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UNITARY TORIC MANIFOLDS, MULTI-FANS AND EQUIVARIANT INDEX

Dedicated to Professor Akio Hattori on his seventieth birthday

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Abstract. We develop the theory of toric varieties from a topological point of view using equivariant cohomology. Indeed, we introduce a geometrical object called a unitary toric manifold and associate a combinatorial object called a multi-fan to it. This generalizes (in one direction) the well-known correspondence between a compact nonsingular toric variety and a (regular) fan. The multi-fan is a collection of cones which may overlap unlike a usual fan. It turns out that the degree of the overlap of cones is essentially the Todd genus of the unitary toric manifold. Since the Todd genus of a compact nonsingular toric variety is one, this explains why cones do not overlap in a usual fan. A moment map relates a unitary toric manifold equipped with an equivariant complex line bundle to a "twisted polytope", and the equivariant Riemann-Roch index for the equivariant line bundle can be described in terms of the moment map. We apply this result to establish a generalization of Pick's formula.

Introduction. The theory of toric varieties says that there is a one-to-one correspondence between toric varieties (an object in algebraic geometry) and fans (an object in combinatorics). This correspondence often brought new insights to combinatorics from algebraic geometry, and vice versa (see [2], [4], [15]).

A compact nonsingular toric variety is called a toric manifold and the corresponding fan is called regular. Toric manifolds are well studied among toric varieties and play an important role in the theory of toric varieties. In this paper we develop the correspondence between toric manifolds and regular fans from a topological point of view. In fact, our geometrical object called a *unitary toric manifold* constitutes a much wider class than that of toric manifolds. A unitary (resp. almost complex) toric manifold M is a compact unitary (resp. almost complex) manifold with an action of a compact torus T having nonempty isolated fixed points, where $2\dim_{\boldsymbol{R}} T=\dim_{\boldsymbol{R}} M$. The Todd genus of a unitary (resp. almost complex) toric manifold takes any (resp. positive) integer, while that of a toric manifold is one.

To a unitary toric manifold M we associate a combinatorial object Δ_M called the *multi-fan* of M using equivariant cohomology. To this end, closed connected real codimension two submanifolds M_i $(i=1,\dots,d)$ of M, left fixed by certain circle subgroups, play an essential role. Each M_i defines an element ξ_i in the equivariant cohomology $H^2_T(M;\boldsymbol{Z})$ through Poincaré duality and ξ_i's are used to associate an element $v_i\in H_2(BT;\boldsymbol{Z})$ to each M_i. To each subset $I\subset\{1,\dots,d\}$ such that $\bigcap_{i\in I} M_i\neq\emptyset$,

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we form a cone in $H_2(BT;\boldsymbol{R})$ spanned by v_i's $(i \in I)$. The multi-fan Δ_M is the collection of these cones (together with two functions on maximal cones).

Whenever M is a toric manifold, the collection of cones in Δ_M agrees with the (usual) fan of M, and in this case cones intersect only at their faces. But, otherwise, cones in Δ_M may overlap in general. The Kosniowski formula about Todd genus tells us that the "multiplicity of overlap" of cones is closely related (often agrees) with the Todd genus of M. Since the Todd genus of a toric manifold is one, this explains why cones in Δ_M do not overlap when M is a toric manifold. One can read other topological properties of M, such as equivariant cohomology and Euler number, from the multi-fan Δ_M.

The theory of toric varieties also says that a toric manifold (or variety) equipped with an (equivariant) ample holomorphic line bundle corresponds to a convex polytope, and the Riemann-Roch-Hirzebruch formula for the line bundle can be used to count the number of lattice points on the convex polytope. The ample line bundle over a toric manifold defines a moment map with the convex polytope as its image. Karshon and Tolman [11] studied the equivariant Riemann-Roch index for an arbitrary equivariant line bundle over a toric manifold from the viewpoint of symplectic topology. They described the equivariant index in terms of a moment map associated with the equivariant line bundle. A notable phenomenon in their study is that the image of the moment map is no longer a convex polytope unless the line bundle is ample. It turns out that their study fits well in our setting. To be more specific, let M be a unitary toric manifold. Then an equivariant Gysin homomorphism

$$\pi_! : K_T(M) \to K_T \text{ (point)} = R(T)$$

is defined in equivariant K-theory for the map π collapsing M to a point. The equivariant Riemann-Roch index for a complex T-line bundle L over M is then given by $\pi_!(L)$, which equals the Todd genus of M when L is trivial. Associated to L there is defined a moment map

$$\Phi_L' : M \to \mathrm{Lie}(T)^* = H^2(BT;\boldsymbol{R})$$

shifted using the "canonical" line bundle of M. Under certain conditions the orbit space M/T becomes a smooth manifold with boundary, and Φ_L' induces a map

$$\bar{\Phi}_L' : \partial(M/T) \to H^2(BT;\boldsymbol{R}) .$$

It turns out that the multiplicity with which an irreducible T-module $u \in \mathrm{Hom}(T, S^1) \cong H^2(BT;\boldsymbol{Z}) \subset H^2(BT;\boldsymbol{R})$ occurs in the equivariant index $\pi_!(L)$ agrees with the winding number of $\bar{\Phi}_L'$ around u.

A classical formula called Pick's formula describes the number of lattice points on the domain bounded by an integral *simple* plane polygon P in terms of the area of the bounded domain and the number of lattice points on P. As is well-known, it can be reproved, when the bounded domain is convex by applying the Riemann-Roch formula to an ample line bundle over a toric manifold. It turns out that Pick's formula can be generalized to any integral plane polygon which may have self-intersections. We will prove it by applying the above result for the Riemann-Roch index to a line bundle over a unitary toric manifold.

This paper is organized as follows. In Section 1 we introduce a unitary toric manifold M and associate the element $v_i \in H_2(BT; \boldsymbol{Z})$ to each M_i. Lemma 1.5 is the key of our study. In Section 2 we introduce a subring $A_T^*(M)$ of $H_T^*(M; \boldsymbol{Z})$ (they often agree) and study its relation with combinatorics. In Section 3 we compute the equivariant Chern class of M and study $A_T^*(M)$ in relation to equivariant Chern class. The multi-fan Δ_M of M is introduced in Section 4, and its relation with the topology of M is studied. We also give a negative answer to a question asked by Guillemin [5, p. 2]. In Section 5 we provide examples of almost complex toric manifolds of real dimension 4 whose Todd genera take any positive integer. A moment map associated with a complex T-line bundle is discussed in Section 6, and the extension of the result of Karshon-Tolman mentioned above is established in Section 7. Section 8 treats the generalization of Pick's formula. Throughout this paper all homology and cohomology groups are taken with $\boldsymbol{Z}$ coefficients unless otherwise stated.

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1. Unitary toric manifolds and characteristic submanifolds. A unitary (or weakly almost complex) manifold M is a smooth manifold endowed with a complex structure on the stable tangent bundle of M. If the complex structure is given on the tangent bundle TM of M, M is called an almost complex manifold. A unitary manifold M is oriented in the following way. Suppose there is given a complex structure on $TM \oplus \underline{\boldsymbol{R}}^l$, where $\underline{\boldsymbol{R}}^l$ denotes the product bundle $M \times \boldsymbol{R}^l$. Then $TM \oplus \underline{\boldsymbol{R}}^l$ is oriented as a complex vector bundle and $\underline{\boldsymbol{R}}^l$ is also oriented in the usual way. These orientations determine an orientation on M.

If a Lie group G acts on a unitary (resp. an almost complex) manifold M and the differential of each element of G preserves the given complex structure on $TM \oplus \underline{\boldsymbol{R}}^l$ (resp. TM), then M is called a unitary (resp. an almost complex) G-manifold. Let T be a compact torus and M a unitary T-manifold. Then each component of the fixed point set of a subgroup of T is again a unitary T-manifold, and its normal bundle to M is a complex T-vector bundle with the complex structure induced from the one on $TM \oplus \underline{\boldsymbol{R}}^l$. In particular, the tangent space T_pM at an isolated T-fixed point p is a complex T-module. Note also that l must be even if there is an isolated T-fixed point.

DEFINITION. A closed, connected, unitary (resp. almost complex) T-manifold M is called a *unitary* (resp. an *almost complex*) *toric* manifold if $\dim_{\boldsymbol{R}} M = 2\dim_{\boldsymbol{R}} T$ and the T-fixed point set M^T is non-empty and isolated. (Note that if M is a unitary T-manifold with $\dim_{\boldsymbol{R}} M = 2\dim_{\boldsymbol{R}} T$ and the T-action is effective, then M^T is necessarily isolated unless M^T is empty.)

Throughout this article, M will denote a unitary toric manifold and the T-action on M will be effective unless otherwise stated. We set

$$n = \dim_{\boldsymbol{R}} T = \frac{1}{2}\dim_{\boldsymbol{R}} M .$$

A closed, connected, real codimension two submanifold of M is called *characteristic* if it is a fixed point set component by a certain circle subgroup of T and contains at least one T-fixed point. One easily sees that M has only finitely many characteristic submanifolds. Let M_i $(i=1, \dots, d)$ be the characteristic submanifolds of M, let v_i be its normal bundle, and let T_i be the circle subgroup which fixes M_i pointwise. For $p \in M^T$ we set

$$I(p) := \{i \mid p \in M_i\} .$$

Then

(1.1) $$T_pM = \bigoplus_{i \in I(p)} v_i|_p \qquad \text{as complex } T\text{-modules},$$

where $v_i|_p$ denotes the restriction of v_i to p. This, in particular, shows that $I(p)$ consists of n elements. Since both M_i and M are oriented, the inclusion map defines an equivariant Gysin homomorphism: $H_T^*(M_i) \to H_T^{*+2}(M)$ (see [12] for example). Let $\xi_i \in H_T^2(M)$ be the image of the identity in $H_T^0(M_i)$. As is well-known,

(1.2) $\xi_i|_p$ agrees with the equivariant Euler class of $v_i|_p$.

LEMMA 1.3. (1) *The set* $\{\xi_i|_p \mid i \in I(p)\}$ *forms a basis of* $H_T^2(p) = H^2(BT)$.

(2) *Let* $j \in I(p)$. *Then* $\mathrm{Res}_{T_j}(\xi_i|_p) \neq 0$ *if and only if* $j = i$, *where* Res_{T_j} *denotes the restriction map from* $H^2(BT)$ *to* $H^2(BT_j)$.

(3) *If* p *and* q *are points in* M_i, *then* $\mathrm{Res}_{T_i}(\xi_i|_p) = \mathrm{Res}_{T_i}(\xi_i|_q)$.

PROOF. (1) Since the T-action on M is assumed to be effective, the one-dimensional T-modules $v_i|_p$ $(i \in I(p))$ form a basis of the free abelian group $\mathrm{Hom}(T, S^1)$ consisting of homomorphisms from T to S^1. Since the equivariant Euler class gives an isomorphism between $\mathrm{Hom}(T, S^1)$ and $H^2(BT)$, (1.1) and (1.2) imply (1).

(2) The identity (1.1) shows that $v_i|_p$ is non-trivial, when restricted to T_j, if and only if $j = i$. The statement (2) immediately follows from this observation.

(3) Since M_i is connected and fixed pointwise by T_i, $v_i|_p$ is isomorphic to $v_i|_q$ as complex T_i-modules. This implies (3). □

The map π which collapses M to a point induces a homomorphism $\pi^*: H^*(BT)\to H_T^*(M)$, which is injective since M^T is non-empty, and makes $H_T^*(M)$ an algebra over $H^*(BT)$. We often regard $H^*(BT)$ as a subset of $H_T^*(M)$ through the collapsing map. Let S be the subset of $H^*(BT)$ generated multiplicatively by non-zero elements in $H^2(BT)$. The localization theorem (see [10, p. 40]) says that the restriction map: $H_T^*(M)\to H_T^*(M^T)$ becomes an isomorphism when localized by S. Since $H_T^*(M^T)=H^*(BT)\otimes H^*(M^T)$ has no S-torsion, this implies

LEMMA 1.4. *The restriction map induces an injection*: $H_T^*(M)/S$*-torsions*$\to H_T^*(M^T)$.

To implify notation we set

$$\hat{H}_T^*(M):=H_T^*(M)/S\text{-torsions}\,.$$

We also use the same notation for an element in $H_T^*(M)$ as well as for its image in $\hat{H}_T^*(M)$. Remember that $H^*(BT)$ is regarded as a subset of $H_T^*(M)$ through the collapsing map π.

LEMMA 1.5. *For each* $i\in\{1,\dots,d\}$ *there exists a unique element* $v_i\in H_2(BT)$ *such that*

$$u=\sum_{i=1}^{d}\langle u, v_i\rangle\xi_i \qquad \text{in } \hat{H}_T^2(M) \text{ for any } u\in H^2(BT)\,,$$

where $\langle\, ,\rangle$ *denotes the usual pairing between cohomology and homology.*

PROOF. Let $p\in M^T$. It follows from Lemma 1.3(1) that there is a unique element $v_i(p)\in H_2(BT)$ for each $i\in I(p)$ such that

$$u=\sum_{i\in I(p)}\langle u, v_i(p)\rangle\xi_i|_p \qquad \text{in } H_T^2(p)=H^2(BT) \text{ for any } u\in H^2(BT)\,.$$

We shall show that $v_i(p)$ is independent of p. Let $q\in M^T$ be another point. For this, we have the same identity as above with $I(p)$ replaced by $I(q)$. Suppose $I(p)\cap I(q)\neq\varnothing$. Let $i\in I(p)\cap I(q)$ and restrict the two identities above for p and q to the circle subgroup T_i. Then it follows from Lemma 1.3(2), (3) that $\langle u, v_i(p)\rangle \operatorname{Res}_{T_i}(\xi_i|_p)=\operatorname{Res}_{T_i}u=\langle u, v_i(q)\rangle\operatorname{Res}_{T_i}(\xi_i|_q)$ and $\operatorname{Res}_{T_i}(\xi_i|_p)=\operatorname{Res}_{T_i}(\xi_i|_q)\neq 0$. This shows that $\langle u, v_i(p)\rangle=\langle u, v_i(q)\rangle$ for any $u\in H^2(BT)$ and hence $v_i(p)=v_i(q)$.

Now we take $v_i=v_i(p)$. By construction the element $\sum\langle u, v_i\rangle\xi_i$ in $\hat{H}_T^2(M)$ restricts to u in $H_T^2(M^T)$. Since, u, viewed as an element of $\hat{H}_T^2(M)$, also restricts to u and the restriction map is injective by Lemma 1.4, the identity in the lemma follows. □

REMARK. When M is a toric manifold, the elements v_i are the edge vectors used to define the fan of M. In fact, Lemma 1.5 is a counterpart in equivariant cohomology to the lemma in [4, p. 61] stated in terms of invariant divisors.

EXAMPLE 1.6. Let $n=1$ (hence $T=S^1$) and $M=S^2$ with the standard effective

T-action which has two fixed points. We shall give M two different unitary structures. One is the usual complex structure on $\boldsymbol{C}P^1$. The elements v_1 and v_2 are then unit vectors with opposite direction. The other unitary structure is defined as follows. We view M as the unit sphere of $\chi \oplus \boldsymbol{R}$, where χ denotes the standard one dimensional complex T-module. Mapping positive unit vectors in $\underline{\boldsymbol{R}}$ to outward unit normal vectors to M in $\underline{\chi} \oplus \underline{\boldsymbol{R}}$ induces an isomorphism from $TM \oplus \underline{\boldsymbol{R}}$ to $\underline{\chi} \oplus \underline{\boldsymbol{R}}$. Adding $\underline{\boldsymbol{R}}$ to them and identifying $\boldsymbol{R}^2$ with $\boldsymbol{C}$ in a natural way, we obtain an isomorphism from $TM \oplus \underline{\boldsymbol{R}}^2$ to $\underline{\chi} \oplus \underline{\boldsymbol{C}}$. This makes M a unitary toric manifold. The tangential representations at the two fixed points are both χ, and the elements v_1 and v_2 in Lemma 1.5 are both the unit vector with "positive" direction. (See Lemma 1.7 below.)

LEMMA 1.7. *The set* $\{v_i \mid i \in I(p)\}$ *is the dual basis of* $\{\xi_i|_p \mid i \in I(p)\}$ *for each* $p \in M^T$. *In particular,* $\xi_i|_p = \xi_i|_q$ *for any* i *if* $I(p) = I(q)$.

PROOF. By Lemma 1.5, $u = \sum \langle u, v_i \rangle \xi_i$ for any u. Take $u = \xi_j|_p$ and restrict the identity to p. It reduces to

$$\xi_j|_p = \sum_{i \in I(p)} \langle \xi_j|_p, v_i \rangle \xi_i|_p ,$$

because $\xi_i|_p = 0$ unless $i \in I(p)$. This together with Lemma 1.3(1) implies the lemma. □

Remember that there is a canonical isomorphism: $\mathrm{Hom}(T, S^1) \cong H^2(BT)$. We denote by χ^u the element in $\mathrm{Hom}(T, S^1)$ corresponding to $u \in H^2(BT)$. We also have an isomorphism: $\mathrm{Hom}(S^1, T) \cong H_2(BT)$ and denote by $\lambda_v \in \mathrm{Hom}(S^1, T)$ the element corresponding to $v \in H_2(BT)$. Note that

$$\chi^u(\lambda_v(z)) = z^{\langle u, v \rangle} \qquad \text{for } z \in S^1 . \tag{1.8}$$

LEMMA 1.9. *Let* $p \in M^T$. *By Lemma* 1.7 *any element* $v \in H_2(BT)$ *can be written as* $v = \sum_{i \in I(p)} b_i v_i$ *with integers* b_i. *We view* T_pM *as an* S^1*-module through the homomorphism* λ_v. *Then the weights of the* S^1*-module are all positive if and only if all* b_i *are positive.*

PROOF. It follows from (1.1) and (1.2) that $T_pM = \bigoplus_{i \in I(p)} \chi^{\xi_i|_p}$. The weight of $\chi^{\xi_i|_p}$ restricted to the S^1-subgroup $\lambda_v(S^1)$ is $\langle \xi_i|_p, v \rangle$ by (1.8), and is equal to b_i by Lemma 1.7, proving the lemma. □

Here is a geometrical meaning of v_i.

LEMMA 1.10. $\lambda_{v_i}(S^1) = T_i$.

PROOF. Let $p \in M_i^T$. Lemma 1.7 together with (1.1) implies that T_pM_i is the subspace of T_pM left fixed by $\lambda_{v_i}(S^1)$. It follows that M_i is fixed pointwise by $\lambda_{v_i}(S^1)$, since M_i is connected, proving the lemma. □

The element $v_i \in H_2(BT)$ is primitive, that is, it is not of the form $v_i = av_i'$, $a \neq \pm 1 \in \boldsymbol{Z}$

and $v_i' \in H_2(BT)$. There are two primitive elements in $H_2(BT)$ which are associated with T_i, and Lemma 1.10 says that v_i is one of them. (The other one is $-v_i$.) One finds that the argument developed in this section works once M and all the M_i's are oriented and that if the orientation on M_i is reversed, then v_i becomes $-v_i$. The unitary (toric) structure on M is used to assign orientations to them in a consistent way.

Let $\iota^* : H_T^*(M) \to H^*(M)$ be the restriction map. Then $\iota^* \xi_i \in H^2(M)$ is the Poincaré dual of the homology class in $H_{2n-2}(M)$ represented by M_i.

LEMMA 1.11. *If M is an almost complex toric manifold, then $\iota^*\xi_i$ is primitive.*

PROOF. It suffices to find a closed submanifold of real dimension two which intersects M_i transversely at only one point. Let $p \in M_i^T$. The connected component N of $\bigcap_{j \neq i \in I(p)} M_j$ containing p is an almost complex manifold of real dimension two. In fact, N is diffeomorphic to S^2, because it supports a non-trivial T_i-action with non-empty fixed point set. (Note that $p \in N^{T_i}$.) We note that M_i intersects N transversely because they are components of the fixed point sets of subgroups of T. Clearly $p \in M_i \cap N$. Suppose $M_i \cap N \neq \{p\}$ and let $q \in M_i \cap N \setminus \{p\}$. Then $T_pN \cong T_qN$ as complex T_i-modules because p and q are in the same T_i-fixed point set component M_i. However, since N is almost complex and diffeomorphic to S^2, those complex tangential representations are not isomorphic as is well-known. Therefore $M_i \cap N = \{p\}$, proving the lemma. □

2. Face rings.

We set

$$\Gamma_M := \{I \subset \{1, \dots, d\} \mid \phi \neq I \subset I(p) \text{ for some } p \in M^T\}.$$

This is an (abstract) simplicial complex. We also set

$$A_T^*(M) := \text{the subring of } H_T^*(M) \text{ generated by } \xi_i\text{'s},$$

$$\hat{A}_T^*(M) := \text{the image of } A_T^*(M) \text{ in } \hat{H}_T^*(M).$$

In this section we will study these from the viewpoint of combinatorics.

Consider a polynomial ring $\boldsymbol{Z}[x_1, \dots, x_d]$ in d-variables and a map

$$\varphi : \boldsymbol{Z}[x_1, \dots, x_d] \to \hat{A}_T^*(M),$$

which sends x_i to ξ_i. Clearly φ is surjective.

PROPOSITION 2.1. *The kernel of φ is the ideal generated by monomials $\prod_{i \in I} x_i$ for all $I \notin \Gamma_M$. In other words, $\hat{A}_T^*(M)$ is isomorphic to the face ring (or Stanley-Reisner ring) of the simplicial complex Γ_M.*

PROOF. We first introduce some notation. Let $\mathscr{I}$ denote a finite set which consists of elements in $\{1, \dots, d\}$ taken with multiplicity, i.e., elements in $\{1, \dots, d\}$ may appear in $\mathscr{I}$ repeatedly. Set $\xi_{\mathscr{I}} := \prod_{i \in \mathscr{I}} \xi_i$ and denote by $r(\mathscr{I})$ the subset of $\{1, \dots, d\}$ consisting of elements appearing in $\mathscr{I}$. Then the proposition is equivalent to the statement: a finite sum $\sum a_{\mathscr{I}}\xi_{\mathscr{I}}$ ($a_{\mathscr{I}}\neq 0\in \boldsymbol{Z}$) vanishes in $\hat{H}^*_T(M)$ if and only if $r(\mathscr{I})\notin \Gamma_M$ for all $\mathscr{I}$. This equivalent statement follows from the following three observations:

(1) $\sum a_{\mathscr{I}}\xi_{\mathscr{I}}=0$ in $\hat{H}^*_T(M)$ if and only if $\sum a_{\mathscr{I}}\xi_{\mathscr{I}}|_p=0$ for all $p\in M^T$ by Lemma 1.4.

(2) $\sum a_{\mathscr{I}}\xi_{\mathscr{I}}|_p=0$ if and only if $\xi_{\mathscr{I}}|_p=0$ for all $\mathscr{I}$, since $\xi_i|_p\neq 0$ if and only if $i\in I(p)$, $H^*_T(p)=H^*(BT)$ is a polynomial ring and $a_{\mathscr{I}}\neq 0$.

(3) $\xi_{\mathscr{I}}|_p=0$ if and only if $r(\mathscr{I})\not\subseteq I(p)$. Hence $\xi_{\mathscr{I}}|_p=0$ for all $p\in M^T$ if and only if $r(\mathscr{I})\notin \Gamma_M$. □

For $0\leq k\leq n-1$ we denote by f_k the number of k-simplices in Γ_M. The vector $(f_0,\dots,f_{n-1})$ is called the *f-vector*. Observe that $f_0=d$. The f-vector is associated with the so-called h-vector $(h_0,\dots,h_n)$ defined by

$$\sum_{k=0}^{n} h_k s^{n-k}=\sum_{k=0}^{n} f_{k-1}(s-1)^{n-k},$$

where $f_{-1}=1$ and s is an indeterminate. Note the following relations

$$h_1=f_0-n=d-n\,,\qquad \sum_{k=0}^{n} h_k=f_{n-1}\,. \tag{2.2}$$

We define the Hilbert series of $\hat{A}^*_T(M)$ by

$$F(\hat{A}^*_T(M),s):=\sum_{q=0}^{\infty}(\mathrm{rank}_{\boldsymbol{Z}}\hat{A}^{2q}_T(M))s^{2q},$$

where we omit the odd degree terms in $\hat{A}^*_T(M)$, since they vanish by definition. Since $\hat{A}^*_T(M)$ is the face ring of the simplicial complex Γ_M, it follows from [17, Theorem 1.4 in p. 54] that

$$F(\hat{A}^*_T(M),s)=\frac{1}{(1-s^2)^n}\sum_{k=0}^{n} h_k s^{2k}\,. \tag{2.3}$$

The following lemmas show that the h-vector is closely related to the Betti numbers of M.

LEMMA 2.4. *$\sum_{k=0}^{n} h_k\leq \chi(M)$, where $\chi(M)$ denotes the Euler number of M, and the equality holds if and only if $M_{I(p)}:=\bigcap_{i\in I(p)}M_i=\{p\}$ for any $p\in M^T$.*

PROOF. We know that $\sum_{k=0}^{n} h_k=f_{n-1}$ by (2.2). By definition any $(n-1)$-simplex in Γ_M is of the form $I(p)$ for some $p\in M^T$. Therefore

$$f_{n-1}\leq \text{the number of points in } M^T=\chi(M^T)=\chi(M)\,,$$

and the equality holds if and only if $I(p)\neq I(q)$ for any distinct points p and q in M^T. The latter is equivalent to saying that $M_{I(p)}=\{p\}$ for any $p\in M^T$ because $I(p)=I(q)$ for any $q\in M_{I(p)}$. □

LEMMA 2.5. *$h_1\leq \mathrm{rank}_{\boldsymbol{Z}} H^2(M)$ and $n\leq d\leq n+\mathrm{rank}_{\boldsymbol{Z}} H^2(M)$.*

PROOF. One easily sees that the kernel of the restriction map $\iota^*: H_T^2(M)\to H^2(M)$ is $H^2(BT)$. Therefore

$$\operatorname{rank}_{\mathbf{Z}} H_T^2(M)\leq n+\operatorname{rank}_{\mathbf{Z}} H^2(M)\,.$$

On the other hand, we have

$$\operatorname{rank}_{\mathbf{Z}} H_T^2(M)\geq \operatorname{rank}_{\mathbf{Z}} \hat{H}_T^2(M)\geq \operatorname{rank}_{\mathbf{Z}} \hat{A}_T^2(M)=n+h_1=d\,,$$

where the last two identities follow from (2.2) and (2.3). These inequalities prove the lemma. □

LEMMA 2.6. *Suppose $H^{\mathrm{odd}}(M)=0$ and $A_T^*(M)=H_T^*(M)$. Then*

(1) *$h_k=\operatorname{rank}_{\mathbf{Z}} H^{2k}(M)$. In particular, $\sum h_k=\chi(M)$.*

(2) *Γ_M is a Cohen-Macaulay complex, i.e., for all $I\in\Gamma_M$ (possibly $I=\varnothing$) and all $q\neq\dim(\operatorname{lk} I)$, $\tilde{H}_q(\operatorname{lk} I)=0$, where $\operatorname{lk} I=\{J\in\Gamma_M \mid I\cup J\in\Gamma_M,\ I\cap J=\varnothing\}$. In particular, $\tilde{H}_q(\Gamma_M)=0$ unless $q=n-1$.*

REMARK. When M is a toric manifold, the assumption in Lemma 2.6 is satisfied and the geometric realization of Γ_M is homeomorphic to a sphere of dimension $n-1$.

PROOF. (1) Since $H^{\mathrm{odd}}(M)$ vanishes, one has that $H_T^*(M)\cong H^*(BT)\otimes H^*(M)$ as $H^*(BT)$-modules. In particular, $H_T^*(M)$ is a free $H^*(BT)$-module. Therefore $\hat{H}_T^*(M)=H_T^*(M)=A_T^*(M)=\hat{A}_T^*(M)$, and

$$F(H_T^*(M),s)=\frac{1}{(1-s^2)^n}\sum_{k=0}^{n}\operatorname{rank}_{\mathbf{Z}} H^{2k}(M)s^{2k}\,.$$

One concludes that $h_k=\operatorname{rank}_{\mathbf{Z}} H^{2k}(M)$ by comparing the above identity with (2.3).

(2) Let $\boldsymbol{F}$ denote a field of prime order. As before, one can view $H^*(BT;\boldsymbol{F})$ as a subset of $H_T^*(M;\boldsymbol{F})$. On the other hand, since $H^{\mathrm{odd}}(M;\boldsymbol{F})$ vanishes, $H_T^*(M;\boldsymbol{F})$ is a free module over $H^*(BT;\boldsymbol{F})$ as in the proof of (1). These remarks show that $H_T^*(M;\boldsymbol{F})$ is a Cohen-Macaulay ring. Therefore $\tilde{H}_q(\operatorname{lk} I;\boldsymbol{F})=0$ for all $q\neq\dim(\operatorname{lk} I)$ by Reisner's theorem (see [17, p. 60]). Since $\boldsymbol{F}$ is a field of arbitrary prime order, the statement (2) follows from the universal coefficient theorem for homology groups. □

3. Equivariant Chern classes. Remember that the equivariant Chern class $c^T(E)$ of a complex T-vector bundle E over M sits in $H_T^*(M)$ and it restricts to the ordinary Chern class $c(E)$ through the restriction map $\iota^*: H_T^*(M)\to H^*(M)$ (see [12] for example). One should note that the equivariant Chern class $c^T(E)$ is computable by means of the localization theorem, once one knows the complex fiber T-modules E_p over T-fixed points p. Applying this idea to $E=TM$, we obtain

THEOREM 3.1. *Let M be a unitary toric manifold. Then $c^T(M)=\prod_{i=1}^d(1+\xi_i)$ in $\hat{H}_T^*(M)$.*

PROOF. When restricted to M^T, both sides of the identity coincide by (1.1) and (1.2), so the theorem follows from Lemma 1.4. □

LEMMA 3.2. $\hat{A}_T^2(M)=\hat{H}_T^2(M)$.

PROOF. By [9] any element in $H_T^2(M)$ is represented as $c_1^T(L)$ for some complex T-line bundle L over M, so it suffices to show that $c_1^T(L)$ viewed in $\hat{H}_T^2(M)$ is a linear combination of ξ_i's over integers.

Let $p\in M^T$. By Lemma 1.3(1) one can write

$$c_1^T(L)\,|_p=\sum_{i\in I(p)} a_i(p)\xi_i\,|_p \tag{3.3}$$

with integers $a_i(p)$. For another point $q\in M^T$ we have the same identity as above with $I(p)$ replaced by $I(q)$. Suppose $i\in I(p)\cap I(q)$. This means that both p and q sit in M_i. Since M_i is connected and fixed pointwise by the T_i-action, $\mathrm{Res}_{T_i}(c_1^T(L)\,|_p)=\mathrm{Res}_{T_i}(c_1^T(L)\,|_q)$. Therefore, restricting (3.3) for p and q to $H^2(BT_i)$ and using Lemma 1.3(2) (3), we see that $a_i(p)=a_i(q)$. This shows that $a_i(p)$ is independent of p, so we may set $a_i=a_i(p)$. Clearly, the restrictions of $c_1^T(L)$ and $\sum a_i\xi_i$ to $H_T^2(M^T)$ coincide, so $c_1^T(L)=\sum a_i\xi_i$ in $\hat{H}_T^2(M)$ by Lemma 1.4. This proves the lemma. □

PROPOSITION 3.4. (1) *If* $H^{\mathrm{odd}}(M)=0$, *then* $A_T^2(M)=H_T^2(M)$ *and* $d=n+\mathrm{rank}_{\mathbf{Z}}H^2(M)$.

(2) *If* $H^*(M)$ *is generated by degree* 2 *elements as ring, then* $A_T^*(M)=H_T^*(M)$, $M_{I(p)}=\{p\}$ *for any* $p\in M^T$, *and* Γ_M *is a Cohen-Macaulay complex.*

PROOF. (1) Since $H^{\mathrm{odd}}(M)=0$, $H_T^*(M)\cong H^*(BT)\otimes H^*(M)$ as $H^*(BT)$-modules. Hence $\hat{H}_T^*(M)=H_T^*(M)$, $\hat{A}_T^*(M)=A_T^*(M)$, and their odd degree terms vanish. This together with Lemma 3.2 shows that $A_T^2(M)=H_T^2(M)$.

Since $H^1(M)=0$ and $M^T\neq\varnothing$, we have a short exact sequence:

$$0\longrightarrow H^2(BT)\xrightarrow{\pi^*}H_T^2(M)\xrightarrow{\iota^*}H^2(M)\longrightarrow 0\,.$$

In particular, $\mathrm{rank}_{\mathbf{Z}}H_T^2(M)=n+\mathrm{rank}_{\mathbf{Z}}H^2(M)$, since $\mathrm{rank}_{\mathbf{Z}}H^2(BT)=n$. On the other hand, it follows from (2.2) and (2.3) that $\mathrm{rank}_{\mathbf{Z}}A_T^2(M)=d$. These prove the desired identity, because $A_T^2(M)=H_T^2(M)$.

(2) Since $A_T^2(M)=H_T^2(M)$ by (1), it suffices to show that $H_T^*(M)$ is generated by ξ_i's as ring. We shall prove this by induction on the degree of cohomology. Suppose that $H_T^*(M)$ is generated by ξ_i's up to $*\leq 2k-2$ as ring. Since $H_T^*(M)\cong H^*(BT)\otimes H^*(M)$ as $H^*(BT)$-modules, the kernel of the restriction map $\iota^*:H_T^{2k}(M)\to H^{2k}(M)$ is additively generated by products of elements in $H_T^*(M)$ for $*\leq 2k-2$ and positive degree elements in $H^*(BT)$. The latter sets are both generated by ξ_i's by induction assumption and by Lemma 1.5, respectively. On the other hand, the image of $\iota^*:H_T^{2k}(M)\to H^{2k}(M)$ is

generated by $\iota^*\xi_i$'s because $\iota^*: H_T^2(M) \to H^2(M)$ is surjective, $H_T^2(M)$ $(=A_T^2(M))$ is additively generated by ξ_i's, and $H^*(M)$ is generated by degree 2 elements by assumption. Thus $H_T^{2k}(M)$ is generated by ξ_i's, and the induction step has been completed. This establishes the first statement of (2) in the proposition. The latter two statements then follow from Lemmas 2.4 and 2.6. □

4. Multi-fans. In this section we introduce the notion of multi-fan and see how it is related to the topology of a unitary toric manifold.

We begin with a notation. Let Γ_M^k be the set of $(k-1)$-simplices in Γ_M. If $I \in \Gamma_M^k$, then $M_I = \bigcap_{i \in I} M_i$ is a compact unitary T-manifold of real dimension $2(n-k)$; in particular, M_I is a finite subset of M^T when $k=n$.

The tangent space T_pM at $p \in M^T$ has two orientations: the one induced from the orientation of M and the other induced from the complex structure on T_pM. They coincide whenever M is almost complex, but otherwise may differ. We set

$$\varepsilon(p) := +1 \qquad \text{or} \qquad -1$$

according as those orientations coincide or not, and define two functions on Γ_M^n: for $I \in \Gamma_M^n$

$$\varepsilon_M^+(I) := \text{the number of } \{p \in M_I \mid \varepsilon(p) = +1\},$$

$$\varepsilon_M^-(I) := \text{the number of } \{p \in M_I \mid \varepsilon(p) = -1\}.$$

Note that if M is an almost complex toric manifold, then $\varepsilon_M^-(I) = 0$ for all $I \in \Gamma_M^n$, and if M is a toric manifold, then $\varepsilon_M^+(I) = 1$ and $\varepsilon_M^-(I) = 0$ for all $I \in \Gamma_M^n$. To each $I \in \Gamma_M$ we associate a cone $\angle v_I$ in $H_2(BT; \boldsymbol{R})$:

$$\angle v_I := \left\{ \sum_{i \in I} b_i v_i \mid b_i \in \boldsymbol{R} \text{ and } b_i \geq 0 \text{ for all } i \in I \right\}.$$

DEFINITION. The collection of cones $\angle v_I$ indexed by Γ_M together with the two functions $\varepsilon_M^{\pm}$ *on* Γ_M^n is called the *multi-fan* of M, which we denote by Δ_M.

Apparently, Δ_M contains more information than Γ_M. For instance, Proposition 2.1 shows that Γ_M determines the *ring* structure of $\hat{A}_T^*(M)$, while it together with Lemma 1.5 shows that Δ_M (in fact, Γ_M and v_i's) determines the *algebra* structure of $\hat{A}_T^*(M)$ over $H^*(BT)$.

We note that $\chi(M_I) = \varepsilon_M^+(I) + \varepsilon_M^-(I) > 0$. Since $\chi(M) = \chi(M^T)$ and M^T is the disjoint union of M_I over $I \in \Gamma_M^n$, we have

$$\begin{aligned} \chi(M) &= \sum_{I \in \Gamma_M^n} (\varepsilon_M^+(I) + \varepsilon_M^-(I)) \\ &\geq \text{the number of } n\text{-dimensional cones in } \Delta_M, \end{aligned} \tag{4.1}$$

and the equality holds if and only if $\varepsilon_M^+(I) + \varepsilon_M^-(I) = 1$ for all $I \in \Gamma_M^n$, that is, if and only if $M_{I(p)}=\{p\}$ for any $p\in M^T$.

Here is a relation of Δ_M with the Todd genus of M.

THEOREM 4.2. *Let M be a unitary toric manifold and $v\in H_2(BT;\boldsymbol{R})$ be generic, i.e., v does not lie in any hyperplane spanned by v_i's. Then the Todd genus $T[M]$ of M is given by*

$$T[M]=\sum_{I:v\in\angle v_I}(\varepsilon_M^+(I)-\varepsilon_M^-(I)).$$

In particular, if M is an almost complex toric manifold, then $T[M]>0$, and if $M_{I(p)}=\{p\}$ for all $p\in M^T$ in addition, then $T[M]$ equals the number of maximal cones which contain v.

PROOF. We may assume that v is integral, i.e., $v\in H_2(BT)$. Since v is generic, the fixed point set of the restricted action of the S^1-subgroup $\lambda_v(S^1)$ agrees with M^T. Then the Kosniowski formula for unitary manifolds ([8], [13]) tells us that $T[M]$ equals the sum of $\varepsilon(p)$ over such $p\in M^T$ that the weights occuring in T_pM viewed as the $\lambda_v(S^1)$-module are all positive. This together with Lemma 1.9 proves the theorem. □

COROLLARY 4.3. *If $T[M]\neq 0$ (e.g., if M is an almost complex toric manifold), then*

(1) *the union of cones in Δ_M cover the whole space $H_2(BT;\boldsymbol{R})$,*

(2) $\chi(M)\geq n+1$,

(3) *M is cohomologically symplectic, i.e., there is an $x\in H^2(M)$ such that $x^n\neq 0$.*

PROOF. (1) If the union of cones in Δ_M does not cover the whole space $H_2(BT;\boldsymbol{R})$, then one can take a generic element $v\in H_2(BT;\boldsymbol{R})$ outside the union, so that $T[M]=0$ by Theorem 4.2. This contradicts the assumption and proves (1).

(2) One easily sees that at least $n+1$ number of n-dimensional cones are necessary to cover the whole space $H_2(BT;\boldsymbol{R})$, so (2) follows from (4.1).

(3) We note that since $\iota^*\xi_i$ is the Poincaré dual of M_i, $\prod_{i\in I}\iota^*\xi_i$ evaluated on the fundamental class of M is equal to $\varepsilon_M^+(I)-\varepsilon_M^-(I)$ for any $I\in\Gamma_M^n$. On the other hand, since $T[M]\neq 0$, there is an $I_0\in\Gamma_M^n$ such that $\varepsilon_M^+(I_0)-\varepsilon_M^-(I_0)\neq 0$ by Theorem 4.2. Therefore $\prod_{i\in I_0}\iota^*\xi_i$ is not a torsion element. Then it is not difficult to see that the n-th power of a certain linear combination of $\iota^*\xi_i$ $(i\in I_0)$ over $\boldsymbol{Z}$ is not zero. □

REMARK. The assumption in Corollary 4.3 cannot be dropped. For instance, take M to be the unit sphere of the direct sum of a complex n-dimensional faithful T-module and $\boldsymbol{R}$. It becomes a unitary toric manifold with the unitary structure described in Example 1.6 for $n=1$. One sees that $T[M]=0$ and the multi-fan of M has two n-dimensional cones when $n=1$ and has only one n-dimensional cone when $n>1$. In case $n=1$, the two n-dimensional cones are the same half line. Therefore the union of cones does not cover the whole space $H^2(BT;\boldsymbol{R})$ in either case.

When M is an almost complex toric manifold, we can make a rather stronger statement than the statement (1) in Corollary 4.3.

LEMMA 4.4. *Suppose that M is an almost complex toric manifold. Then, to each $J\in\Gamma_M^{n-1}$, there exist I and I' in Γ_M^n such that $\angle v_I\cap\angle v_{I'}=\angle v_J$, that is, $\angle v_J$ is not one-sided.*

PROOF. By the definition of Γ_M, there is $p\in M^T$ such that $I(p)\supset J$, i.e., $p\in M_J$. Since $J\in\Gamma_M^{n-1}$, M_J is of real 2-dimension. Let S be the connected component of M_J containing p. The induced T-action on S is non-trivial because of the effectiveness of the T-action on M, so the T-fixed point set S^T consists of exactly two points: one is p and the other one we denote by p'.

We claim that $I(p)$ and $I(p')$ are the desired I and I'. Since S admits an almost complex structure induced from the one on M, the weights w and w' of the one-dimensional complex T-modules T_pS and $T_{p'}S$ are related with $w'=-w$. We note that

$$\langle w, v_j\rangle=0=\langle w', v_j\rangle \quad \text{for all} \quad j\in J, \tag{4.5}$$

which follows from the fact that S is contained in M_J. Now let i and i' be the unique elements in $I(p)\setminus J$ and $I(p')\setminus J$, respectively. It follows from Lemma 1.7 and (4.5) that $\langle w, v_i\rangle=1=\langle w', v_{i'}\rangle$. But, since $w'=-w$, this means that v_i and $v_{i'}$ lie in the defferent regions separated by the hyperplane "orthogonal" to w, while $\angle v_J$ lies in the hyperplane by (4.5), proving the lemma. □

We can define a multi-fan Δ in the context of combinatorics from these three data:

(1) An abstract simplicial complex Γ with vertices $\{1\},\dots,\{d\}$.
(2) Elements $v_1,\dots,v_d$ in $H_2(BT)$.
(3) A pair of functions $\varepsilon^\pm$ on the subset Γ^n of $(n-1)$-simplices in Γ. (Motivated by the multi-fans of unitary toric manifolds, we may require that $\varepsilon^\pm$ take values on nonnegative integers and $\varepsilon^+(I)+\varepsilon^-(I)>0$ for any $I\in\Gamma^n$.)

We may call the multi-fan Δ *nonsingular* if v_i's ($i\in I$) form a basis of $H_2(BT)$ for each $I\in\Gamma^n$, and *complete* if $\sum_{I:v\in\angle v_I}(\varepsilon^+(I)-\varepsilon^-(I))$ is independent of the choice of a generic element $v\in H_2(BT;\mathbf{R})$, where the sum is understood to be zero unless v lies in the union of all cones $\angle v_I$. Then the multi-fan Δ_M of a unitary toric manifold M is nonsingular by Lemma 1.7 and complete by Theorem 4.2. It would be an interesting problem to characterize multi-fans obtained geometrically from unitary (or almost complex) toric manifolds. Proposition 3.4(2) and Lemma 4.4 suggest that there might be other constraints than nonsingularity and completeness on the multi-fans geometrically obtained.

Finally we shall give a rigidity theorem. See [7], [14] and [16] for related results.

THEOREM 4.6. *Let M be an almost complex toric manifold. If $H^*(M)\cong H^*(\mathbf{C}P^n)$ as groups, then $H^*(M)\cong H^*(\mathbf{C}P^n)$ as rings, $c(M)=(1+x)^{n+1}$ and $x^n[M]=T[M]=1$ for a suitable generator $x\in H^2(M)$.*

PROOF. We have $d=n+1$ by Proposition 3.4(1) and $\iota^*\xi_i=\pm x$ by Lemma 1.11. Hence $c(M)=(1+x)^{n+1-q}(1-x)^q$ for some $0\leq q\leq n+1$ by Theorem 3.1. An elementary computation of Todd genus using the Hirzebruch-Riemann-Roch formula shows that $T[M]=0$ unless $q=0$ or $q=n+1$ (see [6, Proposition 4.3] or [12, §49]). However $T[M]>0$ by Theorem 4.2, since M is an almost complex toric manifold. Therefore $q=0$ or $n+1$, and in either case $c(M)=(1+x)^{n+1}$, by replacing x with $-x$ if necessary. Then, an elementary computation of Todd genus again shows that $T[M]=x^n[M]$. Since $c_n(M)=(n+1)x^n$ and $c_n(M)$ agrees with the Euler class of M, we obtain $\chi(M)=(n+1)x^n[M]$. On the other hand, $\chi(M)=n+1$ because $H^*(M)\cong H^*(CP^n)$ as groups. These remarks show that $x^n[M]=1$ and hence $H^*(M)=\boldsymbol{Z}[x]/(x^{n+1})$. □

The proof above shows that if $H^{\mathrm{odd}}(M)=0$ and $H^2(M)\cong\boldsymbol{Z}$, then $c(M)=(1+x)^{n+1}$ and $x^n[M]=T[M]$ for a suitable generator $x\in H^2(M)$.

As is well-known, the complex quadric

$$Q_n=\{[z_0,\dots,z_{n+1}]\in CP^{n+1}\,|\,z_0^2+\cdots+z_{n+1}^2=0\}$$

satisfies the above weakened cohomology condition when $n\geq 3$, and $x^n[Q_n]=2$ for a generator $x\in H^2(Q_n)$. One sees that Q_n cannot be an almost complex toric manifold when $n\geq 3$. In fact, if Q_n $(n\geq 3)$ becomes an almost complex toric manifold, then $\chi(Q_n)=c_n(Q_n)[Q_n]=(n+1)x^n[Q_n]=2(n+1)$, while $\chi(Q_n)=n+1$ if n is odd and $n+2$ if n is even, as is well-known. But this is impossible.

However, Q_n admits an action of $T^{[(n+2)/2]}$ such that the k-th S^1-factor of the torus rotates the coordinates (z_{2k-2}, z_{2k-1}) via 2×2 rotation matrices. This action has a finite number of fixed points and preserves the Kähler form on Q_n, induced from the Fubini-Study form on CP^{n+1}, so that the action is Hamiltonian. This gives a negative answer to the first part of the following question [5, lines 11–9 from the bottom in p. 2]: *Given a Hamiltonian action of a torus on a compact symplectic manifold M with finite fixed point set, is M a Delzant space (=a toric manifold)? If not, can one obtain M from a Delzant space by a series of "blowing-ups" and "blowing-downs"?*

5. Examples. In this section we provide examples of almost complex toric manifolds of real dimension 4 using equivariant plumbing technique. It turns out that their Todd genera take any positive integer, so most of these almost complex toric manifolds are not isomorphic to toric manifolds because the Todd genus of a toric manifold is one. The reader will find that a similar method developed in this section may produce unitary toric manifolds of real 4-dimension whose Todd genera take any integer. One can produce higher dimensional unitary (or almost complex) toric manifolds by taking products of the 4-dimensional examples with toric manifolds or taking projective bundles over the 4-dimensional examples. The author believes that smooth T-manifolds constructed in [3] will provide more essential examples of higher dimensional unitary toric manifolds.

In what follows we take $n=2$. We fix a decomposition of $T=T^2$ into $S^1\times S^1$ and identify $H^2(BT)$ with $\boldsymbol{Z}^2$ through the decomposition. The purpose of this section is to prove the following.

THEOREM 5.1. *Let $v_1,\dots,v_d$ $(d\geq 3)$ be a sequence of vectors of $\boldsymbol{Z}^2$ in counter-clockwise order such that each successive pair v_{i-1} and v_i is a basis of $\boldsymbol{Z}^2$ for $i\in\{1,\dots,d\}$, where $v_0=v_d$. Then*

(1) *there is an almost complex toric manifold M of real dimension 4 whose multi-fan is the collection of cones spanned by successive pairs v_{i-1} and v_i $(i=1,\dots,d)$ together with the functions $\varepsilon_M^+(I)=1$ and $\varepsilon_M^-(I)=0$ for $I\in\Gamma_M^n$,*

(2) *$T[M]=(3d+\sum_{i=1}^d a_i)/12$, where a_i's are the integers defined by $v_{i-1}+v_{i+1}+a_iv_i=0$.*

Combining Theorem 4.2 with Theorem 5.1(2), we obtain

COROLLARY 5.2. *The rotation number of the sequence of the vectors $v_1,\dots,v_d$ in Theorem 5.1 is given by $(3d+\sum_{i=1}^d a_i)/12$.*

The rest of this section is devoted to the proof of Theorem 5.1. We prepare some notation. For an integer a, let $\mathcal{O}(a)$ denote the holomorphic line bundle over $\boldsymbol{C}P^1$ such that the self-intersection number of the zero section is a. The total space of $\mathcal{O}(a)$ is realized as the quotient of $(\boldsymbol{C}^2-0)\times\boldsymbol{C}$ by the $\boldsymbol{C}^*$-action defined by $g(z_1,z_2,w)=(gz_1,gz_2,g^aw)$, where $g\in\boldsymbol{C}^*$ and $(z_1,z_2,w)\in(\boldsymbol{C}^2-0)\times\boldsymbol{C}$. Let $t=(t_1,t_2)\in T^2=T$. For $k=(k_1,k_2)\in\boldsymbol{Z}^2$ we abbreviate $t_1^{k_1}t_2^{k_2}$ as t^k. For a basis $\{l,m\}$ of $\boldsymbol{Z}^2$ we define a T-action on $\mathcal{O}(a)$ by

$$t[z_1,z_2,w]=[z_1,t^lz_2,t^mw],$$

where $[z_1,z_2,w]$ denotes the equivalence class of (z_1,z_2,w). This action is effective and makes $\mathcal{O}(a)$ a holomorphic T-line bundle. Denote by $D_a(l,m)$ the total space of the disk bundle of $\mathcal{O}(a)$ with this T-action. It is a real 4-dimensional T-manifold with boundary and has two fixed points $p=[1,0,0]$ and $q=[0,1,0]$. The tangential T-representations at these points are respectively

$$T_pD_a(l,m)=\chi^l+\chi^m, \qquad T_qD_a(l,m)=\chi^{m-al}+\chi^{-l},$$

where χ^u denotes the T-representation defined by $t\to t^u$ as before. The relation between the weights of the tangential T-representations are expressed as

$$\begin{pmatrix} m-al \\ -l \end{pmatrix}=\begin{pmatrix} -a & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} l \\ m \end{pmatrix}. \tag{5.3}$$

Let v_i and a_i be as in Theorem 5.1. The relation $v_{i-1}+v_{i+1}+a_iv_i=0$ may be written as

$$(v_i,v_{i+1})=(v_{i-1},v_i)\begin{pmatrix} 0 & -1 \\ 1 & -a_i \end{pmatrix}.$$

One then sees that the integers a_i must satisfy

$$\begin{pmatrix} 0 & -1 \\ 1 & -a_1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -a_2 \end{pmatrix}\cdots\begin{pmatrix} 0 & -1 \\ 1 & -a_d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $i\in\{1,\dots,d\}$, let $\{u_1^{(i)}, u_2^{(i)}\}$ be the dual basis of $\{v_{i-1}, v_i\}$. Then we have

$$\begin{pmatrix} u_1^{(i+1)} \\ u_2^{(i+1)} \end{pmatrix}=\begin{pmatrix} -a_i & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u_1^{(i)} \\ u_2^{(i)} \end{pmatrix}, \tag{5.4}$$

where $u_j^{(d+1)}=u_j^{(1)}$ for $j=1, 2$.

Now we consider T-manifolds $D_{a_i}(u_1^{(i)}, u_2^{(i)})$ for each $i\in\{1,\dots,d\}$. As observed above, each $D_{a_i}(u_1^{(i)}, u_2^{(i)})$ has two fixed points $p_i=[1, 0, 0]$ and $q_i=[0, 1, 0]$, and the tangential representations at these points are related as in (5.3). Therefore (5.4) ensures that one can plumb $D_{a_i}(u_1^{(i)}, u_2^{(i)})$ and $D_{a_{i+1}}(u_1^{(i+1)}, u_2^{(i+1)})$ at q_i and p_{i+1} equivariantly. We plumb the T-manifolds $D_{a_i}(u_1^{(i)}, u_2^{(i)})$ $(i=1,\dots,d)$ at all T-fixed points in this way to get a connected compact smooth T-manifold N of real dimension 4. The boundary ∂N of N is connected, on which T acts freely. The orbit space $\partial N/T$ is a connected closed manifold of real dimension one, and hence is a circle, so that principal T-bundle $\partial N\to\partial N/T$ is trivial. Hence ∂N bounds $D^2\times T$ equivariantly, the T-action on D^2 being trivial and that on T by multiplication. We paste N and $D^2\times T$ together along the boundary equivariantly to get a closed connected smooth T-manifold M of real dimension 4.

We shall show that M becomes an almost complex toric manifold. Remember that $D_{a_i}(u_1^{(i)}, u_2^{(i)})$ is the total space of the disk bundle of a holomorphic T-line bundle. In particular, the interior of $D_{a_i}(u_1^{(i)}, u_2^{(i)})$ is a complex T-manifold. Since the plumbing construction does not destroy the complex structures, we may assume that the tangent bundle TM admits a T-invariant complex structure over the interior of N. Pushing N a bit into its interior equivariantly, we may assume that the complex structure on $TM|(N-\partial N)$ extends to $TM|N$. Since the T-action on ∂N is free and the complex structure on $TM|\partial N$ is T-invariant, the quotient vector bundle $(TM|\partial N)/T\to\partial N/T$ inherits a complex structure from $TM|\partial N$, and the complex T-vector bundle $TM|\partial N\to\partial N$ is isomorphic to the pullback of the quotient bundle by the quotient map from ∂N to $\partial N/T$. Similarly, since the T-action on $D^2\times T$ is free, the real T-vector bundle $TM|D^2\times T\to D^2\times T$ is also isomorphic to the pullback of the quotient bundle $(TM|D^2\times T)/T\to D^2\times T/T=D^2$ by the quotient map from $D^2\times T$ to D^2. Thus it suffices to show

CLAIM. *Let $E\to D^2$ be a real vector bundle of dimension 4. Then any complex structure on $E|\partial D^2$ extends to a complex structure on E.*

PROOF. The complex structure on $E|\partial D^2$ is classified by a continuous map from ∂D^2 to $GL_4(\mathbf{R})/GL_2(\mathbf{C})$. Here the homogeneous space is homotopy equivalent to the disjoint union of two copies of S^2, so the map extends to a map from D^2. This implies the claim.

The characteristic submanifolds M_i of the almost toric manifold M constructed above are the zero sections of the disk bundles $D_{a_i}(u_1^{(i)}, u_2^{(i)})$. One can check that the element $v_i \in H_2(BH)$ in Lemma 1.7 associated to M_i is the given vector v_i in $\boldsymbol{Z}^2$ through the identification of $H_2(BT)$ with $\boldsymbol{Z}^2$. Since M is almost complex, the function ε_M^- identically vanishes and it is obvious from the construction of M that $\varepsilon_M^+(I)=1$ for all $I \in \Gamma_M^n$. One also sees that $H^{\mathrm{odd}}(M)=0$ and $H^2(M)\cong \boldsymbol{Z}^{d-2}$.

It remains to prove (2) of Theorem 5.1. Since $D_{a_i}(u_1^{(i)}, u_2^{(i)})$ is the disk bundle of a holomorphic line bundle $\mathcal{O}(a_i)$ over $\boldsymbol{C}P^1$ and M_i is its zero section, the self-intersection number of M_i is a_i. On the other hand, it follows from Theorem 3.1 that $c(M)=\prod_{i=1}^d (1+\iota^*\xi_i)$, where $\iota^*: H_T^*(M)\to H^*(M)$ denotes the restriction map as before. Noting that $\iota^*\xi_i$ is the Poincaré dual of M_i and $(\iota^*\xi_i \cup \iota^*\xi_j)[M]$ is the intersection number of M_i and M_j, one sees that

$$(\iota^*\xi_i \cup \iota^*\xi_j)[M]=\begin{cases} 1 & \text{if } |i-j|=1 \text{ or } \{i,j\}=\{1,d\}, \\ a_i & \text{if } i=j, \\ 0 & \text{otherwise}. \end{cases}$$

Putting these into the Riemann-Roch formula $T[M]=\langle (c_1(M)^2+c_2(M))/12, [M]\rangle$, we obtain the statement (2) in Theorem 5.1.

REMARKS. (1) Instead of D^2 above, one can use a compact orientable surface of genus g with a circle boundary to get an almost complex toric manifold M_g. An elementary computation shows that $H^1(M_g)\cong H^3(M_g)\cong \boldsymbol{Z}^{2g}$ and $H^2(M_g)\cong \boldsymbol{Z}^{d-2+4g}$. The multi-fan of M_g is the same as that of M. This shows that unlike the theory of toric varieties the correspondence between unitary (or almost complex) toric manifolds and multi-fans is not bijective.

(2) Using unitary structures on S^2 described in Example 1.6, one can produce unitary toric manifolds M of real dimension 4, where the successive pair v_{i-1} and v_i is a basis of $\boldsymbol{Z}^2$ for each $i \in \{1, \dots, d\}$ as before, but the vectors $v_1, \dots, v_d$ are not necessarily in counterclockwise order, i.e., they may go back and forth. One checks that $\varepsilon_M^+(I)=1$ and $\varepsilon_M^-(I)=0$ (resp. $\varepsilon_M^+(I)=0$ and $\varepsilon_M^-(I)=1$) for $I \in \Gamma_M^n$ if v_{i-1} and v_i $(I=\{i-1, i\})$ are in counterclockwise order (resp. in clockwise order).

6. Moment maps. Henceforth we will use the following identification. First we identify $\boldsymbol{R}$ with the Lie algebra Lie(S^1) of S^1 through the differential of the exponential map from $\boldsymbol{R}$ to S^1 which sends $r \in \boldsymbol{R}$ to $\exp(2\pi\sqrt{-1}r)$. Similarly, Hom$(\boldsymbol{R}, T)$ is identified with Lie(T). Hom(S^1, T) is viewed as a lattice of Hom$(\boldsymbol{R}, T)$ through the exponential map from $\boldsymbol{R}$ to S^1, and it is naturally isomorphic to $H_2(BT)$ as remarked before. This induces an identification of Lie(T) with $H_2(BT; \boldsymbol{R})$ and that of Lie$(T)^*$ (the dual of Lie(T)) with $H^2(BT; \boldsymbol{R})$.

Let $L \to M$ be a complex T-line bundle. With this it is associated a principal S^1-bundle

P over M. We may think of P as the unit circle bundle of L. Let θ be an invariant connection 1-form on P, i.e., θ is a smooth 1-form on P which satisfies these three properties:

$$\text{(6.1)}\qquad \begin{aligned} &(1)\quad i_{\underline{r}}\theta=r &&\text{for}\quad r\in \boldsymbol{R}=\mathrm{Lie}(S^1)\,,\\ &(2)\quad s^*\theta=\theta &&\text{for}\quad s\in S^1\,,\\ &(3)\quad t^*\theta=\theta &&\text{for}\quad t\in T\,,\end{aligned}$$

where $\underline{r}$ denotes the fundamental vector field on P associated with r. The commutativity of the actions of T and S^1 on P and the S^1-invariance (2) above imply that for $v\in H_2(BT;\boldsymbol{R})=\mathrm{Lie}(T)$ the function $i_{\underline{v}}\theta$ on P descends to a function on M. The descendent function is T-invariant by (3). Thus it produces a T-invariant map $\Phi_L: M\to H^2(BT;\boldsymbol{R})=\mathrm{Lie}(T)^*$ such that

$$\text{(6.2)}\qquad \langle \Phi_L(p), v\rangle=(i_{\underline{v}}\theta)(\tilde{p})\,,$$

where $\tilde{p}\in P$ is any point in the fiber over p. The map Φ_L is called a *moment map* (associated to L).

By Lemma 3.2 one can write

$$\text{(6.3)}\qquad c_1^T(L)=\sum_{i=1}^{d} c_i\xi_i \qquad \text{in}\quad \hat{H}_T^2(M)$$

with integers c_i. Let $I\in\Gamma_M^n$ and $p\in M_I$, i.e., $I=I(p)$. By Lemma 1.7, $\xi_i|_p=\xi_i|_q$ if $q\in M_I$. This together with (6.3) shows that $c_1^T(L)|_p=c_1^T(L_p)$ depends only on I, so we denote $c_1^T(L_p)$ by u_I, i.e., $L_p=\chi^{u_I}$.

LEMMA 6.4. *Let $I\in\Gamma_M^n$ and $p\in M_I$. Then $\Phi_L(p)=u_I$.*

PROOF. Let $\tilde{p}\in P$ be any point in the fiber over p. Since $L_p=\chi^{u_I}$, we have $\lambda_v(z)\tilde{p}=z^{\langle u_I,v\rangle}\tilde{p}$ for $z\in S^1$ by (1.8). This together with (6.2) and (6.1)(1) implies that

$$\langle \Phi_L(p), v\rangle=(i_{\underline{v}}\theta)(\tilde{p})=\theta(\underline{v}_{\tilde{p}})=\langle u_I, v\rangle\,.$$

Since $v\in H_2(BT)$ is arbitrary, this proves the lemma. □

For $i\in\{1,\dots,d\}$ we set

$$F_i:=\{u\in H^2(BT;\boldsymbol{R})\,|\,\langle u, v_i\rangle=c_i\}\,.$$

LEMMA 6.5. (1) *For $I\in\Gamma_M^n$, $u_I\in\bigcap_{i\in I}F_i$.*
(2) *$\Phi_L(M_i)\subset F_i$.*

PROOF. (1) It follows from (6.3) and Lemma 1.7 that

$$\langle u_I, v_i\rangle=\langle c_1^T(L)|_p, v_i\rangle=\left\langle \sum_{i=1}^{d} c_i\xi_i|_p, v_i\right\rangle=c_i\,,$$

proving (1).

(2) It follows from the T-invariance (6.1)(3) of θ that the Lie derivative $\mathscr{L}_{\underline{v}}\theta$ vanishes for any $v \in H^2(BT; \boldsymbol{R})$ and hence $di_{\underline{v}}\theta = -i_{\underline{v}}d\theta$ by the Cartan formula $\mathscr{L}_{\underline{v}} = di_{\underline{v}} + i_{\underline{v}}d$. Therefore, taking the exterior derivative at (6.2), we obtain

$$d\langle \Phi_L, v \rangle = -i_{\underline{v}}d\theta\,. \tag{6.6}$$

Here $d\theta$ is the curvature form of the connection θ and can be viewed as a 2-form on M. Therefore $i_{\underline{v_i}}d\theta$ vanishes on M_i because the S^1-subgroup determined by v_i fixes M_i pointwise. This together with (6.6) means that $\langle \Phi_L, v_i \rangle$ is constant on M_i. On the other hand, we know that $\Phi_L(M_i^T)$ is contained in F_i by Lemma 6.4 and (1) above. These imply (2). □

In the following we will make this assumption which is satisfied for toric manifolds: *all isotropy subgroups of M are subtori of T and each fixed point set component of subtori contains at least one T-fixed point.* Then the union $\bigcup M_i$ is the set of points with nontrivial isotropy subgroups, and it follows from the slice theorem (see [1] or [12]) that the orbit space M/T becomes a compact smooth manifold of dimension n with $\bigcup M_i/T$ as boundary (after we round corners).

LEMMA 6.7. *M/T is orientable.*

PROOF. We note that M/T is diffeomorphic to M/T with an open collar of the boundary removed. Therefore it suffices to prove that if X is an orientable smooth manifold with free T-action, then X/T is also orientable. Furthermore, it reduces to the case when $T = S^1$ because X/T is an iterated orbit space of free S^1-actions. It follows from the Wang sequence of the S^1-bundle $\pi: X \to X/S^1$ that $\pi^*: H^1(X/S^1; \boldsymbol{Z}/2) \to H^1(X; \boldsymbol{Z}/2)$ is injective. On the other hand, since $TX = \pi^*(T(X/S^1)) \oplus T^f X$ where $T^f X$ denotes the tangent bundle along the fiber, $w_1(TX) = \pi^* w_1(T(X/S^1)) + w_1(T^f X)$ where w_1 denotes the first Stiefel-Whitney class. Here both $w_1(TX)$ and $w_1(T^f X)$ vanish because X is orientable and $T^f X$ is a trivial real line bundle, since the free S^1-action on X defines a nowhere zero cross section of $T^f X$. Thus $\pi^* w_1(T(X/S^1)) = 0$ and hence $w_1(T(X/S^1)) = 0$, because π^* is injective. This completes the proof of the lemma. □

Since Φ_L is T-invariant, it factors through the quotient:

$$M \to M/T \xrightarrow{\bar{\Phi}_L} H^2(BT; \boldsymbol{R}) \cong \boldsymbol{R}^n\,.$$

We orient M/T in the following way. Choose any orientation for T and give an orientation on M/T so that the orientation on T followed by that of M/T is equal to that of M times $(-1)^{n(n-1)/2}$. The orientation on T induces an orientation on $H^2(BT; \boldsymbol{R})$. If $u \in H^2(BT; \boldsymbol{R}) \setminus \bigcup F_i$, then $\bar{\Phi}_L$ induces a homomorphism $\bar{\Phi}_{L*}: H_n(M/T, \partial(M/T)) \to H_n(H^2(BT; \boldsymbol{R}), H^2(BT; \boldsymbol{R}) \setminus \{u\})$ by Lemma 6.5(2). The fundamental classes are specified in the above homology groups, since M/T and $H^2(BT; \boldsymbol{R})$ are oriented. We define a function.

$$d_L: H^2(BT; \boldsymbol{R}) \setminus \bigcup F_i \to \boldsymbol{Z}$$

by

(6.8) $$d_L(u) = \text{the mapping degree of } \bar{\Phi}_{L^*}.$$

Karshon-Tolman [11] establishes the following facts when M is a toric manifold, but their proof actually works in our setting.

LEMMA 6.9. (1) *The function d_L is locally constant.*

(2) *Let F be one of F_i's. Let u_1 and u_2 be elements in $H^2(BT; \boldsymbol{R}) \setminus \bigcup F_i$ such that the interval $\overline{u_1 u_2}$ intersects the wall F transversely at w, and does not intersect any other $F_j \neq F$. Then*

$$d_L(u_2) - d_L(u_1) = \sum_{F_i = F} \operatorname{sign}\langle u_1 - u_2, v_i \rangle d_{L|M_i}(w),$$

where $d_{L|M_i}$ is the degree function defined with respect to the map $\Phi_L | M_i : M_i \to F_i$.

PROOF. See [11, Remark 6.5] for (1) and (2). The statement (1) also follows from our Lemma 6.5(2). □

Let K be the "canonical" complex T-line bundle of M, i.e., K is the dual of $(n+l/2)$-th exterior product of the complex T-vector bundle $TM \oplus \underline{\boldsymbol{R}}^l$ (l: even). One has a moment map $\Phi_K : M \to H^2(BT; R)$ associated to K. Let

$$\Phi'_L := \Phi_L - \frac{1}{2}\Phi_K : M \to H^2(BT; \boldsymbol{R}).$$

Since $c_1^T(K) = -\sum_{i=1}^d \xi_i$ in $\hat{H}_T^2(M)$ by Theorem 3.1, it follows from Lemma 6.5(2) that $\Phi'_L(M_i)$ is contained in the affine hyperplane

$$F'_i := \left\{ u \in H^2(BT; \boldsymbol{R}) \,|\, \langle u, v_i \rangle = c_i + \frac{1}{2} \right\}.$$

Similarly to Φ_L, Φ'_L induces a map $\bar{\Phi}'_L : M/T \to H^2(BT; \boldsymbol{R})$ and defines a degree function

$$d'_L : H^2(BT; \boldsymbol{R}) \setminus \bigcup F'_i \to \boldsymbol{Z},$$

which depends only on L. Since the union $\bigcup F'_i$ misses the lattice $H^2(BT)$, d'_L is defined for any lattice point.

LEMMA 6.10. *$d'_L(u) = d_L(u)$ for any $u \in H^2(BT) \setminus \bigcup F_i$.*

PROOF. The map $\bar{\Phi}_L - (s/2)\bar{\Phi}_K$: $(M/T, \partial(M/T)) \to (H^2(BT; \boldsymbol{R}), H^2(BT; \boldsymbol{R}) \setminus \{u\})$ $(0 \leq s \leq 1)$ gives a homotopy between $\bar{\Phi}_L$ and $\bar{\Phi}'_L$, which implies the lemma. □

7. Equivariant index. Since we have a T-invariant complex structure on $TM \oplus \underline{\boldsymbol{R}}^l$ (l: even), the map π collapsing M to a point induces, in equivariant K-theory, an equivariant Gysin homomorphism

$$\pi_! : K_T(M) \to K_T(\text{point}) = R(T),$$

where $R(T)$ denotes the character ring of T. The Todd genus $T[M]$ of M is known to be $\pi_!(1)$. The purpose of this section is to describe $\pi_!(L)$ in terms of the shifted moment map Φ'_L associated with a complex T-line bundle L. To be more specific, we express

$$\pi_!(L) = \sum_{u \in H^2(BT)} m_L(u)\chi^u \tag{7.1}$$

with integers $m_L(u)$. The function m_L vanishes for all but finitely many elements u, since $\pi_!(L)$ is an element of $R(T)$. The following theorem is an extension of the result of Karshon-Tolman [11], where M was a toric manifold.

THEOREM 7.2.[1] *Let L be a complex T-line bundle over a unitary toric manifold M. Suppose that M satisfies the assumption stated just before Lemma* 6.7. *Then $m_L = d'_L$ on $H^2(BT)$, where d'_L is the degree function* (*defined in Section* 6) *of the "shifted" moment map $\bar{\Phi}'_L$ associated with L.*

REMARK. Since $\bar{\Phi}'_{L\otimes K} = \bar{\Phi}_{L\otimes K} - \bar{\Phi}_K/2 = \bar{\Phi}_L + \bar{\Phi}_K/2$ and $\bar{\Phi}'_{L^{-1}} = \bar{\Phi}_{L^{-1}} - \bar{\Phi}_K/2 = -\bar{\Phi}_L - \bar{\Phi}_K/2$, we have $\bar{\Phi}'_{L\otimes K} = -\bar{\Phi}'_{L^{-1}}$. This together with Theorem 7.2 implies the identity $\pi_!(L\otimes K) = (-1)^n \pi_!(L^{-1})^*$, where $*$ denotes the complex conjugate of a character. The identity may be viewed as an equivariant index version of the Serre duality. As a matter of fact, the identity directly follows from the Lefschetz formula of the equivariant index, but our observation gives an explanation of shifting a moment by $\bar{\Phi}_K/2$.

EXAMPLE 7.3. We shall illustrate Theorem 7.2 with an example when $n = 1$. As mentioned before, Theorem 7.2 is established by Karshon-Tolman [11] when M is a toric manifold, so we shall take another unitary structure on M $(= S^2)$ described in Example 1.6, which does not come from the complex structure on $\boldsymbol{C}P^1$.

Remember that M is viewed as the unit sphere of $\chi \oplus \boldsymbol{R}$. The fixed points are $p = (0, 1)$ and $q = (0, -1)$, $T_pM = T_qM = \chi$ and $\varepsilon(p) = 1$, $\varepsilon(q) = -1$ (see Section 4 for ε). Let L be a complex T-line bundle over M. Then $L_p = \chi^\alpha$, $L_q = \chi^\beta$ for some integers α, β. It follows from the Lefschetz formula that

$$\pi_!(L) = \frac{\chi^\alpha}{1-\chi^{-1}} + \frac{-\chi^\beta}{1-\chi^{-1}} = \frac{\chi^{\beta+1} - \chi^{\alpha+1}}{1-\chi}$$

[1] After writing this paper, the author was informed by Professor Karshon that the results of [11] is extended to Spin^c-manifolds by Grossberg-Karshon "Equivariant index and the moment map for completely integrable torus actions", Adv. in Math. 133 (1998), 185–223.

$$=\begin{cases}\chi^{\beta+1}+\chi^{\beta+2}+\cdots+\chi^{\alpha} & \text{if } \beta<\alpha\,,\\ 0 & \text{if } \beta=\alpha\,,\\ -\chi^{\alpha+1}-\chi^{\alpha+2}-\cdots-\chi^{\beta} & \text{if } \beta>\alpha\,.\end{cases}$$

On the other hand, the orbit space M/T is an interval with p and q as boundary. Our orientation convention on M/T, mentioned in the paragraph above Lemma 6.9, says that it is oriented from q to p. By Lemma 6.4 we have $\bar{\Phi}_L(p)=\alpha$ and $\bar{\Phi}_L(q)=\beta$. Since the vectors v_1 and v_2 are positive unit vectors as remarked in Example 1.6, $\bar{\Phi}'_L(p)=\alpha+1/2$ and $\bar{\Phi}'_L(q)=\beta+1/2$. One sees that unless $\beta=\alpha$, we have that for $u\in \boldsymbol{Z}$

$$d'_L(u)=\begin{cases}1 & \text{if } \beta+1/2<u<\alpha+1/2\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

in case $\beta<\alpha$, and

$$d'_L(u)=\begin{cases}-1 & \text{if } \alpha+1/2<u<\beta+1/2\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

in case $\beta>\alpha$. Thus Theorem 7.2 is confirmed for our M. There are other unitary toric structures on M, but the same argument as above may apply to confirm Theorem 7.2 as well.

The rest of this section is devoted to the proof of Theorem 7.2. The key of the proof is to show that the function m_L behaves in the same fashion as d_L, that is, to establish Lemmas 7.7 and 7.8 below. Karshon-Tolman [11] establish them when M is a toric manifold, but their proof uses an explicit construction of toric manifolds and does not work in our setting. Instead we make use of the Lefschetz formula for the equivariant Riemann-Roch index to see the behavior of the function m_L.

Let $u\in H^2(BT)$ and $v\in H_2(BT)$ with $\langle u, v\rangle\neq 0$. We will use the following convention of an expansion

$$\frac{1}{1-\chi^{-u}}=\begin{cases}1+\chi^{-u}+\chi^{-2u}+\cdots & \text{if } \langle u,v\rangle<0\,,\\ -\chi^{u}-\chi^{2u}-\cdots & \text{if } \langle u,v\rangle>0\,,\end{cases} \tag{7.4}$$

and call it the *Laurent expansion with respect to* v. This expansion is motivated by the following observation. The left-hand side of (7.4) is a function on T. We restrict it to the S^1-subgroup determined by v. It turns into $1/(1-z^{-\langle u,v\rangle})$ by (1.8). Although $z\in S^1$, we regard z as a variable of $\boldsymbol{C}$. Then the Laurent expansion of $1/(1-z^{-\langle u,v\rangle})$ on $0<|z|<1$ is

$$\begin{matrix}1+z^{-\langle u,v\rangle}+z^{-2\langle u,v\rangle}+\cdots & \text{if } \langle u,v\rangle<0\,,\\ -z^{\langle u,v\rangle}-z^{2\langle u,v\rangle}-\cdots & \text{if } \langle u,v\rangle>0\,,\end{matrix}$$

which corresponds to the right-hand side of (7.4).

Let $I\in\Gamma^n_M$ and $p\in M_I$. We write

$$L_p = \chi^{u_I}, \qquad T_p M = \sum_{i \in I} \chi^{w_{I,i}}$$

with u_I, $w_{I,i} \in H^2(BT)$. The Lefschetz formula (see [7] for example) applied to $\pi_!(L)$ tells us that

$$\pi_!(L) = \sum_{I \in \Gamma_M^n} \frac{(\varepsilon_M^+(I) - \varepsilon_M^-(I))\chi^{u_I}}{\prod_{i \in I}(1 - \chi^{-w_{I,i}})}. \tag{7.5}$$

(See Section 4 for $\varepsilon_M^{\pm}(I)$.)

LEMMA 7.6. *Let $v \in H_2(BT)$ such that $\langle w_{I,i}, v\rangle \neq 0$ for all weights $w_{I,i}$. (This is equivalent to v being generic.) Then the Laurent expansion of the right-hand side of (7.5) with respect to v agrees with $\sum m_L(u)\chi^u$.*

PROOF. Restrict (7.5) to the S^1-subgroup of T determined by v. It follows from (1.8) that (7.5) together with (7.1) turns into

$$\sum_u m_L(u) z^{\langle u, v\rangle} = \sum_I \frac{(\varepsilon_M^+(I) - \varepsilon_M^-(I)) z^{\langle u_I, v\rangle}}{\prod_{i \in I}(1 - z^{-\langle w_{I,i}, v\rangle})}.$$

Although $z \in S^1$, we may regard the above as the identity of rational functions of z. Then the Laurent expansion of the right-hand side above on $0 < |z| < 1$ is equal to the left-hand side. Since the identity holds for any generic v, one concludes that the Laurent expansion of the right-hand side of (7.5) with respect to a generic v agrees with $\sum m_L(u)\chi^u$. □

LEMMA 7.7. *$m_L(u) = m_L(u')$ if u and u' lie in the same region of $H^2(BT) \setminus \bigcup F_i'$.*

PROOF. Expand the term $\chi^{u_I}/\prod_{i \in I}(1 - \chi^{-w_{I,i}})$ in (7.5) with respect to a generic element $v \in H_2(BT)$ and look at the coefficient of χ^u for $u = u_I + \sum_{i \in I} \alpha_i w_{I,i}$, where α_i are integers. By Lemma 1.7, $\{w_{I,i} \mid i \in I\}$ is a basis of $H^2(BT)$ dual to $\{v_i \mid i \in I\}$. We note that u and $u' = u_I + \sum_{i \in I} \alpha_i' w_{I,i}$ lie in the same region of $H^2(BT) \setminus \bigcup F_i'$ if and only if the integers α_i and α_i' lie in the same half line separated at $1/2$ for all $i \in I$, since $\langle u_I, v_i\rangle = c_i$ and F_i' is the affine hyperplane defined by $\langle u, v_i\rangle = c_i + 1/2$.

Suppose $\langle w_{I,i}, v\rangle > 0$ for all i, e.g., $v = \sum_{i \in I} v_i$. Then it follows from (7.4) that the coefficient of χ^u in the expansion is $(-1)^n$ if $\alpha_i \geq 1$ for all i, and 0 otherwise. This shows that the coefficient of χ^u does not change as long as u stays in the same region of $H^2(BT) \setminus \bigcup F_i'$, and this assertion holds even if $\langle w_{I,i}, v\rangle < 0$ for some i. The lemma follows from this observation. □

LEMMA 7.8. *Let F be one of F_i's. Let u_1, u_2 be elements in $H^2(BT) \setminus \bigcup F_i$ such that the interval $\overline{u_1 u_2}$ intersects the wall F transversely at $w \in H^2(BT)$, and does not intersect any other $F_j \neq F$. Then*

$$m_L(u_2)-m_L(u_1)=\sum_{F_i=F}\operatorname{sign}\langle u_1-u_2, v_i\rangle m_{L|M_i}(w)\,.$$

PROOF. For simplicity, we treat a special case where $F=F_i$ for only one i. One finds that the same idea works in the general case. Consider the Laurent expansion of the right-hand side of (7.5) with respect to a generic element $v\in H_2(BT)$. The difference $m_L(u_2)-m_L(u_1)$ arises from the terms $(\varepsilon_M^+(I)-\varepsilon_M^-(I))\chi^{u_I}/\prod_{j\in I}(1-\chi^{-w_{I,j}})$ for $I\in\Gamma_M^n$ containing the i. By Lemma 1.7 we may assume that $\langle w_{I,j}, v_i\rangle=0$ for all $j\neq i\in I$ and $\langle w_{I,i}, v_i\rangle=1$. We split the term for I into

$$\frac{(\varepsilon_M^+(I)-\varepsilon_M^-(I))\chi^{u_I}}{\prod_{j\neq i\in I}(1-\chi^{-w_{I,j}})}\times\frac{1}{1-\chi^{-w_{I,i}}}\,.$$

The sum over I of the first factor above containing the i is nothing but the Lefschetz formula for the equivariant index of the restricted T-line bundle $L\,|\,M_i$. Therefore the coefficient of χ^w in the Laurent expansion of the sum with respect to v is equal to $m_{L|M_i}(w)$ by Lemma 7.6. On the other hand, the second factor above has two expressions (7.4) according to the sign of $\langle w_{I,i}, v\rangle$. Nothing that $\langle w_{I,i}, v_i\rangle=1$, one sees that

$$m_L(u_2)-m_L(u_1)=\operatorname{sign}\langle u_1-u_2, v_i\rangle m_{L|M_i}(w)$$

in either case. □

PROOF OF THEOREM 7.2. Step 1. We prove that $m_L=d_L$ on $H^2(BT)\setminus\bigcup F_i$ by induction on n. The case where $n=1$ is treated in Example 7.3, so we suppose that the above identity holds for M of dimension $\leq n-1$. Since both m_L and d_L are constant on each region of $H^2(BT)\setminus F_i$ by Lemma 6.9(1) and Lemma 7.7, it suffices to show that $m_L(u)=d_L(u)$ for one element $u\in H^2(BT)$ in each region. Moreover, since $m_{L^k}(ku)=m_L(u)$ and $d_{L^k}(ku)=d_L(u)$ for any positive integer k, we may assume that each region has a lattice point and that for any adjacent regions there exist lattice points u_1 and u_2 as in Lemma 7.8. Remember that $m_L(u)=0$ for all but finitely many elements $u\in H^2(BT)$ and $d_L(u)=0$ for u far away from the origin because the image $\Phi_L(M)$ is compact. This means that $m_L=d_L(=0)$ on some region of $H^2(BT)\setminus\bigcup F_i$. Lemma 6.9(2) and Lemma 7.8 show that the functions m_L and d_L change in the same fashion when they across walls F_i's. Since $m_{L|M_i}=d_{L|M_i}$ on $F_i\cap(H^2(BT)\setminus\bigcup_{j\neq i}F_j)$ by induction assumption, it follows that $m_L=d_L$ on $H^2(BT)\setminus\bigcup F_i$. (To be precise, the identity $m_{L|M_i}=d_{L|M_i}$ is not immediate from the induction assumption because the action of $T=T^n$ on $L\,|\,M_i$ does not reduce to an action of T^{n-1}. In fact, we take a tensor product of $L\,|\,M_i$ with a T-module χ^u for a suitable $u\in H^2(BT)$ so that the action of T on $L\,|\,M_i\otimes\chi^u$ reduces to an action of T^{n-1}, and apply the induction assumption to $L\,|\,M_i\otimes\chi^u$ to get the desired identity.)

Step 2. Step 1 together with Lemma 6.10 establishes $m_L=d'_L$ on $H^2(BT)\setminus\bigcup F_i$, so it remains to prove the equality on $(\bigcup F_i)\cap H^2(BT)$. Let $u_0\in(\bigcup F_i)\cap H^2(BT)$. Define

$$\tilde{c}_i = \begin{cases} c_i & \text{if } u_0 \notin F_i\,, \\ c_i + 1 & \text{if } u_0 \in F_i\,, \end{cases}$$

and consider a complex T-line bundle $\tilde{L}$ with $c_1^T(\tilde{L}) = \sum \tilde{c}_i \xi_i$ in $\hat{H}_T^2(M)$. Then $u_0 \in H^2(BT) \setminus \bigcup \tilde{F}_i$, where $\tilde{F}_i := \{u \in H^2(BT; \boldsymbol{R}) \mid \langle u, v_i \rangle = \tilde{c}_i\}$, so $m_{\tilde{L}}(u_0) = d_{\tilde{L}}(u_0)$ by Step 1. It is clear from the proof of Lemma 6.10 and Lemma 7.7 that $d_{\tilde{L}}(u_0) = d'_L(u_0)$ and $m_{\tilde{L}}(u_0) = m_L(u_0)$, respectively. Thus $d'_L(u_0) = m_L(u_0)$. Since u_0 is arbitrary, this completes the proof of the theorem. □

8. A generalized Pick's formula. In this section we establish a generalization of Pick's formula as an application of the result in Section 7.

Let $\mathscr{P}$ be an integral oriented polygon in $\boldsymbol{R}^2$ with sign assigned to each side, where "integral" means that the vertices lie in the lattice $\boldsymbol{Z}^2 \subset \boldsymbol{R}^2$ and "polygon" means a piecewise linear closed curve. We allow $\mathscr{P}$ to have self-intersections but do not allow that consecutive three vertices lie on a line. Denote the oriented sides of $\mathscr{P}$ by s_i $(i = 1, \ldots, d)$, where they are numbered so that the next side of s_i in $\mathscr{P}$ is s_{i+1}. The assigned sign of s_i is denoted by $\mathrm{sgn}(s_i)$. Let n_i $(i = 1, \ldots, d)$ be a normal vector to s_i such that the 90 degree rotation of $\mathrm{sgn}(s_i)n_i$ has the same direction as s_i. The winding number of $\mathscr{P}$ around a point in $\boldsymbol{R}^2 \setminus \mathscr{P}$ defines a locally constant function $d_{\mathscr{P}}$ on $\boldsymbol{R}^2 \setminus \mathscr{P}$. We introduce three invariants of $\mathscr{P}$:

$$A(\mathscr{P}) := \text{the integral of } d_{\mathscr{P}} \text{ over } \boldsymbol{R}^2\,,$$

$$B(\mathscr{P}) := \sum_{i=1}^{d} \mathrm{sgn}(s_i) |s_i|\,,$$

$$C(\mathscr{P}) := \text{the rotation number of the sequence of normal vectors } n_1, \ldots, n_d\,,$$

where $|s_i|$ denotes the relative length of s_i, i.e., one plus the number of lattice points in the interior of s_i. We say that $\mathscr{P}$ is *simple* if $\mathscr{P}$ has no self-intersection, $\mathrm{sgn}(s_i)$ is positive for any i, and $\mathscr{P}$ is oriented so that the domain bounded by $\mathscr{P}$ lies on the left-hand side of $\mathscr{P}$ when moving in the direction of the orientation of $\mathscr{P}$. If $\mathscr{P}$ is simple, then $A(\mathscr{P})$ is the area of the domain bounded by $\mathscr{P}$, $B(\mathscr{P})$ is the number of lattice points on $\mathscr{P}$, and $C(\mathscr{P}) = 1$.

We now define an integer $\sharp\mathscr{P}$ which coincides with the number of lattice points on the domain bounded by $\mathscr{P}$ when $\mathscr{P}$ is simple. Let $\mathscr{P}'$ be an oriented polygon in $\boldsymbol{R}^2$ obtained from $\mathscr{P}$ by translating each s_i slightly in the direction of n_i. It misses lattice points, so that the winding number $d_{\mathscr{P}'}(u)$ is defined for any lattice point u. We define

$$\sharp\mathscr{P} := \sum_{u \in \boldsymbol{Z}^2} d_{\mathscr{P}'}(u)\,.$$

The main result of this section is the following, which reduces to Pick's formula (see

[4]) when $\mathscr{P}$ is simple.

THEOREM 8.1. $\sharp\mathscr{P}=A(\mathscr{P})+(1/2)B(\mathscr{P})+C(\mathscr{P})$.

REMARKS. (1) Let $\mathscr{P}^\circ$ be $\mathscr{P}$ with reversed signs on the sides of $\mathscr{P}$. Then $\sharp\mathscr{P}^\circ=A(\mathscr{P})-(1/2)B(\mathscr{P})+C(\mathscr{P})$. When $\mathscr{P}$ is simple, $\sharp\mathscr{P}^\circ$ coincides with the number of lattice points on the interior of the domain bounded by $\mathscr{P}$.

(2) Given a positive integer m, one can expand $\mathscr{P}$ by multiplying by m. Denote the expanded polygon by $m\mathscr{P}$. Since $A(m\mathscr{P})=A(\mathscr{P})m^2$, $B(m\mathscr{P})=B(\mathscr{P})m$ and $C(m\mathscr{P})=C(\mathscr{P})$, it follows from Theorem 8.1 that

$$\sharp m\mathscr{P}=A(\mathscr{P})m^2+\frac{1}{2}B(\mathscr{P})m+C(\mathscr{P})\,.$$

This may be viewed as an Ehrhart polynomial of $\mathscr{P}$. We also have

$$\sharp m\mathscr{P}^\circ=A(\mathscr{P})m^2-\frac{1}{2}B(\mathscr{P})m+C(\mathscr{P})\,,$$

so Ehrhart's reciprocity law holds for $\mathscr{P}$.

Theorem 8.1 can be proved in an elementary way, but we shall give a proof which uses the result in Section 7.

We identify $\boldsymbol{R}^2$ (resp. $\boldsymbol{Z}^2$) with $H^2(BT;\boldsymbol{R})$ (resp. $H^2(BT)$) through a decomposition $T=S^1\times S^1$, and view $\mathscr{P}$ as a polygon in $H^2(BT;\boldsymbol{R})$. To each i ($i=1,\dots,d$), there are two primitive elements in $H_2(BT)$ which are constant on s_i. We denote by v_i the one such that $\mathrm{sgn}(s_i)v_i$ is positive on the right-hand side of s_i, and denote by c_i the constant which v_i takes on s_i. The constants c_i are integers because v_i's and the vertices of $\mathscr{P}$ are integral. One can recover $\mathscr{P}$ from the datum $\mathscr{L}=\{(v_1,c_1),\dots,(v_d,c_d)\}$ and may think of $\mathscr{P}'$ as the polygon obtained from a datum $\mathscr{L}'=\{(v_1,c_1'),\dots,(v_d,c_d')\}$ where $c_i'=c_i+1/2$.

Each successive pair v_{i-1} and v_i is not necessarily a basis of $H_2(BT)$. We add vectors v's between v_{i-1} and v_i so that each successive pair of vectors is a basis of $H_2(BT)$ (see [4, Section 2.6]). This provides a new datum $\tilde{\mathscr{L}}$ by adding $(v, v(s_{i-1}\cap s_i))$'s to $\mathscr{L}$. The polygon obtained from $\tilde{\mathscr{L}}$ is the same as $\mathscr{P}$, but the shifted polygon $\tilde{\mathscr{P}}'$ obtained from $\tilde{\mathscr{L}}$ is not the same as $\mathscr{P}'$. However one checks that $d_{\tilde{\mathscr{P}}'}=d_{\mathscr{P}'}$ on the lattice $\boldsymbol{Z}^2=H^2(BT)$. Therefore we may assume that each successive pair v_{i-1} and v_i is a basis of $H_2(BT)$ in the sequel.

Let M be a unitary toric manifold of real dimension 4 whose multi-fan is the collection of cones spanned by successive pairs v_{i-1} and v_i ($i=1,\dots,d$). We may assume that the T-action on M is effective and $H^{\mathrm{odd}}(M)$ vanishes. Let L be a complex T-line bundle over M with $c_1^T(L)=\sum c_i\xi_i$, whose existence is ensured by Lemma 3.2. Then the moment map Φ_L associated to L can take the place of $\mathscr{P}$. It follows from Theorem 7.2 together with the Riemann-Roch formula that

$$\sharp\mathscr{P}=\langle e^{c_1(L)}\mathscr{T}(M),[M]\rangle,$$

where $\mathscr{T}(M)$ denotes the Todd class of M. Since M is of real dimension 4 and $\mathscr{T}(M)=1+c_1(M)/2+\cdots$, the identity above reduces to

$$\sharp\mathscr{P}=\frac{1}{2}\langle c_1(L)^2,[M]\rangle+\frac{1}{2}\langle c_1(L)\cup c_1(M),[M]\rangle+T[M].$$

The formula (5.2) in [11] implies that the first term at the right-hand side of the above identity agrees with $A(\mathscr{P})$. (They state the formula for a toric manifold M, but their proof works in our setting with no change.) We know that $T[M]=C(\mathscr{P})$ by our Section 5. Thus it remains to prove that

$$\langle c_1(L)\cup c_1(M),[M]\rangle=B(\mathscr{P}).$$

Since $c_1(M)=\sum\iota^*\xi_i$ by Theorem 3.1, where ι^* is the restriction map from $H^2_T(M)$ to $H^2(M)$, and $\iota^*\xi_i$ is the Poincaré dual of M_i, we have

$$\langle c_1(L)\cup c_1(M),[M]\rangle=\langle c_1(L),c_1(M)\cap[M]\rangle=\sum_{i=1}^{d}\langle c_1(L\,|\,M_i),[M_i]\rangle.$$

Thus it suffices to prove that

$$\langle c_1(L\,|\,M_i),[M_i]\rangle=\operatorname{sgn}(s_i)|s_i|. \tag{8.2}$$

Set $u_i=s_i\cap s_{i+1}$, so u_{i-1} and u_i are the endpoints of s_i. Let p_i and q_i be the T-fixed points in M_i. We may assume that q_i (resp. p_i) maps to u_{i-1} (resp. u_i) by the moment map Φ_L. Let $\varphi:q_i\to M_i$ be the inclusion map. We give the usual point orientation on q_i and consider an element $\varphi_!(1)\in H^2_T(M_i)$, where $\varphi_!:H^0_T(q_i)\to H^2_T(M_i)$ is the equivariant Gysin map and 1 denotes the unit element of $H^0_T(q_i)$. Since M_i is fixed pointwise under the circle subgroup T_{v_i} of T determined by v_i, and v_i is constant on s_i, $\varphi_!(1)|_{q_i}\in H^2(BT)$ viewed as a vector is parallel to s_i. Moreover, the effectiveness of the T-action on M implies that $\varphi_!(1)|_{q_i}$ is primitive. Therefore there is a unique integer k_i such that

$$k_i\varphi_!(1)|_{q_i}=u_{i-1}-u_i. \tag{8.3}$$

Note that $k_i=|s_i|$ up to sign because $\varphi_!(1)|_{q_i}$ is primitive. On the other hand, we have

$$\varphi_!(1)|_{p_i}=0,\quad c_1^T(L\,|\,M_i)|_{q_i}=c_1^T(L)|_{q_i}=u_{i-1},\quad c_1^T(L\,|\,M_i)|_{p_i}=c_1^T(L)|_{p_i}=u_i,$$

where the first identity follows from the fact that q_i and p_i have no intersection and the latter two identities follows from Lemma 6.4. These observations show that $k_i\varphi_!(1)$ and $c_1^T(L\,|\,M_i)-u_i$ restrict to the same element in $H^2_T(M_i^T)$. Since the restriction map is injective, one concludes that

$$c_1^T(L\,|\,M_i)-u_i=k_i\varphi_!(1)\qquad\text{in}\quad H^2_T(M_i).$$

Now we restrict this identity to $H^2(M_i)$. The element $\varphi_!(1)$ restricts to $\varepsilon(q_i)$ times the cofundamental class of M_i (see Section 4 for $\varepsilon(q_i)$) and u_i restricts to zero. Therefore we obtain

$$\langle c_1(L \mid M_i), [M_i]\rangle = \varepsilon(q_i)k_i .$$

This verifies (8.2) up to sign, since $\varepsilon(q_i) = \pm 1$ and $k_i = |s_i|$ up to sign.

It remains to check that $\varepsilon(q_i)k_i$ and $\operatorname{sgn}(s_i)$ have the same sign. We know by (1.2) and Lemma 1.7 that the tangential representation at a T-fixed point is determined by v_i's. In our case, the T-module $T_{q_i}M$ is determined by v_{i-1} and v_i. Suppose that $\operatorname{sgn}(s_i)$ is positive. When v_{i-1} and v_i are in counterclockwise order, $\varepsilon(q_i) = +1$ and $\varphi_!(1)|_{q_i}$ has the same direction as $u_{i-1} - u_i$; so $\varepsilon(q_i)k_i > 0$ by (8.3). When v_{i-1} and v_i are in clockwise order, $\varepsilon(q_i) = -1$ and $\varphi_!(1)|_{q_i}$ has the opposite direction to $u_{i-1} - u_i$; so $\varepsilon(q_i)k_i > 0$ by (8.3) as well. The same observation shows that if $\operatorname{sgn}(s_i)$ is negative, then $\varepsilon(q_i)k_i < 0$. In either case $\varepsilon(q_i)k_i$ and $\operatorname{sgn}(s_i)$ have the same sign. This completes the proof of Theorem 8.1.

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