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UNITARY ANALOGS OF GENERALIZED RAMANUJAN SUMS

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The multiplicative properties of a certain type of generalized Ramanujan sum have been studied by several authors. In this paper we investigate the multiplicative properties of the unitary analog of this function.

Cohen [2] defined the unitary product of two arithmetic functions f and g, by

$$f \times g(n) = \sum_{d \| n} f(d) g(n/d) , \tag{1}$$

where $d \| n$ indicates that d is a unitary divisor of n, i.e., $d | n$ and $(d, n/d) = 1$. He also defined a unitary analog of Ramanujan's sum $c_k(n)$ by

$$c_k^*(n) = \sum_{\substack{(j,k)_*=1 \\ j \bmod k}} \exp(2\pi i j n/k) \tag{2}$$

where $(j, k)_*$ denotes the largest divisor of j which is a unitary divisor of k. Cohen then demonstrated that, paralleling the Dirichlet product result, we have

$$c_k^*(n) = \sum_{\substack{d | n \\ d \| k}} d\mu^*(k/d) . \tag{3}$$

Here μ^* is the unitary Möbius function and $\mu^* = 1^{-1}$ with respect to the unitary product ($1(n) = 1$ for all n). The function μ^* is multiplicative and $\mu^*(1) = 1$, $\mu^*(p^k) = -1$ for all primes p and positive integers k. It is easy to see that (3) may be rewritten

$$c_k^*(n) = \sum_{d \| (n,k)_*} d\mu^*(k/d) . \tag{4}$$

Cohen also defined $\phi^*(n) = c_n^*(0)$, and paralleling the Dirichlet case showed that $\phi^*(n)$ counts the number of integers unitarily semi-prime to n, i.e., the number of integers k such that $(k, n)_* = 1$. He also showed that $\phi^*(n) = i \times \mu^*(n)$, where i is the identity function, which is also analogous to the well known Dirichlet result.

Anderson and Apostol [1] defined a more general Ramanujan type sum by

$$s_k(n) = \sum_{d | (n,k)} f(d) g(k/d) ,$$

and studied the multiplicative properties of this new function. In this paper we study the multiplicative properties of the unitary analog of $s_k(n)$, defined as follows.

DEFINITION 1. For arithmetic functions f and g, let

$$s_k^*(n) = \sum_{d \| (n,k)_*} f(d)g(k/d) .$$

The proof of the following lemma is straightforward.

LEMMA 2. *If* $(a, k) = (b, m) = 1$ *then* $(ab, mk)_* = (a, m)_*(b, k)_*$ *and* $((a, m)_*, (b, k)_*) = 1$.

THEOREM 3. *If f and g are multiplicative then $s_k^*(n)$ has the following multiplicative properties*:

(i) $s_{mk}^*(ab) = s_m^*(a)s_k^*(b)$ *whenever* $(a, k) = (b, m) = 1$

(ii) $s_m^*(ab) = s_m^*(a)$ *whenever* $(b, m) = 1$

(iii) $s_{mk}^*(a) = s_m^*(a)g(k)$ *whenever* $(a, k) = 1$.

Proof. Suppose $(a, k) = (b, m) = 1$. Then

$$\begin{aligned} s_{mk}^*(ab) &= \sum_{d \| (ab,mk)_*} f(d)g(mk/d) = \sum_{d \| (a,m)_*(b,k)_*} f(d)g(mk/d), \text{ by Lemma 2,} \\ &= \sum_{d_1 \| (a,m)_*} f(d_1)g(m/d_1) \sum_{d_2 \| (b,k)_*} f(d_2)g(k/d_2), \text{ since } (d_1, d_2) = 1 \\ &= s_m^*(a)s_k^*(b). \end{aligned}$$

This proves (i). Now let $k = 1$.

$s_m^*(ab) = s_m^*(a)s_1^*(b) = s_m^*(a)$ which is (ii). Not let $b = 1$ in (i)
$s_{mk}^*(a) = s_m^*(a)s_k^*(a) = s_m^*(a)g(k)$.

The function $s_k^*(n)$ is multiplicative in another sense.

THEOREM 4. *If f and g are multiplicative then $s_k^*(n)$ is multiplicative in k for each fixed n.*

Proof. Suppose $(k, m) = 1$ and n is fixed. Then

$$\begin{aligned} s_k^*(n)s_m^*(n) &= \sum_{d_1 \| (n,k)_*} f(d_1)g(k/d_1) \sum_{d_2 \| (n,m)_*} f(d_2)g(m/d_2) \\ &= \sum_{d_1 \| (n,k)_*} \sum_{d_2 \| (n,m)_*} f(d_1d_2)g(km/d_1d_2) = \sum_{d \| (n,km)_*} f(d)g(km/d) \\ &= s_{km}^*(n) . \end{aligned}$$

The case $s_k^*(n) = c_k^*(n)$ was proved by Cohen [2].

THEOREM 5. *If f and g are multiplicative, and $g(n) = \pm 1$ for all n, then for fixed k the function $g(k)s_k^*(n)$ is multiplicative in the variable n.*

Proof. Choose $(n, m) = 1$ and fix k. Now

$$g(k)s_k^*(n)g(k)s_k^*(m) = s_k^*(n)s_k^*(m), \text{ since } g^2(k) = 1 .$$

Since both sides of the equality

$$s_k^*(n)s_k^*(m) = g(k)s_k^*(nm)$$

are multiplicative in k (by the previous theorem), it is enough to prove the same when k is a prime power.

$$s_k^*(n)s_k^*(m) = \sum_{d_1 \| (n,k)_*} f(d_1)g(k/d_1) \sum_{d_2 \| (m,k)_*} f(d_2)g(k/d_2)$$

but since k is a prime power either d_1 or d_2 is 1, so $g(k/d_1)g(k/d_2) = g(k)g(k/d_1d_2)$ and

$$\begin{aligned} s_k^*(n)s_k^*(m) &= \sum_{d_1 d_2 \| (nm,k)_*} f(d_1d_2)g(k)g(k/d_1d_2) \\ &= g(k) \sum_{d \| (nm,k)_*} f(d)g(k/d) = g(k)s_k^*(nm) . \end{aligned}$$

In particular,

COROLLARY 6. *For fixed k, the function $\mu^*(k)c_k^*(n)$ is multiplicative in the variable n.*

The Dirichlet analog of Corollary 6 was proved by Donovan and Rearick [4].

Theorem 4 is also useful in the proof of another unitary version of a Dirichlet result [1]. A somewhat weaker theorem of this type was proved by V. Sitah Ramaiah [6].

THEOREM 7. *Suppose g and f are multiplicative and $F(n) = f \times g(n) \neq 0$ for all n. Then*

$$(5) \qquad s_k^*(n) = \frac{F(k)g(N)}{F(N)}$$

where $N = k/(n, k)_*$.

Proof. After Theorem 4 it is sufficient to show that the right hand side of (5) is multiplicative in k and demonstrate the equality when k is a prime power. But F is multiplicative [2, Theorem 2.1]. Using this and the fact that $(n, k)_*(n, m)_* = (n, km)_*$ if $(k, m) = 1$, it is easy to see that the right hand side of (5) is indeed multiplicative. So without loss of generality we may assume $k = p^\nu = P$, a prime power. If $P \nmid n$, then $(n, P)_* = 1$ and $F(k)g(N)/F(N)$ reduces to $g(P)$. If $P | n$ then $(n, P)_* = P$ and the right hand side of (5) reduces to $f(1)g(P) + f(P)g(1)$. In either case the value obtained is the value of $s_p^*(n)$, thus establishing the theorem.

COROLLARY 8. $c_k^*(n) = \phi^*(k)\mu^*(k/(n, k)_*)/\phi^*(k/(n, k)_*)$.

Proof. As stated earlier $\phi^*(k) = i \times \mu^*(k)$.

This particular special case of Theorem 7 has been proved by several authors [3], [5], and [7].

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