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Unitary Equivalence of Fock Representations on the Weyl Algebra

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Abstract. A necessary and sufficient condition for unitary equivalence of pure quasi-free states over the Weyl algebra is proved. Some partial results on states over the Weyl algebra are formulated in Theorem 1, and Lemmas 1, 4, 5 and 6.

I. Introduction

As early as 1931 von Neumann [1] proved the uniqueness of the Schrödinger representation, for Boson-systems with a finite number of degrees of freedom. Afterwards a number of people [2] proved that for Boson-systems with an infinite number of degrees of freedom this theorem fails and that there exists a lot of inequivalent representations.

Kastler [3] gave for the first an algebraic formulation of this problem and proved von Neumann's theorem [1] in a more general form. He defined the underlying C^*-algebra for a free Boson-system roughly speaking generated by the Boson creation and annihilation operators, and formulated the problem of equivalence in terms of states on this algebra.

In this work we follow the same method and prove a necessary and sufficient condition (see Theorem 2 below) in order that two pure quasi-free states on the Boson C^*-algebra are unitarily equivalent.

The essential technical difficulty which we had to solve to derive the proof of the criterium, is the construction of finite symplectic subspaces of a sympletic space H which are invariant under two different complex structures on H. This problem is solved in the case that the product of the complex structures has a pure point spectrum.

Some other partial results on states over the Weyl algebra are formulated as remarks following the lemmas, the proofs being trivial extensions of the proofs of the lemmas.

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II. Pure Quasi-free States on the Weyl algebra $\overline{\Delta(H, \sigma)}$

Let (H, σ) be a separable symplectic space, i.e. a real vector space H, equipped with a regular, antisymmetric, real bilinear form. Hence H is a locally convex topological space equipped with the topology defined by the semi-norms

$$\varrho_\varphi : \psi \to |\sigma(\varphi, \psi)|$$

and we suppose that H is complete for this topology, we call H σ-complete.

Let $\Delta(H, \sigma)$ be the algebra generated by finite linear combinations of the functionals $\delta : \psi \in H \to \delta_\psi$ defined by

$$\begin{aligned} \delta_\psi(\varphi) &= 0 \quad \text{if} \quad \psi \neq \varphi \\ &= 1 \quad \text{if} \quad \psi = \varphi \end{aligned}$$

with the product law:

$$\delta_\psi \delta_\varphi = e^{-i\sigma(\psi, \varphi)} \, \delta_{\psi+\varphi} \, .$$

The mapping $\delta_\psi \to \delta_\psi^* = \delta_{-\psi}$ is an involution and

$$\left\| \sum_{i=1}^{n} a_i \delta_{\psi_i} \right\| = \sum_{i=1}^{n} |a_i| \quad \text{with} \quad a_i \in \mathbb{C}$$

is a norm on $\Delta(H, \sigma)$ such that $\Delta(H, \sigma)$ turns out to be a normed *-algebra.

The set of representations π of $\Delta(H, \sigma)$ such that the mapping $\lambda \in \mathbb{R} \to \pi(\delta_{\lambda\psi})$ is strongly continuous, determines a unique C^*-algebra norm on $\Delta(H, \sigma)$. Its closure $\mathfrak{A} = \overline{\Delta(H, \sigma)}$ is a C^*-algebra, which we call the Weyl algebra. For more details see [4].

A state on the Weyl algebra is a positive linear functional, normalized to one.

Any operator J on H satisfying

$J^+ = -J$ ("+" adjoint with respect to σ)

$J^2 = -\mathbf{1}$ ($\mathbf{1}$ unit operator)

$s_J(\psi, \psi) \equiv -\sigma(J\psi, \psi) > 0$ for all $\psi \neq 0$, $\psi \in H$

defines a complex structure on the real space H. Given such a complex structure J on H, the linear functional on $\Delta(H, \sigma)$ defined by

$$\omega_J(\delta_\psi) = \exp\{-\tfrac{1}{2} s_J(\psi, \psi)\}$$

extends to a state ω_J on $\mathfrak{A} = \overline{\Delta(H, \sigma)}$. In fact ω_J is a pure quasi-free state (see [5]).

In what follows we denote by J and K two such complex structures, by $s_J(\psi, \varphi) = -\sigma(J\psi, \varphi)$ and $s_K(\psi, \varphi) = -\sigma(K\psi, \varphi)$ the corresponding real scalar products and by ω_J respectively ω_K the corresponding pure states; further by $\pi_J(\pi_K)$, $\mathscr{H}_J(\mathscr{H}_K)$ and $R_J = \pi_J(\mathfrak{A})''$ ($R_K = \pi_K(\mathfrak{A})''$) the GNS representation, representation space and the associated von Neumann algebra induced by the state $\omega_J(\omega_K)$.

Consider a net $\mathcal{G} = \{H_\alpha\}_{\alpha \in I}$ of symplectic subspaces of H satisfying:

(i) $JH_\alpha \subseteqq H_\alpha$; $KH_\alpha \subseteqq H_\alpha$ for each $H_\alpha \in \mathcal{G}$,

(ii) each H_α is a finite regular symplectic subspace of H,

(iii) to all pairs $H_\alpha, H_\beta \in \mathcal{G}$ there is a $H_\gamma \in \mathcal{G}$ with $H_\alpha \cup H_\beta \subseteqq H_\gamma$,

(iv) $\bigcup_{\alpha \in I} H_\alpha$ generates H.

Then $\mathcal{I} = \{\mathfrak{A}_\alpha = \overline{\Delta(H_\alpha, \sigma)}\}_{\alpha \in I}$ is a net for $\mathfrak{A}$: i.e. it is a collection of C^*-subalgebras of $\mathfrak{A}$ satisfying:

(i) to all pairs $\mathfrak{A}_\alpha, \mathfrak{A}_\beta \in \mathcal{I}$ there is a $\mathfrak{A}_\gamma \in \mathcal{I}$ with $\mathfrak{A}_\alpha \cup \mathfrak{A}_\beta \subseteqq \mathfrak{A}_\gamma$,

(ii) the unit of $\mathfrak{A}$ is contained in all $\mathfrak{A}_\alpha \in \mathcal{I}$,

(iii) for any continuous representation π (i.e. $\psi \to \pi(\delta_\psi)$ is continuous) $\bigcup_{\alpha \in I} \pi(\mathfrak{A}_\alpha)$ is weakly dense in $\pi(\mathfrak{A})''$.

By analogous techniques as in [6] we prove:

Theorem 1. *The pure states ω_J and ω_K on $\overline{\Delta(H, \sigma)}$ are quasi-equivalent (hence unitarily equivalent) if and only if there is a $\mathfrak{A}_\alpha \in \mathcal{I}$ such that*

$$\|(\omega_J - \omega_K) | \mathfrak{A}_\alpha^c\| < \varepsilon \quad \textit{for} \quad \varepsilon > 0 .$$

$\mathfrak{A}_\alpha^c$ is the set of elements of $\mathfrak{A}$ commuting with $\mathfrak{A}_\alpha$.

Proof. Suppose ω_J and ω_K equivalent, then it follows immediately from [6, Corollary to Propositions 2 and 3] that there is a $\mathfrak{A}_\alpha \in \mathcal{I}$ such that $\|(\omega_J - \omega_K) | \mathfrak{A}_\alpha^c\| < \varepsilon$. The converse is proved by a slight modification of the proof of Proposition 13 of [6] and using von Neumann's theorem [1] yielding that for all $H_\alpha \in \mathcal{G}$ the restrictions $\omega_J | \mathfrak{A}_\alpha$ and $\omega_K | \mathfrak{A}_\alpha$ are type I states, hence that ω_J and ω_K are locally normal states in this sence. We omit further details of the proof. Q.E.D.

III. Unitary Equivalence

Now we will apply Theorem 1 to establish the theorem of unitary equivalence. First we prove a number of Lemma's.

Lemma 1. *If ω_J and ω_K are unitary equivalent states, then the operator $1 + JK$ is a compact operator with respect to the metric defined by s_K or by s_J.*

Proof. We prove it for s_J. For s_K the same proof can be repeated. Now ω_J and ω_K being equivalent factor states, for every $\varepsilon > 0$ there is a finite subspace H_0 such that [6; Corollary to Propositions 2 and 3]

$$\|(\omega_J - \omega_K) | \overline{\Delta(H_0, \sigma)}^c\| = \sup_{\substack{x \in \overline{\Delta(H_0, \sigma)}^c \\ \|x\| \leqq 1}} |\omega_J(x) - \omega_K(x)| < \varepsilon .$$

This follows from the definitions of the nets $\mathscr{G}$ and $\mathscr{I}$. The condition of invariance for J and K is irrelevant for this statement. Hence

$$\|(\omega_J-\omega_K)|\overline{\Delta(H_0^\perp,\sigma)}\|<\varepsilon$$

where $H_0^\perp = H\ominus H_0$ is the orthogonal complement of H_0 with respect to σ. Moreover H_0 can be taken as an invariant subspace for the operator J (not a priori invariant for J and K).

In particular for all $\psi\in H_0^\perp$ such that $s_J(\psi,\psi)=1$

$$\sup_{\psi\in H_0^\perp}|\omega_J(\delta_\psi)-\omega_K(\delta_\psi)|<\varepsilon\,.$$

Hence

$$\sup_{\psi\in H_0^\perp}|1-e^{\frac{1}{2}\sigma((J-K)\psi,\psi)}|<\varepsilon e^{\frac{1}{2}}\,.$$

Since the function $x\to\exp x, x\in\mathbb{R}$ is continuous, it follows from the previous inequality that for any $\delta>0$ there is a finite subspace H_0 such that

$$\sup_{\substack{\psi\in H_0^\perp\\ s_J(\psi,\psi)=1}}|\sigma((J-K)\psi,\psi)|<\delta$$

which is equivalent with

$$\sup_{\substack{\psi\in H_0^\perp\\ s_J(\psi,\psi)=1}}|s_J((\mathbf{1}+JK)\psi,\psi)|<\delta\,.$$

Let E_0 be the orthogonal projection on H_0 then we have that

$$\|(\mathbf{1}+JK)-E_0(\mathbf{1}+JK)-(\mathbf{1}+JK)E_0-E_0(\mathbf{1}+JK)E_0\|_{s_J}<\delta$$

where

$$\|A\|_{s_J}=\sup_{\psi\in H}\left(\frac{s_J(A\psi,A\psi)}{s_J(\psi,\psi)}\right)^{\frac{1}{2}}.$$

Since E_0 is of finite rank, this inequality proves that the operator $\mathbf{1}+JK$ can be approximated in norm by finite rank operators and hence is compact. In particular the operator JK has a pure point spectrum. Q.E.D.

Remark 1. Without proof we remark that an analogous result as in Lemma 1 can be proved for all quasi-free states on the Weyl algebra.

Lemma 2. *Let E be a non-zero finite projection, commuting with J and K. Consider the operator $A=-\frac{1}{2}(\mathbf{1}+JK)E$. Then A is hermitian with respect to s_J and we have the following inequalities.*

$$1\leqq 1+\tfrac{1}{2}\operatorname{Tr}A\leqq \det_E(\mathbf{1}+A)\leqq\exp(\tfrac{1}{2}\operatorname{Tr}A)$$

where $\det_E(\mathbf{1}+A)$ *is defined as the product of the eigenvalues of* $(\mathbf{1}+A)E$ *on* EH.

Proof. First remark that

$$\begin{aligned} s_J((-JK)\psi,\varphi) &= -\sigma(K\psi,\varphi)=\sigma(\psi,K\varphi)=-\sigma(J\psi,(-JK)\varphi) \\ &= s_J(\psi,(-JK)\varphi)=s_K(\psi,\varphi) \end{aligned}$$

and that s_K is a scalar product, hence the operator $-JK$ is strictly positive with respect to s_J.

Since $[E,J]_-=[E,K]_-=0$, the operator $-JKE$ is a hermitian finite rank operator on EH which can be diagonalized. Let ψ be an eigenvector in EH of $(-JKE)$ with eigenvalue λ then $\lambda>0$ and

$$(-JK)K\psi=J\psi=-K(KJ)\psi=-K(JK)^{-1}\psi=\frac{1}{\lambda}K\psi$$

hence $1/\lambda$ is an eigenvalue of $-JK$ with eigenvector $K\psi\in EH$.

Consequently, if $\{\lambda_i|i=1,\dots,n\}$ are the eigenvalues of $-JKE$, then $\{1/\lambda_i|i=1,\dots,n\}$ is exactly the same set of numbers. Therefore

$$\operatorname{Tr}A=\sum_{i=1}^{n}\left(\frac{\lambda_i-1}{2}\right)=\sum_{\lambda_i\geqq 1}\left(\frac{\lambda_i+\frac{1}{\lambda_i}-2}{2}\right).$$

Otherwise

$$\det_E(\mathbf{1}+A)=\prod_{i=1}^{n}\left(\frac{1+\lambda_i}{2}\right)=\prod_{\lambda_i\geqq 1}\left(1+\frac{\lambda_i+\frac{1}{\lambda_i}-2}{4}\right).$$

Let

$$\mu_i=\frac{1}{2}\left(\lambda_i+\frac{1}{\lambda_i}-2\right),$$

since $\lambda_i>0$ we have $\mu_i\geqq 0$ hence

$$1\leqq 1+\frac{1}{2}\sum_{\lambda_i>1}\mu_i\leqq\prod_{\lambda_i>1}\left(1+\frac{\mu_i}{2}\right)\leqq\exp\left(\frac{1}{2}\sum_{\lambda_i>1}\mu_i\right)$$

which is equivalent to

$$1\leqq 1+\frac{1}{2}\operatorname{Tr}A\leqq\det_E(\mathbf{1}+A)\leqq\exp\frac{1}{2}\operatorname{Tr}A.$$

Q.E.D.

Lemma 3. *Let* $\{H_\alpha\}_{\alpha\in I}$ *be an increasing and absorbing net of finite,* J*-invariant subspaces of* H *and let the real dimension of* H_α *be* $2n_\alpha$ *for all*

$\alpha \in I$. *Then the elements*

$$P(H_\alpha) = \left(\frac{1}{\pi}\right)^{n_\alpha} \int_{\psi \in H_\alpha} \omega_J(\delta_\psi)\, \delta_\psi d\psi$$

where $d\psi$ *is the Lebesgue measure on* H *induced by the metric determined by* s_J, *form a decreasing net of projections in* $\overline{\Delta(H, \sigma)}$

Proof. Since $\|\delta_\psi\| = 1$ for all $\alpha \in I$

$$\|P(H_\alpha)\| \leqq \left(\frac{1}{\pi}\right)^{n_\alpha} \int_{\psi \in H_\alpha} \omega_J(\delta_\psi)\, d\psi < \infty$$

and further $\overline{\Delta(H_\alpha, \sigma)}$ is a simple type 1 C^*-algebra, hence $P(H_\alpha) \in \overline{\Delta(H, \sigma)}$. Further $\omega_J(\delta_\psi) = \omega_J(\delta_{-\psi})$ and so

$$P(H_\alpha)^* = \left(\frac{1}{\pi}\right)^{n_\alpha} \int_{\psi \in H_\alpha} \omega_J(\delta_\psi)\, \delta_{-\psi} d\psi = \left(\frac{1}{\pi}\right)^{n_\alpha} \int_{\varphi \in H_\alpha} \omega_J(\delta_\varphi)\, \delta_\varphi d\varphi = P(H_\alpha)\,.$$

It remains to prove the inclusion, if $H_\alpha \subsetneqq H_\beta$ then

$$P(H_\alpha)\, P(H_\beta) = P(H_\beta)$$

because for $H_\alpha = H_\beta$ this proves also that $P(H_\alpha)$ is a projection.

Consider now

$$\begin{aligned} \pi^{(n_\alpha + n_\beta)} P(H_\alpha)\, P(H_\beta) &= \int_{\psi \in H_\alpha} \int_{\varphi \in H_\beta} \omega_J(\delta_\psi)\, \omega_J(\delta_\varphi)\, \delta_\psi \delta_\varphi\, d\psi\, d\varphi \\ &= \int_{\psi \in H_\alpha} \int_{\xi \in H_\beta} \omega_J(\delta_\psi)\, \omega_J(\delta_{\xi - \psi})\, \delta_\xi e^{-i\sigma(\psi, \xi)}\, d\psi\, d\xi \\ &= \int_{\xi \in H_\beta} \exp(-\tfrac{1}{2} s_J(\xi, \xi))\, \delta_\xi\, I(\xi)\, d\xi \end{aligned}$$

where

$$I(\xi) = \int_{\psi \in H_\alpha} \exp\left[-s_J(\psi, \psi) + s_J(\xi, \psi) - i\sigma(\psi, \xi)\right] d\psi\,.$$

We prove now that $I(\xi) = \pi^{n_\alpha}$, yielding the result. Consider the symplectic basis[1] of $H_\beta : \{e_i, Je_i \mid i = 1, \ldots, n_\beta\}$ with respect to J, such that $\{e_i, Je_i \mid i = 1, \ldots, n_\alpha\}$ is a symplectic basis of H_α with respect to J, then

$$\psi = \sum_{i=1}^{n_\alpha} (x_i e_i + y_i J e_i)\,,$$

$$\xi = \sum_{i=1}^{n_\beta} (\lambda_i e_i + \mu_i J e_i)\,,$$

$$x_i, y_i, \lambda_i, \mu_i \in \mathbb{R} \text{ (real numbers)}$$

[1] $\{e_i, Je_i\}$ is a symplectic basis if it is a basis and if $\{e_i\}$ is an orthonormal set for s_J.

and one checks that

$$I(\xi) = \prod_{i=1}^{n_\alpha} \int_{-\infty}^{+\infty} \exp\left[-(x_i + \lambda_i - i\mu_i)^2 - (y_i + \mu_i + i\lambda_i)^2\right] dx_i \, dy_i$$

hence $I(\xi) = \pi^{n_\alpha}$. Q.E.D.

Lemma 4. *Suppose H_0 is a J-invariant finite subspace of H, let E_0 be the orthogonal projection on H_0, then with the same notations as in Lemma 3:*

$$\omega_K(P(H_0)) = \left[\det_{E_0}\left(\frac{1 - JK}{2}\right)\right]^{-\frac{1}{2}}$$

Proof. By definition of $P(H_0)$ we have

$$\omega_K(P(H_0)) = \left(\frac{1}{\pi}\right)^{n_0} \int_{\psi \in H_0} \exp\left[-\frac{1}{2} s_J(\psi, \psi) - \frac{1}{2} s_K(\psi, \psi)\right] d\psi .$$

For all $\psi \in H_0$

$$\frac{1}{2} s_J(\psi, \psi) + \frac{1}{2} s_K(\psi, \psi) = s_J(Q\psi, \psi)$$

where $Q = E_0 \left(\frac{1 - JK}{2}\right) E_0$ is a strictly positive finite rank operator on (H_0, s_J). There exists an orthonormal basis $\{\psi_i | i = 1, \ldots, 2n_0\}$ with respect to s_J such that

$$Q\psi_i = \lambda_i \psi_i .$$

Let $\psi = \sum_{i=1}^{2n_0} x_i \psi_i$ then

$$s_J(Q\psi, \psi) = \sum_{i=1}^{2n_0} \lambda_i x_i^2$$

and

$$\omega_K(P(H_0)) = \prod_{i=1}^{2n_0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\lambda_i x_i^2} dx_i$$

$$= \prod_{i=1}^{2n_0} \frac{1}{\sqrt{\lambda_i}} = [\det_{E_0} Q]^{-\frac{1}{2}} .$$

Q.E.D.

Lemma 5. *With the same notations and assumptions as in Lemma 3 the following holds true:*

(i) *for all subspaces H_α one has $\omega_J(P(H_\alpha)) = 1$,*

(ii) *if ω is any state on $\overline{\Delta(H, \sigma)}$ such that for all H_α, $\omega(P(H_\alpha)) = 1$ then $\omega = \omega_J$.*

Proof. (i) Follows immediately from Lemma 4 putting $K=J$. To prove (ii) for each H_α consider the restrictions $\omega_\alpha=\omega|\mathfrak{A}_\alpha$ and $\omega_J^\alpha=\omega_J|\mathfrak{A}_\alpha$, because the union $\bigcup_\alpha \pi(\mathfrak{A}_\alpha)$ is dense in $\pi(\mathfrak{A})$ for any continuous representation π it is sufficient to prove that $\omega_\alpha=\omega_J^\alpha$ for each H_α. This will now be proved.

Let $\pi_J^\alpha, \Omega_J^\alpha, \mathscr{H}_J^\alpha$ be respectively the representation, cyclic vector and representation space induced by ω_J^α, then it follows from (i) that

$$\pi_J^\alpha(P(H_\alpha))\,\Omega_J^\alpha=\Omega_J^\alpha .$$

Furthermore the set $\{\pi_J^\alpha(\delta_\varphi)\,\Omega_J^\alpha\,|\,\varphi\in H_\alpha\}$ is dense in the representation space $\mathscr{H}_J^\alpha$ and an easy calculation (see e.g. the proof of Lemma 3) shows that

$$\pi_J^\alpha(P(H_\alpha))\,\pi_J^\alpha(\delta_\varphi)\,\Omega_J^\alpha=(\Omega_J^\alpha,\pi_J^\alpha(\delta_\varphi)\,\Omega_J^\alpha)\,\Omega_J^\alpha .$$

Hence $\pi_J^\alpha(P(H_\alpha))$ is the projection operator on $\Omega_J^\alpha\in\mathscr{H}_J^\alpha$.

Each H_α being finite dimensional it follows from von Neumann's theorem [1] that

$$\omega_\alpha=\mathrm{Tr}_{\mathscr{H}_J^\alpha}\,\varrho^\alpha\circ\pi_J^\alpha$$

where ϱ^α is a density matrix ($0\leqq\varrho^\alpha\leqq\mathbf{1}$, $\mathrm{Tr}\,\varrho^\alpha=1$). From $\omega(P(H_\alpha))=1$ we get $\omega_\alpha(P(H_\alpha))=1$ and $\mathrm{Tr}_{\mathscr{H}_J^\alpha}\varrho^\alpha\pi_J^\alpha(P(H_\alpha))=1$. Hence $(\Omega_J^\alpha,\varrho^\alpha\Omega_J^\alpha)=1$ and ϱ^α being a density matrix, it follows that ϱ^α is the projection operator on Ω_J^α and $\omega_\alpha=\omega_J^\alpha$. Q.E.D.

Corollary 1. *With the same notations and assumptions as in Lemma 3 we have that* $\|\omega_J-\omega_K\|=2(1-q)^{\frac{1}{2}}$ *where* $q=\inf_{H_\alpha}\omega_K(P(H_\alpha))$.

Proof. It follows from Lemma 3 and 5 that Lemma 2.5 of [7] holds true for the net of projections $\{P(H_\alpha)\}_{\alpha\in I}$ as defined in Lemma 3. Hence ω_K being pure, we have the result. Q.E.D.

Lemma 6. *Suppose that ω_J and ω_K are pure quasi-free states on the Weyl algebra $\overline{\Delta(H,\sigma)}$ and that H_0 is a finite, J and K invariant subspace of H, then*

$$\|(\omega_J-\omega_K)|\overline{\Delta(H_0,\sigma)}^c\|=\|(\omega_J-\omega_K)|\overline{\Delta(H_0^\perp,\sigma)}\| .$$

Proof. Let $\Psi=\pi_J\oplus\pi_K$ be the direct sum of the representations π_J and π_K; Ψ is a representation on $\mathscr{H}=\mathscr{H}_J\oplus\mathscr{H}_K$. Let $x=\Omega_J\oplus\mathbf{0}$, $y=\mathbf{0}\oplus\Omega_K$ and denote by ω_z, $z\in\mathscr{H}$ the vector state $\omega_z(A)=(z,Az)$ on $B(\mathscr{H})$. Then

$$\|(\omega_J-\omega_K)|\overline{\Delta(H_0^\perp,\sigma)}\|=\|(\omega_x-\omega_y)|\Psi(\overline{\Delta(H_0^\perp,\sigma)})\|$$

and

$$\|(\omega_J-\omega_K)|\overline{\Delta(H_0,\sigma)}^c\|=\|(\omega_x-\omega_y)|(\overline{\Delta(H_0,\sigma)})^c\| .$$

Applying Kaplansky density theorem we get

$$\|(\omega_J - \omega_K)\,\overline{\Delta(H_0^\perp, \sigma)}\| = \|(\omega_x - \omega_y) \,|\, \Psi(\overline{\Delta(H_0^\perp, \varrho})''\|$$

and

$$\|(\omega_J - \omega_K)\,\overline{\Delta(H_0, \sigma)}^c\| = \|(\omega_x - \omega_y)\,|\, \Psi(\overline{\Delta(H_0, \sigma)}^c)''\| \,.$$

It remains to prove that

$$\Psi(\overline{\Delta(H_0^\perp, \sigma)})'' = \Psi(\overline{\Delta(H_0, \sigma)}^c)'' \,.$$

H_0 being J and K invariant, the restrictions $\pi_J|\overline{\Delta(H_0, \sigma)}$, $\pi_K|\overline{\Delta(H_0, \sigma)}$ and hence $\Psi|\overline{\Delta(H_0, \sigma)}$, $\Psi|\overline{\Delta(H_0^\perp, \sigma)}$ [8; Prop. 5.4. 12] are irreducible representations. Hence the von Neumann algebras: $P = \Psi(\overline{\Delta(H_0^\perp, \sigma)})''$, $Q = \Psi(\overline{\Delta(H_0, \sigma)})''$, and $R = \Psi(\overline{\Delta(H, \sigma)})''$ are of type I. Moreover it is clear that the von Neumann algebra generated by P and Q equals R and that $P \subseteqq Q'$. Hence P and Q is a factorization of R. Since P and Q are type I factors, it follows that this factorization must be paired [9] i.e. P and Q are each others commutants:

$$P = Q'$$

and

$$P = \Psi(\overline{\Delta(H_0, \sigma)})' \supseteqq \Psi(\overline{\Delta(H_0, \sigma)}^c)'' \supseteqq P \,.$$

Hence

$$\Psi(\overline{\Delta(H_0, \sigma)}^c)'' = \Psi(\overline{\Delta(H_0^\perp, \sigma)})''$$

Q.E.D.

Remark 2. Without proof we mention that Lemma 6 can also be proved for locally normal states on the Weyl algebra.

Theorem 2. *The pure quasi-free states ω_J and ω_K on the Weyl algebra $\overline{\Delta(H, \sigma)}$ are unitary equivalent if and only if the operator $(J-K)^+(J-K) = [J, K]_+ + 2$ is a trace class operator with respect to s_J.*

Proof. Suppose first that $[J, K]_+ + 2$ is a trace class operator with respect to s_J. Then $JK + KJ = JK + (J'K)^{-1}$ has a pure point spectrum and hence JK has this property; JK being hermitian with respect to s_J, there exists an orthonormal basis of eigenvectors $\{\psi_i\}_i$ of H. To any finite subset $\{\psi_{n_i} | i = 1, \ldots, k\}$ corresponds a finite subspace generated by $\{\psi_{n_i}, J\psi_{n_i} | i = 1, \ldots, k\}$. If $JK\psi_{n_i} = \lambda_{n_i}\psi_{n_i}$ then $K\psi_{n_i} = -\lambda_{n_i} J\psi_{n_i}$ and $KJ\psi_{n_i} = \dfrac{1}{\lambda_{n_i}}\psi_{n_i}$ hence this subspace is invariant for J and K, and the orthogonal projection on this subspace commutes with J and K.

Consider the net $\{H_\alpha\}_\alpha$ of all such subspaces of H. Each H_α is invariant under J and K and their union $\bigcup_\alpha H_\alpha$ generates H. Let $\{E_\alpha\}_\alpha$ be the corresponding net of projections; each E_α commutes with J and K.

Since $[J, K]_+ + 2$ is a trace class operator, for every $\varepsilon > 0$ there is an index α_0 such that for all $E_\alpha < \mathbf{1} - E_{\alpha_0}$

$$|\operatorname{Tr}(2 + [J, K]_+) E_\gamma| < \varepsilon$$

or

$$|\operatorname{Tr}(\mathbf{1} + JK) E_\alpha| < \frac{\varepsilon}{2}.$$

By Lemma 2:

$$\det_{E_\alpha}\left(\frac{\mathbf{1} - JK}{2}\right) \leqq \exp\frac{\varepsilon}{8}$$

and by Lemma 4

$$\omega_K(P(H_\alpha)) \geqq \exp\left(-\frac{\varepsilon}{16}\right).$$

Let $p = \inf_{E_\alpha < (\mathbf{1} - E_{\alpha_0})} \omega_K(P(H_\alpha))$ then $1 \geqq p \geqq \exp\left(-\frac{\varepsilon}{16}\right)$ and from Corollary 1 and Lemma 6 we get

$$\|(\omega_J - \omega_K) | \overline{\Delta(H_{\alpha_0}, \sigma)^c} \| = 2(1 - p)^{\frac{1}{2}}.$$

Clearly if ε tends to zero, then $(1 - p)$ tends to zero and from Theorem 1 it follows that ω_J and ω_K are unitary equivalent.

Conversely, suppose that ω_J and ω_K are unitary equivalent, then by Lemma 1, the operator $\mathbf{1} + JK$ is compact and JK has a pure point spectrum. Again we consider the net $\{H_\alpha\}_\alpha$ of subspaces and the net $\{E_\alpha\}_\alpha$ of projections as defined above. Moreover we consider the corresponding net $\{\mathfrak{A}_\alpha = \overline{\Delta(H_\alpha, \sigma)}\}$ of C^*-subalgebras of $\mathfrak{A}$. By Theorem 1 and Lemma 6 there is a subspace $H_{\alpha_0} \in \{H_\alpha\}_\alpha$ such that

$$\|(\omega_J - \omega_K) | \Delta(H_{\alpha_0}^\perp, \sigma)\| < \varepsilon.$$

From Corollary 1 it follows that

$$\inf_{\substack{E_\gamma < (\mathbf{1} - E_{\alpha_0}) \\ E_\gamma \in \{E_\alpha\}_\alpha}} \omega_K(P(H_\gamma)) \geqq 1 - \frac{\varepsilon^2}{4}.$$

By Lemma 4

$$\sup_{E_\gamma < (\mathbf{1} - E_{\alpha_0})} \det_{E_\gamma}\left(\frac{\mathbf{1} - JK}{2}\right) \leqq \left(1 - \frac{\varepsilon^2}{4}\right)^{-2}.$$

By Lemma 2

$$1 \leqq 1 + \sup_{E_\gamma < (\mathbf{1} - E_{\alpha_0})} \operatorname{Tr}\left[-\frac{(\mathbf{1} + JK)}{4} E_\gamma\right] \leqq \left(1 - \frac{\varepsilon^2}{4}\right)^{-2}$$

or

$$1 \leqq 1 + \frac{1}{8} \operatorname{Tr}[-(2 + [J, K]_+)(\mathbf{1} - E_{\alpha_0})] \leqq \left(1 - \frac{\varepsilon^2}{4}\right)^{-2}$$

Since H_{α_0} is a finite subspace the operator $2 + [J, K]_+$ is a trace class operator Q.E.D.

Remark 3. The condition appearing in Theorem 2 can of course also be expressed in terms of the scalar product s_K. The problem is completely symmetric in s_J and s_K. It is just a matter of choosing a symplectic basis in H with respect to J or to K.

Remark 4. After the preparation of this work we were informed about an analogous result by Courbage, Miracle-Sole and Robinson [10] who gave a characterization of all states quasi-equivalent with the Fock representation of the canonical commutation relations. An essential difference with our work is a different choice of underlying C^*-algebra and method of derivation. Furthermore our criterium is given directly in terms of the states we are considering.

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