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UNIVALENT ANALYTIC FUNCTIONS AND THE POINCARÉ METRIC

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Abstract

Let Ω be a hyperbolic domain in the complex plane C, let ρ_Ω be the density of the Poincaré metric in Ω, and let $\beta_\Omega = 1/\rho_\Omega$. For g analytic in Ω we set $\|g\|_\Omega = \sup \beta_\Omega(w)|g(w)|$, $w \in \Omega$. Let $S(\Omega)$ be the family of functions f analytic and univalent in Ω. Criteria in terms of the partial derivatives of β_Ω for Ω to satisfy $\sup \|f''/f'\|_\Omega < +\infty$, where f ranges over $S(\Omega)$, are given. For example, $\sup \beta_\Omega(w)|(\beta_\Omega)_{w\bar{w}}(w)| < +\infty$, $w \in \Omega$. If $f \in S(\Omega)$ is isolated in the sense that there is an $\varepsilon > 0$ such that $0 < \|f''/f' - g''/g'\|_\Omega < \varepsilon$ for no $g \in S(\Omega)$, then $C \backslash f(\Omega)$ is of zero area. The domain Ω is simply connected if $\sup \beta_\Omega(w)|(\beta_\Omega)_{w\bar{w}}(w)| \leqq 1$, $w \in \Omega$, and Ω is convex (hence simply connected) if and only if $\sup |(\beta_\Omega)_{\bar{w}}(w)| = 1$, $w \in \Omega$.

1. Introduction.

By Ω we always mean a subdomain of the complex plane $C = \{|z| < +\infty\}$ such that the complement contains at least two points. Let ρ_Ω be the density of the Poincaré metric $\rho_\Omega(w)|dw|$ in Ω so that $\rho_D(w) = (1-|w|^2)^{-1}$ if w is in the disk $D = \{|z| < 1\}$. We shall call $\beta_\Omega = 1/\rho_\Omega$ the weight function which appears in

$$\|f\|_\Omega = \sup_{w \in \Omega} \beta_\Omega(w)|f(w)|$$

for f analytic in Ω. For g analytic in Ω and locally univalent in Ω, namely, $g'(w) \neq 0$ at each $w \in \Omega$, we set $\lambda(g) = g''/g'$. Let $S(\Omega)$ be the family of functions f analytic and univalent in Ω. We shall call Ω of finite type if

$$a(\Omega) = \sup_{f \in S(\Omega)} \|\lambda(f)\|_\Omega$$

is finite. We have $a(\Omega) \leqq 8$ for each simply connected Ω, $a(D) = 6$ and $a(D \setminus \{0\}) = +\infty$; see [**6**, Theorem 2] and [**13**, Theorem 1 and p. 452].

We begin with weight function criteria for Ω to be of finite type. For a complex function $g(w)$ of $w = u + iv \in \Omega$ we recall the definition of the partial derivatives:

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$$g_w=2^{-1}(g_u-ig_v) \quad\text{and}\quad g_{\bar{w}}=2^{-1}(g_u+ig_v).$$

If g is real-valued and of C^2 further, then we have

$$g_{w\bar{w}}=4^{-1}\Delta g,\quad \overline{(g_{\bar{w}})}=g_w \quad\text{and}\quad \overline{(g_{\bar{w}\bar{w}})}=g_{ww}.$$

THEOREM 1. *The following are equivalent, where $\beta=\beta_\Omega$ for simplicity.*

(1.1) *Ω is of finite type.*
(1.2) *β_w is bounded in Ω.*
(1.3) *$\beta\beta_{ww}$ is bounded in Ω.*
(1.4) *$\beta\beta_{w\bar{w}}$ is bounded in Ω.*

Note that B. G. Osgood proved essentially (1.2)$\Rightarrow$(1.1), and it is not difficult to observe that (1.1)$\Rightarrow$(1.2) [**13**, Theorem 5]. For the completeness we include the proof of (1.1)$\Leftrightarrow$(1.2).

For f nonconstant and meromorphic in Ω we define the Schwarzian derivative of f by

$$\sigma(f)=(f''/f')'-2^{-1}(f''/f')^2.$$

It is known that if f is meromorphic and univalent in Ω, then

$$\|\sigma(f)\|_\Omega^*=\sup_{w\in\Omega}\beta_\Omega(w)^2|\sigma(f)(w)|\leqq 12;$$

see [**4**, Theorem 1]; by a rotation of the Riemann sphere the meromorphic case is reduced to the analytic case. The notation $\|\sigma(f)\|_\Omega^*$ will be used also for f analytic and locally univalent in Ω. We shall prove

THEOREM 2. *If f is analytic and locally univalent in Ω of finite type, then*

$$\|\sigma(f)\|_\Omega^*\leqq\|\lambda(f)\|_\Omega\{K+c(\Omega)+2^{-1}\|\lambda(f)\|_\Omega\},$$

where $8/e\leqq K<10/3$ and $c(\Omega)\geqq 0$ is a constant with

$$c(\Omega)\leqq 2\{1+\sup_{w\in\Omega}|(\beta_\Omega)_w(w)|\}. \tag{1.5}$$

This is an Ω-analogue of the P. L. Duren, H. S. Shapiro and A. L. Shields estimate in D:

$$\|\sigma(f)\|_D^*\leqq 4\|\lambda(f)\|_D+2^{-1}(\|\lambda(f)\|_D)^2;$$

see [**6**, p. 251]. Thus, for example, if Ω is of finite type, then $\|\sigma(f)\|_\Omega^*$ becomes smaller as $\|\lambda(f)\|_\Omega$ becomes smaller.

In view of (1.2) and (1.5), the quantity $\omega(\Omega)=\sup|(\beta_\Omega)_w(w)|$, $w\in\Omega$, is important. We shall investigate this in Section 6.

Returning to general Ω we call $f\in S(\Omega)$ isolated if there exists $\varepsilon>0$ such that $0<\|\lambda(f)-\lambda(g)\|_\Omega<\varepsilon$ for no function $g\in S(\Omega)$. A set $E\subset \boldsymbol{C}$ is called of full measure if $\boldsymbol{C}\setminus E$ is of measure zero. The "measure" always means the twodimensional Lebesgue measure.

THEOREM 3. *If $f\in S(\Omega)$ is isolated, then $f(\Omega)$ is of full measure. The converse is false if Ω is simply connected.*

See [**14**] for the study of f meromorphic and univalent in Ω with $\sigma(f)$ and $\|\cdot\|_{\Omega}^{*}$ instead of $\lambda(f)$ and $\|\cdot\|_{\Omega}$. W. P. Thurston [**17**, p. 191] (see [**3**] also) found an Ω such that each Möbius transformation is isolated: There exists $\varepsilon>0$ such that $0<\|\sigma(f)\|_{\Omega}^{*}<\varepsilon$ for no f meromorphic and univalent in Ω. It is open to find Ω such that $S(\Omega)$ contains an isolated point in our sense.

Let $SD(\Omega)$ be the family of $f\in S(\Omega)$ with finite Dirichlet integral:

$$\iint_{\Omega}|f'(w)|^2 dudv<+\infty.$$

Theorem 3 shows in particular that each $f\in SD(\Omega)$ is not isolated. We can prove this fact in a somewhat stronger form in

THEOREM 4. *For each $f\in SD(\Omega)$ and each $\varepsilon>0$ we can find distinct functions $f_k\in SD(\Omega)$ $(k=1, 2)$ such that*

$$0<\|\lambda(f)-\lambda(f_k)\|_{\Omega}<\varepsilon \ (k=1, 2) \quad \text{and} \quad f=2^{-1}(f_1+f_2).$$

In the proofs of Theorems 3 and 4, X. U. Nguyen's existence theorem of Lipschitz functions and the method of D. H. Hamilton for constructing univalent functions are fundamental; see [**12**] and [**7**].

My esteemed colleagues, Hisao Sekigawa and Toshihiro Nakanishi gave me invaluable informations on the paper [**13**]. I wish to express my sincere gratitude to them.

2. A short survey on domains of finite type.

For each universal covering projection φ from D onto Ω we have

$$(1-|z|^2)|\varphi'(z)|=\beta_{\Omega}(w) \tag{2.1}$$

at each $z\in D$ with $w=\varphi(z)$; see [**1**, Chapter 1] for example. In particular, φ' never vanishes in D and β_{Ω} is of C^{∞}. Set

$$\delta(w)=\inf_{z\in\partial\Omega}|w-z|, \ w\in\Omega, \quad \text{and} \quad b(\Omega)=\sup_{w\in\Omega}\beta_{\Omega}(w)\delta(w)^{-1}.$$

Osgood proved that $2b(\Omega)\leqq a(\Omega)\leqq 4b(\Omega)$ and

$$a(\Omega)<+\infty \Longleftrightarrow \|\lambda(\varphi)\|_{D}<+\infty; \tag{2.2}$$

see [**13**, the proofs of Theorems 2 and 6]. Note that if $\|\lambda(\varphi)\|_{D}<+\infty$ for a φ, then $\|\lambda(\varphi)\|_{D}<+\infty$ for each φ; for the proof, see the forthcoming exprêssion (3.2).

In the special case $\partial\Omega$ is unbounded, C. Pommerenke proved that

$$\|\lambda(\varphi)\|_D<+\infty \Longleftrightarrow b(\Omega)<+\infty\,;$$

see [**16**, Corollary 1, p. 195 and (4.2) in p. 196]. Actually, Pommerenke observed that the unbounded $\partial\Omega$ is uniformly perfect if and only if $\|\lambda(\varphi)\|_D<+\infty$. Here $\partial\Omega$ is called uniformly perfect if there exists a constant $0<c\leqq 1$ such that $\partial\Omega$ contains a point of the set $\{w\,;\,cr\leqq|w-z|\leqq r\}$ for each $z\in\partial\Omega$ and each $0<r<+\infty$. If $\partial\Omega$ contains an isolated point, then $\partial\Omega$ is not uniformly perfect, so that Ω is not of finite type.

If Ω is of finite type, then the image $f(\Omega)$ by an $f\in S(\Omega)$ is again of finite type [**13**, Corollary 1, p. 457]. Actually, by the conformal invariance of the Poincaré metric one obtains

$$\beta_{f(\Omega)}(z)=|f'(w)|\beta_\Omega(w), \qquad z=f(w).$$

Taking the logarithms of the both sides and partially differentiating them by w we have

$$(2.3) \qquad (\beta_{f(\Omega)})_z(z)=\{|f'(w)|/f'(w)\}\{2^{-1}\beta_\Omega(w)\lambda(f)(w)+(\beta_\Omega)_w(w)\}\,.$$

Therefore one obtains

$$\sup_{z\in f(\Omega)}|(\beta_{f(\Omega)})_z(z)|\leqq 2^{-1}a(\Omega)+\sup_{w\in\Omega}|(\beta_\Omega)_w(w)|\,,$$

which, combined with (1.1)$\Leftrightarrow$(1.2), proves the property.

Let $w_0\in\partial\Omega$. Then $f(w)=(w-w_0)^{-1}$ is in $S(\Omega)$. Thus, Ω is of finite type if and only if $\partial f(\Omega)$ is uniformly perfect.

3. Proof of Theorem 1.

Taking the logarithms of the both sides of (2.1), and then partially differentiating them by z, we obtain

$$(3.1) \qquad \lambda(\varphi)(z)=2\bar{z}/(1-|z|^2)+2\varphi'(z)(\beta_w/\beta)(w)\,.$$

Although φ may not be in $S(D)$, the computation is essentially the same as that for (2.3). Setting $\chi_\varphi(z)=\varphi'(z)/|\varphi'(z)|$ we now have

$$(3.2) \qquad (1-|z|^2)\lambda(\varphi)(z)=2\{\bar{z}+\chi_\varphi(z)\beta_w(w)\}\,.$$

Therefore,

$$\|\lambda(\varphi)\|_D<+\infty \Longleftrightarrow \sup_{w\in\Omega}|\beta_w(w)|<+\infty\,,$$

which implies (1.1)$\Leftrightarrow$(1.2) in view of (2.2).

Eliminating $\varphi''(z)$ from (3.1) and from the right hand side of

$$\lambda(\varphi)'(z)=2\bar{z}^2/(1-|z|^2)^2+2\varphi''(z)(\beta_w/\beta)(w)+2\varphi'(z)^2(\beta_w/\beta)_w(w)\,,$$

we have

$$\sigma(\varphi)(z)=2\varphi'(z)^2\{(\beta_w/\beta)_w(w)+(\beta_w/\beta)^2(w)\},$$

whence

$$(1-|z|^2)^2\sigma(\varphi)(z)=2\chi_\varphi(z)^2\beta(w)\beta_{ww}(w). \tag{3.3}$$

It is known that $\|\lambda(\varphi)\|_D<+\infty$ if and only if $\|\sigma(\varphi)\|_D^*<+\infty$; see [**19**, Theorem 2], [**18**], [**20**] and [**21**]. We thus have (1.1)⇔(1.3).

We recall the Gauss curvature identity in terms of β:

$$\beta^2\Delta\log\beta\equiv-4 \quad \text{or} \quad \beta\beta_{w\bar{w}}-\beta_w\beta_{\bar{w}}\equiv-1. \tag{3.4}$$

This follows from the partial differentiation of (3.1) by $\bar{z}$. Then, $|\beta_w|^2=\beta_w\beta_{\bar{w}}$ is bounded in Ω if and only if $\beta\beta_{w\bar{w}}$ is bounded in Ω.

Remark. If $\varphi\in S(D)$, then $\|\sigma(\varphi)\|_D^*\leqq 6$. Suppose that $\|\sigma(\varphi)\|_D^*\leqq 2q$. If $q=1$, then $\varphi\in S(D)$, while if $q<1$, then φ is the restriction of a $(1+q)/(1-q)$-quasiconformal mapping from $\boldsymbol{C}\cup\{\infty\}$ onto $\boldsymbol{C}\cup\{\infty\}$. See [**8**], [**11**] and [**2**]. Combining these with (3.3) we have the obvious criteria in terms of β_Ω for the property of Ω. For example, *if*

$$\sup_{w\in\Omega}\beta_\Omega(w)|(\beta_\Omega)_{ww}(w)|\leqq 1,$$

then Ω is simply connected.

In case Ω is simply connected, we set $\delta(\Omega)=\|\sigma(f)\|_\Omega^*$, where $f:\Omega\to D$ is an onto conformal homeomorphism; $\delta(\Omega)$ is independent of the choice of f and is called the distance of Ω from a disk [**9**, p. 61]. We can choose $f=\varphi^{-1}$ and we have $\|\sigma(\varphi^{-1})\|_\Omega^*=\|\sigma(\varphi)\|_D^*$. A known result [**9**, Theorem 2.1, p. 63], together with (3.3), now shows: *If Ω is the image of a convex domain by a Möbius transformation, then*

$$\sup_{w\in\Omega}\beta_\Omega(w)|(\beta_\Omega)_{ww}(w)|\leqq 1.$$

The equality holds for the domains specified in the cited theorem. See Proposition 2 in the forthcoming Section 6.

4. Proof of Theorem 2.

LEMMA 1. *For g analytic in D we have*

$$\sup_{z\in D}(1-|z|^2)^2|g'(z)|\leqq K\|g\|_D, \tag{4.1}$$

where K is an absolute constant with $8/e\leqq K<10/3$.

This is due to *M.-C.* Liu [**10**, Theorem]; he actually proved that

$$K\leqq 2^{-1}(\sqrt{5}+1)(\sqrt{5}+2)^{1/2}=3.3301\cdots \qquad [\mathbf{10}, \text{ p. } 207].$$

For the proof of Theorem 2 we let g be a branch of $\log(f'\circ\varphi)$ in D. Then,

$$\|g'\|_D=\|\lambda(f)\|_\Omega,$$

which, combined with (4.1), yields

$$\sup_{z\in D}(1-|z|^2)^2|g''(z)|\leqq K\|\lambda(f)\|_\Omega. \tag{4.2}$$

On the other hand,

$$\varphi'^2\lambda(f)'\circ\varphi=g''-\varphi'\cdot(\lambda(f)\circ\varphi)\lambda(\varphi). \tag{4.3}$$

It then follows from (4.2) and (4.3) that

$$\|\sigma(f)\|_\Omega^*\leqq K\|\lambda(f)\|_\Omega+\|\lambda(f)\|_\Omega\|\lambda(\varphi)\|_D+2^{-1}(\|\lambda(f)\|_\Omega)^2.$$

Setting

$$c(\Omega)=\sup_\varphi\|\lambda(\varphi)\|_D,$$

we now have (1.5) from (3.2).

5. Proofs of Theorems 3 and 4.

LEMMA 2 (X. U. Nguyen [12]). *For each compact set E of positive measure, there exists a nonconstant analytic function F in the open set $E^c=(C\cup\{\infty\})\setminus E$ such that F is bounded in E^c and F satisfies the Lipschitz condition in $C\setminus E$:*

$$\Lambda(F)=\sup_{\substack{z,w\in C\setminus E\\ z\neq w}}|F(z)-F(w)|/|z-w|<+\infty.$$

Note that, then $|F'(w)|\leqq\Lambda(F)$ for each $w\in C\setminus E$.

LEMMA 3. *If f is analytic, bounded and bounded away from zero: $0<A\leqq|f|\leqq B<+\infty$ in D, then*

$$\|f'/f\|_D\leqq(2/\pi)\log(B/A).$$

Proof. Apply the Schwarz-Pick lemma

$$(1-|z|^2)|h'(z)|/(1-|h(z)|^2)\leqq 1, \qquad z\in D,$$

to $h=(iH+1)/(iH-1)$, where H is defined by

$$H=\exp[\{(\pi i)/\log(B/A)\}\log(f/A)].$$

Proof of Theorem 3. Suppose that $f(\Omega)$ is not of full measure. Let $E\subset C\setminus f(\Omega)$ be a compact set of positive measure and consider F of Lemma 2 with $\Lambda=\Lambda(F)$. Given $\varepsilon>0$ we choose $\gamma>0$ such that

$$\gamma\Lambda<(e^{\pi\varepsilon/2}-1)/(e^{\pi\varepsilon/2}+1). \tag{5.1}$$

For a complex number α, $0<|\alpha|\leqq\gamma$, we set

$$g_{\alpha}=f+\alpha F\circ f.$$

Then $g_{\alpha}\in S(\Omega)$ because

$$|g_{\alpha}(w)-g_{\alpha}(z)|\geqq(1-\gamma\Lambda)|f(w)-f(z)|>0 \quad \text{for} \quad w, z\in\Omega,\ w\neq z.$$

We shall show that

$$0<\|\lambda(f)-\lambda(g_{\alpha})\|_{\Omega}<\varepsilon, \tag{5.2}$$

so that f is not isolated. Set

$$h_{\alpha}=1+\alpha F'\circ f \quad \text{and} \quad G_{\alpha}=h_{\alpha}\circ\varphi.$$

Then G_{α} is nonconstant and

$$0<1-\gamma\Lambda<|G_{\alpha}|<1+\gamma\Lambda \qquad \text{in } D,$$

so that, by Lemma 3, together with (5.1), we have

$$0<\|G_{\alpha}'/G_{\alpha}\|_{D}<\varepsilon.$$

Therefore, (5.2) follows from $\lambda(g_{\alpha})-\lambda(f)=h_{\alpha}'/h_{\alpha}$.

Suppose that Ω is simply connected and set $\psi=\varphi^{-1}$, the inverse map. Then, for $0<p\leqq 2$, each single-valued branch

$$f_{p}=\{(1+\psi)/(1-\psi)\}^{p}$$

is in $S(\Omega)$. Furthermore,

$$\|\lambda(f_{p})-\lambda(f_{2})\|_{\Omega}=2(2-p)\longrightarrow 0 \qquad \text{as} \quad p\to 2-0.$$

Therefore, $f_{2}(\Omega)$ is of full measure, yet f_{2} is not isolated.

Corollary to Theorem 3. *If Ω is not of full measure, then no linear function $L(z)=Az+B$ ($A\neq 0$ is isolated.*

Proof. Suppose that $L(z)=Az+B$ is isolated. Then $L(\Omega)$ is of full measure, so that $\Omega=L^{-1}\circ L(\Omega)$ is of full measure.

Proof of Theorem 4. Since $f(\Omega)$ is not of full measure, we can construct g_{α} as in the proof of Theorem 3. Set

$$f_{1}=g_{\gamma} \quad \text{and} \quad f_{2}=g_{-\gamma}.$$

Then, $f_{k}\in S(\Omega)$ and $0<\|\lambda(f)-\lambda(f_{k})\|_{\Omega}<\varepsilon$, $k=1, 2$. Since

$$|f_{k}'|=|f'|\cdot|1\pm\gamma F'(f)|\leqq(1+\gamma\Lambda)|f'|, \qquad k=1, 2,$$

it follows that $f_{k}\in SD(\Omega)$. Apparently, $f=2^{-1}(f_{1}+f_{2})$.

6. The order of a locally nnivalent function.

Let g be a function analytic and locally univalent in D. The order $ord_D(g)$ of g is the supremum of $|a_2(z)|$, $z\in D$, where $a_2(z)$ is the Taylor coefficient in the expansion

$$\frac{g((\zeta+z)/(1+\bar{z}\zeta))-g(z)}{(1-|z|^2)g'(z)}=\zeta+a_2(z)\zeta^2+\cdots, \qquad \zeta\in D.$$

By simple computation, we have

$$ord_D(g)=\sup_{z\in D}|-\bar{z}+2^{-1}(1-|z|^2)\lambda(g)(z)|.$$

Set $A=ord_D(g)$. Then $A\leqq 2$ for $g\in S(D)$ by the coefficient theorem. Since for general g,

$$|(\partial/\partial|z|)\log\{(1-|z|^2)g'(z)\}|\leqq 2A/(1-|z|^2), \qquad z\in D,$$

it follows from the familiar manipulation in the univalent function theory that

$$(1-|z|)^{A-1}/(1+|z|)^{A+1}\leqq|g'(z)/g'(0)|, \qquad z\in D.$$

The minimum modulus principle for g', never vanishing in D, yields that $A\geqq 1$. Furthermore,

$$\mathrm{Re}\{1+z\lambda(g)(z)\}\geqq(1-2A|z|+|z|^2)/(1-|z|^2)>0$$

if $|z|<A-(A^2-1)^{1/2}$, so that $A=1$ implies $g\in S(D)$ and $g(D)$ is convex. Conversely, if $g\in S(D)$ and $g(D)$ is convex, then by the coefficient theorem, $|a_2(z)|\leqq 1$, for g, $z\in D$, so that $A\leqq 1$, or $A=1$. See [**5**, pp. 33, 42 and 45] and [**15**, pp. 116, 117 and 133].

We note that $(\beta_D)_z(z)=-\bar{z}$ by $\beta_D(z)=1-|z|^2$. For f analytic and locally univalent in Ω, we set

$$ord_\Omega(f)=\sup_{w\in\Omega}|(\beta_\Omega)_w(w)+2^{-1}\beta_\Omega(w)\lambda(f)(w)|$$

and call it the order of f in Ω. It then follows from (3.1), together with $\lambda(f\circ\varphi)=\varphi'\lambda(f)\circ\varphi+\lambda(\varphi)$, that

$$ord_\Omega(f)=ord_D(f\circ\varphi). \tag{6.1}$$

In particular, if $f_0(w)\equiv w$, *then*

$$\omega(\Omega)\equiv\sup_{w\in\Omega}|(\beta_\Omega)_w(w)|=ord_\Omega(f_0)=ord_D(\varphi)\geqq 1. \tag{6.2}$$

Thus, Ω is of finite type if and only if $\omega(\Omega)<+\infty$. Furthermore, $2^{-1}c(\Omega)-1\leqq\omega(\Omega)\leqq 2^{-1}c(\Omega)+1$; the left hand side is (1.5). In view of (6.1) it is now easy to prove the following

PROPOSITION 1. *Let f be analytic and locally univalent in Ω. Then $ord_{\Omega}(f)=1$ if and only if $f\in S(\Omega)$ and $f(\Omega)$ is convex.*

"Only if". It follows from (6.1) that $f\circ\varphi\in S(D)$ and $f(\Omega)=f\circ\varphi(D)$ is convex. For the proof of the univalency of f in Ω, we let $w_1\neq w_2$, both in Ω. Then there exist $z_1\neq z_2$, both in D, such that $w_k=\varphi(x_k)$, $k=1, 2$. Therefore, $f(w_1)=(f\circ\varphi)(z_1)\neq(f\circ\varphi)(z_2)=f(w_2)$. Since $f(\Omega)$ is simply connected, Ω is simply connected. "If". If $f(\Omega)$ by $f\in S(\Omega)$ is convex, then Ω must be simply connected. Thus, $f\circ\varphi\in S(D)$, and $ord_{\Omega}(f)=1$ follows from (6.1) with $ord_D(f\circ\varphi)=1$.

As a consequence of Proposition 1 we have: *If Ω is not simply connected, then $ord_{\Omega}(f)>1$ for each f locally univalent in D.* It would be of interest to have a convex domain criterion in terms of $\omega(\Omega)$.

PROPOSITION 2. *A domain Ω is convex (and hence, simply connected) if and only if $\omega(\Omega)=1$.*

This is a consequence of (6.2) with $\varphi(D)=\Omega$. Proposition 2 has the following corollary: *A domain Ω is convex if and only if β_{Ω} is a superharmonic function in Ω.* Remember that $\beta\beta_{w\bar{w}}=|\beta_w|^2-1$ ($\beta=\beta_{\Omega}$; see (3.4)). "Only if". It follows from $\omega(\Omega)=1$ that $4^{-1}\Delta\beta=\beta_{w\bar{w}}\leqq 0$. "If". It follows from $\beta_{w\bar{w}}\leqq 0$ that $|\beta_w|\leqq 1$, whence $\omega(\Omega)\leqq 1$, or $\omega(\Omega)=1$.

If Ω is simply connected or $\varphi\in S(D)$, then $ord_D(\varphi)\leqq 2$, which, combined with (6.2), shows that $\omega(\Omega)\leqq 2$. For the Koebe function κ we have $ord_D(\kappa)=2$. These are observed in [**13**, Theorem 3, (14), p. 454].

It would be interesting to have an upper bound of $a(\Omega)$ by $\omega(\Omega)$.

PROPOSITION 3. *For each Ω we have*

$$a(\Omega)\leqq 8\omega(\Omega). \tag{6.3}$$

Actually, the same proof as that of [**15**, the left half of (1.11), p. 115], together with the property of the projection φ, that is, $\delta(w)=d(\varphi(z))$, $w=\varphi(z)$, teaches us that

$$\beta_{\Omega}(w)/\{2\,ord_D(\varphi)\}\leqq\delta(w), \qquad w\in\Omega,$$

which, combined with $a(\Omega)\leqq 4b(\Omega)$ observed in Section 2, yields (6.3).

In the specified case where Ω is simply connected we have a better estimate $a(\Omega)\leqq 8$ than that in Proposition 3. In the "convex" case we have

PROPOSITION 4. *If Ω is convex, then $a(\Omega)\leqq 6$.*

As we observed the equality holds for $\Omega=D$. For each $w\in\Omega$ we choose a conformal homeomorphism φ from D onto Ω such that $w=\varphi(0)$. Then, for $f\in S(\Omega)$,

$$\beta_{\Omega}(w)|\lambda(f)(w)|=|\lambda(f\circ\varphi)(0)-\lambda(\varphi)(0)|.$$

Since $f\circ\varphi\in S(D)$, we have $|\lambda(f\circ\varphi)(0)|\leqq 4$, while, since $\varphi\in S(D)$ and $\varphi(D)$ is convex, we have $|\lambda(\varphi)(0)|\leqq 2$ [**5**, p. 45]. This completes the proof.

PROPOSITION 5.

$$\sup_{f\in S(\Omega)}|\omega(f(\Omega))-\omega(\Omega)|\leqq 2^{-1}a(\Omega). \tag{6.4}$$

Actually, in view of (2.3) one has

$$|(\beta_{f(\Omega)})_z(z)|\leqq|(\beta_\Omega)_w(w)|+2^{-1}\beta_\Omega(w)|\lambda(f)(w)|$$

and

$$|(\beta_\Omega)_w(w)|\leqq|(\beta_{f(\Omega)})_z(z)|+2^{-1}\beta_\Omega(w)|\lambda(f)(w)|.$$

It is now easy to have (6.4).

Propositions 3 and 5 yield

$$\sup_{f\in S(\Omega)}|\omega(f(\Omega))-\omega(\Omega)|\leqq 4\omega(\Omega).$$

If Ω is simply connected, then

$$\sup_{f\in S(\Omega)}|\omega(f(\Omega))-\omega(\Omega)|\leqq 1; \tag{6.5}$$

actually since $f(\Omega)$ is simply connected, we have $1\leqq\omega(\Omega)\leqq 2$ and $1\leqq\omega(f(\Omega))\leqq 2$. Note that $\omega(\kappa(D))-\omega(D)=1$. The sharp estimate (6.5) is unchanged if we further assume that Ω is convex. The Koebe function and D again show the sharpness.

We show that the set

$$\{\omega(\Omega);\ \Omega \text{ simply connected}\}=\{ord_D(f);\ f\in S(D)\}$$

is precisely the closed interval [1, 2]. Set

$$f_\alpha(z)=\{(1+z)/(1-z)\}^\alpha \qquad (f_\alpha(0)=1,\ 1\leqq\alpha\leqq 2).$$

Then, $f_\alpha\in S(D)$ and it suffices to prove that $ord_D(f_\alpha)=\alpha$. By a simple calculation we have

$$ord_D(f_\alpha)=\sup_{z\in D}F(z)^{1/2},$$

where

$$F(z)=\{\alpha^2(1-|z|^2)^2+4(\operatorname{Im} z)^2\}/\{(1-|z|^2)^2+4(\operatorname{Im} z)^2\}.$$

Then $F(z)\leqq\alpha^2$ and the equality holds for real z.

If Ω is simply connected, then again

$$\{ord_\Omega(f);\ f\in S(\Omega)\}=[1, 2]$$

by (6.1). It would be an interesting problem to determine $\{ord_\Omega(f);\ f\in S(\Omega)\}$ for Ω of finite type yet not simply connected; as was remarked, this set does not contain 1, so that (6.1) shows that this set is contained in the interval $(1, +\infty)$.

Added in Proof. On the basis of the result of K.-J. Wirths in his paper: Über holomorphe Funktionen, die einer Wachstumsbeschränkung unterliegen; *Archiv der Mathematik* **30** (1978), 606-612, the constant K in Teorem 2 should be $K=(13\sqrt{3}+55\sqrt{11})/64=3.20204\cdots$. On the basis of the fact the author of [**12**] called himself N. X. Uy in his reference in the recent paper: A removable set for Lipschitz harmonic functions; *Mschigan Mathematical Journal* **37** (1990), 45-51, the precise reference of Lemma 2 should be Uy's theorem.

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