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UNITARY EQUIVALENCE IN PAIRWISE SPECTRAL ANALYSIS

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A word of Dedication

It is a privilege to share in the honoring of Professor Katuzi Ono on this occasion. It would be enough to be simply putting in evidence my high regard for a good and dear friend. But there is so very much more than that here. It is an opportunity for me to declare my esteem for the kind of mathematician who looks at his field from the outside as well as the inside; who sees the richness of all experience and seeks its instruction to know better and better the meaning of mathematics; who by the example of his own life urges young, coming scientists to the more productive path of an uncompartmentalized personality. Katuzi Ono is all of these.

§1. Introduction.

In the pairwise spectral analysis of two projections F and G (see [1, §4] of the References) there are determined two resolutions of the identity, $\mathbf{E}_F$ and $\mathbf{E}_G$, which are spectral measures on the Borel interval $[0,1]$, and which are, in fact, the spectral measures of the Hermitian operators S_F and S_G of (1. 6) below. The derived measures $\hat{\mathbf{E}}_F$ and $\hat{\mathbf{E}}_G$, defined by

$$(1.\ 1)\qquad \hat{\mathbf{E}}_F(\mathbf{A}) = \begin{cases} \mathbf{E}_F(\mathbf{A}) - (I - F), & \text{if } 0 \in \mathbf{A}, \\ \mathbf{E}_F(\mathbf{A}), & \text{if } 0 \notin \mathbf{A} \end{cases}$$

and

$$(1.\ 2)\qquad \hat{\mathbf{E}}_G(\mathbf{A}) = \begin{cases} \mathbf{E}_G(\mathbf{A}) - (I - G), & \text{if } 0 \in \mathbf{A}, \\ \mathbf{E}_G(\mathbf{A}), & \text{if } 0 \notin \mathbf{A}, \end{cases}$$

are resolutions of F and G, respectively, and they enjoy the relationship

$$(1.\ 3)\qquad \begin{cases} G\hat{\mathbf{E}}_F(\mathbf{A}) = \hat{\mathbf{E}}_G(\mathbf{A})F, \\ F\hat{\mathbf{E}}_G(\mathbf{A}) = \hat{\mathbf{E}}_F(\mathbf{A})G, \end{cases}$$

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for every Borel subset $\mathbf{A}$ of $[0,1]$. From (1. 3) other identities are derivable; for example, if f is a bounded, measurable function on $[0,1]$, then

$$G\cdot\int_{[0,1]} f(\lambda)\hat{\mathbf{E}}_F(d\lambda)=\int_{[0,1]} f(\lambda)\hat{\mathbf{E}}_G(d\lambda)\cdot F \tag{1. 4}$$

and

$$F\cdot\int_{[0,1]} f(\lambda)\hat{\mathbf{E}}_G(d\lambda)=\int_{[0,1]} f(\lambda)\hat{\mathbf{E}}_F(d\lambda)\cdot G. \tag{1. 5}$$

For a set $\mathbf{A}\subseteqq(0,1]$ the ranges of corresponding projections $\mathbf{E}_F(\mathbf{A})$ and $\mathbf{E}_G(\mathbf{A})$ are isometric. In [1] we gave explicit partial isometries that establish this, namely, $V_{F;\mathbf{A}}$ and $V_{G;\mathbf{A}}$. Let us recall their definition. Consistent with our standard notation

$$\begin{cases} S_F=FGF=\int_{[0,1]}\lambda\mathbf{E}_F(d\lambda),\\ S_G=GFG=\int_{[0,1]}\lambda\mathbf{E}_G(d\lambda),\end{cases} \tag{1. 6}$$

we set

$$\begin{cases} S_F^{-\frac{1}{2}}=\int_{[0,1]}\frac{1}{\sqrt{\lambda}}\mathbf{E}_F(d\lambda),\\ S_G^{-\frac{1}{2}}=\int_{[0,1]}\frac{1}{\sqrt{\lambda}}\mathbf{E}_G(d\lambda),\end{cases} \tag{1. 7}$$

and then define

$$\begin{cases} V_F=S_G^{-\frac{1}{2}}GF,\\ V_G=S_F^{-\frac{1}{2}}FG.\end{cases} \tag{1. 8}$$

V_F is proved to be a partial isometry whose initial domain is the range of $\mathbf{E}_F((0,1])$ and whose terminal domain is the range of $\mathbf{E}_G((0,1])$. Exactly the reverse is true for V_G, so that, indeed, we have $V_G=V_F^*$. With these we finally define, for any $\mathbf{A}\subseteqq(0,1]$,

$$\begin{cases} V_{F;\mathbf{A}}=V_F\mathbf{E}_F(\mathbf{A}),\\ V_{G;\mathbf{A}}=V_G\mathbf{E}_G(\mathbf{A}),\end{cases} \tag{1. 9}$$

and we find that $V_{F;\mathbf{A}}$ is a partial isometry with initial domain the range of $\mathbf{E}_F(\mathbf{A})$ and terminal domain the range of $\mathbf{E}_G(\mathbf{A})$; and *vice versa* for $V_{G;\mathbf{A}}$.

Some of the relations satisfied by these partial isometries are

$$\begin{cases} V_F = V_{F;(0,1]}, \\ V_G = V_{G;(0,1]}; \end{cases} \tag{1. 10}$$

$$V_{G;\mathbf{A}} = V^*_{F;\mathbf{A}}; \tag{1. 11}$$

$$\begin{cases} V_{F;\mathbf{A}}V_G = V_F V_{G;\mathbf{A}} = V_{F;\mathbf{A}}V_{G;\mathbf{A}} = \mathbf{E}_G(\mathbf{A}), \\ V_{G;\mathbf{A}}V_F = V_G V_{F;\mathbf{A}} = V_{G;\mathbf{A}}V_{F;\mathbf{A}} = \mathbf{E}_F(\mathbf{A}). \end{cases} \tag{1. 12}$$

The question readily suggests itself whether all the partial isometries $V_{F;\mathbf{A}}$, $\mathbf{A} \subseteqq (0,1]$, derive from a single unitary equivalence; that is, whether there exists a unitary operator U_F such that

$$\mathbf{E}_G(\mathbf{A})\, U_F = U_F\, \mathbf{E}_F(\mathbf{A}) = V_{F;\mathbf{A}} \quad \textit{for all } \mathbf{A} \subseteqq (0,1]. \tag{1. 13}$$

Notice that if this is so, then the unitary operator $U_G = U_F^*$ satisfies

$$\mathbf{E}_F(\mathbf{A})U_G = U_G\mathbf{E}_G(\mathbf{A}) = V_{G;\mathbf{A}} \quad \text{for all } (\mathbf{A}) \subseteqq (0,1]. \tag{1. 14}$$

In his recent doctoral dissertation, [2], Stephen Strasen presents a construction of a unitary operator for which he proves that the first equation in (1. 13) holds. It is, moreover, immediately evident from the construction that (1. 13) as a whole is verified. Hence, the immediate question is answered. But here in this present paper we will report on another approach to a unitary operator satisfying (1. 13). Our formulation is altogether different form that given by Strasen: we construct an explicit spectral measure on the circle, and the unitary operator having this spectral measure proves to fulfill (1. 13). By contrast, Strasen's approach avails itself of the interrelationships between certain linear manifolds that are involved and thereby succeeds in avoiding a great deal of explicit detail. We can describe this approach in a few words. Recall the operator V_F introduced above, which is a partial isometry with initial domain the range of $\mathbf{E}_F((0,1])$ and terminal domain the range of $\mathbf{E}_G((0,1])$. Let H and J denote the projections $\mathbf{E}_F(\{0\})$ and $\mathbf{E}_G(\{0\})$, respectively. Let an analysis be carried out as above with H and J in the places of F and G, respectively. We come up, then, with spectral measures $\mathbf{E}_H$ and $\mathbf{E}_J$, and with a partial isometry V_H having the range of $\mathbf{E}_H((0,1])$ for its initial domain and the range of $\mathbf{E}_J((0,1])$ for its terminal domain. It is a fact, as proved by Strasen, that $\mathbf{E}_H((0,1]) = \mathbf{E}_F(\{0\})$, that $\mathbf{E}_J((0,1]) = \mathbf{E}_G(\{0\})$, and that $V_F + V_H$ is a unitary operator satisfying (1. 13).

There appears to be near at hand a tying together of these two approaches, with the result that one would have explicitly the spectral form of the unitary operator $V_F + V_H$.

For the sake of a bit greater clarity of exposition we shall divide our work into three sections. In the first, that is, in Section 2, we shall define the operators $W_F^{(+)}(\mathbf{A})$, $W_F^{(-)}(\mathbf{A})$, $W_F^{(+)}$, $W_F^{(-)}$ and develop their properties. In Section 3 we shall go on to define and elaborate the spectral measure $\mathbf{K}_F$. And then finally, in Section 4, we present our unitary operator U_F and show that it fulfills (1. 13).

§2. The *W*-operators.

We shall let ϑ denote the function on the closed unit interval defined by

(2. 1)
$$\vartheta(\lambda) \overset{(\mathrm{def.})}{=} \cos^{-1}\sqrt{\lambda}\,.$$

For the two projections F and G—which shall remain fixed throughout our discussion—we consider all the appertinent quantities as defined in Section 1, and we go directly to making the following definition:

(2. 2)
$$W_F^{(+)}(\mathbf{A}) \overset{(\mathrm{def.})}{=} \frac{1}{\sqrt{2}}\left[G\int_{\mathbf{A}}\frac{1}{\sqrt{\lambda(1-\lambda)}}\,\mathbf{E}_F(d\lambda) - \int_{\mathbf{A}}\frac{e^{i\vartheta(\lambda)}}{\sqrt{1-\lambda}}\,\mathbf{E}_F(d\lambda)\right],$$
$$\text{for}\quad \mathbf{A} \subseteqq (\alpha,\beta),\ \ 0<\alpha<\beta<1.$$

It is now an excellent tour of practice in the operational calculus, enhanced in our case by the relationships (1. 4) and (1. 5), to perform the evaluation of the product of $(W_F^{(+)}(\mathbf{A}))^*$ into $W_F^{(+)}(\mathbf{A})$. Carrying through all the details of this we find the very simple result that

(2. 3)
$$(W_F^{(+)}(\mathbf{A}))^* W_F^{(+)}(\mathbf{A}) = \mathbf{E}_F(\mathbf{A}),$$
$$A \subseteqq (\alpha,\beta),\ \ 0<\alpha<\beta<1.$$

This implies in particular that, for any element ψ of the underlying Hilbert space $\mathfrak{H}$,

(2. 4)
$$\|W_F^{(+)}(\mathbf{A})\psi\|^2 = \|\mathbf{E}_F(\mathbf{A})\psi\|^2,$$
$$\mathbf{A} \subseteqq (\alpha,\beta),\ \ 0<\alpha<\beta<1.$$

From this in turn it follows that $\|W_F^{(+)}(\mathbf{A})\varphi\|$ is uniformly arbitrarily small for sufficiently small $\sigma > 0$ if $\mathbf{A}$ is bounded away from 0 and lies in $(0, \sigma]$ or is bounded away from 1 and lies in $[1-\sigma, 1)$. And this fact asserts that there exists the strong limit of $W_F^{(+)}([\varepsilon_1, 1-\varepsilon_2])$ as ε_1 and ε_2 tend to 0 through positive values, and that this limit operator has domain $\mathfrak{H}$. We define

$$W_F^{(+)} = \mathbf{E}_F(\{1\}) + \lim_{\varepsilon_1, \varepsilon_2 \downarrow 0} W_F^{(+)}([\varepsilon_1, 1-\varepsilon_2]). \tag{2. 5}$$

If $\mathbf{A} \subseteqq (\alpha, \beta)$, with $0 < \alpha < \beta < 1$, we readily calculate that the following relation holds:

$$W_F^{(+)}(\mathbf{A}) = W_F^{(+)} \cdot \mathbf{E}_F(\mathbf{A}). \tag{2. 6}$$

We now take this equation as definitional of $W_F^{(+)}(\mathbf{A})$ *for all Borel subsets* $\mathbf{A}$ *of the closed interval* $[0, 1]$. Accordingly we see that $W_F^{(+)}(\cdot)$ is completely additive and that we have the following particular evaluations

$$\begin{cases} W_F^{(+)}(\{0\}) = \mathbf{O} \\ W_F^{(+)}(\{1\}) = \mathbf{E}_F(\{1\}), \\ W_F^{(+)}((0,1]) = W_F^{(+)}. \end{cases} \tag{2. 7}$$

The limit of $W_F^{(+)}([\varepsilon_1, 1-\varepsilon_2])$, as $\varepsilon_1 \downarrow 0$ and $\varepsilon_2 \downarrow 0$, is $W_F^{(+)}((0,1))$ according to (2. 5) and (2. 7), and it follows that the limit of the product $(W_F^{(+)}[\varepsilon_1, 1-\varepsilon_2])^* \cdot W_F^{(+)}([\varepsilon_1, 1-\varepsilon_2])$ is $(W_F^{(+)}((0,1)))^* \cdot W_F^{(+)}((0,1))$. On the other hand, the latter limit is $\mathbf{E}_F((0,1))$, by (2. 3). Hence, we have the equality of these last two expressions, or, equivalently,

$$W_F^{*(+)} W_F^{(+)} = \mathbf{E}_F((0,1]). \tag{2. 8}$$

With this result we immediately obtain from (2. 6) the following result generalizing (2. 3):

$$(W_F^{(+)}(\mathbf{A}))^* W_F^{(+)}(\mathbf{B}) = \mathbf{E}_F(\mathbf{A} \cap \mathbf{B} - \{0\}), \quad \mathbf{A}, \mathbf{B} \subseteqq [0,1]. \tag{2. 9}$$

Along with (2. 2) we introduce another definition:

$$W_F^{(-)}(\mathbf{A}) \overset{(\text{def.})}{=} \frac{1}{\sqrt{2}}\left[G\int_{\mathbf{A}} \frac{1}{\sqrt{\lambda(1-\lambda)}} \mathbf{E}_F(d\lambda) - \int_{\mathbf{A}} \frac{e^{-i\vartheta(\lambda)}}{\sqrt{1-\lambda}} \mathbf{E}_F(d\lambda)\right] \tag{2. 10}$$

for $\mathbf{A} \subseteqq (\alpha, \beta)$, $0 < \alpha < \beta < 1$.

The same calculation as in the case of $W_F^{(+)}(\mathbf{A})$ shows that here too we have the relation

$$(2.\ 11)\qquad (W_F^{(-)}(\mathbf{A}))^* W_F^{(-)}(\mathbf{A}) = \mathbf{E}_F(\mathbf{A}),\quad \mathbf{A} \subseteqq (\alpha, \beta),\ 0 < \alpha < \beta < 1,$$

and consequently also the existence of the strong limit operator, with domain $\mathfrak{H}$,

$$(2.\ 12)\qquad W_F^{(-)} = \lim_{\varepsilon_1, \varepsilon_2 \downarrow 0} W_F^{(-)}([\varepsilon_1, 1-\varepsilon_2]).$$

As before, we find that the following equation is verified for $\mathbf{A} \subseteqq (\alpha, \beta)$ with $0 < \alpha < \beta < 1$, and we take the equation as definitional for all other Borel subsets of $[0, 1]$:

$$(2.\ 13)\qquad W_F^{(-)}(\mathbf{A}) = W_F^{(-)} \cdot \mathbf{E}_F(\mathbf{A}).$$

In particular, then,

$$(2.\ 14)\qquad \begin{cases} W_F^{(-)}(\{0\}) = W_F^{(-)}(\{1\}) = \mathbf{O}, \\ W_F^{(-)}((0,1)) = W_F^{(-)}; \end{cases}$$

and also

$$(2.\ 15)\qquad W_F^{(-)*} W_F^{(-)} = \mathbf{E}_F((0,1)).$$

The relations (2. 13) and (2. 15) combine to give

$$(2.\ 16)\qquad (W_F^{(-)}(\mathbf{A}))^* W_F^{(-)}(\mathbf{B}) = \mathbf{E}_F(\mathbf{A} \cap \mathbf{B} - \{0\} - \{1\}),\quad \mathbf{A},\ \mathbf{B} \subseteqq [0, 1].$$

In addition to (2. 9) and (2. 16) we have the formula for mixed products:

$$(2.\ 17)\qquad (W_F^{(+)}(\mathbf{A}))^* W_F^{(-)}(\mathbf{B}) = \mathbf{O},\quad \mathbf{A},\ \mathbf{B} \subseteqq [0, 1].$$

To obtain this result one again first evaluates the product for $\mathbf{A}, \mathbf{B} \subseteqq (\alpha, \beta)$, $0 < \alpha < \beta < 1$, using (2. 2) and (2. 10), and finds the product to vanish identically. One then determines that $W_F^{(+)*} W_F^{(-)} = \mathbf{O}$ by taking limits. And finally (2. 6) and (2. 13) give the full result (2. 17).

Notice that by taking adjoints one finds that (2. 17) holds just as well with the (+) and (−) interchanged.

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§3. The spectral measure $\mathbf{K}_F$.

It will be convenient to introduce the following notational devices: If $\mathbf{c}$ is any subset of the cell $(-1,1]$, we set

$$\mathbf{c}^+ = (0,1] \cap \mathbf{c}, \tag{3. 1}$$
$$\mathbf{c}^- = (-1,0) \cap \mathbf{c}$$

and

$$-\mathbf{c} = \{x \in (-1,1] \mid -x \in \mathbf{c}\}. \tag{3. 2}$$

Now we make the following definition for any Borel subset $\mathbf{c}$ of $(-1,1]$:

$$\mathring{\mathbf{K}}_F(\mathbf{c}) \overset{(\text{def.})}{=} W_F^{(+)}\mathbf{E}_F(\mathbf{c}^+)W_F^{(+)*} + W_F^{(-)}\mathbf{E}_F(-\mathbf{c}^-)W_F^{(-)*}, \tag{3. 3}$$
$$\mathbf{c} \subseteqq (-1,1].$$

By (2. 6) and (2. 13) we can just as well write

$$\mathring{\mathbf{K}}_F(\mathbf{c}) = W_F^{(+)}(\mathbf{c}^+)(W_F^{(+)}(\mathbf{c}^+))^* + W_F^{(-)}(-\mathbf{c}^-)W_F^{(-)}(-\mathbf{c}^-))^*. \tag{3. 4}$$

From either (3. 3) or (3. 4) we see directly that $\mathring{\mathbf{K}}_F(\mathbf{c})$ is Hermitian. It is also idempotent, as we find by using (2. 8), (2. 15) and (2. 17). Thus, for each $\mathbf{c}$, $\mathring{\mathbf{K}}_F(\mathbf{c})$ is a projection. From (3. 3) we have, moreover, that $\mathring{\mathbf{K}}_F$ is completely additive. Hence, $\mathring{\mathbf{K}}_F$ is a spectral measure on $(-1,1]$. This does not mean, however, that necessarily $\mathring{\mathbf{K}}_F((-1,1]) = I$; indeed we shall see that this is not true in general.

We now evaluate $\mathring{\mathbf{K}}_F$. By virtue of the additivity we may reduce the general computation to separate computations for $\mathbf{c} \subseteqq (0,1]$ and $\mathbf{c} \subseteqq (-1,0)$. By (3. 4) we see that in either of these cases $\mathring{\mathbf{K}}_F(\mathbf{c})$ is of the form $W_F^{(\pm)}(\mathbf{A})$ $(W_F^{(\pm)}(\mathbf{A}))^*$ for some $\mathbf{A} \subseteqq (0,1]$. Since the latter products are actually additive (see (2. 6) and (2. 13) again) we may accomplish their general evaluation by calculating them separately for $\mathbf{A} = \{1\}$ and $\mathbf{A} \subseteqq (\alpha,\beta)$, $0 < \alpha < \beta < 1$. This last case, of course, enables the calculation for any $\mathbf{A} \subseteqq (0,1)$ through the continuity implied by the complete additivity.

From (2. 7) we have immediately

$$W_F^{(+)}(\{1\})(W_F^{(+)}(\{1\}))^* = \mathbf{E}_F(\{1\}), \tag{3. 5}$$

and from (2. 14) we have

$$W_F^{(-)}(\{1\})(W_F^{(-)}(\{1\}))^* = \mathbf{O}. \tag{3. 6}$$

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By computing again from the original defining formulas (2. 2) and (2. 10), we find

$$(3.7)\quad W_F^{(\pm)}(\mathbf{A})(W_F^{(\pm)}(\mathbf{A}))^* = \frac{1}{2}\left[\begin{array}{l}\int_{\mathbf{A}}\frac{1}{1-\lambda}\mathbf{E}_G(d\lambda)+\int_{\mathbf{A}}\frac{1}{1-\lambda}\mathbf{E}_F(d\lambda)\\ -F\int_{\mathbf{A}}\frac{e^{\pm i\vartheta(\lambda)}}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_G(d\lambda)-G\int_{\mathbf{A}}\frac{e^{\mp i\vartheta(\lambda)}}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_F(d\lambda)\end{array}\right],$$

$$\mathbf{A}\subseteqq(\alpha,\beta),\ 0<\alpha<\beta<1.$$

(Notice here that for the first time the other basic spectral measure, $\mathbf{E}_G$, has entered explicitly into our formulas). According to the discussion in the preceding paragraph formulas (3. 5), (3. 6), and (3. 7) are sufficient now to enable us to calculate $\mathring{\mathbf{K}}_F(\mathbf{C})$ for any $\mathbf{C}\subseteqq(-1,1]$.

For $\mathbf{C}=\{0\}$ both $\mathbf{C}^+$ and $\mathbf{C}^-$ are the empty set, and (3. 3) therefore tells that

$$(3.8)\qquad \mathring{\mathbf{K}}_F(\{0\})=\mathbf{O}.$$

For $\mathbf{C}=\{1\}$ we apply (3. 5) to (3. 4) and we get

$$(3.9)\qquad \mathring{\mathbf{K}}(\{1\})=\mathbf{E}_F(\{1\}).$$

Completing a basic set of computational formulas for $\mathring{\mathbf{K}}_F$ are the following two, derived from (3. 7) and (3. 4):

$$(3.10)\quad \mathring{\mathbf{K}}_F(\mathbf{C}) = \frac{1}{2}\left[\begin{array}{l}\int_{\mathbf{C}}\frac{1}{1-\lambda}\mathbf{E}_G(d\lambda)+\int_{\mathbf{C}}\frac{1}{1-\lambda}\mathbf{E}_F(d\lambda)\\ -F\int_{\mathbf{C}}\frac{e^{i\vartheta(\lambda)}}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_G(d\lambda)-G\int_{\mathbf{C}}\frac{e^{-i\vartheta(\lambda)}}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_F(d\lambda)\end{array}\right]$$

$$\mathbf{C}\subseteqq(\alpha,\beta),\ 0<\alpha<\beta<1;$$

$$(3.11)\quad \mathring{\mathbf{K}}_F(-\mathbf{C}) = \frac{1}{2}\left[\begin{array}{l}\int_{\mathbf{C}}\frac{1}{1-\lambda}\mathbf{E}_G(d\lambda)+\int_{\mathbf{C}}\frac{1}{1-\lambda}\mathbf{E}_F(d\lambda)\\ -F\int_{\mathbf{C}}\frac{e^{-i\vartheta(\lambda)}}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_G(d\lambda)-G\int_{\mathbf{C}}\frac{e^{i\vartheta(\lambda)}}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_F(d\lambda)\end{array}\right]$$

$$\mathbf{C}\subseteqq(\alpha,\beta),\ 0<\alpha<\beta<1.$$

From these last two formulas—and referring, of course, to (3. 4)—we determine, for example,

$$(3.12)\quad \mathring{\mathbf{K}}_F((-1,1)) = \lim_{\varepsilon\downarrow 0}\left[(I-F)\int_{\varepsilon}^{1-\varepsilon}\frac{1}{1-\lambda}\mathbf{E}_G(d\lambda)+(I-G)\int_{\varepsilon}^{1-\varepsilon}\frac{1}{1-\lambda}\mathbf{E}_F(d\lambda)\right].$$

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From this we find that

$$F\mathring{\mathbf{K}}_F((-1,1)) = \mathring{\mathbf{K}}_F((-1,1))F = \mathbf{E}_F((0,1)); \tag{3. 13}$$

and therefore furthermore, by (3. 9),

$$F\mathring{\mathbf{K}}_F((-1,1]) = \mathring{\mathbf{K}}_F((-1,1])F = \mathbf{E}_F((0,1]). \tag{3. 14}$$

This relationship shows that $\mathring{\mathbf{K}}_F((-1,1])$ is not in general the identity operator. We therefore now define, for all Borel subsets $\mathbf{c}$ of $(-1,1]$,

$$\mathbf{K}_F(\mathbf{c}) \overset{\text{(def.)}}{=} \begin{cases} \mathring{\mathbf{K}}_F(\mathbf{c}) & \textit{if} \quad 0 \notin \mathbf{c}, \\ \mathring{\mathbf{K}}_F(\mathbf{c}) + [I - \mathring{\mathbf{K}}_F((-1,1])] & \textit{if} \quad 0 \in \mathbf{c}. \end{cases} \tag{3. 15}$$

$\mathbf{K}_F$ is clearly a resolution of the identity on $(-1,1]$.

If F and G are interchanged in all of the analysis we have performed then we come out with a spectral measure $\mathbf{K}_G$ on $(-1,1]$. It is of interest to note the symmetry of the right-hand side of (3. 12) and consequently the fact that

$$\mathbf{K}_G(\{0\}) = \mathbf{K}_F(\{0\}). \tag{3. 16}$$

We notice furthermore that the right-hand sides of (3. 10) and (3. 11) pass into each other when F and G are interchanged. This fact, together with (3. 16) implies the general result that

$$\mathbf{K}_G(\mathbf{c}) = \mathbf{K}_F(-\mathbf{c}), \quad \mathbf{c} \subseteqq (-1,1]. \tag{3. 17}$$

§4. The unitary equivalence.

We now finally define the operator U_F:

$$U_F \overset{\text{(def.)}}{=} \int_{(-1,0)} e^{i\vartheta(-\lambda)}\mathbf{K}_F(d\lambda) + \int_{[0,1]} e^{-i\vartheta(\lambda)}\mathbf{K}_F(d\lambda). \tag{4. 1}$$

Let us note right at the start that if U_G is defined similarly, with $\mathbf{K}_G$ in place of $\mathbf{K}_F$ in (4. 1), then (3. 17) immediately produces the result that

$$U_G = U_F^*. \tag{4. 2}$$

A straightforward calculation shows that

$$U_F U_F^* = U_F^* U_F = I, \tag{4. 3}$$

so that U_F is indeed a unitary operator. Similarly, U_G is unitary.

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We may obtain a more explicit evaluation of U_F, as follows. By (3. 15) we have

$$(4.4)\qquad \int_{\{0\}} e^{-i\vartheta(\lambda)}\mathbf{K}_F(d\lambda) = -i[I-\mathring{\mathbf{K}}_F((-1,1])].$$

With (3. 9) we obtain further

$$(4.5)\qquad \int_{\{1\}} e^{-i\vartheta(\lambda)}\mathbf{K}_F(d\lambda) = \mathbf{E}_F(\{1\}).$$

Then, from (3. 11) we get

$$(4.6)\qquad \int_{(-1,0)} e^{i\vartheta(-\lambda)}\mathbf{K}_F(d\lambda) = \frac{1}{2}\lim_{\varepsilon\downarrow 0}\left\{\begin{array}{l}\displaystyle\int_\varepsilon^{1-\varepsilon}\frac{e^{i\vartheta(\lambda)}}{1-\lambda}\mathbf{E}_G(d\lambda)+\int_\varepsilon^{1-\varepsilon}\frac{e^{i\vartheta(\lambda)}}{1-\lambda}\mathbf{E}_F(d\lambda)\\ \displaystyle -F\int_\varepsilon^{1-\varepsilon}\frac{1}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_G(d\lambda)-G\int_\varepsilon^{1-\varepsilon}\frac{e^{2i\vartheta(\lambda)}}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_F(d\lambda)\end{array}\right\}$$

and from (3. 10)

$$(4.7)\qquad \int_{(0,1)} e^{-i\vartheta(\lambda)}\mathbf{K}_F(d\lambda) = \frac{1}{2}\lim_{\varepsilon\downarrow 0}\left\{\begin{array}{l}\displaystyle\int_\varepsilon^{1-\varepsilon}\frac{e^{-i\vartheta(\lambda)}}{1-\lambda}\mathbf{E}_G(d\lambda)+\int_\varepsilon^{1-\varepsilon}\frac{e^{-i\vartheta(\lambda)}}{1-\lambda}\mathbf{E}_F(d\lambda)\\ \displaystyle -F\int_\varepsilon^{1-\varepsilon}\frac{1}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_G(d\lambda)-G\int_\varepsilon^{1-\varepsilon}\frac{e^{-2i\vartheta(\lambda)}}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_F(d\lambda)\end{array}\right\}.$$

Now taking the sum of (4. 4)-(4. 7) inclusive, we obtain

$$(4.8)\qquad U_F = \mathbf{E}_F(\{1\}) - i[I-\mathring{\mathbf{K}}_F((-1,1])] + \lim_{\varepsilon\downarrow 0}\left\{\begin{array}{l}\displaystyle\int_\varepsilon^{1-\varepsilon}\frac{\sqrt{\lambda}}{1-\lambda}\mathbf{E}_G(d\lambda)+\int_\varepsilon^{1-\varepsilon}\frac{\sqrt{\lambda}}{1-\lambda}\mathbf{E}_F(d\lambda)\\ \displaystyle -F\int_\varepsilon^{1-\varepsilon}\frac{1}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_G(d\lambda)-G\int_\varepsilon^{1-\varepsilon}\frac{2\lambda-1}{\sqrt{\lambda}\,(1-\lambda)}\mathbf{E}_F(d\lambda)\end{array}\right\}.$$

With this expression for U_F we are now able to carry out the calculations to verify that this operator provides the desired unitary equivalence; that is, that it satisfies (1. 13). Indeed, let (4. 8) be multiplied on the right by $\mathbf{E}_F((0,1])$. The result of this multiplication into $I-\mathring{\mathbf{K}}_F((-1,1])$ is $\mathbf{O}$, as can be seen by (3. 14). The product with the remaining terms evaluates to

$$(4.9)\qquad \left(\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1\frac{1}{\sqrt{\lambda}}\mathbf{E}_G(d\lambda)\right)F$$

which is identical with (see (1. 7)) the product $S_F^{-\frac{1}{2}}GF$ which is (see (1. 8)) precisely V_F. Thus we have

(4. 10) $$U_F \mathbf{E}_F((0,1]) = V_F.$$

Multiplying this on the right by $\mathbf{E}_F(\mathbf{A})$, for $\mathbf{A} \subseteqq (0,1]$, gives (see (1. 9))

(4. 11) $$U_F \mathbf{E}_F(\mathbf{A}) = V_{F;\mathbf{A}}.$$

This is half of the relationship (1. 13). To achieve the other half consider our analysis having been carried through with F and G interchanged. In this way we should obtain, corresponding to (4. 10), the equation

(4. 12) $$U_G \mathbf{E}_G((0,1]) = V_G.$$

Taking adjoints in this equation, and remembering that V_F and V_G are adjoints of each other (see (1. 11)) and likewise U_F and U_G (see (4. 2)) we get

(4. 13) $$\mathbf{E}_G((0,1])U_F = V_F.$$

Multiplying this on the left by $\mathbf{E}_G(\mathbf{A})$ gives (utilizing (1. 11) and the adjoint of the second equation in (1. 9))

(4. 14) $$\mathbf{E}_G(\mathbf{A})U_F = V_{F;\mathbf{A}}.$$

which is the second half of (1. 13).

We have thus fully established that our unitary operator U_F, given by (4. 1), fulfills (1. 13).

Of course, U_G satisfies a corresponding pair of equalities involving $V_{G;\mathbf{A}}$.

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