Units of Measurement and the Stone Index in Demand System Estimation

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The Stone index typically used in estimating linear almost ideal demand systems is not invariant to changes in units of measurement, which may seriously affect the approximation properties of the model. A modification to the Stone index, or use of a regular price index instead, are both desirable practices in estimating linear AI models.

Key words: almost ideal demand system, index numbers.

One of the most commonly used specifications in applied demand analysis is the almost ideal (AI) demand system proposed by Deaton and Muellbauer. A primary reason for the popularity of this model is that, while satisfying a number of desirable properties, it can be approximated at the estimation stage by a linear form. This linear AI model specification typically utilizes a "Stone" price index. Because this linear AI model is not itself derived from a well-specified representation of preferences, this system is of interest only as an approximation to the (integrable) nonlinear AI model. Hence, it is important to ensure good approximation properties for the linear AI model. Unfortunately, as this paper will show, such approximation properties may be seriously affected by the fact that the Stone index is not invariant to the (arbitrary) choice of units of measurement for prices and quantities.

AI and Linear AI Demand Systems

Deaton and Muellbauer's AI demand system can be written in share form as

$$(1) \qquad w_{it} = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log(p_{jt}) + \beta_i \log\left(\frac{x_t}{P_t}\right)$$

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where w denotes shares, p denotes prices, x is total expenditure, (i,j) index the goods, and t indexes time. To satisfy homogeneity, adding-up, and Slutsky symmetry, the parameters of the model are constrained by $\Sigma_i \alpha_i = 1$, $\Sigma_i \gamma_{ij} = \Sigma_j \gamma_{ij} = 0$, $\Sigma_i \beta_i = 0$, and $\gamma_{ij} = \gamma_{ji}$. Finally, P is a translog price index defined by

$$(2) \qquad \log(P_t) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \log(p_{it}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij} \log(p_{it}) \log(p_{jt}).$$

The AI demand system possesses some desirable properties: it is a "flexible" representation of an arbitrary demand system, and it can satisfy exact aggregation across consumers. However, one of the main reasons for its popularity is that the price aggregator P in equation (2) can be replaced by a price index so that one obtains a linear demand system at the estimation stage. To implement the linear AI demand system, Deaton and Muellbauer suggest replacing P in (1) by the Stone price index P^* defined as

$$(3) \qquad \log(P_t^*) \equiv \sum_{i=1}^{n} w_{it} \log(p_{it}).$$

The linear AI demand system with the Stone index has been used extensively in demand analysis (agricultural economics applications include Blanciforti and Green; Chalfant; Eales and Unnevehr; Gould, Cox, and Perali; Moschini and Meilke).

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Units of Measurement and the Stone Index

A somewhat intriguing point, which seems to have escaped the attention of most practitioners, is that index number theory makes no mention of the Stone index (see, e.g., Diewert 1987). The reason is that the Stone index does not satisfy a fundamental property of index numbers, what Diewert calls the "commensurability" property. In other words, the Stone index is not invariant to changes in the units of measurement of prices. This problem is apparent in equation (3). For example, by changing the units of the first good from pounds to kilograms, one would scale the corresponding price by the conversion rate between the two units (approximately 2.19). Because such a change does not affect expenditure shares, the Stone index would apply unchanged weights to (arbitrarily) scaled prices.

Why this feature of the Stone index turns out to be very important for the linear AI model is readily illustrated. Suppose that demand preferences are described by (1) and (2), with the corresponding linear AI model written as

$$(4) \quad w_{it} = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log(p_{jt}) + \beta_i \log(x_t) - \beta_i \sum_{j=1}^{n} w_{jt} \log(p_{jt}).$$

Now define a new set of prices $\{\bar{p}_i\}$ according to some other units of measurement, such that $p_i \equiv \theta_i \bar{p}_i$ (for some constants $\theta_i > 0$). With these scaled prices the demand system becomes

$$(5) \quad w_{it} = \tilde{\alpha}_i + \sum_{j=1}^{n} \gamma_{ij} \log(\bar{p}_{jt}) + \beta_i \log(x_t) - \beta_i \sum_{j=1}^{n} w_{jt} \log(\bar{p}_{jt})$$

where

$$(6) \quad \tilde{\alpha}_i = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log(\theta_j) - \beta_i \sum_{j=1}^{n} w_{jt} \log(\theta_j).$$

Note that $\tilde{\alpha}_i$ is not a constant, so that the model with scaled prices (5) is not equivalent to the original one in (4). Because such intercepts are treated as constants in estimating any given linear AI demand system, it is clear that estimation of the behavioral parameters $\{\gamma_{ij}, \beta_i\}$ in equation (5) will generally be biased.

It is useful to underscore that the units-of-measurement problem does not plague the nonlinear AI model. Suppose again that demand preferences are described by equation (1) and (2), and define the new set of prices $\{\bar{p}_i\}$ as described above. However, in this case the demand system, in terms of the scaled prices, is equivalent to (1) and (2); that is

$$(7) \quad w_{it} = \bar{\alpha}_i + \sum_{j=1}^{n} \gamma_{ij} \log(\bar{p}_{jt}) + \beta_i \log\left(\frac{x_t}{\bar{P}_t}\right)$$

where

$$(8) \quad \log(\bar{P}_t) = \bar{\alpha}_0 + \sum_{i=1}^{n} \bar{\alpha}_i \log(\bar{p}_{it}) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \log(\bar{p}_{it}) \log(\bar{p}_{jt}).$$

Note that the constants $\{\bar{\alpha}_0, \bar{\alpha}_i\}$ are related to the original parameters of (1) and (2) as

$$(9) \quad \bar{\alpha}_0 = \alpha_0 + \sum_{i=1}^{n} \alpha_i \log(\theta_i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \log(\theta_i) \log(\theta_j)$$

$$(10) \quad \bar{\alpha}_i = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log(\theta_j).$$

Thus, as in the case of the translog model (Christensen and Manser, pp. 50–52), changing the units of measurement for the nonlinear AI system only affects the intercepts, which does not have any significant consequences for the properties of the demand model.[1]

[1] Rescaling of such constants is automatically accounted for in estimation. However, if one follows the suggestion of Deaton and Muellbauer of fixing the intercept of the translog price index prior to estimating the parameters of the nonlinear AI model, then a value to this parameter can be assigned meaningfully only for a given set of units of measurement, and such a value would need to be changed as these units change.

Deaton and Muellbauer named the index in equation (3) after Sir Richard Stone. There is an important and crucial difference, however, between Stone's use of the index in (3) and the practice of linear AI demand system estimation initiated by Deaton and Muellbauer. Stone (p. 277) introduced the index in (3) to justify, from a theoretical point of view, the interpretation of the price coefficients of his model as compensated elasticities. However, in empirical estimation he used indices that are invariant to the choice of units of measurement.[2] For the AI model the strategy is reversed. The Stone index is used as an empirical approximation to a theoretical translog price index. As shown here, this seemingly innocuous switch can have important consequences.

Possible Solutions

The foregoing analysis indicates that changes are needed in the way the AI model is commonly used in empirical applications. One solution is to discard the linear AI model in favor of the nonlinear AI model, in light of the fact that estimation of nonlinear systems can readily be performed by most econometric packages. However, there are circumstances in which the linear AI model is still appealing. For example, if there are other sources of nonlinearities in the model, starting with linear demand equations allows for simplification of the final model. Such a situation arises, for example, in the dynamic demand models of Anderson and Blundell, or in the separable models of Moschini, Moro, and Green.

Use of the linear AI demand system for empirical applications requires modification of the typical procedure. Following the discussion in the preceding section, note that changes in units of measurement will be inconsequential for the linear AI demand system if P is approximated by an index P^I which is invariant to changes in units of measurement up to a multiplicative constant. In other words, in the notation used earlier, changes in units are allowed to affect the index only as $P^I(p) = \kappa P^I(\bar{p})$, where κ is an arbitrary constant.[3] Hence, use of any one of the regular price indices will be sufficient because such indices are invariant to changes in units of measurement (for these indices $\kappa = 1$). An obvious choice is the Tornqvist index, which is a superlative index for the translog function in (2) (Diewert 1976). The Tornqvist index P^T, viewed as a discrete approximation to the Divisia index, is

$$(11) \quad \log\left(P_t^T\right) = \frac{1}{2}\sum_{i=1}^{n}\left(w_{it} + w_i^0\right)\log\left(\frac{p_{it}}{p_i^0}\right)$$

where the zero superscript denotes base period values.[4]

The form of (11) suggests two alternatives that retain some features of the Stone index: the loglinear analogue of the Paasche price index and the loglinear analogue of the Laspeyres price index.[5] The loglinear analogue of the Paasche price index, labeled P^S and referred to as the "corrected" Stone index, is written as

$$(12) \quad \log\left(P_t^S\right) = \sum_{i=1}^{n} w_{it} \log\left(\frac{p_{it}}{p_i^0}\right).$$

Note that, in certain situations, use of the corrected index in (12) is equivalent to employing the Stone index P^*. Specifically, this applies when the prices are themselves indices (e.g., Deaton and Muellbauer), or when prices are scaled by their mean (e.g., Moschini and Meilke).[6] The loglinear analogue of the Laspeyres price index is obtained by replacing w_{it} in (12) with the base shares w_i^0. In this case, however, the fact that the weights w_i^0 are constant allows further simplification. As noted earlier, only invariance up to a multiplicative constant is necessary for the approximating index. Hence, one could simply use the following geometrically weighted average of prices:[7]

$$(13) \quad \log\left(P_t^C\right) = \sum_{i=1}^{n} w_i^0 \log(p_{it}).$$

[2] Stone used a fixed-weights cost-of-living index as the index for all prices. Also, when aggregating goods and prices, Stone relied on the chained Fisher's "ideal" index, which is invariant to units of measurement (Stone, pp. 419–20).

[3] Hence, $\log[P^I(p)] = \log(\kappa) + \log[P^I(\bar{p})]$, and changes in units of measurement will simply affect the intercept of the linear AI system.

[4] One can use a specific observation period for the base (say, the first observation) or, perhaps better, mean values can be used for the base. Alternatively, values at time $(t-1)$ could provide the base for time t, with the resulting bilateral indices linked using the "chain" principle (Diewert 1987).

[5] These two indices were considered in Fisher's pioneering work on index numbers.

[6] Unfortunately, many applied demand studies deal with disaggregated commodities whose prices are measured in natural units and are not scaled to a common base.

[7] This form appears to have been used by Heien and Pompelli, although no justification for that choice was given.

Table 1. Estimated Expenditure Elasticities with Alternative AI Models

Elasticity	True Value	Nonlinear AI System	Linear AI System With P^*	With P^T	With P^S	With P^C
ε_1	1.20	1.20	1.32	1.20	1.20	1.20
		(0.09)	(0.15)	(0.09)	(0.09)	(0.09)
ε_2	0.90	0.90	0.94	0.90	0.90	0.90
		(0.13)	(0.16)	(0.13)	(0.13)	(0.13)
ε_3	0.60	0.61	1.01	0.61	0.61	0.62
		(0.49)	(0.57)	(0.49)	(0.49)	(0.49)
ε_4	0.99	0.99	0.96	0.99	0.99	0.99
		(0.02)	(0.04)	(0.02)	(0.02)	(0.02)

Note: Entries for AI systems are averages of 1,000 estimates and, below them in parentheses, the corresponding RMSEs.

Note that this index differs from the loglinear analogue of the Laspeyres price index by an additive constant that is inconsequential in estimating a linear AI model.

Finally, it is worth noting that although P^T and P^S (and the logarithmic Laspeyres index underlying P^C) are invariant to the choice of units of measurement (as all proper index numbers should), they are not unique in that they depend on the particular base chosen (as all index numbers do).

An Example

A simulation was performed to illustrate the type of problems that may arise with the use of the Stone index. Specifically, 1,000 samples were generated from a four-good nonlinear AI model, as in equations (1) and (2). For each sample, five models were estimated: the true nonlinear AI model, the standard linear AI model that uses P^*, the linear AI using the Tornqvist index P^T, the linear AI using the "corrected" Stone index P^S, and the linear AI using the constant-weights index P^C.[8] To give a concrete flavor to the exercise, the variables used to generate the 1,000 replications are actual U.S. annual data for the period 1958 to 1985 for beef, pork, chicken, and other food. Beef, pork, and chicken quantity data are expressed in pounds per capita (retail-weight-equivalent for beef and pork, and ready-to-cook weight for chicken) and prices are in $/lb. Expenditures on these three meats are obtained as the product of the aforementioned prices and quantities. Subtracting the sum of these expenditures from per-capita expenditures on food gives expenditures on the other-foods group. For this composite group, clearly, prices are not in natural units; rather, we use the price index for food (which equals 100 in 1967).[9]

The parameters of the model used to generate the observations were calibrated so that the implied elasticities at the shares' mean equal certain preselected values.[10] Letting ε_{ij} denote Marshallian price elasticities, and ε_i denote expenditure elasticities, then the preselected elasticities are $\varepsilon_{11} = -1.50$, $\varepsilon_{12} = 0.10$, $\varepsilon_{13} = 0.10$, $\varepsilon_{22} = -1.20$, $\varepsilon_{23} = 0.10$, $\varepsilon_{33} = -0.80$, $\varepsilon_1 = 1.20$, $\varepsilon_2 = 0.90$, and $\varepsilon_3 = 0.60$.[11] Given these calibrated parameters, 1,000 samples of shares were generated by adding, to the deterministic model of equations (1) and (2), random draws from a multivariate normal process with mean zero and the following variance-covariance matrix Ω:

$$(14)\quad \Omega \equiv 10^{-5}\begin{bmatrix} 0.50 & -0.15 & -0.10 \\ -0.15 & 0.40 & -0.05 \\ -0.10 & -0.05 & 0.30 \end{bmatrix}.$$

For each draw and each estimated model, relevant elasticities were calculated (at the sample's mean point). For our purposes, we can concentrate on expenditure (income) elasticities, which best represent the essence of the AI model (nonlinear Engel curves, while satisfying some aggregation properties). For all models considered, income elasticities are given by

[8] Mean values were used for the base of these indices, so that p_i^0 represents the mean of p_{it} and w_i^0 is the mean of w_i.

[9] The data set used is reported in the appendix.

[10] The means of the shares are $w_1^0 = 0.119$, $w_2^0 = 0.070$, $w_3^0 = 0.026$, and $w_4^0 = 0.785$.

[11] The remaining true elasticities are implied by symmetry, homogeneity, and adding-up. With all that, the parameter α_0 is still unconstrained, and was set equal to zero.

$$(15) \quad \varepsilon_i = 1 + \frac{\beta_i}{w_i}$$

where the mean value of w_i is used. Table 1 summarizes the results by reporting, in addition to the true elasticity values, the average (over 1,000 draws) of the estimated expenditure elasticities, as well as the square roots of the corresponding mean square errors (RMSE). It is clear that the nonlinear AI model is quite successful in estimating the true elasticity parameters. On the other hand, the linear AI with the standard Stone index P^* performs rather poorly, in particular producing extremely biased estimates of ε_1 and ε_3. (Also, all estimates for this model have higher RMSE than those of the nonlinear AI model.) As expected, the Tornqvist index P^T, the "corrected" Stone index P^S, and the fixed-weights index P^C vastly improve the linear AI model, producing virtually the same results as the true nonlinear AI model.

The bias in elasticity estimates is not confined to income elasticities. Rather than reporting the estimated price elasticities for each model, it suffices to note that Marshallian price elasticities are related to income elasticities by the homogeneity condition

$$(16) \quad \sum_{j=1}^{n} \varepsilon_{ij} = -\varepsilon_i .$$

It follows that the bias in income elasticities attributable to the use of the standard Stone index P^* must spill over to price elasticities as well.

Conclusion

This article suggests caution in using the Stone index to estimate a linear approximation to the AI demand model. The significance of the problem was illustrated through a simulation exercise. The consequences of relying on the usual Stone index can be serious, depending on the nature of the data. Similarly, it can be shown that disregarding the pitfalls associated with the standard Stone index has serious implications for hypothesis testing as well. Hence, for the purpose of estimating the linear version of the AI model, the standard Stone index P^* should be avoided. A better approach would be to approximate the index in (2) with a proper price index, such as the Tornqvist index or the modified Stone price index discussed here. The simulation results of this article also indicate that the linear AI model can approximate the nonlinear AI model well, provided a proper price index is used.[12] Thus, some of the concerns that have been raised about the approximation properties of the linear AI model may be due to neglect of the simple but important units-of-measurement problem of the Stone index.

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[12] This is consistent with, and reinforces, the evidence reported by Moschini and Vissa, who found that linear and nonlinear AI models gave virtually identical results when fitted with data generated by a linear expenditure system.

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Appendix

Data Used in Simulation

q_1	q_2	q_3	q_4	p_1	p_2	p_3	p_4
61.6	56.0	28.1	3.82	0.80	0.64	0.47	89
61.9	62.7	28.7	3.98	0.82	0.56	0.42	87
64.2	60.3	27.8	3.99	0.80	0.56	0.43	88
65.8	57.7	29.9	3.99	0.78	0.58	0.38	89
66.2	59.1	29.8	3.97	0.82	0.59	0.41	90
69.9	61.0	30.8	3.97	0.79	0.57	0.40	91
73.9	61.0	31.2	4.12	0.76	0.56	0.38	92
73.6	54.7	33.3	4.25	0.80	0.66	0.39	94
77.0	54.4	35.6	4.27	0.82	0.74	0.41	99
78.8	60.0	36.5	4.32	0.83	0.67	0.38	100
81.2	61.4	36.7	4.47	0.87	0.67	0.40	104
82.0	60.5	38.4	4.42	0.96	0.74	0.42	109
84.0	62.3	40.4	4.50	0.99	0.78	0.41	115
83.4	68.3	40.3	4.44	1.04	0.70	0.41	118
85.4	62.9	41.8	4.45	1.14	0.83	0.41	124
80.5	57.3	40.5	4.26	1.36	1.10	0.60	141
85.6	61.8	40.7	4.20	1.39	1.09	0.56	162
87.9	50.7	40.1	4.26	1.46	1.35	0.63	176
94.4	53.7	42.8	4.46	1.39	1.34	0.60	181
91.8	55.8	44.2	4.74	1.38	1.25	0.60	192
87.2	55.9	46.7	4.59	1.82	1.44	0.67	211
78.0	63.8	50.6	4.61	2.26	1.44	0.68	235
76.5	68.3	50.1	4.72	2.38	1.39	0.72	254
77.1	65.9	51.6	4.75	2.39	1.52	0.74	274
77.2	59.0	53.1	4.86	2.42	1.75	0.72	286
78.7	62.2	53.8	5.03	2.38	1.70	0.73	292
78.5	61.8	55.6	5.17	2.40	1.62	0.81	303
79.1	62.0	58.0	5.27	2.33	1.62	0.76	310