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Available at: http://www.pmf.ni.ac.rs/filomat
Filomat **23:3** (2009), 223–229 DOI:10.2298/FIL0903223F

UNIVALENCE OF TWO GENERAL INTEGRAL OPERATORS

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Abstract

In this paper, we give some sufficient conditions for general two integral operators to be univalent in the open unit disk.

1 Introduction and definitions

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ defined in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$ .Recently, Breaz and Breaz [6] and Breaz et al. [10] introduced and studied the integral operators

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \ldots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{1}$$

and

$$F_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1} \ldots \left(f_n'(t)\right)^{\alpha_n} dt \tag{2}$$

where $f_i \in \mathcal{A}$ and for $\alpha_i > 0$, for all $i = 1, \ldots, n$ (see also [3, 4, 5, 7, 9]).

Breaz and Güney [8] considered the above integral operators and they obtained their properties on the classes $\mathcal{S}^*_\alpha(b)$, $\mathcal{C}_\alpha(b)$ of starlike and convex functions of complex order b and type α introduced and studied by Frasin [11].

Very recently, Frasin [12] obtained some sufficient conditions for the above integral operators to be in the classes $\mathcal{S}^*$, $\mathcal{C}(\alpha)$ and $\mathcal{UCV}$, where $\mathcal{C}(\alpha)$ and $\mathcal{UCV}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, close -to-convex of order $\alpha (0 \le \alpha < 1)$ in $\mathcal{U}$ and uniformly convex functions.

2000 *Mathematics Subject Classifications.* 30C45.
Key words and Phrases. Analytic functions, univalent functions, integral operator.
Received: April 10, 2009
Communicated by Jelena Manojlović

In the present paper, we obtain some sufficient conditions for the above integral operators $F_n(z)$ and $F_{\alpha_1,\ldots,\alpha_n}(z)$ to be univalent in $\mathcal{U}$.

In order to derive our main results, we have to recall here the following lemma:

Lemma 1.1. *([1]) Let $f \in \mathcal{A}$, $\beta \in \mathbb{C}, Re(\beta) > 0$. If for some $\theta \in [0, 2\pi]$ the inequality*

$$\mathrm{Re}\left\{e^{i\theta}\frac{zf''(z)}{f'(z)}\right\} \leq \begin{cases} \frac{1}{2}\mathrm{Re}(\beta) & for \;\; 0 < \mathrm{Re}(\beta) < 1 \\ \frac{1}{4} & for \;\; \mathrm{Re}(\beta) \geq 1 \end{cases} \quad (z \in \mathcal{U})$$

is valid, then the function

$$G_\beta(z) = \left\{\beta \int_0^z u^{\beta-1} f'(u) du\right\}^{1/\beta}$$

is in $\mathcal{S}$, for all $\theta \in [0, 2\pi]$.

2 Main results.

Theorem 2.1. *Let $\alpha_j > 0$ be real numbers for all $j = 1, 2, \ldots, n$, $\beta \in \mathbb{C}$, $Re(\beta) > 0$. If $f_j \in \mathcal{A}$ for all $j = 1, 2, \ldots, n$ satisfies*

$$\mathrm{Re}\left(e^{i\theta}\frac{zf_j'(z)}{f_j(z)}\right) \leq \begin{cases} \frac{\mathrm{Re}(\beta)}{2\sum\limits_{j=1}^{n}\alpha_j} + \cos\theta & for \;\; 0 < \mathrm{Re}(\beta) < 1 \\ \frac{1}{4\sum\limits_{j=1}^{n}\alpha_j} + \cos\theta & for \;\; \mathrm{Re}(\beta) \geq 1 \end{cases} \tag{3}$$

for all $z \in \mathcal{U}$ and for some $\theta \in [0, 2\pi]$, then the function

$$\left\{\beta \int_0^z u^{\beta-1} \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{\alpha_j} du\right\}^{1/\beta} \in \mathcal{S}$$

for all $\theta \in [0, 2\pi]$.

Proof. From (1) we observe that $F_n \in \mathcal{A}$, i.e. $F_n(0) = F_n'(0) - 1 = 0$. On the other hand, it is easy to see that

$$F_n'(z) = \prod_{j=1}^{n} \left(\frac{f_j(z)}{z}\right)^{\alpha_j}$$

and

$$\left(\frac{zF_n''(z)}{F_n'(z)}\right) = \sum_{j=1}^{n}\alpha_j\left(\frac{zf_j'(z)}{f_j(z)}\right) - \sum_{j=1}^{n}\alpha_j$$

thus we have

$$\left(e^{i\theta}\frac{zF_n''(z)}{F_n'(z)}\right) = \sum_{j=1}^{n}\alpha_j\left(e^{i\theta}\frac{zf_j'(z)}{f_j(z)}\right) - e^{i\theta}\sum_{j=1}^{n}\alpha_j. \tag{4}$$

It follows from (4) and the hypothesis (3) that

$$\begin{aligned}\mathrm{Re}\left(e^{i\theta}\frac{zF_n''(z)}{F_n'(z)}\right) &= \sum_{j=1}^{n}\alpha_j \mathrm{Re}\left(e^{i\theta}\frac{zf_j'(z)}{f_j(z)}\right) - (\cos\theta)\sum_{j=1}^{n}\alpha_j \\ &\le \begin{cases}\frac{1}{2}\mathrm{Re}(\beta) & for\ \ 0<\mathrm{Re}(\beta)<1 \\ \frac{1}{4} & for\ \ \mathrm{Re}(\beta)\ge 1\end{cases}\end{aligned}$$

for all $z\in\mathcal{U}$ and for some $\theta\in[0,2\pi]$. Applying Lemma 1.1, we have

$$\left\{\beta\int_0^z u^{\beta-1}F_n'(u)du\right\}^{1/\beta}\in\mathcal{S}$$

or, equivalently

$$\left\{\beta\int_0^z u^{\beta-1}\prod_{j=1}^{n}\left(\frac{f_j(u)}{u}\right)^{\alpha_j}du\right\}^{1/\beta}\in\mathcal{S}$$

for all $\theta\in[0,2\pi]$.
This completes the proof. □

Letting $n=1$, $\alpha_1=\alpha$ and $f_1=f$ in Theorem 2.1, we have

Corollary 2.2. *Let $\alpha>0$ be real number, $\beta\in\mathbb{C}$, $Re(\beta)>0$. If $f\in\mathcal{A}$ satisfies*

$$Re\left(e^{i\theta}\frac{zf'(z)}{f(z)}\right)\le\begin{cases}\frac{Re(\beta)}{2\alpha}+\cos\theta & for\ \ 0<Re(\beta)<1 \\ \frac{1}{4\alpha}+\cos\theta & for\ \ Re(\beta)\ge 1\end{cases}$$

for all $z\in\mathcal{U}$ and for some $\theta\in[0,2\pi]$, then the function

$$\left\{\beta\int_0^z u^{\beta-1}\left(\frac{f(u)}{u}\right)^{\alpha}du\right\}^{1/\beta}\in\mathcal{S}$$

for all $\theta\in[0,2\pi]$.
Letting $\alpha=1$ in Corollary 2.2, we have

Corollary 2.3. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$. If $f \in \mathcal{A}$ satisfies*

$$Re\left(e^{i\theta}\frac{zf'(z)}{f(z)}\right) \leq \begin{cases} \frac{Re(\beta)}{2} + \cos\theta & for \quad 0 < Re(\beta) < 1 \\ \frac{1}{4} + \cos\theta & for \quad Re(\beta) \geq 1 \end{cases}$$

for all $z \in \mathcal{U}$ and for some $\theta \in [0, 2\pi]$,then the function

$$\left\{\beta \int_0^z u^{\beta-2} f(u) du\right\}^{1/\beta} \in \mathcal{S}$$

for all $\theta \in [0, 2\pi]$.

Letting $\beta = 1$ in Corollary 2.3, we have

Corollary 2.4. *If $f \in \mathcal{A}$ satisfies*

$$Re\left(e^{i\theta}\frac{zf'(z)}{f(z)}\right) \leq \frac{1}{4} + \cos\theta$$

for all $z \in \mathcal{U}$ and for some $\theta \in [0, 2\pi]$,then the function

$$\int_0^z \frac{f(u)}{u} du \in \mathcal{S}$$

for all $\theta \in [0, 2\pi]$.

Next, we have

Theorem 2.5. *Let $\alpha_j > 0$ be real numbers for all $j = 1, 2, \ldots, n$, $\beta \in \mathbb{C}$, $Re(\beta) > 0$. If $f_j \in \mathcal{A}$ for all $j = 1, 2, \ldots, n$ satisfies*

$$\mathrm{Re}\left(e^{i\theta}\frac{zf_i''(z)}{f_i'(z)}\right) \leq \begin{cases} \frac{\mathrm{Re}(\beta)}{2\sum_{j=1}^{n}\alpha_j} & for \quad 0 < \mathrm{Re}(\beta) < 1 \\ \frac{1}{4\sum_{j=1}^{n}\alpha_j} & for \quad \mathrm{Re}(\beta) \geq 1 \end{cases} \tag{5}$$

for all $z \in \mathcal{U}$ and for some $\theta \in [0, 2\pi]$,then the function

$$\left\{\beta \int_0^z u^{\beta-1} \prod_{j=1}^{n} \left(f_j'(u)\right)^{\alpha_j} du\right\}^{1/\beta} \in \mathcal{S}$$

for all $\theta \in [0, 2\pi]$.

Proof. It follows from (2) that $F_{\alpha_1,\ldots,\alpha_n}(0) = F'_{\alpha_1,\ldots,\alpha_n}(0) - 1 = 0$. Also a simple computation yields

$$\left(\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}\right) = \sum_{j=1}^{n} \alpha_j \left(\frac{zf''_j(z)}{f'_j(z)}\right). \tag{6}$$

Thus we have

$$\mathrm{Re}\left(e^{i\theta}\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}\right) = \sum_{j=1}^{n} \alpha_j \mathrm{Re}\left(e^{i\theta}\frac{zf''_j(z)}{f'_j(z)}\right). \tag{7}$$

Since f_j satisfies the condition (5) for every $j = 1, \ldots, n$, then from (7), we obtain

$$\mathrm{Re}\left(e^{i\theta}\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}\right) \leq \begin{cases} \frac{1}{2}\mathrm{Re}(\beta) & for \;\; 0 < \mathrm{Re}(\beta) < 1 \\ \frac{1}{4} & for \;\; \mathrm{Re}(\beta) \geq 1 \end{cases}$$

for all $z \in \mathcal{U}$ and for some $\theta \in [0, 2\pi]$. Lemma 1.1 implies that

$$\left\{\beta \int_0^z u^{\beta-1} F'_{\alpha_1,\ldots,\alpha_n}(u)du\right\}^{1/\beta} \in \mathcal{S}$$

or, equivalently

$$\left\{\beta \int_0^z u^{\beta-1} \prod_{j=1}^{n} \left(f'_j(u)\right)^{\alpha_j} du\right\}^{1/\beta} \in \mathcal{S}$$

for all $\theta \in [0, 2\pi]$. □

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.5, we have

Corollary 2.6. *Let $\alpha > 0$ be real number, $\beta \in \mathbb{C}$, $Re(\beta) > 0$. If $f \in \mathcal{A}$ satisfies*

$$Re\left(e^{i\theta}\frac{zf''(z)}{f'(z)}\right) \leq \begin{cases} \frac{Re(\beta)}{2\alpha} & for \;\; 0 < Re(\beta) < 1 \\ \frac{1}{4\alpha} & for \;\; Re(\beta) \geq 1 \end{cases}$$

for all $z \in \mathcal{U}$ and for some $\theta \in [0, 2\pi]$,then the function

$$\left\{\beta \int_0^z u^{\beta-1} \left(f'(u)\right)^{\alpha} du\right\}^{1/\beta} \in \mathcal{S}$$

for all $\theta \in [0, 2\pi]$.

Letting $\alpha = 1$ in Corollary 2.6, we have

Corollary 2.7. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$. If $f \in \mathcal{A}$ satisfies*

$$Re\left(e^{i\theta}\frac{zf''(z)}{f'(z)}\right) \leq \begin{cases} \frac{Re(\beta)}{2} & for \;\; 0 < Re(\beta) < 1 \\ \frac{1}{4} & for \;\; Re(\beta) \geq 1 \end{cases}$$

for all $z \in \mathcal{U}$ and for some $\theta \in [0, 2\pi]$,then the function

$$\left\{\beta \int_0^z u^{\beta-1} f'(u)du\right\}^{1/\beta} \in \mathcal{S}$$

for all $\theta \in [0, 2\pi]$.

Letting $\beta = 1$ in Corollary 2.7, we obtain the following result of Blezu and Pascu [2].

Corollary 2.8. *([2])If $f \in \mathcal{A}$ satisfies*

$$Re\left(e^{i\theta}\frac{zf''(z)}{f'(z)}\right) \leq \frac{1}{4}$$

for all $z \in \mathcal{U}$ and for some $\theta \in [0, 2\pi]$,then $f \in \mathcal{S}$ for all $\theta \in [0, 2\pi]$.

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