UNITARITY AND CAUSALITY IN A RENORMALIZABLE FIELD THEORY WITH UNSTABLE PARTICLES

M. Veltman

CERN, Geneva

ABSTRACT

The problems of unitarity, causality and renormalizability are treated in a field theory containing an unstable particle. Perturbation theory is suitably modified and leads to an implicit equation for complete propagators. The S matrix constructed with these propagators and connecting stable particle states only is shown to be unitary, renormalizable and causal. It is also shown to give rise to interpolating Heisenberg fields which verify the original field equations.

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1. Introduction.

In recent years many authors discussed unstable particles in the framework of quantum field theory. In particular, Matthews and Salam 1) gave suitable definitions of mass and lifetime of an unstable particle in terms of its field theoretical propagator. As shown by Jacob and Sachs 2) these definitions are in agreement with the experimental situation.

In the present paper we study another aspect of a field theory with unstable particles, namely the questions of unitarity, causality and renormalization in perturbation theory. To be more explicit suppose that we have a situation where an unstable scalar particle, say A-particle, can decay into two identical stable scalar particles, say φ-particles. In setting up perturbation theory for such a model one starts by introducing the "bare"fields A and φ, obeying the Klein-Gordon equation and coupled to each other in some way specified by an interaction Lagrangian. Ordinary perturbation theory leads then, however, to a very undesirable feature, namely the unstable A-particles appear at infinite times in incoming and outgoing states. A realistic theory cannot have this feature, but if one just removes the unstable particle states from the in- and out states one is faced with the problem of unitarity of the resulting truncated S-matrix. The problem can now be stated as follows: consider the Hilbert space of stable particle states. Is it then possible to construct by suitable modification of perturbation theory an S-matrix which is unitary in this Hilbert space. The answer is yes, and the present paper gives the necessary modification of perturbation theory. Of course, such an S-matrix still has to meet in addition the requirements of renormalizability and causality (where one can use for the latter for instance the definitions of Lehmann-Symanzik-Zimmermann 3) or Bogoliubov-Shirkov-Medvedev-Polivanov 4)), but it will be shown that in this respect no particular difficulties arise in the present case as compared with a theory describing stable particles only.

There is still a further point of interest in connection with the above mentioned S-matrix. From the fact that no unstable particles appear in ingoing or outgoing states one expects that it should be possible to reformulate the theory in terms of the stable field operators only, and indeed this will turn out to be the case. Such a treatment, however, leads to a Lagrangian density involving fields at different space time points (non-local interaction) and again in setting up perturbation theory in this new framework one encounters difficulties with respect to unitarity 5). Clearly our treatment of the unstable particle case may then be of help in understanding non-local theories, in the sense that we have a concrete model of a theory with non-local interaction, which, in a non-trivial way, satisfies the requirements of unitarity, causality and renormalizability.

Our method will now be as follows: for a simple model of unstable A-particles and stable φ-particles we construct an S-matrix which can be expected to satisfy the requirements formulated above. This S-matrix will be given in terms of diagrams, with propagators which satisfy the Lehmann-representation 6) and a certain implicit equation derived in Section 3. It will be shown then that this S-matrix is unitary in the Hilbert space of stable particle states, that it is a solution of the field equations of the model, and finally that it is causal in the sense of Bogoliubov. Throughout the work renormalisation is taken care of.

In Section 2 we derive a general identity valid for diagrams with propagators obeying the Lehmann representation. In Section 3 the model mentioned above is introduced explicitly, and the difficulties resulting in a perturbation expansion from the unstability of the A-particle are exhibited. As a result perturbation theory is reformulated, leading to an implicit equation for the propagators involved. Next an S-matrix involving stable particles only is constructed in Section 4 and its unitarity properties are established with the help of the identity derived in Section 2. Section 5 gives the connection of this S-matrix with the field equations of the model. Finally, in Section 6, using the techniques of Section 2 and 5, one shows that causality holds in the sense of Bogoliubov.

2. <u>The cutting formula.</u>

This formula describes certain properties of diagrams arising from the properties of field propagators. The complete propagator associated with a field $\varphi(x)$ (which for simplicity we suppose to be a real scalar field) is defined by:

$$\tilde{\Delta}_F(x) = \theta(x_o)\tilde{\Delta}^+(x) + \theta(-x_o)\tilde{\Delta}^-(x) \qquad (2.1)$$

$$\tilde{\Delta}^+(x) = i\langle o|\varphi(x)\varphi(o)|o\rangle \qquad \tilde{\Delta}^-(x) = i\langle o|\varphi(o)\varphi(x)|o\rangle$$

$$\tilde{\Delta}^-(x) = -(\tilde{\Delta}^+(x))^* \qquad \theta(u) = \begin{cases} 1 & u > o \\ \frac{1}{2} & u = o \\ o & u < o \end{cases}$$

Inserting intermediate states in the expressions for $\tilde{\Delta}^+$ one observes that $\tilde{\Delta}^+$ is a positive frequency function with respect to time and consequently $\tilde{\Delta}^-$ a negative frequency function. Following Lehmann 6) we write (we take the metric so that $kx = \vec{k}\vec{x} - k_o x_o$):

$$\tilde{\Delta}^{\pm}(x) = \frac{i}{(2\pi)^3}\int d_4k\; e^{ikx}\theta(\pm k_o)\int_o^\infty d\kappa^2\delta(k^2+\kappa^2)\rho(-\kappa^2) \qquad (2.2)$$

with $\rho(-\kappa^2) \geqslant o$.

Using (2.1), (2.2) and some structure properties of diagrams we will be able to prove the cutting formula. In its simplest form, for a diagram consisting of two points x and x' and one propagator, it reduces to

$$\tilde{\Delta}_F(x-x') - \tilde{\Delta}^+(x-x') - \tilde{\Delta}^-(x-x') - \tilde{\Delta}^*_F(x-x') = 0$$

or:

$$2i\mathrm{Im}.\tilde{\Delta}_F(x-x') = \tilde{\Delta}^+(x-x') + \tilde{\Delta}^-(x-x')$$

which clearly is an immediate consequence of (2.1). In much the same way for a general diagram the cutting formula gives an expression for the imaginary part of the diagram. It will be used later on to prove unitarity. Further it will give some information concerning the space-time behaviour of diagrams which we will show to be related to causality.

We will start now by proving the cutting formula without explicit reference to diagrams. In order to simplify matters we leave the space variables aside and in the rest of this paragraph x will stand for time only.

Let there be given a number of functions*) $f_{\underline{l}}^{+}(x)$, $f_{\underline{l}}^{-}(x)$ so that $(f_{\underline{l}}^{+}(x))^{*} = - f_{\underline{l}}^{-}(x)$, and so that $f_{\underline{l}}^{+}(x)$ contains only positive frequencies while $f_{\underline{l}}^{-}(x)$ contains only negative ones i.e. they can be written in the form:

$$f_{\underline{l}}^{\pm}(x) = \int e^{-ipx}\Theta(\pm p)f_{\underline{l}}^{\pm}(p)dp \qquad (2.3)$$

This is clearly compatible with the reality requirement above. We define functions $f_{\underline{l}}(x)$ by:

$$f_{\underline{l}}(x) = \Theta(x)f_{\underline{l}}^{+}(x) + \Theta(-x)f_{\underline{l}}^{-}(x)$$

so that (2.4)

$$f_{\underline{l}}^{*}(x) = - \Theta(x)f_{\underline{l}}^{-}(x) - \Theta(-x)f_{\underline{l}}^{+}(x)$$

The analogy of these equations with (2.1), (2.2) is obvious. For $x = o$ we have $f_{\underline{l}}(o) = - f_{\underline{l}}^{*}(o)$. The inversion formulae of (2.4) are

$$f_{\underline{l}}^{+}(x) = \Theta(x)f_{\underline{l}}(x) - \Theta(-x)f_{\underline{l}}^{*}(x)$$

(2.5)

$$f_{\underline{l}}^{-}(x) = - \Theta(x)f_{\underline{l}}^{*}(x) + \Theta(-x)f_{\underline{l}}(x)$$

We now consider a function F of n variables $x_1 \ldots x_n$, which can be written as a product of $f_{\underline{l}}(x)$ functions, with as arguments of these functions only differences $x_i - x_j$:

$$F(x_1 \ldots x_n) = \prod_r f_{\underline{l}_r}(x_{\alpha_{1r}} - x_{\alpha_{2r}}) \qquad (2.6)$$

Any function $f_{\underline{l}}$ can appear several times. As for the variables $x_1 \ldots x_n$ we assume that each appears at least once and that $\alpha_{1r} \neq \alpha_{2r}$ for each r.

*) Whenever confusion can arise the character ℓ is denoted by $\underline{l}$.

We will now replace in a systematic way in $F(x_1 \ldots x_n)$ functions f by f^+, f^- or f^*. By definition the function

$$F_m^+(x_1 \ldots x_{i_1-1}\ x_{i_1+1} \ldots\ x_{i_m-1}\ x_{i_m+1} \ldots x_n | x_{i_1} \ldots x_{i_m}) \qquad (2.7)$$

is constructed from $F(x_1 \ldots x_n)$ by replacing a function $f_{\underline{1}_r}(x_{\alpha_{1r}} - x_{\alpha_{2r}})$ by

$f^+_{\underline{1}_r}(x_{\alpha_{1r}} - x_{\alpha_{2r}})$ if $x_{\alpha_{1r}}$ appears on the left and $x_{\alpha_{2r}}$ on the right of the bar

$f^-_{\underline{1}_r}(x_{\alpha_{1r}} - x_{\alpha_{2r}})$ if $x_{\alpha_{1r}}$ appears on the right and $x_{\alpha_{2r}}$ on the left of the bar

$- f^*_{\underline{1}_r}(x_{\alpha_{1r}} - x_{\alpha_{2r}})$ if $x_{\alpha_{1r}}$ and $x_{\alpha_{2r}}$ appear on the right of the bar

If both $x_{\alpha_{1r}}$ and $x_{\alpha_{2r}}$ appear on the left hand side of the bar we leave the function $f_{\underline{1}_r}(x_{\alpha_{1r}} - x_{\alpha_{2r}})$ unchanged. Suppose now that x_a is smaller than or equal to any of the other x. One sees very simply with the help of (2.4) or (2.5) that

$$F_m^+(x_1 \ldots x_{i_1-1}\ x_{i_1+1} \ldots x_a \ldots x_n | x_{i_1} \ldots x_{i_m}) = \qquad (2.8)$$

$$F_{m+1}^+(x_1 \ldots x_{i_1-1}\ x_{i_1+1} \ldots x_n | x_a\ x_{i_1} \ldots x_{i_m}) \qquad x_a \leqq x_1 \ldots x_n$$

In words: the functions F_m^+ do not change if the smallest x_a of all x is moved from the right to the left of the bar and vice versa. Take now a subset $x_{j_1} \ldots x_{j_p}$ containing x_a. We write for brevity

$$x_{j_q} = y_q$$

From (2.8) follows

$$\sum_{q=0}^{p} \sum_{\{q\}} (-1)^q F_{m+q}^+(x_1 \ldots y_1 \ldots x_{i_1-1}\ x_{i_1+1} \ldots y_{s_1-1}\ y_{s_1+1} \ldots$$

$$\ldots y_p \ldots x_n | x_{i_1} \ldots x_{i_m}\ y_{s_1} \ldots y_{s_q}) = 0 \qquad (2.9)$$

$\{q\}$ means summation over all choices of q numbers $s_1 \ldots s_q$ out of $1 \ldots p$ with $s_1 < s_2 \ldots < s_q$. The functions F_i^+ in this equation differ only in the position of the y with respect to the bar. The result follows by observing that each function F_i^+ with the smallest variable to the left of the bar cancels a function F_{i+1}^+ with the smallest variable to the right of the bar.

Suppose now that (2.9) is integrated over a number of variables from - to + infinity. (2.9) can be true in the whole region of integration only if the subset $y_1 = x_{j_1} \ldots y_p = x_{j_p}$ contains all integration variables as well as the smallest (or one of the smallest) of the variables which are not integrated over. In this way we arrive at the <u>cutting formula</u>:

$$\int dy_{\alpha_1} \ldots dy_{\alpha_\nu} \sum_{q=0}^{p} \sum_{\{q\}} (-1)^q F^+_{m+q}(x_1 \ldots y_1 \ldots x_{i_1 -1}\; x_{i_1+1} \ldots$$
$$\ldots y_{s_1 -1}\; y_{s_1+1} \ldots y_p \ldots x_n \mid x_{i_1} \ldots x_{i_m}\; y_{s_1} \ldots y_{s_q}) = 0 \qquad (2.10)$$

It holds if there exist an i with $i \neq \alpha_1 \ldots \alpha_\nu$ such that

$$y_i \leq x_i \text{ for all } i \neq j_1, \ldots j_p$$

One easily proves a similar formula for a function F^-, defined from F^+ by interchange of f^+ and f^- functions, if one replaces $\leq$ by $\geq$ in the inequalities just given for the y_i.

Up to now we have not made use of the fact that f^+ and f^- contain only positive or negative frequencies (see (2.3)). We note that due to (2.3) many terms in (2.10) actually vanish. This can be understood by introducing (2.3) into (2.10) and performing the integrations over the integration variables y_{α_i} (of course also for the f and f^* functions fourier representations must be inserted). Each integration gives rise to a δ-function of an (algebraic) sum of variables p_β. Every f^+ or f^- function containing a y_{α_i} gives rise to a $\Theta(p_\beta)$ or $\Theta(-p_\beta)$, and it may well happen that the product of these Θ functions and the δ function is always zero. This happens for instance for the product $\delta(p_1+p_2)\Theta(p_1)\Theta(p_2)$. As can easily be established in general with the definition given above for the F^+ functions, a term in (2.10) will be zero if from the set of integration variables $y_{\alpha_1}\dots y_{\alpha_\nu}$ a subset $y_{\beta_1}\dots y_{\beta_\mu}$ can be found which is entirely situated on one side of the bar and such that every function depending on the difference of one of the y_β and one of the other variables (y as well as x) is a $f_{\underline{1}}^+$ or $f_{\underline{1}}^-$.

We will now picture (2.10) with the help of diagrams (consisting of lines and vertices). The functions F can be pictured in the normal way: any variable x corresponds with a vertex (integration variables correspond to what will be called "internal vertices", whereas any other vertices will eventually be end-point of an external line, reason why we will call them "external vertices") and any function $f_{\underline{1}}(x_j-x_i)$ with a line connecting x_i and x_j. For simplicity we assume that the functions $f_{\underline{1}}(x)$ are symmetrical with respect to the change $x \to -x$, otherwise an arrow would have to be attached to all lines. This symmetry implies $f_{\underline{1}}^+(-x) = f_{\underline{1}}^-(x)$. It holds for the propagators (2.1), (2.2). For the functions F_m^+ we use the same diagrams with the difference that the vertices that appear in F_m^+ to the right of the bar are marked with a small circle. We henceforth will speak accordingly of marked and unmarked vertices. A line between a marked and an unmarked vertex corresponds to a f^+ or f^- function, whereas a line between two marked vertices corresponds to a $-f^*$ function. Lines between unmarked vertices are f functions. We now remark that the functions f^+ and f^- can always be written in the form $f^+(+x) = f^-(-x)$ with $x = z - y$ where z is marked and y unmarked. One sees now that if, as is usually done, we define energy as the variable associated with x in fourierspace, viz.p in (2.3), it will always be positive.

Equation (2.10) now gives a relation between a set of diagrams which differ from each other by some of the unmarked vertices being replaced by marked ones and vice versa. As a result of energy conservation (expressed by the aforementioned appearance of δ-functions) all diagrams having a set of marked internal vertices that is connected with the rest of the diagram through lines representing f^+ or f^- functions only are zero. This leads to a new method of picturing the terms F^+: any term F^+ is represented by a diagram in which a number of lines are cut by a line (called the "cut") that is shaded on one side. All vertices on the shaded side of the cut are to be considered as marked vertices, all vertices on the unshaded side as unmarked vertices. Or: lines intersected by the cut represent f^+ or f^- functions, lines on the shaded side $-f^*$ functions, lines on the unshaded side f functions. In going from unshaded to shaded region energy has to be positive. A simple example of identity (2.10) is represented in fig.1. In fig.2 we see how this can be pictured in a simple way for a diagram with 3 external vertices and an unspecified number of internal vertices.

Let us now consider (2.10) for some special cases. A very important case is that where the subset $x_{j_1} \ldots x_{j_p}$ $(= y_1 \ldots y_p)$ contains all x. In (2.10) the term with q = o is exactly the original function F from (2.7), while the term with q = n is up to a sign the complex conjugate of F. One finds then that the real or imaginary part of a diagram equals the sum of all cut diagrams (with the appropriate signs). This will be the form in which the cutting formula is best suited to prove unitarity. Another important case is provided when the subset $x_{j_1} \ldots x_{j_p}$ contains all x except one external vertex x_b, and when one of the $x_{j_1} \ldots x_{j_p}$ is an external vertex x_a with $x_a \lesssim x_b$.

One has then an equation stating that the sum of all cut diagrams with x_a and x_b unmarked equals minus the sum of all cut diagrams with x_b unmarked and x_a marked. In this form the cutting formula will be used in connection with causality.

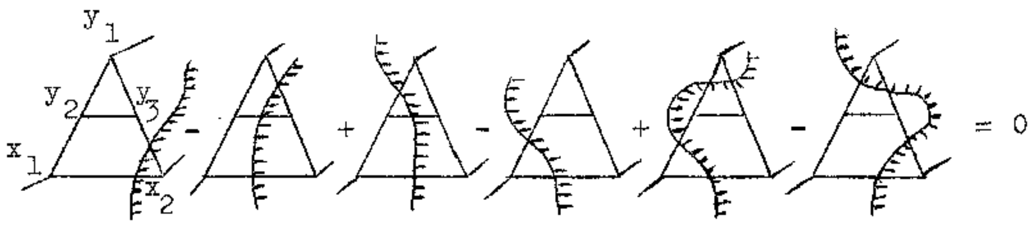


Fig.1.

y_2 and y_3 are internal, x_1, x_2 and y_1 external vertices. $y_1 \leq x_1$ and x_2. Note that x_1 is unmarked and x_2 marked in each term of the identity; the marking of the y's changes from term to term.

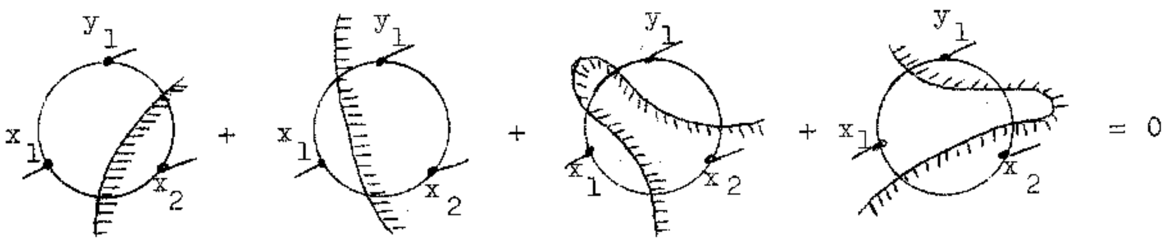


Fig.2.

$y_1 \leq x_1$, x_2. The circle stands for a general diagram, and we think all vertices and lines located inside the circle. Only the external vertices are indicated. The first diagram stands for all diagrams with x_1 and y_1 unmarked, x_2 marked, and such that one can go from x_1 to y_1 along lines and vertices without crossing the cut. The second diagram has y_1 and x_2 marked with a path from x_2 to y_1 that does not cross the cut. The third and fourth diagram have y_1 unmarked and marked respectively and each path connecting any two vertices out of y_1, x_1, x_2 crosses the cut.

3. Implicit equation for propagators.

We will now discuss the propagators of a field theory with unstable particles in somewhat more detail. To this purpose we introduce a model, namely a field theory with two kinds of particles, uncharged scalar bosons for simplicity, with an interaction Lagrangian density:

$$L_I = \frac{g}{2}(\varphi^2(x)A(x) + A(x)\varphi^2(x)) \qquad (3.1)$$

The physical masses M of the A-field and m of the φ-field shall be such that the A-field be unstable, i.e.:

$$M > 2m \qquad (3.2)$$

We now note that for $g = o$ the A-particle can be expected to exist at all times, whereas for $g \neq o$ it is unstable, which means that no A-particle will be found at infinite times, i.e. in asymptotic states. We expect therefore difficulties in the application of perturbation theory to this model, for instance with respect to unitarity of the S-matrix. Furthermore, as will be seen later in greater detail, the A-field can be eliminated from the theory whereby the Lagrangian density becomes non-local (involves fields at different space-time points) and it is well known that perturbation theory for a non-local interaction Lagrangian gives rise to difficulties with respect to unitarity 5).

Fig. 3

In the following we will see how these difficulties arise from an unjustified handling of perturbation theory. The Feynman-rules for the present model are: any diagram consists of two kinds of lines, namely φ-lines and A-lines which we shall respectively represent by solid and dotted lines. An internal line connects two vertices, an external one is connected to one vertex only. A vertex is endpoint of two φ-lines and one A-line. To a φ or A-line between the vertices x_i and x_j belongs respectively a factor (propagator):

$$i\Delta_F(x_i-x_j) = i(2\pi)^{-4}\int d_4k\ e^{ik(x_i-x_j)}(k^2+m^2-io)^{-1} \tag{3.3}$$

or a factor

$$i\Delta'_F(x_i-x_j) = i(2\pi)^{-4}\int d_4k\ e^{ik(x_i-x_j)}(k^2+M^2-io)^{-1}$$

A diagram has a factor

$$\frac{g^n i^n}{n!}$$

where n is the number of vertices in the diagram. Furthermore there are simple combinatorial factors due to the undistinguishability of particles. For instance the second order diagram of fig.3 gets a factor 2.

We now investigate the complete propagator for the A-field. To begin with we consider the lowest order selfenergy-diagram of fig.3. Neglecting terms that vanish after renormalization one finds:

$$F(x_1-x_2) = (2\pi)^{-4}\int d_4k\ e^{ik(x_1-x_2)}\Big\{(k^2+M^2)^2R_2(k^2) + \\ + i\Theta(-k^2-4m^2)I_2(k^2)\Big\} \tag{3.4}$$

In this $R_2(k^2)$ and $I_2(k^2)$ are real functions of k^2, of order g^2. This lowest order diagram can be inserted 0, 1, 2 ... times in a A-line. The fourier transform of the sum of the resulting expressions is a geometrical series with argument:

$$\frac{(k^2+M^2)^2R_2(k^2) + i\Theta(-k^2-4m^2)I_2(k^2)}{k^2+M^2-io} \tag{3.5}$$

Because of (3.2) there will be a neighbourhood of $k^2 = -M^2$ where (3.5) in absolute value is larger then unity, for $I_2(-M^2) \neq o$ for arbitrary $g \neq o$. Therefore this series will not converge and perturbation theory is not valid in this region. It is exactly this divergence of perturbation theory which is responsible for the difficulties with unstable particles and non-local field-theories.

The way whereby this difficulty can be circumvented is well-known 7).

One uses perturbation theory in the region where the series converges and uses then the analytical continuation of the result in the difficult regions. We will adopt the same technique here. The underlying idea is that single selfenergy diagrams alone are meaningless quantities, but that repeated insertion, summation and analytical continuation leads to a reasonable result. We will do this for the A and φ-particles. Consider for a certain particle all selfenergy diagrams that do not contain any further selfenergy parts in the internal lines. We will call such selfenergy diagrams simple proper diagrams (s.p.d.) for the given particle. Their fourier transforms depend on the square of the four-momentum and on the propagators Δ_F, Δ_F' of φ and A-particle respectively, see (3.3). The sum of the contributions for all s.p.d. of the φ or A-particle respectively will be denoted by

$$E(\Delta_F, \Delta_F', k^2) \qquad , \qquad E'(\Delta_F, \Delta_F', k^2)$$

E, E' are thus functionals of the functions $\Delta_F(x)$ and $\Delta_F'(x)$, or of their fourier transforms $\Delta_F(k^2)$ and $\Delta_F'(k^2)$. We will always adopt the convention

$$\Delta(x) = (2\pi)^{-4} \int e^{ikx} \Delta(k) d_4 k$$

Let us for the moment neglect renormalisation terms. In the region where perturbation theory converges we define functions $\Delta_F^{(n)}(k^2)$, $\Delta_F'^{(n)}(k^2)$ by iteration.

$$i\Delta_F^{(n)}(k^2) = i\Delta_F(k^2) + \sum_{m=1}^{\infty} (i\Delta_F(k^2))^{m+1} E^m(\Delta_F^{(n-1)}, \Delta_F'^{(n-1)}, k^2) =$$

$$= i\Delta_F(k^2)\left[1 - i\Delta_F(k^2) E(\Delta_F^{(n-1)}, \Delta_F'^{(n-1)}, k^2)\right]^{-1} \qquad (3.6)$$

$$i\Delta_F'^{(n)}(k^2) = i\Delta_F'(k^2)\left[1 - i\Delta_F'(k^2) E'(\Delta_F^{(n-1)}, \Delta_F'^{(n-1)}, k^2)\right]^{-1}$$

Here $\Delta_F^{(o)}(k^2) = \Delta_F(k^2)$, $\Delta_F'^{(o)}(k^2) = \Delta_F'(k^2)$, see (3.3). Outside the region of convergence the functions $\Delta_F^{(n)}(k^2)$, $\Delta_F'^{(n)}(k^2)$ are defined by analytical continuation of (3.6). Assuming that the limits exist we have for the complete propagators

$$\tilde{\Delta}_F(k^2) = \lim_{n\to\infty} \Delta_F^{(n)}(k^2)$$
$$\tilde{\Delta}'_F(k^2) = \lim_{n\to\infty} \Delta'^{(n)}_F(k^2) \qquad (3.7)$$

From (3.6) and (3.7) follows the implicit equations for complete propagators:

$$\tilde{\Delta}_F(k^2) = \Delta_F(k^2)\left[1 - i\Delta_F(k^2)E(\tilde{\Delta}_F,\tilde{\Delta}'_F,k^2)\right]^{-1}$$
$$\tilde{\Delta}'_F(k^2) = \Delta'_F(k^2)\left[1 - i\Delta'_F(k^2)E'(\tilde{\Delta}_F,\tilde{\Delta}'_F,k^2)\right]^{-1} \qquad (3.8)$$

Taking renormalization into account we subtract purely imaginary terms $iZ_2\delta m^2$, $iZ'_2\delta M^2$, $i(Z_2-1)(k^2+m^2)$ and $i(Z'_2-1)(k^2+M^2)$ in such a way that a development of the imaginary parts of E and E' in powers of (k^2+m^2) and (k^2+M^2) respectively contains only second or higher order powers in these quantities. In this Z_2, Z'_2, δm^2 and δM^2 are constants and the notation used is the standard one.

In our case the $\tilde{\Delta}_F(k^2)$ will take the form:

$$\tilde{\Delta}_F(k^2) = \left[k^2+m^2 - (k^2+m^2)^2R(k^2) - i\Theta(-k^2-9m^2)I(k^2)\right]^{-1}$$
$$\Delta'_F(k^2) = \left[k^2+M^2 - (k^2+M^2)^2R'(k^2) - i\Theta(-k^2-4m^2)I'(k^2)\right]^{-1} \qquad (3.9)$$

M and m are the physical masses. R, I, R', and I' are real functions of second and higher order in g, analogous to R_2 and I_2 in (3.4), (3.5). If the A-particle is unstable, i.e. $M > 2m$, the propagator $\tilde{\Delta}'_F(k^2)$ will have no pole at the point $k^2 + M^2 = o$. In section 4 where unitarity is considered we will see that it is exactly the absence of this pole which will ensure unitarity of the theory without A-particles in the asymptotic in- or out-states.

The equations (3.8) are quite similar to equations obtained by Symanzik 8) for the cases of retarded and timeordered functions. By means of them one can separate renormalisation effects from the S-matrix. One uses then an S-matrix without selfenergy parts, but with complete propagators that have to satisfy the implicit equation. If no vertex divergences occur (as in the super-renormalisable model considered here) all renormalisation problems are separated from the actual S-matrix calculation. In case of vertex-divergences one has to go further than the two-point function and also the three point function has to be treated. The situation is then more complicated due to overlapping divergences but can be handled following the rules of renormalization theory as developed for stable particles. For further details we refer to Symanzik's work. In the following we will use (3.8) as a starting point without reference to its derivation. We will suppose that (3.8) has a solution that obeys Lehmann's representation

$$\tilde{\Delta}_F(k^2) = \int_0^\infty \rho(-\mathbf{k}^2)\Delta_F(k^2,\mathbf{k}^2)\,d\mathbf{k}^2$$

$$\tilde{\Delta}'_F(k^2) = \int_0^\infty \rho'(-\mathbf{k}^2)\Delta_F(k^2,\mathbf{k}^2)\,d\mathbf{k}^2 \qquad (3.10)$$

$$\Delta_F(k^2,\mathbf{k}^2) = (k^2+\mathbf{k}^2)_P^{-1} + i\pi\delta(k^2+\mathbf{k}^2) = (k^2+\mathbf{k}^2-io)^{-1}$$

The weight functions ρ, ρ' are non-negative and vanish for positive argument. They are related to the functions I, I' in (3.9) as indicated in (3.11).

Our task will be to prove that such a solution gives rise to a unitary S-matrix connecting φ-particle states only. We close this section by writing down for the complete propagators the formulae needed in order to apply the cutting formula in the following section. From (3.10)

$$\tilde{\Delta}_F(x) = \Theta(x_o)\tilde{\Delta}^+(x) + \Theta(-x_o)\tilde{\Delta}^-(x)$$

$$\tilde{\Delta}'_F(x) = \Theta(x_o)\tilde{\Delta}'^+(x) + \Theta(-x_o)\tilde{\Delta}'^-(x)$$

with:

$$\tilde{\Delta}^\pm(x) = i(2\pi)^{-3}\int d_4k\; e^{ikx}\Theta(\pm k_o)\rho(k^2)$$

$$\tilde{\Delta}'^\pm(x) = i(2\pi)^{-3}\int d_4k\; e^{ikx}\Theta(\pm k_o)\rho'(k^2)$$

Clearly these equations coincide with (2.1), (2.2).
We note also

$$\begin{aligned}\tilde{\Delta}^{\pm}(k^2) &= 2\pi i\Theta(\pm k_o)\rho(k^2) = 2i\Theta(\pm k_o)\,\mathrm{Im}\,\tilde{\Delta}_F(k^2)\\ \tilde{\Delta}'^{\pm}(k^2) &= 2\pi i\Theta(\pm k_o)\rho'(k^2) = 2i\Theta(\pm k_o)\,\mathrm{Im}\,\tilde{\Delta}'_F(k^2)\end{aligned} \tag{3.11}$$

4. Unitarity of the S-matrix.

We start now by defining the S-matrix of our model in terms of diagrams. It shall consist of all possible diagrams constructed according to the rules given in section 3, with the exclusion of diagrams that contain selfenergy parts and/or outgoing A-lines. The correspondence between the diagrams and the analytical expressions associated with them will be as before, with the exception that internal A and φ lines now correspond to the complete propagators $\tilde{\Delta}_F'$ and $\tilde{\Delta}_F$ that are solutions of the implicit equation (3.8) and obey the Lehmann representation. The complete analytical expression for the S-matrix can be written in the form

$$S = 1 + \sum_{n=3}^{\infty} \int \dots \int dx_1 \dots dx_n f_n(x_1 \dots x_n) : \varphi_1 \dots \varphi_n : \qquad (4.1)$$

The term n=3 is of course zero because of energy momentum conservation. Nevertheless we retain it explicitly for future use. In this $\varphi_i \equiv \varphi_{in}(x_i)$ is the free in-field corresponding to the φ-particle. The commutator of φ_i with itself is just the free-field commutator Δ for a particle with mass m. The functions $f_n(x_1 \dots x_n)$ are the sums of the contributions arising from all diagrams with n vertices $x_1 \dots x_n$ as external vertices. In the vertices $x_1 \dots x_n$ one internal A-line, one internal φ-line and one external φ-line come together. The n external φ-lines are not included in the evaluation of $f_n(x_1 \dots x_n)$, but correspond to the operators $\varphi_1 \dots \varphi_n$ in (4.1), where: $\varphi_1 \dots \varphi_n$: is the usual ordered product.

Equation (4.1) gives us a well defined expression for the S-matrix, for which we must prove unitarity. If in (4.1) we separate the unit term from the rest by writing S = 1 + T unitarity takes the form

$$T + T^+ = - TT^+ \qquad (4.2)$$

T is given by the sum over n in the right hand side of (4.1). Let us write down TT^+:

$$TT^{+} = \sum_{n=3}^{\infty} \sum_{n'=3}^{\infty} \int dx_1 \ldots dx_n dx'_1 \ldots dx'_{n'} f_n(x_1 \ldots x_n) f^*_{n'}(x'_1 \ldots x'_{n'}) \cdot$$

$$:\varphi_1 \ldots \varphi_n: \; :\varphi_{1'} \ldots \varphi_{n'}:$$

By standard techniques 9) one can reduce the product of the two ordered products to a sum of ordered products by introducing the contraction of two field operators:

$$\overline{\varphi_i \varphi_{i'}} \equiv \langle o| \varphi_{in}(x_i) \varphi_{in}(x_{i'}) |o\rangle = - i\Delta^{+}(x_i - x_{i'})$$

$$= \frac{1}{(2\pi)^3} \int d_4k \; e^{ik(x_i - x_{i'})} \Theta(k_o) \delta(k^2 + m^2) \qquad (4.3)$$

Clearly TT^{+} can be associated with diagrams obtained by connecting, through lines carrying Δ^{+} functions, the various diagrams of T (propagators $\widetilde{\Delta}_F, \widetilde{\Delta}'_F$) with the various diagrams of T^{+} (propagators $\widetilde{\Delta}^*_F, \widetilde{\Delta}'^*_F$).

Next we inspect $T + T^{+}$. Consider a diagram of T with m vertices $y_1 \ldots y_m$, among which ν internal vertices $y_{\alpha_1} \ldots y_{\alpha_\nu}$ and $n = m - \nu$ external vertices which we call $x_1 \ldots x_n$. Its contribution to the function $f_n(x_1 \ldots x_n)$ can be written in the form:

$$(i)^{m+a+b} \int d_4y_{\alpha_1} \ldots d_4y_{\alpha_\nu} \; F(y_1 \ldots y_m) \qquad (4.4)$$

In this a is the number of internal φ-lines and b the number of internal A-lines. We separated out the factor i attached to each internal line (see (3.3)) in order to be able to apply the cutting formula directly to $F(y_1 \ldots y_m)$, which is seen to be of the form (2.6). We apply now (2.10) for the case that the subset $x_{j_1} = y_1 \ldots x_{j_p} = y_p$ consists of all vertices ($p = m$) and get

$$(i)^{m+a+b} \int d_4y_{\alpha_1} \ldots d_4y_{\alpha_\nu} \sum_{q=o}^{m} \sum_{\{q\}} (-1)^q F_q(y_1 \ldots y_{i_1-1} y_{i_1+1} \ldots$$

$$\ldots y_m | y_{i_1} \ldots y_{i_q}) = o \qquad (4.5)$$

The term with $q = o$ equals (4.4) while the term with $q = m$ equals

$$(-i)^{m+a+b}\int d_4y_{\alpha_1}\cdots d_4y_{\alpha_\nu}\, F^*(y_1\cdots y_m) \tag{4.6}$$

where we used

$$F(\{y_1\cdots y_m) = (-1)^{a+b}F^*(y_1\cdots y_m)$$

as can be inferred from the rules for the construction of the modified F functions (2.7). Obviously the term with q = m can be interpreted as a contribution to T^+. As described in section 2 the terms with $1<q<m$ can be pictured as cut diagrams, where to the cut lines correspond $\tilde{\Delta}^{\pm}$ or $\tilde{\Delta}'^{\pm}$ functions, while all internal lines on the shaded side give rise to $-\tilde{\Delta}_F^*$ or $-\tilde{\Delta}_F'^*$ functions. If now in the cutting formula (4.5) we move all terms with $1<q<m$ to the right-hand side, we see that the left hand side, containing only (4.4) and (4.6), exactly corresponds to the left hand side of the unitarity equation (4.2). The right-hand side will reduce to the right-hand side of (4.2) if we are able to achieve two aims: eliminate all diagrams with cuts which cross A-lines, and replace in the other diagrams $\tilde{\Delta}^{\pm}$ by Δ^+ for all cut φ-lines. This is done as follows. We first convert all $\tilde{\Delta}^-$ and $\tilde{\Delta}'^-$ functions into $\tilde{\Delta}^+$ and $\tilde{\Delta}'^+$ functions with the help of the identities valid in the case of symmetrical propagators:

$$\tilde{\Delta}^+(x) = \tilde{\Delta}^-(-x) \ , \ \tilde{\Delta}'^+(x) = \tilde{\Delta}'^-(-x)$$

(there is no loss of generality here, one can if necessary circumvent this step). We next use the identities (3.8), (3.9) and (3.11); going over to the fourier representation we have:

$$\begin{aligned}\tilde{\Delta}^+(k^2) &= \Theta(k_o)\Big[\,|\tilde{\Delta}_F(k^2)|^2 2i\mathrm{Re}E(\tilde{\Delta}_F,\tilde{\Delta}_F',k^2) + 2\pi i\delta(k^2+m^2)\Big] \\ \tilde{\Delta}'^+(k^2) &= \Theta(k_o)\,|\tilde{\Delta}_F'(k^2)|^2 2i\mathrm{Re}E'(\tilde{\Delta}_F,\tilde{\Delta}_F',k^2)\end{aligned} \tag{4.7}$$

The crucial point in these equation is that $\tilde{\Delta}^+(k^2)$, belonging to the stable φ-particle, contains $\Delta^+(k^2) = 2\pi i\Theta(k_o)\delta(k^2+m^2)$ as a separate term whereas $\tilde{\Delta}'^+(k^2)$ contains no such contribution. The following step is to express ReE and ReE' in terms of cut selfenergy diagrams. Consider E and E'. They are the sums of all s.p.d. with $\tilde{\Delta}_F$ and $\tilde{\Delta}'_F$ propagators, are thus themselves sums of contributions corresponding with diagrams, and accordingly we can apply the cutting formula to them, always taking the case where the subset $y_1 \ldots y_p$ consists of all vertices. One finds that $|\tilde{\Delta}_F|^2 2\text{ReE}$ (and $|\tilde{\Delta}'_F|^2\, 2\text{ReE}'$) can be regarded as corresponding to the sum of all cuttings of all s.p.d. with $\tilde{\Delta}_F, \tilde{\Delta}'_F$ ($\tilde{\Delta}^*_F, \tilde{\Delta}'^*_F$) on the unshaded (shaded) side. Returning then to (4.5), and the original diagrams we can now picture the contribution of the terms in ReE and ReE' to a cut line of the original diagrams as obtained by substituting to it a cut selfenergy diagram, more precisely a cut s.p.d.*). This can be repeated ad infinitum, and the only contributions from cut lines which are eventually left are the $2\pi i\Theta(k_o)\delta(k^2+m^2) = \Delta^+(k^2)$. They originate from cut φ-lines, and cut A-lines are left with no contribution at all. The last step in the proof of (4.2) consist in showing that one gets all diagrams of TT^+ by treating in the manner indicated above all diagrams of T. This is a simple combinatorial problem which we do not consider further here.

As an illustration figure 4 gives an example of the reduction procedure for a simple diagram. Factors are not indicated for simplicity

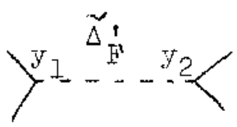


Fig.4a.

Example of a diagram as represented by a formula of the type (4.4). We added external lines.

*) One easily verifies that the shaded area of the original diagram and the shaded area of the s.p.d. substituted into it are both on the same side. This follows from the sign of the energy specified by $\Theta(k_o)$ in (4.5).

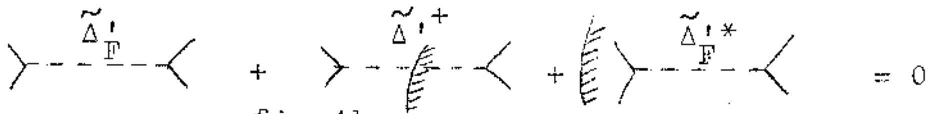


fig.4b

Equation (4.5) for the diagram of figure 4a.

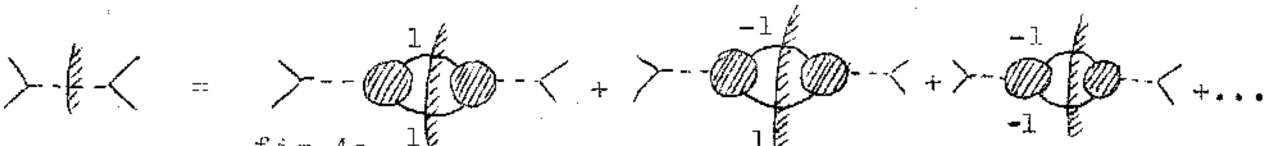


fig.4c

First step in the reduction of $\tilde{\Delta}'^{+}$. A $\Theta(k_0)\delta(k^2+m^2)$ factor is indicated by a cut line with a number 1. $\Theta(k_0)\mathrm{Re}E$ or $\Theta(k_0)\mathrm{Re}E'$ factors are indicated by cut lines with the number -1. The shaded cercles stand for the sum of all diagrams with three lines outgoing as indicated.

5. S-matrix and field equations.

In this section we will show that the S-matrix defined in section 4 is a solution of the field equations belonging to a Lagrangian density

$$L = -\frac{1}{2}\left\{\left(\frac{\partial\varphi}{\partial x_\mu}\right)^2 + m^2\varphi^2 + \left(\frac{\partial A}{\partial x_\mu}\right)^2 + M^2A^2\right\} + L_I \qquad (5.1)$$

where L_I is as given in (3.1) and all renormalization terms are suppressed. In order to prove field equations we define from the S-matrix an interpolating field $\varphi(x)$ with the property that for infinite times its expectation values equal those of the in- or outfields. This interpolating field shall be defined so that the vacuum expectation value of the product of the field with itself equals $\tilde{\Delta}^+$ (see (2.1)), and we will show then that it satisfies the integral form of the field equations derived from L, see (5.11) below. In this form of the equations, A is explicitly given in terms of φ.

Let us start by recalling the definition of the outfield 10)

$$\varphi_{out}(x_i) \equiv \varphi_{\underline{i}} \equiv S^+\varphi_{in}(x_i)S = S^+\varphi_i S = \varphi_i + S^+[\varphi_i, S] \qquad (5.2)$$

From the commutation rule

$$[\varphi_i, \varphi_j] = -i\Delta(x_i - x_j) = (2\pi)^{-3}\int d_4k\ e^{ikx}\delta(k^2+m^2)\mathcal{E}(k_o)$$

$$\mathcal{E}(k_o) = -1 + 2\Theta(k_o)$$

one finds by introducing (4.1) in (5.2):

$$\varphi_{\underline{i}} = \varphi_i - \int d_4x\ \Delta(x_i - x)j(x)$$

where j(x) is given by

$$j(x) = iS^+\left[\sum_{n=3}^{\infty}\ \sum_{j=1}^{n}\int dx_1 \dots \not{d}x_j \dots dx_n\ .\right.$$

$$f_n(x_1 \dots x_{j-1}, x, x_{j+1} \dots x_n)\ :\varphi_1 \dots \not{\varphi}_j \dots \varphi_n: \qquad (5.3)$$

In this we used the notation

$$a_1 \dots a_{i-1}a_{i+1} \dots a_n = a_1 \dots \not{a}_i \dots a_n$$

Because of the fact that the fourier transform of $\Delta(x_i - x)$ contains a $\delta(k^2+m^2)$ function we need up to now the function $f_n(x_1 \ldots x_n)$ only on the mass shell (i.e. of the fourier transform $f_n(k_1 \ldots k_n)$ of $f_n(x_1 \ldots x_n)$ we need only the values for $k_i^2 = -m^2$, $i = 1, \ldots, n$). Of course the f_n given by diagrams and Feynman rules have well defined values off the mass shell also, but in the S-matrix only the mass shell values are required.

We now define $j(x)$ also off the mass shell (i.e. for $k^2 \neq -m^2$, where k is the 4-momentum associated with x):

$$j(x) = iS^+ \left[\sum_{n=3}^{\infty} \sum_{j=1}^{n} \int dx_1 \ldots \not{dx}_j \ldots dx_n \, . \right.$$

$$f_n'(x_1 \ldots x_{j-1}, x, x_{j+1} \ldots x_n) :\varphi_1 \ldots \not{\varphi}_j \ldots \varphi_n:$$

where the fourier transformed $f_n'(k_1 \ldots k_n)$ of $f_n'(x_1 \ldots x_n)$ is related to $f_n(k_1 \ldots k_n)$ as calculated by the ordinary Feynman rules through

$$f_n'(k_1 \ldots k_n) = N(k_1^2) \ldots N(k_n^2) f_n(k_1 \ldots k_n)$$

$$N(k_i^2) \equiv \frac{\tilde{\Delta}_F(k_i^2)}{\Delta_F(k_i^2)} = 1 + iE(\tilde{\Delta}_F, \tilde{\Delta}_F', k_i^2) \tilde{\Delta}_F(k_i^2) \tag{5.4}$$

This definition coincides on the mass shell with (4.1) and (5.3) because of the fact that $N(-m^2) = 1$. In the following we will use the notation:

$$\sum_{n=3}^{\infty} \sum_{j=1}^{n} \int dx_1 \ldots \not{dx}_j \ldots dx_n \, . f_n'(x_1 \ldots x_{j-1}, x, x_{j+1}, \ldots x_n) \, .$$

$$:\varphi_1 \ldots \not{\varphi}_j \ldots \varphi_n: \equiv \frac{\delta S}{\delta \varphi_{in}(x)} = \frac{\delta S}{\delta x} \tag{5.5}$$

We stress the fact that the δ/δ symbols are introduced as a notation rather then indicating functional differentiation with respect to a field. However, one can establish that well-known properties of functional derivatives also hold in this case. For instance defining $\delta S^+/\delta x$ as the hermitian conjugate of (5.5) one has

$$iS^{+}\frac{\delta S}{\delta x} = -\, i\frac{\delta S^{+}}{\delta x}S(=j(x)) \qquad (5.6)$$

or

$$\frac{\delta S^{+}S}{\delta x} = 0$$

The proof of this equation, obviously related to unitarity, is not trivial and needs an interpretation of the factor $N(k^2)$ in terms of diagrams. From (5.4) we see that the factors $N(k_i^2)$ in f_n' can be interpreted as unity plus all (renormalised) s.p.d. connected to the original f_n diagrams by a $\tilde{\Delta}_F$ propagator (fig.5a). The proof of (5.6) is now entirely analogous to the proof of unitarity in section 4: apply the cutting formula to all diagrams contributing to the $f_n'(x_1 \ldots x \ldots x_n)$; reduce all $\tilde{\Delta}^{\pm}$, $\tilde{\Delta}'^{\pm}$ functions in the manner used in the unitarity proof. The terms corresponding to diagrams with x unmarked give the left-hand side of (5.6), the others give the right-hand side. The necessity of the additional diagrams introduced in f_n' as compared to f_n (see an example in fig.5a) is illustrated in fig.5b and 5c. The occurrence of diagrams of the type shown in fig.5b implies diagrams as in fig.5c, which themselves require that the definition of $\delta S/\delta x$ contains diagrams as in fig.5a.

Fig.5a
Diagrams appearing in $\frac{\delta S}{\delta x}$ in addition to those that appear also in S. As indicated in fig.2 the circles stand for general diagrams with external vertices indicated by dots. Their necessity is illustrated in fig.5b and c.

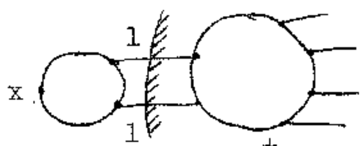


Fig.5b
Diagram giving non zero contribution to $S^{+}\frac{\delta S}{\delta x}$ if the fourier variable associated with x satisfies $k^2 < -4m^2$. As in fig. 4 the 1 indicates a Δ^{+} function.

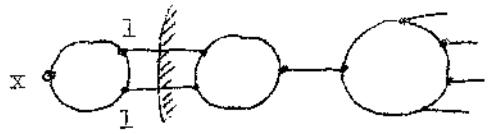


Fig.5c

Type of diagram appearing in $S^+(\delta S/\delta x)$ which requires the consideration in $\delta S/\delta x$ of the extra diagrams as given in fig. 5a

We now define the interpolating field $\varphi(x)$:

$$\varphi(x) = \varphi_{in}(x) - \int d_4x'\Delta_R(x-x')j(x') \qquad (5.7)$$

and we will be concerned with the properties of this field in the remaining part of the present section. As a first point we observe that $j(x')$ is used also off the mass shell because $\Delta_R(k^2) \neq o$ for $k^2 \neq -m^2$:

$$\Delta_R(x) = \Theta(x_o)\Delta(x) = (2\pi)^{-4}\int d_4k\ e^{ikx}\left[k^2+m^2-i\varepsilon(k_o)o\right]^{-1}$$

One verifies further that for arbitrary states $|\alpha\rangle$ and $|\beta\rangle$:

$$\lim_{t\to\pm\infty}\langle\alpha|\varphi(x)|\beta\rangle = \langle\alpha|\varphi_{\substack{out\\in}}(x)|\beta\rangle$$

Also from (5.7) together with $S|1\rangle = |1\rangle$ where $|1\rangle$ is the one particle state one derives

$$\langle o|\varphi(x)|1\rangle = \langle o|\varphi_{in}(x)|1\rangle$$

We prove now

$$i\langle o|\varphi(x)\varphi(x')|o\rangle = \tilde{\Delta}^+(x-x') \qquad (5.8)$$

Inserting (5.7) we have:

$$\langle o|\varphi(x)\varphi(x')|o\rangle = \langle o|\varphi_{in}(x)\varphi_{in}(x')|o\rangle +$$

$$+\int d_4y\ d_4y'\Delta_R(x-y)\Delta_R(x'-y')\langle o|j(y)j(y')|o\rangle$$

Inserting intermediate states into $\langle o|j(y)j(y')|o\rangle$ we see that this function contains positive frequencies only, i.e. it can be written in the form:

$$\int d_4k\ e^{ik(y-y')}\Theta(k_o)g(k^2)$$

This enables us to write

$$i\langle o|\varphi(x)\varphi(x')|o\rangle = i\langle o|\varphi_{in}(x)\varphi_{in}(x')|o\rangle +$$

$$+ i\int d_4y\ d_4y'\Delta_F(x-y)\langle o|j(y)j(y')|o\rangle\Delta_F^*(y'-x') \qquad (5.9)$$

where we used

$$\Delta_R(x) = \Delta_F(x) - \Delta^-(x) = \Delta_F^*(x) + \Delta^+(x) \qquad (5.10)$$

We now study $\langle o|j(y)j(y')|o\rangle$. On substituting

$$j(y) = iS^+\frac{\delta S}{\delta y} \qquad j(y') = -i\frac{\delta S^+}{\delta y'}S$$

one finds with $S|o\rangle = |o\rangle$:

$$\langle o|j(y)j(y')|o\rangle = \sum_{n=6}^{\infty}\sum_{m=3}^{n-3}\sum_{j=m+1}^{n}\sum_{i=1}^{m}\int dx_1\ldots \not{d}x_i\ldots \not{d}x_j\ldots dx_n\ \cdot$$

$$\cdot\ f'_m(x_1\ldots x_{i-1},y,x_{i+1}\ldots x_m)f'^*_{n-m}(x_{m+1}\ldots x_{j-1},y',x_{j+1}\ldots x_n)\cdot$$

$$\cdot\langle o|:\varphi_1\ldots\not{\varphi}_i\ldots\varphi_m::\varphi_{m+1}\ldots\not{\varphi}_j\ldots\varphi_n:|o\rangle$$

The vacuum expectation value can be reduced to a sum of products of Δ^+ functions, and again with the help of cutting formula and reduction technique we find:

$$\langle o|j(y)j(y')|o\rangle = (2\pi)^{-4}\int d_4k e^{ik(y-y')}\Theta(k_o)N(k^2)N^*(k^2)\cdot$$

$$\cdot\ 2\mathrm{Re}E(\widetilde{\Delta}_F,\widetilde{\Delta}'_F,k^2)$$

This result inserted in (5.9) gives together with (5.4), (4.7) and (4.3) the desired result (5.8).

Let us now try to prove that $\varphi(x)$ is a solution of the field equations. These equations, derived from the Lagrangian (5.1) and put in integral equation form, are (Δ'_R is Δ_R with M instead of m)

$$A(x) = -g\int d_4x'\Delta'_R(x-x')\{\varphi^2(x')-\langle o|\varphi^2(x')|o\rangle-\text{ren. terms.}\}$$

$$(5.11)$$

$$\varphi(x) = \varphi_{in}(x)-g\int d_4x'\Delta_R(x-x')\{\varphi(x')A(x')+A(x')\varphi(x')-\text{ren. terms}\}$$

The absence of A_{in} field is due to the instability of the A-particle.

Indeed the matrix-elements of the A-field must vanish for infinite times. Because of the absence of A_{in} field one can eliminate the A-field from the equations (5.11). One has then in the φ-field a non-local theory of the type investigated by Bloch, ref.5. The exact form of the renormalization terms will be given later.

Let us consider the operator $\varphi^2(x)$. With the help of (5.7), (5.6) and the relations

$$\varphi_{in}^2(x) = :\varphi_{in}^2(x): + \langle o|\varphi_{in}^2(x)|o\rangle$$

$$\frac{\delta S^+}{\delta x_j}\frac{\delta S}{\delta x_i} + \frac{\delta S^+}{\delta x_i}\frac{\delta S}{\delta x_j} = - S^+\frac{\delta^2 S}{\delta x_j \delta x_i} - \frac{\delta^2 S^+}{\delta x_j \delta x_i} S$$

(the latter can be proved by the methods described before) one finds:

$$\varphi^2(x) = \tfrac{1}{2}:\varphi_{in}^2(x): + \tfrac{1}{2}\langle o|\varphi_{in}^2(x)|o\rangle +$$

$$+ i\int \Delta_R(x-x')\varphi_{in}(x)\frac{\delta S^+}{\delta x'}S \; d_4x' +$$

$$- \frac{1}{2}\int \Delta_R(x-x')\Delta_R(x-x'')\frac{\delta^2 S^+}{\delta x'\delta x''}S \; d_4x' d_4x''$$

$$+ \text{herm.conjugate} \tag{5.12}$$

We introduce some notations. Let the operator B be a linear combination of the ordered products $:\varphi_1 \dots \varphi_n:$. A product of $\varphi_{in}(x)$ with B can be split up in a part where $\varphi_{in}(x)$ is contracted with a φ_{in} in B, and the rest, and we will adopt a notation exhibiting this splitting explicitly

$$\varphi_{in}(x)B = \overbrace{\varphi_{in}(x)B} + :\varphi_{in}(x)B:$$

For instance with $B = :\varphi_1 \dots \varphi_n:$ we have

$$\overbrace{\varphi_{in}(x)B} = \sum_{j=1}^{n}\langle o|\varphi_{in}(x)\varphi_j|o\rangle :\varphi_1 \dots \not{\varphi_j} \dots \varphi_n:$$

$$:\varphi_{in}(x)B: = :\varphi_{in}(x)\varphi_1 \dots \varphi_n:$$

We write further

$$\frac{\delta S}{\delta x} = \int d_4x' N(x-x') \frac{\delta S}{\delta \underline{x}'}$$

$$N(x) = (2\pi)^{-4} \int d_4k \; e^{ikx} N(k^2)$$

$\delta S / \delta \underline{x}$ is $\delta S / \delta x$ without the factor $N(k^2)$. With these notations and (5.10) we can write

$$\int d_4x' \Delta_R(x-x') \varphi_{in}(x) \frac{\delta S^+}{\delta x'} S = \int d_4x' d_4x'' .$$

$$. \left\{ \tilde{\Delta}_F^*(x-x') + \Delta^+(x-x') \right\} \left\{ - i\Delta^+(x-x'') \frac{\delta^2 S^+}{\delta \underline{x}' \delta \underline{x}''} S + \right.$$

$$\left. + :\varphi_{in}(x) \frac{\delta S^+}{\delta \underline{x}'}: S \right\}$$

Further we have:

$$:\varphi_{in}^2(x): = :\varphi_{in}^2(x): S^+ S = - 2i \int d_4x' \Delta^+(x-x') : \varphi_{in}(x) \frac{\delta S^+}{\delta x'} : S +$$

$$- \int d_4x' d_4x'' \Delta^+(x-x') \Delta^+(x-x'') \frac{\delta^2 S^+}{\delta x' \delta x''} S + :\varphi_{in}^2(x) S^+ : S$$

Insertion of the last two formulae into (5.12) gives after some algebra:

$$\varphi^2(x) = \frac{1}{2} \langle o | \varphi_{in}^2(x) | o \rangle - \frac{i}{2} \, BS + \text{herm.conj.}$$

with

$$B = - 2 \int d_4x' \tilde{\Delta}_F^*(x-x') : \varphi_{in}(x) \frac{\delta S^+}{\delta \underline{x}'} : +$$

$$- i \int d_4x' d_4x'' \tilde{\Delta}_F^*(x-x') \tilde{\Delta}_F^*(x-x'') \frac{\delta^2 S^+}{\delta \underline{x}' \delta \underline{x}''} + i : \varphi_{in}^2(x) S^+ : \quad (5.13)$$

The operator B is very similar to the operator $\delta S^+ / \delta x$. The first term corresponds to all diagrams of the S^+-matrix where one external line is replaced by an internal line ending in the vertex x, where in addition one external φ-line but no A-line is attached (fig.6a). The second term contains all diagrams of S^+ with two outgoing lines replaced by two internal lines ending in

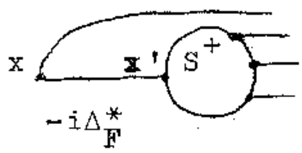


Fig.6a.

Diagrams contributing to the first term of the right hand side of (5.13).

the vertex x to which no A-line is attached (fig.6b). The third term contains all diagrams of S^+ supplemented by an external vertex with two outgoing φ-lines and without A-line (fig.6c). There is also a factor i for the new vertex as required by the Feynman rules given before. Now the second term contains also diagrams from which a part without external φ-line can be separated by cutting one internal line (fig.7). We recognize this part, together with the internal A-line involved, as the analogon of $N(k^2)-1$ for the A-field, and shall call it $N'(k^2)-1$ (fig.8). Taking all terms together we can write:

$$gB = N'^*(k^2)C$$

$$N'(k^2) = \frac{\tilde{\Delta}'_F(k^2)}{\Delta'_F(k^2)}$$

where C contains all diagrams having one external vertex without A-line and from which no part without external lines can be separated by cutting one A-line.

We observe now that because $\tilde{\Delta}'_F(-M^2) \neq \infty$ and $\Delta'_F(-M^2) = \infty$

$$N'(-M^2) = 0$$

This makes it possible to add formally diagrams having outgoing

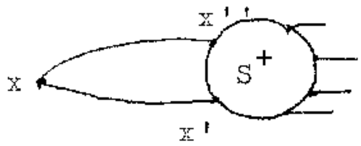


Fig.6b.

Diagrams contributing to the second term of the right-hand side of (5.13).

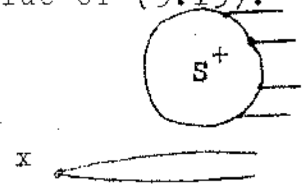


Fig.6c.

Diagrams contributing to the last term of the right hand side of (5.13).

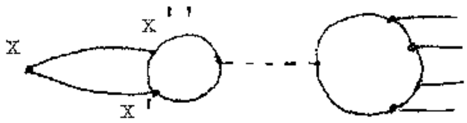


Fig.7.

Diagram contained in fig.6b having the property that a part without external lines can be separated by cutting only one A-line.

Fig.8.

The factor $N'(k^2)-1$. The circle stands for all possible diagrams with three outgoing lines as indicated. Together with the two φ-lines it gives exactly all s.p.d. for the A-field, i.e. a contribution $E'(\tilde{\Delta}_F, \tilde{\Delta}'_F, k^2) i\tilde{\Delta}'_F(k^2)$

A-lines with the factor $N'(k^2)$ to our S-matrix without influencing the physical results (since their contribution vanishes on the mass shell). The operator $g(-\,iBS - i\langle o|\,BS\,|o\rangle)$ can be seen then as

$$- i\frac{\delta S^+}{\delta A_{in}(x)}S - i\langle o|\frac{\delta S^+}{\delta A_{in}(x)}S\,|o\rangle \equiv j'(x)$$

i.e. as a current $j'(x)$ from which an A-field can be generated

$$A(x) = -\int d_4x'\Delta_R'(x-x')j'(x') \qquad (5.14)$$

Again, for this A one can prove

$$i\langle o|\,A(x)A(x')\,|o\rangle = \tilde{\Delta}'^{+}(x-x')$$

Furthermore, because of the important relation

$$\int d_4x'\Delta'(x-x')j'(x') = o$$

characteristic for unstable particles and due to the vanishing of $N'(k^2)$ on the mass shell, we find as expected

$$\lim_{t=\pm\infty}\langle\alpha|\,A(x)\,|\beta\rangle = o$$

We remark that from the diagrams in fig.7 the infinite renormalization terms still have to be subtracted. This is done as indicated after (3.8) and the outcome is that we have proved the first of the field-equations (5.11), namely (5.14) with the following current

$$j'(x) = g\{\varphi^2(x) - \langle o|\,\varphi^2(x)\,|o\rangle\} - \delta M^2 Z_2' A(x) -$$
$$- (Z_2'-1)(\Box^2 - M^2)A(x)$$

The derivation of the other field equation is somewhat simpler due to the absence of the A_{in}-field. The essential reasoning being the same, however, we do not give the proof here.

6. Causality.

If we eliminate the A-field we can regard our theory as an example of a non-local theory in the field φ. Despite its non local character it must fulfil the requirements of causality, and it is of interest to verify this explicitly. To this end we use Bogoliubov's form of causality (ref.4):

$$\frac{\delta}{\delta\varphi_{in}(x_1)}\left\{\frac{\delta S^+}{\delta\varphi_{in}(x_2)}S\right\} = 0 \qquad x_1 \lesssim x_2$$

($x_1 \lesssim x_2$ means spacelike separated x_1 and x_2 or $x_{o1} < x_{o2}$) or

$$\frac{\delta^2 S^+}{\delta\varphi_{in}(x_1)\delta\varphi_{in}(x_2)}S = -\frac{\delta S^+}{\delta\varphi_{in}(x_2)}\frac{\delta S}{\delta\varphi_{in}(x_1)} \qquad (6.1)$$

Fig.9 expresses this causality condition for a general diagram and one immediately recognizes it as deducible from a special case of the cutting formula. The validity of the latter therefore establishes causality. In its most general form the cutting formula gives rise to equations generalizing (6.1) with more derivatives, i.e. more vertices x_i. For an arbitrary number of vertices x_i one has (we write again $\delta/\delta x_i$ for $\delta/\delta\varphi_{in}(x_i)$):

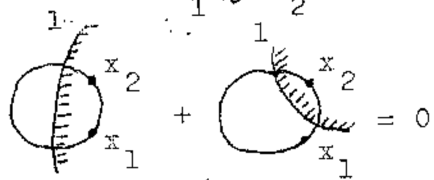
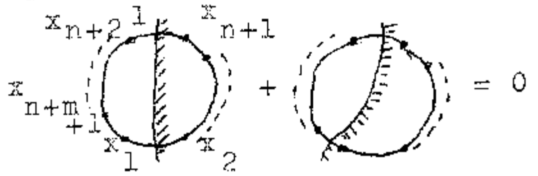


Fig.9.

Diagrams contributing to (6.1). The number 1 added to the cut means that all cut lines give Δ^+ functions. Only the relevant external vertices x_1 and x_2 are explicitly indicated.



$x_1 \lesssim x_i$, $i=1,\dots,n+m+1$.

Fig.10.

Diagram interpretation of (6.2).

$$\frac{\delta}{\delta x_1}\left\{\frac{\delta^{(n)}S^+}{\delta x_2\dots\delta x_{n+1}}\cdot\frac{\delta^{(m)}S}{\delta x_{n+2}\dots\delta x_{n+m+1}}\right\} = 0 \qquad (6.2)$$

if $x_1 \lesssim x_2, \dots x_{n+m+1}$

Fig.10 expresses this in terms of diagrams. It should be remarked, however, that formula (6.2) can be derived from (6.1) and consequently contains no additional information.

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R E F E R E N C E S

1) P.T.Matthews and A.Salam, Phys.Rev.112, (1958), 283.

2) R.Jacob and R.G.Sachs, Phys.Rev.121, (1961), 350.

3) H.Lehmann, K.Symanzik, W.Zimmermann, Nuovo Cimento 1, (1955), 205; ibid.6 (1957), 319.

4) N.Bogoliubov, D.Shirkov, Fortschr.der Physik 3 (1955), 439; N.Bogoliubov, B.Medvedev, M.Polivanov, ibid.6 (1958), 169.

5) C.Bloch, K.Danske, Vidensk.Selsk., Mat.fys.Medd.27 (1952), No.8.

6) H.Lehmann, Nuovo Cimento 11, (1954), 342.

7) F.J.Dyson, Phys.Rev.75, (1949), 486, 1736.

8) K.Symanzik, J.of Math.Phys.1 (1960) 249; Lecture notes at the Summer school for High Energy Physics at Hercegnovi, Yougoslavia, 1961.

9) G.Kallen, Handbuch der Physik V/1, page 248.

10) The following discussion is closely related to the treatment of Bogoliubov c.s. ref.4.