MATHEMATICS OF COMPUTATION
VOLUME 36, NUMBER 153
JANUARY 1981

Unitary Hyperperfect Numbers

By Peter Hagis, Jr.

Abstract. Unitary hyperperfect numbers are generalized unitary perfect numbers. In this paper a list of such numbers is given, and some results concerning them are proved.

1. Introduction. In this paper, lower-case letters denote positive integers, with p and q always representing primes. If $d \mid n$ and $(d, n/d) = 1$, d is said to be a unitary divisor of n. If $\sigma^*(n)$ denotes the sum of the unitary divisors of n, then $\sigma^*(1) = 1$ while

$$\sigma^*(n) = (1 + p_1^{a_1})(1 + p_2^{a_2}) \cdots (1 + p_s^{a_s}) \tag{1}$$

if $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$. It is easy to see that $\sigma^*(n) = \sigma(n)$ (where $\sigma(n)$ denotes the sum of *all* the positive divisors of n) if and only if n is square-free.

Subbarao and Warren [2] have defined n to be a unitary perfect number if $\sigma^*(n) = 2n$. Minoli [1] has defined m to be a hyperperfect number if $m = 1 + t[\sigma(m) - m - 1]$. We shall say that m is a *unitary hyperperfect number* if

$$m = 1 + t\left[\sigma^*(m) - m - 1\right]. \tag{2}$$

When $t = 1$, m is a unitary perfect number. A square-free number is hyperperfect if and only if it is unitary hyperperfect.

2. Some Elementary Results. Suppose that m is a unitary hyperperfect number. From (2) it is immediate that $(m, t) = 1$. Therefore, if $3 \mid t$, then $3 \nmid m$. If $t \equiv -1, 1 \pmod 3$, then, from (2), $\sigma^*(m) \equiv 2, 2m \pmod 3$. Now, assume that $m = \prod_{j=1}^r p_j^{a_j}$ where $p_j \neq 3$. From (1), $\sigma^*(m) = \prod_{j=1}^r (1 + p_j^{a_j})$ and since $\sigma^*(m) \not\equiv 0 \pmod 3$ it follows that $2 \mid a_j$ if $p_j \equiv -1 \pmod 3$ *and* that $\sigma^*(m) \equiv 2^r \pmod 3$. Therefore, $m \equiv 1 \pmod 3$ *and* r is odd. We have proved:

FACT 1. If m is a unitary hyperperfect number and $3 \nmid m$, then $m \equiv 1 \pmod 3$. Also, if $3 \nmid mt$, then m has an odd number of distinct prime factors.

Since $\sigma^*(p^a) = p^a + 1$, we have immediately from (2):

FACT 2. If m is a unitary hyperperfect number, then $m \neq p^a$.

Now, suppose that $m = p^a q^b$ is a unitary t hyperperfect number. Since $\sigma^*(m) = m + p^a + q^b + 1$, it follows easily from (2) that

$$q^b = (tp^a + 1)/(p^a - t), \tag{3}$$

and we see immediately that $p^a > t$. If $p^a - t = 1$, then

$$q^b = p^{2a} - p^a + 1. \tag{4}$$

Received June 27, 1980.
1980 *Mathematics Subject Classification.* Primary 10A20.

A search was made for all solutions of (4) with $2 \leqslant p^a < 10^5$. There were 720. In 695 solutions, $a = b = 1$. Eighteen solutions were of the form p^2q. The "exceptional" cases were $2^4 \cdot 241$, $3^4 \cdot 6481$, $3^6 \cdot 530713$, $5^4 \cdot 390061$, $7^3 \cdot 117307$, $13^4 \cdot 815702161$, and $19 \cdot 7^3$.

If, in (3), $p^a - t = 2$, then

$$2q^b = (p^a - 1)^2 \tag{5}$$

so that $p^a = 2^k + 1$. If $a = 1$, then p is a Fermat prime; if $a > 1$, then $a = 2$, $p = 3$, $k = 3$. Therefore, the only known unitary t hyperperfect numbers of the form $p^a q^b$ with $p^a - t = 2$ are $3 \cdot 2$, $5 \cdot 2^3$, $17 \cdot 2^7$, $257 \cdot 2^{15}$, $65537 \cdot 2^{31}$ and $3^2 \cdot 2^5$.

If $p^a - t = g$ in (3), then it is easy to see that $p^{2a} \equiv -1 \pmod g$. It follows that $4 \nmid g$ and that g has no prime factors of the form $4j + 3$ since -1 is a quadratic nonresidue of all such primes. We have established:

FACT 3. If m is a unitary t hyperperfect number of the form $p^a q^b$, then $p^a - t = 1, 2, 5, 10, 13, 17, 25, 26, 29, 34, \cdots$.

If $m = p^a q^b r^c$ is a unitary t hyperperfect number, then it follows from (2) (see also [3]) that

$$\begin{aligned} &\left[(p^a - t)q^b - t(p^a + 1)\right]\left[(p^a - t)r^c - t(p^a + 1)\right] \\ &\qquad = (p^a - t)(1 + tp^a) + t^2(p^a + 1)^2. \end{aligned} \tag{6}$$

If AB, where $A < B$, is a factorization into positive integers of the right-hand member of (6), then it follows that

$$q^b = \left[t(p^a + 1) + A\right]/(p^a - t), \qquad r^c = \left[t(p^a + 1) + B\right]/(p^a - t) \tag{7}$$

if the right-hand members of (7) are powers of distinct primes. Now suppose that $p^a - t = 1$. Then

$$q^b = p^{2a} - 1 + A, \qquad r^c = p^{2a} - 1 + B, \qquad AB = p^{4a} - p^{2a} - p^a + 2. \tag{8}$$

If $A = 1$, then $q^b = p^{2a}$ which is impossible. A search was made for all solutions of (8) with $2 \leqslant p^a \leqslant 313$. Eight were found. For five of these $a = b = c = 1$ so that m is square-free and therefore hyperperfect (as well as unitary hyperperfect). These hyperperfect numbers were found earlier by te Riele and are given in [3]. The other three solutions yielded the unitary hyperperfect numbers $2 \cdot 5 \cdot 3^2$, $5 \cdot 3^3 \cdot 223$ and $11^2 \cdot 14683 \cdot 4999387$.

3. The Unitary Hyperperfect Numbers Less Than 10^6. Using the CDC Cyber 174 at the Temple University Computing Center, a search was made for all unitary hyperperfect numbers m such that $m \leqslant 10^6$. The search required about one hour and fifteen minutes of computer time, and 36 numbers were found. Four were unitary perfect numbers and were found by Subbarao and Warren [2]. Twenty of the others were square-free and appear in the list of hyperperfect numbers given by

Minoli [1]. The twelve new ones are: $2^3 \cdot 5$ ($t = 3$); $2^2 \cdot 13$ ($t = 3$); $2^5 \cdot 3^2$ ($t = 7$); $3^2 \cdot 73$ ($t = 8$); $2^7 \cdot 17$ ($t = 15$); $2^4 \cdot 241$ ($t = 15$); $2^5 \cdot 173$ ($t = 27$); $7^3 \cdot 19$ ($t = 18$); $5^2 \cdot 601$ ($t = 24$); $3^3 \cdot 5 \cdot 223$ ($t = 4$); $7^3 \cdot 307$ ($t = 162$); $3^4 \cdot 6481$ ($t = 80$).

Department of Mathematics
Temple University
Philadelphia, Pennsylvania 19122

1. D. Minoli, "Issues in nonlinear hyperperfect numbers," *Math. Comp.*, v. 34, 1980, pp. 639–645.

2. M. V. Subbarao & L. J. Warren, "Unitary perfect numbers," *Canad. Math. Bull.*, v. 9, 1966, pp. 147–153.

3. H. J. J. te Riele, "Hyperperfect numbers with three different prime factors," *Math. Comp.*, v. 36, 1981, pp. 297–298.