, we will often use the Einstein summation convention, which says that every index appearing twice in a formula is to be summed over. The indices refer to elements of an orthonormal basis $\{E_1,\dots,E_n\}$.

Remark 1. The unit tangent sphere bundle T_1M can be equipped with a natural contact metric structure $(\xi,\eta,\varphi,g_{cm})$ (see [4], [8]). Obviously, the characteristic vector field ξ has nothing to do with the $(0,2)$-tensor ξ mentioned previously. The metric g_{cm} is up to a homothetic change with factor $1/4$ equal to g_S. In particular, all the results in this paper hold also for the unit tangent sphere bundle T_1M endowed with the metric g_{cm} (up to an occasional factor $1/4$).

3. Constant scalar curvature

In the previous section, we calculated the scalar curvature

$$\bar{\tau}_{|(x,u)} = \tau_x + (n-1)(n-2) - \xi_x(u,u)/4.$$

Suppose next that $\bar{\tau}$ is constant along T_1M. In particular, for fixed $x \in M$, $\bar{\tau}_{|(x,u)}$ does not depend on the choice of unit vector u. This implies that ξ_x is proportional to the metric g_x, and the proportionality constant is necessarily equal to $|R|^2{}_x/n$. Moreover, $\bar{\tau}$ should be independent of the point $x \in M$. This proves

Theorem 3.1. *The unit tangent sphere bundle (T_1M, g_S) has constant scalar curvature $\bar{\tau}$ if and only if on (M,g) it holds*

$$\xi = \frac{|R|^2}{n}\, g, \tag{4}$$

$$4n\tau - |R|^2 = \text{constant}. \tag{5}$$

Remark 2. The *algebraic* condition (4) has appeared in the literature before (see, e.g., [2], [11], [15]), but without a clear *geometric* meaning. In [3, p. 134], an analytic interpretation is given of this condition for the case of a compact manifold: an Einstein metric (or, more generally, a metric with parallel Ricci tensor) is critical for the functional $SR(g) = \int_M |R_g|^2 d\,\mathrm{vol}$ restricted to those metrics g such that $\mathrm{vol}(M) = 1$ if and only if $\xi = \frac{|R|^2}{n} g$. Based on the above theorem, we can now give a nice geometric interpretation of condition (4) for Riemannian manifolds (M,g) such that (5) holds: such manifolds satisfy (4) if and only if their unit tangent sphere bundle (T_1M, g_S) has constant scalar curvature. Clearly, (5) holds for curvature homogeneous manifolds. Hence, in view of our quest for curvature homogeneous manifolds, a *classification of all curvature homogeneous spaces (and, in particular, locally homogeneous spaces) satisfying* $\xi = (|R|^2/n)\, g$ would be very desirable.

The case of (locally) reducible manifolds is now easy to deal with:

Corollary 3.2. *The unit tangent sphere bundle (T_1M, g_S) of a (local) product manifold $(M,g) = (M_1^{n_1}, g_1) \times (M_2^{n_2}, g_2)$ has constant scalar curvature if and only if the unit tangent sphere bundles of both (M_1, g_1) and (M_2, g_2) have constant scalar curvature and, additionally,*

$$\frac{|R_1|^2}{n_1} = \frac{|R_2|^2}{n_2}. \tag{6}$$

P r o o f. Suppose that (T_1M, g_S) has constant scalar curvature $\bar{\tau}$. Condition (4) implies that $\xi_i = \frac{|R_i|^2}{n_i} g_i$ for $i = 1, 2$, and that (6) holds. In particular, $|R_1|^2$ and $|R_2|^2$ are constant. (5) reads $4(n_1 + n_2)(\tau_1 + \tau_2) - |R_1|^2 - |R_2|^2 =$ constant, so also τ_1 and τ_2 are constant. Hence, the tangent unit sphere bundles of (M_1, g_1) and (M_2, g_2) have constant scalar curvature. The converse is immediate. □

Remark 3. Corollary 3.2 can be used for the construction of examples. Suppose that (M_1, g_1) and (M_2, g_2) are such that their unit tangent sphere bundles have constant scalar curvature and that τ_i and $|R_i|^2$, $i = 1, 2$, are separately constant. If either one of these spaces is flat, so must the other be in order that $(T_1(M_1 \times M_2), (g_1 \times g_2)_S)$ have constant scalar curvature. Otherwise, we make a homothetic change of the metric g_1 on M_1 to a new metric $\tilde{g}_1 = cg_1$ with factor $c = \frac{n_2|R_1|^2}{n_1|R_2|^2} > 0$. Then, (6) is satisfied for the Riemannian manifolds $(M_1, \tilde{g}_1)$ and (M_2, g_2), and $(M, g) = (M_1 \times M_2, \tilde{g}_1 \times g_2)$ has a unit tangent sphere bundle with constant scalar curvature.

When looking for further examples, the following Schur-like theorem is useful.

Lemma 3.3. *Let (M^n, g) be a Riemannian manifold of dimension $n \neq 4$ such that its Ricci tensor is a Codazzi tensor (i.e., $(\nabla_X \varrho)(Y, Z) = (\nabla_Y \varrho)(X, Z)$). If $\xi = \lambda g$, then $\lambda = |R|^2/n$ is constant and (T_1M, g_S) has constant scalar curvature $\bar{\tau}$.*

P r o o f. Derive the equality $R_{ijkm}R_{ijk\ell} = \lambda\delta_{m\ell}$ with respect to E_m and sum over $m = 1, \dots, n$. Using the second Bianchi identity and the equality $n\lambda = (R_{ijkm})^2$, we derive

$$8 \sum_{i,j,k} \nabla_i \varrho_{jk} R_{ijk\ell} = (n-4)\nabla_\ell \lambda,$$

or the equivalent formula

$$8 \sum_{i,j,k} (\nabla_i \varrho_{jk} - \nabla_j \varrho_{ik}) \, R_{ijk\ell} = 2(n-4)\nabla_\ell \lambda$$

for all $\ell = 1, \dots, n$. Hence $\lambda = |R|^2/n$ is constant under the hypotheses of the lemma. As is well-known (see, e.g., [13]), a Riemannian manifold with Codazzi Ricci tensor has constant scalar curvature. So, both (4) and (5) are satisfied and the last statement of the lemma follows. □

We are now ready to give some classes of Riemannian manifolds whose unit tangent sphere bundles have constant scalar curvature for the metric g_S.

0. Spaces of constant curvature.
1. Irreducible symmetric spaces and, more generally, isotropy irreducible homogeneous spaces: irreducibility ensures that ξ, an invariant (0,2)-tensor, is proportional to the metric; homogeneity takes care of condition (5).

Furthermore, for reducible symmetric spaces $(M, g) = (M_1, g_1)\times\ldots\times(M_k, g_k)$ with irreducible components (M_i, g_i), we have from the above and condition (4) that (T_1M, g) has constant scalar curvature $\bar{\tau}$ if and only if $|R_1|^2/n_1 = \ldots = |R_k|^2/n_k$.

2. Super-Einstein spaces ([15]): these are Einstein manifolds (M, g) satisfying additionally the condition (4). By Lemma 3.3, we find that $|R|^2$ is constant if the dimension is different from four. For a four-dimensional super-Einstein space we require the constancy of $|R|^2$.
3. Harmonic spaces: because every harmonic space is super-Einstein (see, e.g., [2], [11]). In particular it follows that *the unit tangent sphere bundle of any Damek-Ricci space has constant scalar curvature.* For the definition of Damek-Ricci spaces, their geometric properties and further references, see [1].
4. Sasakian space forms with constant φ-sectional curvature $c = \pm 1$: this follows from a straightforward computation expressing condition (4), where we use the curvature formulas given in [4, p. 97]. For every odd dimension, we single out two Sasakian space forms in this way: one of these ($c = 1$) is locally isometric to a sphere of radius 1 and has a locally homogeneous unit tangent sphere bundle; the unit tangent sphere bundle of the other is not even Ricci-curvature homogeneous. (See [9].)

A Riemannian manifold (M, g) is called a *semi-symmetric space* ([7]) if its curvature tensor R is, at each point, the same as that of some symmetric space. These spaces are characterized by the curvature property $R(X, Y)\cdot R = 0$ for all vector fields X and Y on M, where $R(X, Y)$ acts as a derivation on tensors. For such spaces, we have

Proposition 3.4. *Let (M, g) be a semi-symmetric space. If (T_1M, g_S) has constant scalar curvature $\bar{\tau}$, then (M, g) is locally symmetric.*

Proof. The local structure theorem for semi-symmetric spaces by Z. Szabó ([25]) states that, around the points of a dense open subset, a semi-symmetric space is a local product of symmetric spaces, two-dimensional manifolds, six types of cones and Riemannian manifolds foliated by Euclidean leaves of conullity two. If the semi-symmetric space has a unit tangent sphere bundle with constant scalar curvature, then the same is true for all the factors in this local decomposition by Corollary 3.2.

Next, we note that the nullity distribution for the semi-symmetric spaces of cone type or of foliated type is non-empty. Condition (4) then implies readily that these spaces must be flat if their unit tangent sphere bundles are to have constant scalar curvature. On the other hand, we will show in Proposition 4.1 that a surface whose unit tangent sphere bundle has constant scalar curvature must itself have constant curvature and, hence, is locally symmetric. □

Next, we consider Einstein spaces. We have immediately the following results.

Proposition 3.5. *Let (M^4, g) be a four-dimensional Einstein space. Then (T_1M, g_S) has constant scalar curvature if and only if $|R|^2$ is constant.*

P r o o f. This follows at once from the observation that a four-dimensional Einstein space always satisfies the property $\xi = (|R|^2/n)\, g$ (see [2]). □

From the definition of a super-Einstein space above, we get

Proposition 3.6. *An Einstein space (M, g) is super-Einstein if and only if (T_1M, g_S) has constant scalar curvature.*

Remark 4. As stated in [2, p. 165], it would be worthwhile to find more examples, besides the ones mentioned in 1 and 3 above, of super-Einstein metrics.

Next, suppose (M, g) to be 2-stein, i.e., the manifold is Einstein and $\sum\limits_{i,j} R^2_{uiuj}$ is constant for all unit vectors u at a given point x. This condition implies (4) ([2]). Hence with Lemma 3.3 above, we find

Proposition 3.7. *Let (M^n, g), $n > 4$, be a 2-stein space. Then (T_1M, g_S) has constant scalar curvature.*

A Riemannian manifold is *pointwise Osserman* if the eigenvalues of $R_u = R(\,\cdot, u)u$ are independent of the choice of unit vector u at a given point x. If the eigenvalues are global constants, i.e., also independent of x, the space is said to be *globally Osserman*. See [12] for more information.

If a manifold (M^n, g) is pointwise Osserman, it is also 2-stein ([12]) and hence super-Einstein for $n \neq 4$. If we suppose moreover that $n \neq 2$, then both τ and $|R|^2$ are constant. So, we have

Corollary 3.8. *Let (M^n, g) be a pointwise Osserman space and $n \neq 2, 4$. Then (T_1M, g_S) has constant scalar curvature.*

In dimension four, 2-stein is equivalent to pointwise Osserman ([12]) and we get

Corollary 3.9. *Let (M^4, g) be a four-dimensional pointwise Osserman space. Then (T_1M, g_S) has constant scalar curvature if and only if $|R|^2$ is constant.*

Still in dimension four, an orientable Einstein manifold is 2-stein if and only if it is self-dual or anti-self-dual ([22]). Hence,

Corollary 3.10. *Let (M^4, g) be an orientable four-dimensional Einstein manifold which is self-dual or anti-self-dual. Then (T_1M, g_S) has constant scalar curvature if and only if $|R|^2$ is constant.*

Finally, the global Osserman property implies the 2-stein condition and the constancy of $|R|^2$. So for all dimensions we get

Corollary 3.11. *Let (M, g) be a globally Osserman space. Then (T_1M, g_S) has constant scalar curvature.*

4. Classification in dimension two and three

We now determine all two- and three-dimensional Riemannian manifolds (M, g) whose unit tangent sphere bundles have constant scalar curvature $\bar{\tau}$. We begin with

Proposition 4.1. *(T_1M^2, g_S) has constant scalar curvature $\bar{\tau}$ if and only if (M^2, g) has constant curvature.*

P r o o f. The curvature tensor of a two-dimensional Riemannian manifold is given in the form $R(X,Y)Z = \kappa\,(g(Y,Z)\,X - g(X,Z)\,Y)$. One easily checks that (4) is satisfied. On the other hand, (5) reduces to

$$4\kappa - \kappa^2 = \text{constant}.$$

So κ must be constant. □

The scalar curvature $\bar{\tau}$ is given explicitly as $\bar{\tau} = \kappa\,(4-\kappa)/2$.

Next, we have

Proposition 4.2. *(T_1M^3, g_S) has constant scalar curvature $\bar{\tau}$ if and only if (M^3, g) has constant curvature or (M^3, g) is a curvature homogeneous space with constant Ricci roots $\varrho_1 = \varrho_2 = 0 \neq \varrho_3$.*

P r o o f. In dimension three, the Riemann curvature tensor R can be expressed completely through the Ricci curvature ϱ and the scalar curvature τ (see, e.g., [3]):

$$R = \varrho \owedge g - \frac{\tau}{4}\, g \owedge g \tag{7}$$

where $\owedge$ is the Kulkarni-Nomizu product of symmetric 2-tensors defined as follows:

$$
\begin{aligned}
(h \owedge k)(X,Y,Z,V) = {} & h(X,Z)k(Y,V) + h(Y,V)k(X,Z) \\
& - h(X,V)k(Y,Z) - h(Y,Z)k(X,V).
\end{aligned}
$$

A straightforward calculation using (7) gives

$$
\begin{aligned}
|R|^2 &= 4|\varrho|^2 - \tau^2, \\
\xi(X,Y) &= (2|\varrho|^2 - \tau^2)g(X,Y) + 2\tau\varrho(X,Y) - 2\sum_i \varrho(X,E_i)\varrho(Y,E_i).
\end{aligned}
$$

Condition (4) then reduces to

$$
\frac{1}{3}(|\varrho|^2 - \tau^2)\, g(X,Y) = \sum_i \varrho(X,E_i)\varrho(Y,E_i) - \tau\varrho(X,Y). \tag{8}
$$

Note that both sides are linear in X and Y. So, it is enough to check equality (8) for an orthonormal basis. Take an orthonormal basis $\{E_1, E_2, E_3\}$ consisting of eigenvectors of the Ricci tensor ϱ, i.e., $\varrho(E_i,E_j) = \varrho_i\delta_{ij}$. We find

$$
\varrho_i^2 - \tau\varrho_i - \frac{1}{3}(|\varrho|^2 - \tau^2) = 0. \tag{9}
$$

So, at $x \in M$, $\varrho_i(x)$ is a root of a quadratic equation. Hence, there can be at most two different Ricci roots and there is a root with multiplicity at least two. Put $\varrho_1(x) = \varrho_2(x) = \lambda$, $\varrho_3(x) = \mu$. Then $\tau(x) = 2\lambda + \mu$, $|\varrho|^2(x) = 2\lambda^2 + \mu^2$ and (9) gives

$$
\lambda\,(\mu - \lambda) = 0.
$$

So, either all Ricci roots are the same at x, or two of them are zero and one is non-zero.

Now, let U be the dense open subset of M on which the multiplicity of the Ricci roots is locally constant. If on a connected component of U there is only one root with multiplicity three, then this component has constant curvature; if we have a zero root with multiplicity two, then the non-zero root must be constant on the connected component by condition (5). In particular: the Ricci roots are always locally constant, hence globally constant and $U = M$. $\square$

In the case when (M^3, g) has constant curvature κ, we have $\bar{\tau} = 2 + 6\kappa - \kappa^2$. When (M^3, g) has two Ricci roots equal to zero ($\varrho_1 = \varrho_2 = 0 \neq \varrho_3$), we have $\bar{\tau} = 2 + \varrho_3 - \varrho_3^2/4$.

Remark 5. Three-dimensional curvature homogeneous spaces have been studied extensively ([10], [18], [19] and [7, Chapter 6] for further references). The class of spaces with constant Ricci roots $\varrho_1 = \varrho_2 = 0$ and $\varrho_3 \neq 0$ contains both homogeneous ([20]) and non-homogeneous ones ([18]). O. Kowalski has given explicit examples of three-dimensional curvature homogeneous spaces with two zero Ricci roots and negative third Ricci root ([18, Example 5.8]). As a simple example, $\mathbb{R}^3$ with the metric

$$ds^2 = e^{-2\lambda z} dx^2 + \big(e^{\lambda z} dy + x(e^{-\lambda z} - \lambda^2 y^2 e^{\lambda z}) dx\big)^2 + \big(dz + 2\lambda xy\, dx\big)^2$$

has Ricci roots $\varrho_1 = \varrho_2 = 0$, $\varrho_3 = -2\lambda^2$. Moreover, this example is *not locally homogeneous.*

Remark 6. Three-dimensional curvature homogeneous spaces with two zero Ricci roots and one non-zero constant Ricci root are also non-trivial examples of so-called *O-spaces.* These are Riemannian manifolds (M, g) defined as follows: let $c(t)$ be a circle in (M, g), i.e., $\nabla_{\dot c}\dot c = k\, n$ and $\nabla_{\dot c} n = -k\, \dot c$ for some $k \in \mathbb{R}_0$, and consider the associated curvature operator $R_c(X) := R(\dot c, n)X$. (M, g) is an O-space if, for every circle c, R_c has constant eigenvalues along this circle. See [16] for classification results and for more details.

5. Conformally flat manifolds

The classification of three-dimensional Riemannian manifolds whose unit tangent sphere bundle has constant scalar curvature was possible because we could express the Riemann curvature R using only the Ricci curvature ϱ and the scalar curvature τ. The same is true for conformally flat manifolds. So, suppose that (M^n, g), $n \geqslant 4$, is conformally flat. Its curvature tensor R is given by

$$R = \frac{1}{n-2}\, \varrho \owedge g - \frac{\tau}{2(n-1)(n-2)}\, g \owedge g.$$

A routine calculation gives

$$\begin{aligned}
|R|^2 &= \frac{4}{n-2}\, |\varrho|^2 - \frac{2}{(n-1)(n-2)}\, \tau^2, \\
\xi(X,Y) &= \frac{2}{(n-2)^2}\Big((|\varrho|^2 - \frac{\tau^2}{n-1}) g(X,Y) \\
&\qquad + \frac{2}{n-1} \tau \varrho(X,Y) + (n-4) \sum_i \varrho(X,E_i)\varrho(Y,E_i)\Big),
\end{aligned}$$

and condition (4) reduces to

$$\begin{aligned}(10)\qquad &\frac{1}{n}\big((n-4)|\varrho|^2+\frac{2}{n-1}\tau^2\big)\,g(X,Y)\\ &=\frac{2}{n-1}\,\tau\varrho(X,Y)+(n-4)\sum_i\varrho(X,E_i)\varrho(Y,E_i).\end{aligned}$$

As before, we take an orthonormal basis $\{E_1,\dots,E_n\}$ consisting of eigenvectors of ϱ, i.e., $\varrho(E_i,E_j)=\varrho_i\delta_{ij}$. The equation (10) simplifies to

$$(11)\qquad (n-4)\varrho_i^2+\frac{2\tau}{n-1}\,\varrho_i-\frac{1}{n}\Big((n-4)|\varrho|^2+\frac{2}{n-1}\,\tau^2\Big)=0.$$

First, we consider the four-dimensional situation, $n=4$. In that case, (11) reads $\tau\,(4\varrho_i-\tau)=0$. Let U_1 be the open subset of M where $\tau\neq 0$. Then $\varrho_i=\tau/4$ for all $i=1,\dots,n$, and $\varrho=(\tau/4)\,g$. Hence, (U_1,g) is a space of constant curvature and τ is constant. Denote by U_2 the set of points $x\in M$ such that τ is identically zero on some neighbourhood of x. On the open set U_2 we have $\tau=0$. Condition (5) simply says that $|\varrho|^2$ is constant on U_2. But $U_1\cup U_2$ is an open and dense subset of M and τ is locally constant on this set. So the scalar curvature is globally constant and we have

Proposition 5.1. *Let (M^4,g) be conformally flat. Then (T_1M,g_S) has constant scalar curvature if and only if (M^4,g) has constant curvature or its scalar curvature τ is zero and $|\varrho|^2$ is constant.*

When (M^4,g) has constant curvature κ, $\bar\tau$ is given by $\bar\tau=3\,(4+8\kappa-\kappa^2)/2$. When $\tau=0$ and $|\varrho|^2$ is constant, $\bar\tau=6-(|\varrho|^2/8)$.

Examples of this last class of spaces are the Riemannian products $M=M^2(c)\times M^2(-c)$, where $M^2(c)$ is a two-dimensional space of constant curvature c (cf. Proposition 5.2). It would be worthwhile to classify explicitly the four-dimensional Riemannian manifolds with $\tau=0$ and $|\varrho|^2$ constant.

Returning to the general situation, $n>4$, (11) says that all Ricci roots are solutions of a quadratic equation. So, at each $x\in M$, there are at most two distinct Ricci roots, say λ with multiplicity k and μ with multiplicity $n-k$. Then $\tau=k\lambda+(n-k)\mu$, $|\varrho|^2=k\lambda^2+(n-k)\mu^2$ and (11) gives

$$(12)\qquad (\mu-\lambda)\big((n^2-5n+4+2k)\lambda+(n^2-3n+4-2k)\mu\big)=0.$$

As in the previous section, it follows from condition (5) that the Ricci roots are constants and (M^n,g) is curvature homogeneous.

In [26], H. Takagi gives an explicit classification of conformally flat locally homogeneous spaces of arbitrary dimension. He shows that such a space is locally isometric to one of the following locally symmetric spaces:

1. a space of constant curvature;
2. the Riemannian product of a space of non-zero constant curvature κ and a space of constant curvature $-\kappa$;
3. the Riemannian product of a space of non-zero constant curvature κ and a one-dimensional space.

As his proof uses only curvature homogeneity, this classification is also valid for conformally flat curvature homogeneous spaces. (See also [17].) We check which of the above three classes in the classification are possible in our present situation.

1. A space of constant curvature corresponds to the solution $\lambda = \mu$ in (12).
2. Suppose (M^n, g) is locally a product of the form $M^n = M^k(\kappa) \times M^{n-k}(-\kappa)$, where $M^k(\kappa)$ is a k-dimensional space of constant curvature κ, $\kappa \neq 0$. Then from (12) it follows $n = 2k$.
3. Suppose (M^n, g) is locally a product of the form $M^n = M^{n-1}(\kappa) \times \mathbb{R}$, $\kappa \neq 0$. Then (12) gives the contradiction $(n-1)(n-2) = 0$.

We have proved

Proposition 5.2. *Let (M^n, g), $n > 4$, be conformally flat. Then (T_1M, g_S) has constant scalar curvature if and only if either (M, g) has constant curvature, or if n is even, say $n = 2k$, and (M^n, g) is locally isometric to the product $M^k(\kappa) \times M^k(-\kappa)$, $\kappa \neq 0$.*

In the case when (M^n, g) has constant curvature κ, we have $\bar{\tau} = (n-1)\big(2(n-2) + 2n\kappa - \kappa^2\big)/2$. When (M^{2k}, g) is locally isometric to $M^k(\kappa) \times M^k(-\kappa)$, we have $\bar{\tau} = (n-2)((n-1) - \kappa^2/4)$.

Remark 7. In this section we have only considered conformally flat manifolds of dimension greater than or equal to four. One could ask the question which of the three-dimensional conformally flat manifolds has a unit tangent sphere bundle with constant scalar curvature. Combining Proposition 4.2 with Takagi's result on conformally flat curvature homogeneous spaces mentioned above, we see easily that only the three-dimensional spaces of constant curvature qualify.

6. Einstein unit tangent sphere bundles

The previous sections dealt with unit tangent sphere bundles having constant scalar curvature. In the rest of this paper, we consider some further subclasses.

First, we investigate for which Riemannian manifolds their unit tangent sphere bundle with the metric g_S is Einstein, i.e., $\bar{\varrho} = \alpha\, g_S$ for some constant α.

Theorem 6.1. *(T_1M, g_S) is an Einstein manifold if and only if (M, g) is locally isometric either to a two-dimensional Euclidean space or to a two-dimensional sphere of radius* 1.

P r o o f. From the expressions (2) for the Ricci tensor it follows that (T_1M, g_S) is Einstein if and only if

$$R_{uXij}R_{uYij} = 4(\alpha - n + 2)\big(g(X,Y) - g(X,u)g(Y,u)\big), \tag{13}$$

$$(\nabla_u \varrho)(X,Y) = (\nabla_X \varrho)(u,Y), \tag{14}$$

$$R_{uiXj}R_{uiYj} = 2\varrho(X,Y) - 2\alpha g(X,Y) \tag{15}$$

for some constant α, all unit vectors u and all vectors X and Y tangent to M.

In (13) and (15) we put $X = Y = E_k$ and sum for $k = 1, \ldots, n$ to obtain

$$\xi(u,u) = 4(\alpha - n + 2)(n-1), \tag{16}$$

$$\xi(u,u) = 2\tau - 2n\alpha. \tag{17}$$

As an Einstein manifold of dimension greater than or equal to three has constant scalar curvature, we know from (4) that $\xi(u,u) = |R|^2/n$. Eliminating α from these two equations, we obtain the identity

$$(3n-2)|R|^2 = -4n^2(n-1)(n-2) + 4n(n-1)\tau. \tag{18}$$

Next, put $X = Y = u$ in (15) and integrate over $u \in S^{n-1}(1)$ in T_xM (see the formulas in [11]). This gives

$$\frac{1}{n(n+2)}\Big(\frac{3}{2}\,|R|^2 + |\varrho|^2\Big) = \frac{2\tau}{n} - 2\alpha. \tag{19}$$

We eliminate α from (17) and (19) to find a second identity:

$$(n-4)|R|^2 + 2n|\varrho|^2 = 0. \tag{20}$$

For $n \geqslant 4$, $|\varrho|^2 = 0$ and hence also $\tau = 0$. But then (18) gives a contradiction.

For $n = 3$, (20) reads $|R|^2 = 6|\varrho|^2$. On the other hand, we always have $|R|^2 = 4|\varrho|^2 - \tau^2$ for a three-dimensional manifold. Hence, $\tau = |\varrho|^2 = |R|^2 = 0$ and again (18) cannot hold.

For $n = 2$, $R = (\kappa/2)\, g \owedge g$ and $|R|^2 = 4\kappa^2$, $|\varrho|^2 = 2\kappa^2$, $\tau = 2\kappa$. The only solutions to (18) and (20) are $\kappa = 0$ and $\kappa = 1$.

Conversely, if (M, g) is locally isometric to a two-dimensional Euclidean space, then (T_1M, g_S) is locally flat. If (M, g) is locally isometric to a two-dimensional sphere of radius 1, then (T_1M, g_S) has constant curvature $1/4$ ([8]). In both cases the unit tangent sphere bundle is an Einstein manifold. $\square$

7. Ricci-parallel unit tangent sphere bundles

Next, we consider the case when the unit tangent sphere bundle (T_1M, g_S) is Ricci-parallel, i.e., $\overline{\nabla}\bar{\varrho} = 0$. Reverting to index notation and adopting the Einstein summation convention, we have the following necessary and sufficient conditions, coming from (3):

$$R_{ZXij}R_{uYij} + R_{uXij}R_{ZYij} = 0, \qquad X, Y, Z \perp u, \tag{21}$$

$$2(\nabla_Z\varrho_{XY} - \nabla_X\varrho_{YZ}) - R_{uZYi}(\nabla_u\varrho_{Xi} - \nabla_X\varrho_{ui}) = 0, \qquad X, Z \perp u, \tag{22}$$

$$\begin{aligned} &R_{uZXj}R_{uijk}R_{uiYk} + R_{uZYj}R_{uijk}R_{uiXk} \\ &\quad - 2(R_{uZYi}\varrho_{Xi} + R_{uZXi}\varrho_{Yi}) \\ &\quad - 2(R_{ZiXj}R_{uiYj} + R_{uiXj}R_{ZiYj}) = 0, \qquad Z \perp u, \end{aligned} \tag{23}$$

$$\begin{aligned} &\nabla_Z R_{uXij}R_{uYij} + R_{uXij}\nabla_Z R_{uYij} \\ &\quad - R_{uXZi}(\nabla_u\varrho_{iY} - \nabla_Y\varrho_{iu}) - R_{uYZi}(\nabla_u\varrho_{iX} - \nabla_X\varrho_{iu}) = 0, \end{aligned} \tag{24}$$

$$\begin{aligned} &4(n-2)R_{ZYuX} - 4R_{uXZi}\varrho_{iY} \\ &\quad + 4(\nabla^2_{Zu}\varrho_{XY} - \nabla^2_{ZX}\varrho_{uY}) \\ &\quad + 2R_{uXZj}R_{uijk}R_{uiYk} + R_{ZYuj}R_{ujik}R_{uXik} = 0, \end{aligned} \tag{25}$$

$$\begin{aligned} &4\nabla_Z\varrho_{XY} - 2\nabla_Z R_{uiXj}R_{uiYj} - 2R_{uiXj}\nabla_Z R_{uiYj} \\ &\quad + R_{ZXui}(\nabla_u\varrho_{iY} - \nabla_i\varrho_{uY}) + R_{ZYui}(\nabla_u\varrho_{iX} - \nabla_i\varrho_{uX}) = 0. \end{aligned} \tag{26}$$

Here, u is an arbitrary unit vector, X, Y and Z are arbitrary vectors and the indices i, j, k stand for vectors of an orthonormal basis $\{E_1, \dots, E_n\}$.

Lemma 7.1. *Both τ and $|R|^2$ are constant on the manifold (M, g).*

Proof. As (T_1M, g_S) has constant scalar curvature, we know that the linear combination $4n\tau - |R|^2$ is constant and that $\xi(X, Y) = R_{Xijk}R_{Yijk} = \frac{|R|^2}{n}\, g(X, Y)$. We use this last fact implicitly in all the calculations in this section.

We only need condition (22) to prove the lemma. We note beforehand that by the second Bianchi identity we have

$$\nabla_X \varrho_{YZ} - \nabla_Y \varrho_{XZ} = \nabla_i R_{XYZi}.$$

Using this, we rewrite (22) in the form

$$\begin{aligned} &2(\nabla_Z \varrho_{XY} - \nabla_X \varrho_{YZ}) - R_{uZYi}\nabla_j R_{uXij} \\ &- 2g(u,Z)(\nabla_u \varrho_{XY} - \nabla_X \varrho_{uY}) - 2g(u,X)(\nabla_Z \varrho_{uY} - \nabla_u \varrho_{YZ}) = 0 \end{aligned}$$

which holds for *all* vectors X, Y and Z tangent to M. Integrating this formula over $u \in S^{n-1}(1)$ (using again the formulas in [11]), we find

$$2(n-2)(\nabla_Z \varrho_{XY} - \nabla_X \varrho_{YZ}) - R_{kZYi}\nabla_j R_{kXij} = 0. \tag{27}$$

First, put $X = Z = E_\ell$ and sum over $\ell = 1, \dots, n$. This gives $R_{k\ell Yi}\nabla_j R_{k\ell ij} = 0$ or, after some reworking (using the second Bianchi identity),

$$(n-4)\nabla_Y |R|^2 = 0. \tag{28}$$

Next, we take $X = Y = E_\ell$ in (27) and sum over $\ell = 1, \dots, n$ to obtain

$$(n-2)\nabla_Z \tau - R_{kZ\ell i}\nabla_j R_{k\ell ij} = 0. \tag{29}$$

We rewrite the last term on the left hand side: start by differentiating the equality $R_{kZ\ell i}R_{k\ell ij} = -(|R|^2/2n)\, g(Z,j)$ with respect to E_j and sum over $j = 1, \dots, n$. Then use (4), the first of the identities (30) below and the second Bianchi identity to find $R_{kZ\ell i}\nabla_j R_{k\ell ij} = ((n-4)/8n)\,\nabla_Z |R|^2$. By (28), this is zero. So, we have $(n-4)\nabla_Y |R|^2 = (n-2)\nabla_Z \tau = 0$ and the lemma follows for all dimensions from the constancy of $4n\tau - |R|^2$. □

Next, we derive identities between curvature invariants of order two, four and six from the conditions (23) and (25). The ones of order six that show up are $\langle \varrho, \dot{R} \rangle$, $\langle \varrho \otimes \varrho, \overline{R} \rangle$, $\check{R}$ and $\check{\overline{R}}$. With our sign convention for R, they have the expressions

$$\begin{aligned} \langle \varrho, \dot{R} \rangle &= \varrho_{ij} R_{ipqr} R_{jpqr}, \\ \langle \varrho \otimes \varrho, \overline{R} \rangle &= -\varrho_{ij}\varrho_{k\ell} R_{ikj\ell}, \\ \check{R} &= -R_{ijk\ell} R_{k\ell pq} R_{pqij}, \\ \check{\overline{R}} &= -R_{ikj\ell} R_{kp\ell q} R_{piqj}. \end{aligned}$$

We also need the identities (see [14] for more details)

$$
\begin{aligned}
R_{ijkp}R_{ikjq} &= \frac{1}{2}\xi_{pq},\\
R_{ijk\ell}R_{k\ell pq}R_{piqj} &= -\frac{1}{2}\check{R},\\
R_{ijk\ell}R_{kp\ell q}R_{piqj} &= -\frac{1}{4}\check{R},\\
R_{ijk\ell}R_{jp\ell q}R_{pkqi} &= -\check{\overline{R}} + \frac{1}{4}\check{R},\\
\varrho_{ij}R_{ikpq}R_{jpkq} &= \frac{1}{2}\langle \varrho, \dot{R}\rangle.
\end{aligned}
\tag{30}
$$

We do not go into detail as concerns the computations and only indicate how we obtain the identities between the curvature invariants.

First, we rewrite (23) in a form which is valid for *all* vectors Z:

$$
\begin{aligned}
&R_{uZXj}R_{uijk}R_{uiYk} + R_{uZYj}R_{uijk}R_{uiXk}\\
&- 2(R_{uZYi}\varrho_{Xi} + R_{uZXi}\varrho_{Yi}) - 2(R_{ZiXj}R_{uiYj} + R_{uiXj}R_{ZiYj})\\
&+ 4g(u,Z)R_{uiXj}R_{uiYj} = 0.
\end{aligned}
$$

Then we take $X = u$ and integrate over $u \in S^{n-1}(1)$ in T_xM. Next, we put $Y = Z = E_q$ and sum over $q = 1, \ldots, n$. In this way we find

$$
\langle \varrho \otimes \varrho, \overline{R}\rangle + \check{\overline{R}} - \frac{1}{2}\check{R} + \frac{1}{2}\langle \varrho, \dot{R}\rangle - 2n|\varrho|^2 - (n-4)|R|^2 = 0. \tag{31}
$$

In (25), we first take $Y = u$ and integrate, followed by a summation over $X = Z = E_q$ to obtain

$$
4(n+2)(n-2)\tau - 4(n+2)|\varrho|^2 + 4\langle \varrho, \dot{R}\rangle + 2\langle \varrho \otimes \varrho, \overline{R}\rangle + \frac{3}{2}\check{R} = 0. \tag{32}
$$

Also in (25), we take $Z = u$ and integrate, followed by a summation over $X = Y = E_q$. We get

$$
-4(n+2)(n-2)\tau + 4(n+2)|\varrho|^2 - 3\langle \varrho, \dot{R}\rangle + 2\check{\overline{R}} - \frac{5}{2}\check{R} = 0. \tag{33}
$$

If we now sum equations (32) and (33) and subtract (31) twice, we have the simple relation

$$
(n-4)|R|^2 + 2n|\varrho|^2 = 0. \tag{34}
$$

If $n > 4$, the only solutions to (34) are $|R|^2 = |\varrho|^2 = 0$ and the manifold (M, g) is flat. But then (T_1M, g_S) is locally isometric to $\mathbb{R}^n \times S^{n-1}$, which is locally symmetric, hence Ricci-parallel.

If $n = 4$, $|\varrho|^2 = 0$ by (34), hence also $\tau = 0$. (M, g) is Ricci-flat. In particular, we can use a Singer-Thorpe orthonormal basis $\{E_1, \ldots, E_4\}$ such that

$$\begin{aligned}
&R_{1212} = R_{3434} = \lambda_1, \qquad && R_{1234} = \mu_1, \\
&R_{1313} = R_{2424} = \lambda_2, \qquad && R_{1342} = \mu_2, \\
&R_{2323} = R_{1414} = \lambda_3, \qquad && R_{1423} = \mu_3, \\
&R_{ijkl} = 0 \quad \text{otherwise.}
\end{aligned}$$

(See [2] and [24].) As the Ricci tensor ϱ vanishes in the present case, we have $\lambda_1 + \lambda_2 + \lambda_3 = 0$. By the first Bianchi identity, also $\mu_1 + \mu_2 + \mu_3 = 0$ holds. Take $u = \cos\theta\, E_1 + \sin\theta\, E_2$ and express condition (21) for arbitrary X, Y and Z perpendicular to u. The algebraic equations one obtains for λ_i and μ_i, together with the analogues corresponding to similar choices of u, have only the zero solution. So again, only the flat space is a solution.

If $n = 3$, it always holds $|R|^2 = 4|\varrho|^2 - \tau^2$. Combined with (34), we see that the flat space is the only solution also in this dimension.

Finally, if $n = 2$, an easy calculation shows that the spaces of constant curvature zero or one are the only solutions to the equations (31)–(33). Conversely, their unit tangent sphere bundles also have constant curvature, hence are Ricci-parallel.

We have proved

Theorem 7.2. (T_1M, g_S) *has parallel Ricci curvature tensor if and only if* (M, g) *is either flat or is a surface of constant curvature* 1.

Corollary 7.3. ([5]) (T_1M, g_S) *is locally symmetric if and only if* (M, g) *is either flat or is a surface of constant curvature* 1.

Remark 8. Blair's proof of this last result uses the natural contact metric structure of (T_1M, g_S) in an essential way. Our proof is more basic in that it uses only curvature information.

Remark 9. Our formulas may also be used to determine for which Riemannian manifolds (M, g) the corresponding unit tangent sphere bundle (T_1M, g_S) is conformally flat. As in [6], we find that this is the case if and only if (M, g) is a surface of constant curvature 0 or 1.

References

[1] *J. Berndt, F. Tricerri and L. Vanhecke*: Generalized Heisenberg groups and Damek-Ricci harmonic spaces. Lecture Notes in Math. 1598, Springer-Verlag, Berlin, Heidelberg, New York, 1995.

[2] *A. L. Besse*: Manifolds all of whose geodesics are closed. Ergeb. Math. Grenzgeb. 93, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[3] *A. L. Besse*: Einstein manifolds. Ergeb. Math. Grenzgeb. 3. Folge 10, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[4] *D. E. Blair*: Contact manifolds in Riemannian geometry. Lecture Notes in Math. 509, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[5] *D. E. Blair*: When is the tangent sphere bundle locally symmetric? Geometry and Topology. World Scientific, Singapore, 1989, pp. 15–30.

[6] *D. E. Blair and T. Koufogiorgos*: When is the tangent sphere bundle conformally flat? J. Geom. *49* (1994), 55–66.

[7] *E. Boeckx, O. Kowalski and L. Vanhecke*: Riemannian manifolds of conullity two. World Scientific, Singapore, 1996.

[8] *E. Boeckx and L. Vanhecke*: Characteristic reflections on unit tangent sphere bundles. Houston J. Math. *23* (1997), 427–448.

[9] *E. Boeckx and L. Vanhecke*: Curvature homogeneous unit tangent sphere bundles. Publ. Math. Debrecen *53* (1998), 389–413.

[10] *P. Bueken*: Three-dimensional Riemannian manifolds with constant principal Ricci curvatures $\varrho_1 = \varrho_2 \neq \varrho_3$. J. Math. Phys. *37* (1996), 4062–4075.

[11] *B.-Y. Chen and L. Vanhecke*: Differential geometry of geodesic spheres. J. Reine Angew. Math. *325* (1981), 28–67.

[12] *P. Gilkey, A. Swann and L. Vanhecke*: Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator. Quart. J. Math. Oxford *46* (1995), 299–320.

[13] *A. Gray*: Einstein-like manifolds which are not Einstein. Geom. Dedicata *7* (1978), 259–280.

[14] *A. Gray and L. Vanhecke*: Riemannian geometry as determined by the volumes of small geodesic balls. Acta Math. *142* (1979), 157–198.

[15] *A. Gray and T. J. Willmore*: Mean-value theorems for Riemannian manifolds. Proc. Roy. Soc. Edinburgh Sect. A *92* (1982), 343–364.

[16] *S. Ivanov and I. Petrova*: Riemannian manifolds in which certain curvature operator has constant eigenvalues along each circle. Ann. Global Anal. Geom. *15* (1997), 157–171.

[17] *O. Kowalski*: A note to a theorem by K. Sekigawa. Comment. Math. Univ. Carolin. *30* (1989), 85–88.

[18] *O. Kowalski*: A classification of Riemannian 3-manifolds with constant principal Ricci curvatures $\varrho_1 = \varrho_2 \neq \varrho_3$. Nagoya Math. J. *132* (1993), 1–36.

[19] *O. Kowalski*: An explicit classification of 3-dimensional Riemannian spaces satisfying $R(X,Y) \cdot R = 0$. Czechoslovak Math. J. *46* (1996), 427–474.

[20] *J. W. Milnor*: Curvatures of left invariant metrics on Lie groups. Adv. Math. *21* (1976), 293–329.

[21] *E. Musso and F. Tricerri*: Riemannian metrics on tangent bundles. Ann. Mat. Pura Appl. *150* (1988), 1–20.

[22] *K. Sekigawa and L. Vanhecke*: Volume preserving geodesic symmetries on four-dimensional Kähler manifolds. Differential Geometry Peñiscola, 1985, Proceedings (A. M. Naveira, A. Ferrández and F. Mascaró, eds.). Lecture Notes in Math. 1209, Springer, pp. 275–290.

[23] *I. M. Singer*: Infinitesimally homogeneous spaces. Comm. Pure Appl. Math. *13* (1960), 685–697.

[24] *I. M. Singer and J. A. Thorpe*: The curvature of 4-dimensional Einstein spaces. Global Analysis. Papers in honor of K. Kodaira. Princeton University Press, Princeton, 1969, pp. 355–365.
[25] *Z. I. Szabó*: Structure theorems on Riemannian manifolds satisfying $R(X, Y) \cdot R = 0$, I, Local version. J. Differential Geom. *17* (1982), 531–582.
[26] *H. Takagi*: Conformally flat Riemannian manifolds admitting a transitive group of isometries. Tôhoku Math. J. *27* (1975), 103–110.
[27] *A. Tomassini*: Curvature homogeneous metrics on principal fibre bundles. Ann. Mat. Pura Appl. *172* (1997), 287–295.
[28] *A. L. Yampol'skii*: The curvature of the Sasaki metric of tangent sphere bundles (Russian). Ukrain. Geom. Sb. *28* (1985), 132–145; English translation. J. Sov. Math. *48* (1990), 108–117.

Authors' address: Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium, e-mail: `eric.boeckx@wis.kuleuven.ac.be`, `lieven.vanhecke@wis.kuleuven.ac.be`.