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Univalence Criteria for a General Integral Operator

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Abstract. In this paper the author introduces a general integral operator and determines conditions for the univalence of this integral operator. Also, the significant relationships and relevance with other results are also given.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open unit disk

$\mathcal{U} = \{z : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in $\mathcal{U}$.

Miller and Mocanu [11] have considered the integral operator M_α given by

$$M_\alpha(z) = \left\{ \frac{1}{\alpha} \int_0^z (f(u))^{\frac{1}{\alpha}} u^{-1} du \right\}^\alpha, \quad z \in \mathcal{U} \tag{1}$$

for functions $f(z)$ belonging to the class $\mathcal{A}$ and for some complex numbers α, $\alpha \neq 0$. It is well known that $M_\alpha(z) \in \mathcal{S}$ for $f(z) \in \mathcal{S}^*$ and $\alpha > 0$, where $\mathcal{S}^*$ denotes the subclass of $\mathcal{S}$ consisting of all starlike functions $f(z)$ in $\mathcal{U}$.

In this present investigation, we introduce a general integral operator as follows:

$$\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,\beta}(z) = \left\{ \left(\sum_{i=1}^n \frac{1}{\gamma_i} - n + 1 \right) \int_0^z u^{-n\beta} \prod_{i=1}^n (f_i(u))^{\frac{1}{\gamma_i}+\beta-1} du \right\}^{\frac{1}{\sum_{i=1}^n \frac{1}{\gamma_i} - n + 1}} \tag{2}$$

where $z \in \mathcal{U}$, $f_i \in \mathcal{A}$ and $\gamma_i, \beta \in \mathbb{C}$, $(\gamma_i \neq 0)$, $i = \overline{1,n}$ which is a generalization of integral operator M_α.

From (2), for $n = 1$, $\beta = 1$, $f_1 = f$, $\gamma_1 = \alpha$, we obtain the integral operator M_α.

If $\gamma_i = \gamma$ for each $i = \overline{1,n}$, from (2) we obtain the integral operator

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$$\mathcal{J}_{n,\gamma,\beta}(z) = \left\{ \left(\frac{n(1-\gamma)+\gamma}{\gamma} \right) \int_0^z u^{-n\beta} \prod_{i=1}^{n} (f_i(u))^{\frac{1}{\gamma}+\beta-1} du \right\}^{\frac{\gamma}{n(1-\gamma)+\gamma}}. \tag{3}$$

If $\beta = 0$, from (2) we obtain the integral operator define by

$$\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,0}(z) = \left\{ \left(\sum_{i=1}^{n} \frac{1}{\gamma_i} - n + 1 \right) \int_0^z \prod_{i=1}^{n} (f_i(u))^{\frac{1}{\gamma_i}-1} du \right\}^{\frac{1}{\sum_{i=1}^{n} \frac{1}{\gamma_i} - n+1}}. \tag{4}$$

If $\beta = 1$, from (2) we obtain the integral operator

$$\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,1}(z) = \left\{ \left(\sum_{i=1}^{n} \frac{1}{\gamma_i} - n + 1 \right) \int_0^z u^{-n} \prod_{i=1}^{n} (f_i(u))^{\frac{1}{\gamma_i}} du \right\}^{\frac{1}{\sum_{i=1}^{n} \frac{1}{\gamma_i} - n+1}}. \tag{5}$$

If $n = 1$ and $f_1 = f$, from (2) we obtain the integral operator

$$\mathcal{J}_{\gamma,\beta}(z) = \left\{ \frac{1}{\gamma} \int_0^z u^{-\beta} (f(u))^{\frac{1}{\gamma}+\beta-1} du \right\}^{\gamma}, \quad z \in \mathcal{U} \tag{6}$$

which was introduced and studied by Pescar and Breaz [17].

If $n = 1, \quad f_1 = f, \quad \frac{1}{\gamma_1} = 1$ and $\beta \in \mathbb{C} - \{0, 1\}$, from (2) we obtain the integral operator

$$\mathcal{J}_{\beta}(z) = \int_0^z \left(\frac{f(u)}{u} \right)^{\beta} du, \quad z \in \mathcal{U} \tag{7}$$

which was introduced Kim-Merkes [9].

From (2), for $n = 1, \quad f_1 = f, \quad \frac{1}{\gamma_1} = 1$ and $\beta = 1$ we obtain Alexander integral operator defined by

$$\mathcal{J}(z) = \int_0^z \frac{f(u)}{u} du. \tag{8}$$

Recently the problem of univalence of some generalized integral operators have discussed by many authors such as: (see [1]-[8], [10], [15]-[17], [19] and [20]). In our paper, we consider the general integral operator of the type (2) and obtain some sufficient conditions for this integral operator to be univalent in $\mathcal{U}$. In particular, we obtain simple sufficient conditions for some integral operators which involve special cases n, β and γ_i $(\gamma_i \neq 0)$, $i = \overline{1, n}$ of the integral operator (2).

2. Preliminary Results

To discuss our problems for univalence of the integral operator $\mathcal{J}_{\gamma_1,\gamma_2,...\gamma_n,\beta}$, we need the following lemmas.

Lemma 2.1. *(Pascu [14]). Let $h(z) \in \mathcal{A}$ and $\gamma \in \mathbb{C}$. If $\Re(\gamma) > 0$ and*

$$\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad z \in \mathcal{U}$$

then the function $H_\gamma(z)$ given by

$$\mathcal{H}_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1}h'(t)dt\right)^{1/\gamma}, \quad z \in \mathcal{U} \tag{9}$$

is in the class $\mathcal{S}$.

Lemma 2.2. *(Ozaki and Nunokawa [13]). Let $f \in \mathcal{A}$ satisfy the following condition*

$$\left|\frac{z^2 f'(z)}{f^2(z)}\right| \leq 1, \quad z \in \mathcal{U}$$

then the function f is in the class $\mathcal{S}$.

Let $\mathcal{S}(p)$ denote the class of functions $f \in \mathcal{A}$ which satisfies the conditions

$$\begin{cases} f(z) \neq 0, & \text{for } 0 < |z| < 1 \\ \left|\left(\dfrac{z}{f(z)}\right)''\right| \leq p, & \text{for } z \in \mathcal{U} \end{cases}$$

for some real p with $0 < p \leq 2$. The class $\mathcal{S}(p)$ is defined by Yang and Liu [21]. Sign [18] has shown that if $f \in \mathcal{S}(p)$, then f satisfies

$$\left|\frac{z^2 f'(z)}{f^2(z)}\right| \leq p\,|z|^2, \quad z \in \mathcal{U}. \tag{10}$$

Lemma 2.3. *(General Schwarz Lemma [16]). Let $f(z)$ be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has at $z = 0$ one zero with multiplicity greater than m, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \quad z \in \mathcal{U}_R \tag{11}$$

the equality (in the equality (11) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

Lemma 2.4. *(Caratheodory [1], [12]). Let f be analytic function in $\mathcal{U}$, with $f(0) = 0$. If f satisfies*

$$\Re f(z) \leq M \tag{12}$$

for some $M > 0$, then

$$(1-|z|)\,|f(z)| \leq 2M\,|z|, \quad z \in \mathcal{U}. \tag{13}$$

3. Main Results

Theorem 3.1. *Let γ_i be complex numbers $\Re(\gamma_i) \neq 0$, $\delta = \sum_{i=1}^{n} \Re\left(\frac{1}{\gamma_i}\right) - n + 1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots$, $i = \overline{1,n}$. If*

$$\Re\left\{e^{i\theta}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right\} \leq \frac{\left|\gamma_i\right|\left(\Re(\frac{1}{\gamma_i}) - 1 + \frac{1}{n}\right)}{4\left(1 + \left|\gamma_i\right|\left|\beta - 1\right|\right)}, \quad \left(i = \overline{1,n};\ 0 < \delta < 1\right) \tag{14}$$

or

$$\Re\left\{e^{i\theta}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right\} \leq \frac{\left|\gamma_i\right|}{4n\left(1 + \left|\gamma_i\right|\left|\beta - 1\right|\right)}, \quad \left(i = \overline{1,n};\ \delta \geq 1\right), \tag{15}$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$ and $\beta \in \mathbb{C}$, then the integral operator $\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,\beta}$ defined by (2) is in the class $\mathcal{S}$.

Proof. The integral operator $\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,\beta}$ has the form

$$\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,\beta}(z) = \left\{\left(\sum_{i=1}^{n}\frac{1}{\gamma_i} - n + 1\right)\int_0^z u^{\sum_{i=1}^{n}\frac{1}{\gamma_i} - n}\prod_{i=1}^{n}\left(\frac{f_i(u)}{u}\right)^{\frac{1}{\gamma_i}+\beta-1} du\right\}^{\frac{1}{\sum_{i=1}^{n}\frac{1}{\gamma_i} - n + 1}}. \tag{16}$$

We consider the function

$$g(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(u)}{u}\right)^{\frac{1}{\gamma_i}+\beta-1} du \tag{17}$$

analytic in $\mathcal{U}$.

We have

$$\frac{zg''(z)}{g'(z)} = \sum_{i=1}^{n}\left(\frac{1}{\gamma_i} + \beta - 1\right)\left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{18}$$

Let us consider the function

$$\varphi_i(z) = e^{i\theta}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right), \quad z \in \mathcal{U},\ i = \overline{1,n},\ \theta \in [0, 2\pi] \tag{19}$$

and we observe that $\varphi_i(0) = 0$ for all $i = \overline{1,n}$.

By (14) and Lemma 2.4 for $\delta \in (0, 1)$ we obtain

$$\left|\varphi_i(z)\right| \leq \frac{|z|\left|\gamma_i\right|\left(\Re\left(\frac{1}{\gamma_i}\right) - 1 + \frac{1}{n}\right)}{2\left(1 - |z|\right)\left(1 + \left|\gamma_i\right|\left|\beta - 1\right|\right)}, \quad z \in \mathcal{U},\ \beta \in \mathbb{C},\ i = \overline{1,n}. \tag{20}$$

From (15) and Lemma 2.4, for $\delta \in [1, \infty)$ we have

$$\left|\varphi_i(z)\right| \leq \frac{|z|\left|\gamma_i\right|}{2\left(1 - |z|\right)\left(1 + \left|\gamma_i\right|\left|\beta - 1\right|\right)}, \quad z \in \mathcal{U},\ \beta \in \mathbb{C},\ i = \overline{1,n}. \tag{21}$$

From (18) and (20) we get

$$\frac{1 - |z|^{2\delta}}{\delta}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1 - |z|^{2\delta}}{2\left(1 - |z|\right)}|z|, \quad \delta \in (0, 1). \tag{22}$$

Because $1-|z|^{2\delta}\leq 1-|z|^2$ for $\delta\in(0,1)$, $z\in\mathcal{U}$ from (22) we have

$$\frac{1-|z|^{2\delta}}{\delta}\left|\frac{zg''(z)}{g'(z)}\right|\leq 1 \tag{23}$$

for all $z\in\mathcal{U}$, $\delta\in(0,1)$.

For $\delta\in[1,\infty)$, we have $\frac{1-|z|^{2\delta}}{\delta}\leq 1-|z|^2$, $z\in\mathcal{U}$ and from (18) and (21), we obtain

$$\frac{1-|z|^{2\delta}}{\delta}\left|\frac{zg''(z)}{g'(z)}\right|\leq 1 \tag{24}$$

for all $z\in\mathcal{U}$, $\delta\in[1,\infty)$.

From (17) we have $g'(z)=\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\frac{1}{\gamma_i}+\beta-1}$ and using (23) and (24), by Lemma 2.1 it results that $\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,\beta}$ given by (2) is in the class $\mathcal{S}$. □

Letting $\gamma_i=\gamma$ for $i=\overline{1,n}$ in Theorem 3.1, we have

Corollary 3.2. *Let γ be complex number $\Re(\gamma)\neq 0$, $\delta=n\Re(\frac{1-\gamma}{\gamma})+1>0$, $f_i\in\mathcal{A}$, $f_i(z)=z+a_{2i}z^2+a_{3i}z^3+\ldots$, $i=\overline{1,n}$. If*

$$\Re\left\{e^{i\theta}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right\}\leq\frac{|\gamma|\left(\Re(\frac{1}{\gamma})-1+\frac{1}{n}\right)}{4\left(1+|\gamma|\,|\beta-1|\right)},\quad\left(i=\overline{1,n};\ 0<\delta<1\right)$$

or

$$\Re\left\{e^{i\theta}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right\}\leq\frac{|\gamma|}{4n\left(1+|\gamma|\,|\beta-1|\right)},\quad\left(i=\overline{1,n};\ \delta\geq 1\right),$$

for all $z\in\mathcal{U}$, $\theta\in[0,2\pi]$ and $\beta\in\mathbb{C}$, then the integral operator $\mathcal{J}_{n,\gamma,\beta}$ defined by (3) is in the class $\mathcal{S}$.

Letting $\beta=0$ in Theorem 3.1, we have

Corollary 3.3. *Let γ_i be complex numbers $\Re(\gamma_i)\neq 0$, $\delta=\sum_{i=1}^{n}\Re\left(\frac{1}{\gamma_i}\right)-n+1>0$, $f_i\in\mathcal{A}$, $f_i(z)=z+a_{2i}z^2+a_{3i}z^3+\ldots$, $i=\overline{1,n}$. If*

$$\Re\left\{e^{i\theta}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right\}\leq\frac{|\gamma_i|\left(\Re\left(\frac{1}{\gamma_i}\right)-1+\frac{1}{n}\right)}{4\left(1+|\gamma_i|\right)},\quad\left(i=\overline{1,n};\ 0<\delta<1\right)$$

or

$$\Re\left\{e^{i\theta}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right\}\leq\frac{|\gamma_i|}{4n\left(1+|\gamma_i|\right)},\quad\left(i=\overline{1,n};\ \delta\geq 1\right),$$

for all $z\in\mathcal{U}$, $\theta\in[0,2\pi]$, then the integral operator $\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,0}$ defined by (4) is in the class $\mathcal{S}$.

Letting $\beta=1$ in Theorem 3.1, we have

Corollary 3.4. *Let γ_i be complex numbers $\Re(\gamma_i) \neq 0, \delta = \sum_{i=1}^{n} \Re\left(\frac{1}{\gamma_i}\right) - n + 1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots$, $i = \overline{1,n}$. If*

$$\Re\left\{e^{i\theta}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right\} \leq \frac{\left|\gamma_i\right|\left(\Re\left(\frac{1}{\gamma_i}\right) - 1 + \frac{1}{n}\right)}{4}, \quad \left(i = \overline{1,n};\ 0 < \delta < 1\right)$$

or

$$\Re\left\{e^{i\theta}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right\} \leq \frac{\left|\gamma_i\right|}{4n}, \quad \left(i = \overline{1,n};\ \delta \geq 1\right),$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$, then the integral operator $\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,1}$ defined by (5) is in the class $\mathcal{S}$.

Theorem 3.5. *Let γ_i be complex numbers $\Re(\gamma_i) \neq 0, \delta = \sum_{i=1}^{n} \Re\left(\frac{1}{\gamma_i}\right) - n + 1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots$, $i = \overline{1,n}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq \frac{(2\delta+1)^{\frac{2\delta+1}{2\delta}}}{2n} \frac{\left|\gamma_i\right|}{\left(1 + \left|\gamma_i\right|\left|\beta - 1\right|\right)}, \quad \left(i = \overline{1,n}\right) \tag{25}$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$ and $\beta \in \mathbb{C}$, then the integral operator $\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,\beta}$ defined by (2) is in the class $\mathcal{S}$.

Proof. We observe that $\mathcal{J}_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta}$ has the form (16). Consider the function $g(z)$ defined by (17). We define the function $\psi(z) = \frac{zg''(z)}{g'(z)}$ for all $z \in \mathcal{U}$. From (18) and (25) we have

$$\left|\psi(z)\right| \leq \frac{(2\delta+1)^{\frac{2\delta+1}{2\delta}}}{2}$$

for all $z \in \mathcal{U}$.

The function ψ satisfies the condition $\psi(0) = 0$ and applying Lemma 2.3 we obtain

$$\left|\psi(z)\right| \leq \frac{(2\delta+1)^{\frac{2\delta+1}{2\delta}}}{2} |z|, \quad z \in \mathcal{U}. \tag{26}$$

From (26) we get

$$\frac{1-|z|^{2\delta}}{\delta}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{(2\delta+1)^{\frac{2\delta+1}{2\delta}}}{2} \frac{1-|z|^{2\delta}}{\delta} |z|. \tag{27}$$

Because

$$\max_{|z|\leq 1}\left\{\frac{1-|z|^{2\delta}}{\delta}|z|\right\} = \frac{2}{(2\delta+1)^{\frac{2\delta+1}{2\delta}}}$$

from (27) we obtain

$$\frac{1-|z|^{2\delta}}{\delta}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1. \tag{28}$$

From (28) and because $g'(z) = \prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\frac{1}{\gamma_i}+\beta-1}$, by Lemma 2.1 we obtain that the integral operator $\mathcal{J}_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta}$ is in the class $\mathcal{S}$. □

Letting $\gamma_i = \gamma$ for $i = \overline{1, n}$ in Theorem 3.5, we have

Corollary 3.6. *Let γ be complex number $\Re(\gamma) \neq 0$, $\delta = n\Re(\frac{1-\gamma}{\gamma})+1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots$, $i = \overline{1, n}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq \frac{(2\delta+1)^{\frac{2\delta+1}{2\delta}}}{2n} \frac{|\gamma|}{\left(1 + |\gamma|\,|\beta - 1|\right)}, \quad \left(i = \overline{1, n}\right)$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$ and $\beta \in \mathbb{C}$, then the integral operator $\mathcal{J}_{n,\gamma,\beta}$ defined by (3) is in the class $\mathcal{S}$.

Letting $\beta = 0$ in Theorem 3.5, we have

Corollary 3.7. *Let γ_i be complex numbers $\Re(\gamma_i) \neq 0$, $\delta = \sum_{i=1}^{n} \Re\left(\frac{1}{\gamma_i}\right) - n + 1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots$, $i = \overline{1, n}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq \frac{(2\delta+1)^{\frac{2\delta+1}{2\delta}}}{2n} \frac{|\gamma|}{\left(1 + |\gamma|\right)}, \quad \left(i = \overline{1, n}\right)$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$ and $\beta \in \mathbb{C}$, then the integral operator $\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n}$ defined by (4) is in the class $\mathcal{S}$.

Letting $\beta = 1$ in Theorem 3.5, we have

Corollary 3.8. *Let γ_i be complex numbers $\Re(\gamma_i) \neq 0$, $\delta = \sum_{i=1}^{n} \Re\left(\frac{1}{\gamma_i}\right) - n + 1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots$, $i = \overline{1, n}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq \frac{|\gamma_i|\,(2\delta+1)^{\frac{2\delta+1}{2\delta}}}{2n}, \quad \left(i = \overline{1, n}\right)$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$, then the integral operator $\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,1}$ defined by (5) is in the class $\mathcal{S}$.

Remark 3.9. *For $n = 1$ in Theorem 3.1 and Theorem 3.5, we obtain Theorem 3.2 and Theorem 3.5 in [17] respectively.*

Theorem 3.10. *Let the functions $f_i \in \mathcal{S}(p_i)$, $\left(i = \overline{1, n}\right)$ which satisfy the inequality (10) with $0 < p_i \leq 2$ and $M_i > 0$ for $i = \overline{1, n}$. Furthermore γ_i be complex numbers with $\lambda = \sum_{i=1}^{n} \frac{1}{\gamma_i} - n + 1$ and $\Re(\lambda) > 0$. If*

$$|f_i(z)| \leq M_i \quad \left(z \in \mathcal{U},\ i = \overline{1, n}\right) \tag{29}$$

and

$$\sum_{i=1}^{n} \left|\frac{1}{\gamma_i} + \beta - 1\right| (M_i(p_i + 1)+1) \leq \Re(\lambda) \tag{30}$$

then the integral operator $\mathcal{J}_{\gamma_1,\gamma_2,\ldots\gamma_n,\beta}$ defined by (2) is in the class $\mathcal{S}$.

Proof. We observe that $\mathcal{J}_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta}$ has the form (16). Consider the function $g(z)$ defined in the formula (17). Equation (18), yields

$$\begin{aligned} \left|\frac{zg''(z)}{g'(z)}\right| &\leq \sum_{i=1}^{n} \left|\frac{1}{\gamma_i} + \beta - 1\right| \left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) \\ &\leq \sum_{i=1}^{n} \left|\frac{1}{\gamma_i} + \beta - 1\right| \left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| \left|\frac{f_i(z)}{z}\right| + 1\right). \end{aligned}$$

The hypothesis will then yield $\left|f_i(z)\right| \le M_i \left(z \in \mathcal{U},\ i = \overline{1,n}\right)$. By Genaral Schwarz Lemma, we can obtain that

$$\left|f_i(z)\right| \le M_i |z| \tag{31}$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$. Therefore by using the inequality (31), we obtain

$$\begin{aligned} \left|\frac{zg''(z)}{g'(z)}\right| &\le \sum_{i=1}^{n} \left|\frac{1}{\gamma_i} + \beta - 1\right| \left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| M_i + 1 \right) \\ &\le \sum_{i=1}^{n} \left|\frac{1}{\gamma_i} + \beta - 1\right| \left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2} - 1\right| M_i + M_i + 1 \right). \end{aligned} \tag{32}$$

Since $f \in \mathcal{S}(p_i)$, in view of (30), using (10), (32) may be written as

$$\begin{aligned} \left|\frac{zg''(z)}{g'(z)}\right| &\le \sum_{i=1}^{n} \left|\frac{1}{\gamma_i} + \beta - 1\right| \left(M_i p_i |z|^2 + M_i + 1\right) \\ &\le \sum_{i=1}^{n} \left|\frac{1}{\gamma_i} + \beta - 1\right| (M_i(p_i + 1) + 1) \le \Re(\lambda). \end{aligned} \tag{33}$$

On multiplying (33) by $\dfrac{\left(1 - |z|^{2\Re(\lambda)}\right)}{\Re(\lambda)}$, the following inequality is obtained

$$\frac{1 - |z|^{2\Re(\lambda)}}{\Re(\lambda)} \left|\frac{zg''(z)}{g'(z)}\right| \le 1 - |z|^{2\Re(\lambda)} \le 1 \quad (z \in \mathcal{U}).$$

So, according to the Lemma 2.1, the integral operator $\mathcal{J}_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta}$ belongs to the class $\mathcal{S}$. □

Letting $\gamma_i = \gamma$ for $i = \overline{1,n}$ in Theorem 3.10, we have

Corollary 3.11. *Let the functions* $f_i \in \mathcal{S}(p_i)$, $\left(i = \overline{1,n}\right)$ *which satisfy the inequality (10) with* $0 < p_i \le 2$ *and* $M_i > 0$ *for* $i = \overline{1,n}$. *Furthermore* γ *be complex number with* $\arg\gamma = \theta$ *and* $\Re\left(\dfrac{n(1-\gamma)+\gamma}{\gamma}\right) > 0$. *If*

$$\left|f_i(z)\right| \le M_i \quad \left(z \in \mathcal{U},\ i = \overline{1,n}\right)$$

and

$$\left|1 + \gamma(\beta - 1)\right| \sum_{i=1}^{n} (M_i(p_i + 1) + 1) \le n\cos\theta + (1 - n)\left|\gamma\right|$$

then the integral operator $\mathcal{J}_{n,\gamma,\beta}$ *defined by (3) is in the class* $\mathcal{S}$.

References

[1] D. Blezu, On univalence criteria, Gen. Math. 14 (1) (2006), 87–93.

[2] N. Breaz, D. Breaz, V. Pescar, On the univalence of a certain integral operator, Acta Univ. Apulensis Math. Inform. 26 (2011), 251–256.

[3] D. Breaz, N. Breaz, H. M. Srivastava, An extension of the univalent condition for a family of integral operators, Appl. Math. Lett. 22 (2009), 41-44.

[4] N. Breaz, V. Pescar, D. Breaz, Univalence criteria for a new integral operator, Math. Comput. Modelling 52(1-2) (2010), 241-246.

[5] E. Deniz, H. Orhan, An extension of the univalence criterion for a family of integral operators, Ann. Univ. Mariae Curie-Sklodowska Sect. A 64(2) (2010), 29–35.

[6] E. Deniz, H. Orhan, H.M. Srivastava, Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions, Taiwanese J. Math. 15(2) (2011), 883–917.
[7] B. A. Frasin, Univalence of two general integral operators, Filomat 23(3) (2009), 223–229.
[8] B. A. Frasin, Univalence criteria for general integral operator, Math. Commun. 16(1) (2011), 115–124.
[9] Y.J. Kim, E.P. Merkes, On an integral of powers of a Spirallike function, Kyungpook Math. J. 12 (1972), 249–253.
[10] J. -L. Liu, An univalent condition for a family of integral operators, Tamkang J. Math. 42(4) (2011), 441–444.
[11] S. S. Miller, P.T. Mocanu, Differential Subordinations, Theory and Applications, in: Monographs and Text Books in Pure and Applied Mathematics, vol. 255, Marcel Dekker, New York, 2000.
[12] S. Moldoveanu, N.N. Pascu, R.N. Pascu, On the univalence of an integral operator, Mathematica 43 (2001), 113–116.
[13] S. Ozaki and M. Nunokawa, The Schwarzian derivative and univalent functions, Proc. Amer. Math. Soc. 33 (1972), 392–394.
[14] N.N. Pascu, On a univalence criterion II, Itinerant Seminar Functional Equations, Approximation and Convexity, (Cluj Napoca, 1985), Preprint, University "Babes Bolyai", Cluj Napoca, 85 (1985), 153–154.
[15] V. Pescar, On the univalence of an integral operator, Appl. Math. Lett. 23 (2010), 615–619.
[16] V. Pescar, D. Breaz, The univalence of integral operators, Prof. Marin Drinov Acad. Publishing House, Sofia, 2008.
[17] V. Pescar, D. Breaz, On an integral operator, Appl. Math. Lett. 23 (2010), 625-629.
[18] V. Singh, On a class of univalent functions, Int. J. Math. Math. Sci. 23(12) (2000), 855–857.
[19] H. M. Srivastava, E. Deniz, H. Orhan, Some general univalence critera for a family of integral operators, Appl. Math. Comput. 215 (2010), 3696-3701.
[20] N. Ularu, D. Breaz, Univalence criterion for two integral operators, Filomat 25(3) (2011), 105-110.
[21] D. -G. Yang, J. -L. Liu, On a class of univalent functions, Int. J. Math. Math. Sci. 22(3) (1999), 605-610.