Bulletin of the **S**ection of **L**ogic
Volume 32:1/2 (2003), pp. 19–26

Wojciech Dzik

UNITARY UNIFICATION OF S5 MODAL LOGIC AND ITS EXTENSIONS

Abstract

It is shown that all extensions of S5 modal logic, both in the standard formalization and in the formalization with strict implication, as well as all varieties of monadic algebras have unitary unification.

1. Introduction

Unification and E-unification are important applications of logic in Computer Science, in particular in Automated Deduction as well as in Term Rewriting Systems and Databases (see [1], [8]). There is a classification of equational theories, or varieties of algebras, under unification types.
Given an equational theory E and a finite set of pairs of terms called E-*unification problem*: $(\Pi) : (s_1, t_1), ..., (s_n, t_n)$,
a unifier (a solution) for (Π) is a substitution σ such that
$E \vdash \sigma(s_1) = \sigma(t_1), ..., \sigma(s_n) = \sigma(t_n)$.
(Π) is called *unifiable (solvable)* if there exists at least one unifier.
A substitution σ is *more general* then a substitution τ, $\tau \preceq \sigma$,
if there is a substitution θ such that $E \vdash \theta \circ \sigma = \tau$.
$\preceq$ is reflexive and transitive.
A mgu, the most general unifier can be interpreted as "the best" solution of the unification problem (Π). We consider $\preceq$ between unifiers modulo exchanging of variables.
An equational theory E is said to have *unitary unification* (or a *unification type =1*) if for every two unifiable terms t_1, t_2, there is a mgu σ (more precisely E-mgu) such that $E \vdash \sigma(t_1) = \sigma(t_2)$.

The other unification types are defined by taking “the worst” cases of unifiable terms under an equational theory E: if there exist finitely (infinitely) many maximal w.r.t $\preceq$ unifiers in the standard formalization and in the formalization with strict implication, for some terms then E has a finite (infinite) unification type, if there is no maximal w.r.t. $\preceq$ unifiers then E has the unification type $= 0$ (very bad). Hence ”symbolic” unification type can be =1 (unitary), finite, infinite or $= 0$.

Consider an example of the variety of Boolean algebras. Unifiable terms always have a mgu, i.e. the variety of Boolean algebras has unitary unification. In other words Classical (propositional) Logic has unitary unification. Unification algorithms for finding a mgu in Boolean algebras are described in Martin, Nipkov [7].

Questions on unification and unification types of varieties of algebras can be translated into various logics which correspond to the varieties (see Ghilardi [5]). In this case the unification problem is a single formula A and the unifier for a formula A is a substitution σ such that $\vdash \sigma(A)$ in logic. A formula A is unifiable if such σ exists. It is known that every unifiable formula A in the classical logic has a most general substitution σ such that $\vdash \sigma(A)$.

Intuitionistic logic INT (or, the variety of Heyting algebras) can not have unitary unification. Example: the formula $x \vee \neg x$ has two “maximal” unifiers: $x \mapsto (p \rightarrow p)$ and $x \mapsto \neg(p \rightarrow p)$ but a mgu for $x \vee \neg x$ does not exist. S.Ghilardi, ([5]) showed that INT has finitary unification, i.e. if a formula is unifiable, then there are finitely many ”best” (i.e. maximal w.r.t. $\preceq$) unifiers.

Using algebraic approach (which will be sketched in the next paragraph) S. Ghilardi also showed ([4]) that the variety of distributive lattices and the variety of distributive lattices with pseudocomplement have unification type $= 0$.

We briefly present, after S.Ghilardi [4], [6], an algebraic to unification and unification types. Ghilardi used finitely presented algebras but we will only use locally finite varieties, i.e. such varieties that finitely generated algebras are finite. In this case “finitely presented” is reduced to “finite”. We show, using Ghilardi’s algebraic method, that all varieties of monadic algebras have unitary unification. Moreover we prove, both in algebraic and in syntactical way, that all extensions of modal logic S5, in the standard formalization and in the formalization with strict implication, also have unitary unification.

2. The algebraic approach to unification

In this approach, which is equivalent to the previous "symbolic" one (cf. [4]), for a given locally finite equational theory E, that determines the variety of algebras V_E, E-unification problem (Π) corresponds to a finite algebra $\mathbf{A}$ from V_E.
A *unifier (a solution)* for $\mathbf{A}$ is a pair given by a projective algebra $\mathbf{P}$ and a morphism $u : \mathbf{A} \to \mathbf{P}$. $\mathbf{P}$ is *projective* if for every f and q there is a g such that the diagram Diag.1 commutes

(Diag.1)

Given two unifiers u_1 and u_2 for $\mathbf{A}$, $u_1 : \mathbf{A} \to \mathbf{P}_1$ is *more general* then $u_2 : \mathbf{A} \to \mathbf{P}_2$, $u_2 \preceq u_1$ iff there is a morphism such that the following diagram (Diag.2) commutes:

(Diag.2)

Unification types are defined in the same way as for symbolic unification, i.e. according to the number (1, finite, infinite or 0) of maximal (w.r.t. $\preceq$) unifiers.

Theorem 1. (S.Ghilardi [4]). *For any equational theory E the 'symbolic' and the 'algebraic' unification type coincide.*

Corollary 2. ([4]): *The unification type is a categorical invariant.*

It is known that finite Heyting algebras correspond, by duality, to finite posets (partially ordered sets).
For a locally finite variety $V \subseteq H$, given by a theory E, V_{fin}, the category of finite algebras from V corresponds to F_V, the category of finite poset.
Now E-unification problem corresponds to a finite poset P.

A unifier (solution) for P is a pair given by an injective (see below) poset $I \in F_V$mand an (open) morphism $u : I \to P$,
Given two unifiers u_1 and u_2 for P, $u_1 : I_1 \to P$ is *more general* then $u_2 : I_2 \to P$, $u_2 \preceq u_1$, there is a morphism such that the following diagram (Diag.3) commutes:

(Diag.3)

Example 1. Boolean algebras and Classical Logic (S.Ghilardi [6]). The variety is locally finite; finite (countable) algebras are projective (P.Halmos), except the degenerate one-element algebra which is not unifiable. Hence, unification type is =1, for any finite non-degenerate Boolean algebra A, the identity morphism $i : A \to A$ is a mgu.

For a given poset P there is a Heyting algebra P^* of the *upward closed* subsets, i.e. such $X \subseteq P$, that $(p \in X, p \leq q \Rightarrow q \in X)$.
Upward closed subsets are also called *generated subframes*.
An order preserving map $f : P \to Q$ among posets is said to be *open* iff for $p \in P, q \in Q$, $f(p) \leq q \Rightarrow \exists p_1$ $(p \leq p_1$ and $f(p_1) = q)$.
If an open map f is surjective, then it is called a p-morphism and then Q is a p-morphic image of P.
A poset I is *injective* in F_V iff whenever I is a generated subframe of some $P \in F_V$ then there is a p-morphism $g : P \to I$ which does not change the elements of I.

EXAMPLE 2. Gödel-Dummett's algebras and linear logic of Gödel and Dummett (cf.S.Ghilardi, [6]). Gödel Dummett's algebras are Heyting algebras such that:
$(p \to q) \vee (q \to p) = 1$
The variety is locally finite; finite algebras are dual to the category of locally linear posets, i.e. posets satisfying:
$(p \leq q_1 \textit{ and } p \leq q_2) \Rightarrow (q_1 \leq q_2 \textit{ or } q_2 \leq q_1)$
and open maps.
It can be checked that every nonempty finite locally linear poset P is an injective object, hence the identity morphism $i : P \to P$ is a mgu, and linear logic of Gödel and Dummett has unitary unification.
In this example it is more convenient to deal with the dual category of finite posets then with algebras.

3. Monadic Algebras and Extensions of Modal Logic S5

A topological Boolean algebra $\mathsf{A} = \langle A, \vee, \wedge, -, I, 0, 1\rangle$ (equivalently a closure algebra) is a Boolean algebra $\langle A, \vee, \wedge, -, 0, 1\rangle$ with an additional unary operation I, for "interior" (or C, for "closure") such that $Ia \leq a, I(a \wedge b) = Ia \wedge Ib, IIa = Ia, I1 = 1$; $(Ca = -I - a)$.
Modal logic S4 (or the variety of topological Boolean algebras) does not have unitary unification: the unifiable formula $\Box A \vee \Box\neg A$ does not have a mgu. ('necessity' $\Box$ corresponds to the 'interior' I).
However S.Ghilardi showed that modal logic S4 has finitary unification type.

A topological Boolean algebra is a *monadic* algebra if $I - Ia = -Ia$ holds. Monadic algebras were introduced by Paul Halmos in his algebraization of 1st order logic
An algebra is *simple* if it has only two congruences; in case of a monadic algebra A, this means that $Ia = 0$, for $a \neq 1$.
Let $\mathsf{A_n} = \langle \mathbf{2^n}, \vee, \wedge, -, I, 0, 1\rangle$ be a simple monadic algebra of power 2^n, i.e. with n atoms (it is called *a Henle algebra*), let $\mathbf{M_n}$ be the variety generated by $\mathsf{A_n}$, and let $\mathbf{M}$ be the variety of all monadic algebras. It is known that $\mathbf{M}$ and $\mathbf{M_i}$ for $i = 1, 2, ...$ are locally finite and all subvarieties of $\mathbf{M}$ form a chain: $\mathbf{M_0} \subset \mathbf{M_1} \subset ... \subset \mathbf{M}$.
$\mathbf{M_0}$ is the class of one element monadic algebras. It is also known that

every finite monadic algebra is a direct product of simple algebras and that every n-generated free algebra in $\mathbf{M}$, $Fr_{\mathbf{M}}(n)$, is isomorphic to n-generated free algebra in $\mathbf{M_k}$, $Fr_{\mathbf{M_k}}(n)$, for $k = 2^n$.
The next lemma follows from the fact that a (finite) projective algebra is a retract of a free (finite) algebra and from the remarks (above) on n-generated free algebras.

LEMMA 3. (R.Quackenbush [9]). *Let* $\mathbf{V}$ *be a variety of monadic algebras,* $\mathbf{V} \neq \mathbf{M_0}$. *Then a finite nontrivial algebra* $\mathbf{P}$ *is projective in* $\mathbf{V}$ *iff* $\mathbf{P}$ *has* A_1 *as a homomorphic image.*

EXAMPLE 3. A monadic algebra on the powerset of $X = \{a, b, c, d\}$ with I such that all the open (and closed) sets are $\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}$ is not projective.

THEOREM 4. *The variety of all monadic algebras and its every subvariety have unitary unification.*

PROOF (ALGEBRAIC). Let $\mathbf{V}$ be a variety of monadic algebras, $\mathbf{V} \neq \mathbf{M_0}$. We use algebraic definition of unification. A unifier (a solution) for $\mathbf{A}$ is a pair given by a projective algebra $\mathbf{P}$ and a morphism $u : \mathbf{A} \to \mathbf{P}$. A finite nontrivial algebra $\mathbf{A}$ is unifiable in $\mathbf{V}$ iff there is a projective algebra $\mathbf{P}$ and a homomorphism $u : \mathbf{A} \to \mathbf{P}$. By the above lemma any finite nontrivial algebra $\mathbf{P}$ is projective iff $\mathbf{P}$ has A_1 as a homomorphic image. It follows that $\mathbf{A}$ has A_1 as a homomorphic image, i.e. $\mathbf{A}$ is projective in $\mathbf{V}$. Hence the identity morphism $i : \mathbf{A} \to \mathbf{A}$ is a mgu and $\mathbf{V}$ has unitary unification.

COROLLARY 5. *Modal logic S5 and all its extensions have unitary unification.*

Modal logic S5 corresponds to the variety $\mathbf{M}$ of all monadic algebras and extensions of S5 correspond to subvarieties of the variety $\mathbf{M}$.

REMARKS: The above proof is an application of S. Ghilardi algebraic approach to unification. Unitary unification in the variety of all monadic algebras can also be derived in entirely different way from the fact that discriminator varieties have unitary unification (S. Burris, [1]). S. Ghilardi stated in his lecture (Tarski Centenary Symposium 2001) that modal logic S5 has unitary unification.

Till now we have considered modal logic S5 in the standard formalization based on classical logic with the classical connectives plus 'necessity' operator $\Box$. There is also a formalization of S5 with *strict implication* "$\prec$" (C. A. Meredith) and the other connectives, with the axioms $x \prec (y \prec y)$, $(x \prec y) \prec ((y \prec z) \prec (x \prec z))$, $(((x \prec y) \prec z) \prec (x \prec y)) \prec (x \prec y)$, usual axioms for $\wedge$, $(x \prec y) \prec (\neg y \prec \neg x)$, $x \prec \neg\neg x$, $\neg\neg x \prec x$, $x \wedge \neg(x \wedge \neg y) \prec y$, $x \vee y$ defined by $\neg(\neg x \wedge \neg y)$ and Modus Ponens for $\prec$.

Now we present a syntactic proof of the above corollary. Its advantage is that a unifier is given in an explicit way. It will also be used in the formalization of S5 with strict implication. This formalization does not have well developed algebraic theory.

THEOREM 6. *Modal logic S5 (both in the standard formalization and in the formalization with strict implication) and all its extensions have unitary unification.*

PROOF. a) For the standard formalization. Assume that L is a modal logic extending S5. Let A be a formula unifiable in L, hence there is a ground substitution U_0 such that $\vdash U_0(A)$ (a ground substitution means using only $\top = (x \to x)$ and $\bot = \neg(x \to x)$). Define a substitution (x is a propositional variable):

$$\sigma_A(x) = \begin{cases} \Box A \to x, & \text{if} \quad U_0(x) = \top \\ \Box A \wedge x, & \text{if} \quad U_0(x) = \bot \end{cases}$$

By induction on the length of B we have for every formula B:

$$\sigma_A(B) \rightleftarrows \begin{cases} \Box A \to B, & \text{if} \quad U_0(B) = \top \\ \Box A \wedge B, & \text{if} \quad U_0(B) = \bot \end{cases}$$

We have $\vdash \sigma_A(A)$, and, for every unifier τ of A, $\tau(\sigma_A(x)) = \tau(x)$, i.e. $\sigma_A(x)$ is a mgu for A in any extension L of S5.
b) For the formalization with strict implication the proof is similar to a) but σ_A, for a unifiable formula A, is defined as follows:

$$\sigma_A(x) = \begin{cases} \neg((A \prec A) \prec A) \vee x, & \text{if} \quad U_0(x) = \top \\ ((A \prec A) \prec A) \wedge x, & \text{if} \quad U_0(x) = \bot \end{cases}$$

The rest of the proof is analogous to a).

Remark. It can be proved (cf. [3]) that *pure implicational logic C5*, which is the strict implicational fragment of S5 formalized with strict implication, and all its extensions which are included in logic determined by the 4-element Henle algebra, does not have unitary unification.
Intuitionistic logic is just the opposite case; it does not have unitary unification but its pure implicational fragment and all its extensions have unitary unification.

References

[1] F. Baader, J. H. Siekmann *Unification Theory*, in D.M.Gabbay, C.J.Hogger, J.A.Robinson (eds.) "Handbook of Logic in Artificial Intelligence and Logic Programming", Oxford Univ. Press, 1993, pp. 41–125.

[2] S. Burris, *Discriminator Varieties and Symbolic Computation*, J. of Symbolic Computation, vol 13, 1992, 175–207.

[3] W. Dzik, *Unification in some implicational logics, in preparation.*

[4] S. Ghilardi, *Unification through Projectivity*, J. of Logic and Computation, vol.7, 1997, pp. 733–752.

[5] S. Ghilardi, *Unification in Intuitionistic Logic*, J. of Symbolic Logic, vol.64, N.2, 1999, pp. 859–880.

[6] S. Ghilardi, *Unification, Finite Duality and Projectivity in Varieties of Heyting Algebras*, to appear in Proceedings of Tarski Centenary Symposium 2001, Warszawa.

[7] U. Martin, T.Nipkov, *Boolean unification - the story so far*, J. of Symbolic Computation, vol.7, 1989, pp. 275–293.

[8] J. H. Siekmann, *Unification Theory*, J. of Symbolic Computation, vol.7, 1989, pp. 207–274.

[9] R. W. Quackenbush, *Demi -Semi-Primal Algebras and Malcev-Type Conditions*, Mathematische Zeithschrift vol.122, 1971, pp. 177–188.

University of Silesia
Institute of Mathematics,
Bankowa 12
40 007 Katowice, Poland