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Univalent Holomorphic Functions with Negative and Fixed Finitely Many Coefficients in terms of Generalized Fractional Derivative

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Abstract. A new class of univalent holomorphic functions with fixed finitely many coefficients based on Generalized fractional derivative are introduced. Also some important properties of this class such as coefficient bounds, convex combination, extreme points, Radii of starlikeness and convexity are investigated.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic and univalent in the open disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Also denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0, \quad z \in \Delta \tag{1.1}$$

introduced and studied by Silverman [9]. Also see ([3], [4], [8]).

We shall make use of the following definitions of fractional calculus (that is, fractional integrals and fractional derivatives) (cf. Owa [7]; see also Srivastava and Owa [11]).

Definition 1.1. The fractional integral of order λ is defined, for a function $f(z)$,

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by

$$D_z^{-\lambda} f(z) := \frac{1}{\gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \qquad (\lambda > 0),$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) := \frac{1}{\gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \qquad (0 \leq \lambda < 1), \tag{1.2}$$

where the function $f(z)$ is constrained, and multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as Definition 1.1, above.

The generalization of the fractional derivative of order k for a function in a simply-connected region of $z-$plan containing the origin is denoted by

$$J_{0,z}^{k,\eta,\delta} f(z) \qquad (0 \leq \delta < 1, 0 \leq k < \eta < 1)$$

and defined for a power function as follows:

$$J_{0,z}^{k,\eta,\delta}\ z^p = \frac{\Gamma(p+1)\ \Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\ \Gamma(p-k+\delta+1)}\ z^{p-\eta} \quad (p > \max\{0, \eta-\delta\} - 1). \tag{1.3}$$

By a simple calculation if $f(z) \in T$ then

$$\frac{\Gamma(2-\eta)\ \Gamma(2-k+\delta)}{\Gamma(2-\eta+\delta)}\ z^{\eta}\ J_{0,z}^{k,\eta,\delta} f(z) = z - \sum_{k=2}^{\infty} \phi(k)\ a_k\ z^k, \tag{1.4}$$

where

$$\phi(k) = \frac{(1)_k\ (2-\eta+\delta)_{k-1}}{(2-\eta)_{k-1}(2-k+\delta)_{k-1}} \quad , k \in \{2, 3, ...\}\ \mathit{and}\ (x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} \tag{1.5}$$

Application of fractional calculus on univalent functions was carried out by several authors in ([1], [2], [5], [6], [10]).

Definition 1.3. A function $f(z) \in T$ is said to be in the class $\Lambda(\alpha, \beta, \theta, \nu)$ if it satisfies the inequality

$$\left| \frac{H\ J_{0,z}^{k,\eta,\delta} f(z) - 1}{2\alpha\ H\ J_{0,z}^{k,\eta,\delta} f(z)\ - \beta(1-\theta)\alpha} \right| <\ \nu, \tag{1.6}$$

where $H = \frac{\Gamma(2-\eta)\ \Gamma(2-k+\delta)}{\Gamma(2-\eta+\delta)}\ z^{\eta-1}$, α, β, θ and ν are belong to [0.1).
Now we introduce the class $\Lambda(c_m, \alpha, \beta, \theta, \nu)$ the subclass of $\Lambda(\alpha, \beta, \theta, \nu)$ consisting

of functions of the form

$$f(z) = z - \sum_{m=2}^{n} \frac{\alpha\nu\ (2-\beta(1+\theta))\ c_m}{\phi(m)\ (1+2\alpha\nu)}\ z^m\ - \sum_{k=n+1}^{\infty} a_k\ z^k \tag{1.7}$$

The different cases of such functions were studied earlier by many authors in [2], [4].
For achieving our main results we need the following lemma.

Lemma 2.1. *Let $f(z) \in T$, then $f(z) \in \Lambda(\alpha,\beta,\theta,\nu)$ if and only if*

$$\sum_{k=2}^{\infty} \phi(k)\ (1+2\alpha\nu)\ a_k\ \leq\ \alpha\nu\ (2-\beta(1+\theta)), \tag{1.8}$$

where $\phi(k)$ is defined by (1.5).
The result is sharp for the function

$$G(z) = z - \frac{\alpha\nu\ (2-\beta(1+\theta))(2-\eta)_{k-1}(2-k+\delta)_{k-1}}{(1+2\nu\alpha)\ (1)_k\ (2-\nu+\delta)_{k-1}}\ z^k. \tag{1.9}$$

Proof. Let the inequality (1.8) holds true and suppose $|z| = 1$. Then we obtain

$$\left| H\ J_{0,z}^{k,\eta,\delta} f(z)\ -1 \right| - \nu \left| 2\alpha\ H\ J_{0,z}^{k,\eta,\delta} f(z)\ - \beta(1+\theta)\alpha \right|$$
$$= \left| -\sum_{k=2}^{\infty} \phi(k)\ a_k\ z^{k-1} \right| - \nu \left| 2\alpha\ - 2\alpha \sum_{k=2}^{\infty} \phi(k)\ a_k\ z^{k-1}\ - \beta(1+\theta)\alpha \right|$$
$$\leq \sum_{k=2}^{\infty} \phi(k)\ (1+2\nu\alpha)\ a_k\ - \alpha\nu\ (2-\beta(1+\theta))\ \leq\ 0.$$

Hence, by maximum modulus theorem, we conclude that

$$f(z) \in \Lambda(\alpha,\beta,\theta,\nu).$$

Conversely, let $f(z)$, defined by (1.1), be in the class $\Lambda(\alpha,\beta,\theta,\nu)$. So the condition (1.4) yields

$$\left| \frac{H\ J_{0,z}^{k,\eta,\delta} f(z) - 1}{2\alpha\ H\ J_{0,z}^{k,\eta,\delta} f(z) -\ \beta(1+\theta)\alpha} \right| = \left| \frac{\sum_{k=2}^{\infty} \phi(k)\ a_k\ z^{k-1}}{2\alpha - \sum_{k=2}^{\infty} 2\alpha\ \phi(k)\ a_k\ z^{k-1}\ - \beta(1+\theta)\alpha} \right| < \nu.$$

Since for any z, $Re(z) < |z|$, then

$$Re\left\{\frac{\sum_{k=2}^{\infty}\phi(k)\ a_k\ z^{k-1}}{\alpha(2-\beta(1+\theta))-2\alpha\sum_{k=2}^{\infty}\phi(k)\ a_k\ z^{k-1}}\right\} < \nu.$$

By letting $z \to 1$ through real values, we have

$$\sum_{k=2}^{\infty}\phi(k)\ a_k \le \alpha\nu(2-\beta(1+\theta))-2\alpha\sum_{k=2}^{\infty}\phi(k)\ a_k,$$

and this completes the proof. □

2. Main results

In this section we prove a necessary and sufficient condition for functions to be in the class $\Lambda(c_m,\alpha,\beta,\theta,\nu)$ and obtain some important corollaries.

Theorem 2.1. *Let $f(z)$ is defined by* (1.7)*, then $f(z) \in \Lambda(c_m,\alpha,\beta,\theta,\nu)$ if and only if*

$$\sum_{k=n+1}^{\infty}\frac{\phi(k)(1+2\alpha\nu)}{\alpha\nu(2-\beta(1+\theta))}\ a_k < 1-\sum_{m=2}^{\infty}c_m \tag{2.1}$$

Proof. Let

$$a_m = \frac{\alpha\nu(2-\beta(1+\theta))c_m}{\phi(m)(1+2\alpha\nu)} \quad (m=2,...,n). \tag{2.2}$$

Since $\Lambda(c_m,\alpha,\beta,\theta,\nu) \subseteq \Lambda(\alpha,\beta,\theta,\nu)$, so $f(z) \in \Lambda(c_m,\alpha,\beta,\theta,\nu)$ if and only if

$$\sum_{m=2}^{n}\frac{\phi(m)\ (1+2\alpha\nu)}{\alpha\nu\ (2-\beta(1+\theta))}\ a_m + \sum_{k=n+1}^{\infty}\frac{\phi(k)(1+2\alpha\nu)}{\alpha\nu(2-\beta(1+\theta))}\ a_k < 1,$$

or

$$\sum_{k=n+1}^{\infty}\frac{\phi(k)(1+2\alpha\nu)}{\alpha\nu(2-\beta(1+\theta))}\ a_k < 1-\sum_{m=2}^{n}c_m,$$

and this gives the result. □

Corollary 1. *If $f(z)$, defined by* (1.7)*, be in $\Lambda(c_m,\alpha,\beta,\theta,\nu)$, then for $k \ge n+1$ we have*

$$a_k \le \frac{\alpha\nu\ (2-\beta(1+\theta))(1-\sum_{m=2}^{n}c_m)}{\phi(k)(1+2\alpha\nu)}, \tag{2.3}$$

this result is sharp for $G(z)$ defined by

$$G(z) = z - \sum_{m=2}^{n} \frac{\alpha\nu\ (2\ - \beta(1+\theta))c_m}{\phi(m)(1+2\alpha\nu)}\ z^m\ -\ \frac{\alpha\nu\ (2\ - \beta(1+\theta))(1\ - \sum_{m=2}^{n} c_m)}{\phi(k)(1+2\alpha\nu)}\ z^k.$$

Theorem 2.2. *Let*

$$f_j(z) = z - \sum_{m=2}^{n} \frac{\alpha\nu\ (2\ - \beta(1+\theta))c_m}{\phi(m)(1+2\alpha\nu)}\ z^m\ - \sum_{k=n+1}^{\infty} a_{k,j}\ z^k, \tag{2.4}$$

for $j = 1, 2, ..., t$ be in $\Lambda(c_m, \alpha, \beta, \theta, \nu)$, then the function $F(z) = \Sigma_{j=1}^{t} d_j\ f_j(z)$ is also in $\Lambda(c_m, \alpha, \beta, \theta, \nu)$, where

$$\sum_{j=1}^{m} d_j = 1\ \ ,\ \ 0 \le \sum_{m=2}^{n} c_m \le 1\ \ ,\ \ 0 \le c_m \le 1.$$

Proof. By Theorem 2.1 for every $j = 1, 2, ..., t$, we have

$$\sum_{k=n+1}^{\infty} \frac{\phi(k)(1+2\alpha\nu)}{\alpha\nu(2\ - \beta(1+\theta))}\ a_{k,j}\ <\ 1 - \sum_{m=2}^{n} c_m.$$

But

$$F(z) = \sum_{j=1}^{t} d_j\ f_j(z) = z - \sum_{m=2}^{n} \frac{\alpha\nu\ (2\ - \beta(1+\theta))c_m}{\phi(m)(1+2\alpha\nu)}\ -\ \sum_{k=n+1}^{\infty} \left(\sum_{j=1}^{t} d_j\ a_{k_1 j} \right) z^k.$$

So

$$\begin{aligned} \sum_{k=n+1}^{\infty} \frac{\phi(k)(1+2\alpha\nu)}{\alpha\nu(2\ - \beta(1+\theta))} \left(\sum_{j=1}^{t} d_j\ a_{k_1 j} \right) &= \sum_{j=1}^{t} \left(\sum_{k=n+1}^{\infty} \frac{\phi(k)(1+2\alpha\nu)}{\alpha\nu(2\ - \beta(1+\theta))}\ a_{k_1 j} \right) d_j \\ &<\ \sum_{j=1}^{t} \left(1 - \sum_{m=2}^{n} c_m \right) d_j \\ &= 1 - \sum_{m=2}^{n} c_m. \end{aligned}$$

□

Corollary 1. *If $f(z)$ be in the class $\Lambda(c_m, \alpha, \beta, \theta, \nu)$, then the function $F(z) = \frac{1}{2}(f_1(z)\ +\ f_2(z))$ is also in $\Lambda(c_m, \alpha, \beta, \theta, \nu)$.*

Corollary 2. *The class $\Lambda(c_m, \alpha, \beta, \theta, \nu)$ is a convex set.*

3. Extreme points of the class $\Lambda(c_m, \alpha, \beta, \theta, \nu)$

Now we obtain the extreme points of the class $\Lambda(c_m, \alpha, \beta, \theta, \nu)$. The following theorem is required in our investigation.

Theorem 3.1. *Let*

$$f_n(z) = z - \sum_{m=2}^{n} \frac{\alpha\nu\ (2\ -\beta(1+\theta))c_m}{\phi(m)(1+2\alpha\nu)}\ z^m, \tag{3.1}$$

and for $k \geq n+1$,
(3.2)

$$f_k(z) = z - \sum_{m=2}^{n} \frac{\alpha\nu\ (2\ -\beta(1+\theta))c_m}{\phi(m)(1+2\alpha\nu)}\ z^m\ -\ \frac{\alpha\nu\ (2\ -\beta(1+\theta))(1-\sum\limits_{m=2}^{n} c_m)}{\phi(k)(1+2\alpha\nu)}\ z^k.$$

Then the function $F(z)$ *is in* $\Lambda(c_m, \alpha, \beta, \theta, \nu)$ *if and only if it can be expressed in the form*

$$F(z) = \sum_{k=n}^{\infty} d_k\ f_k(z),$$

where $d_k \geq 0\ \ (k \geq n)$ *and* $\sum\limits_{k=n}^{\infty} d_k = 1.$

Proof. Let $F(z) = \sum\limits_{k=n}^{\infty} d_k\ f_k(z)$, then we obtain

$$\begin{aligned} F(z) &= d_n\ f_n(z)\ +\ \sum_{k=n+1}^{\infty} d_k\ f_k(z) \\ &= z - \sum_{m=2}^{n} \frac{\alpha\nu\ (2-\beta(1+\theta))}{\phi(m)(1+2\alpha\nu)} c_m z^m - \sum_{k=n+1}^{\infty} \frac{(1-\sum\limits_{m=2}^{n} c_m)\alpha\nu(2-\beta(1+\theta))}{\phi(k)(1+2\alpha\nu)} d_k\ z^k. \end{aligned}$$

Finally we have

$$\begin{aligned} &\sum_{k=n+1}^{\infty} \frac{\phi(k)(1+2\alpha\nu)(1-\sum\limits_{m=2}^{n} c_m)\ \alpha\nu(2\ -\beta(1+\theta))}{\alpha\nu\ (2\ -\beta(1+\theta))\phi(k)(1+2\alpha\nu)}\ d_k \\ &= (1-\sum_{m=2}^{n} c_m)\Big(\sum_{k=n+1}^{\infty} d_k\Big) \\ &= (1-\sum_{m=2}^{n} c_m)(1-d_n) < 1 - \sum_{m=2}^{n} c_m. \end{aligned}$$

Thus $F(z) \in \Lambda(c_m, \alpha, \beta, \theta, \nu)$.

Conversely suppose $F(z) \in \Lambda(c_m, \alpha, \beta, \theta, \nu)$. So

$$F(z) = z - \sum_{m=2}^{n} \frac{\alpha\nu \ (2 \ - \beta(1+\theta))c_m}{\phi(m)(1+2\alpha\nu)} \ z^m \ - \ \sum_{k=n+1}^{\infty} a_k \ z^k.$$

By putting

$$d_k = \frac{\phi(k)(1+2\alpha\nu)}{\alpha\nu \ (2 \ - \beta(1+\theta))(1 - \sum_{m=2}^{n} c_m)} \ a_k, \qquad (k \geq n+1)$$

we have $d_k \geq 0$ and if we put $d_k = 1 - \sum_{k=n+1}^{\infty} d_k$, we obtain

$$\begin{aligned} F(z) &= z - \sum_{m=2}^{n} \frac{\alpha\nu \ (2-\beta(1+\theta))}{\phi(m)(1+2\alpha\nu)} \ z^m - \sum_{k=n+1}^{\infty} \frac{\alpha\nu \ (2-\beta(1+\theta))(1 - \sum_{m=2}^{n} c_m)}{\phi(k)(1+2\alpha\nu)} \ d_k \ z^k \\ &= f_n(z) \ - \ \sum_{k=n+1}^{\infty} \left(z - \sum_{m=2}^{n} \frac{\alpha\nu \ (2 \ - \beta(1+\theta))c_m}{\phi(m)(1+2\alpha\nu)} \ z^m \ - \ f_k(z) \right) d_k \\ &= f_n(z) \ - \ \sum_{k=n+1}^{\infty} \left(f_n(z) \ - \ f_k(z) \right) d_k \\ &= \left(1 \ - \ \sum_{k=n+1}^{\infty} d_k \right) f_n(z) \ + \ \sum_{k=n+1}^{\infty} d_k \ f_k(z) \\ &= \sum_{k=n}^{\infty} d_k \ f_k(z). \end{aligned}$$

□

Corollary. *The extreme points of the class* $\Lambda(c_m, \alpha, \beta, \theta, \nu)$ *are the functions* $f_k(z)$ $(k \geq n)$ *defined by* (3.1) *and* (3.2).

4. Radii of starlikeness and convexity

In the last section we obtain the radii of starlikeness and convexity for the elements of the class $\Lambda(c_m, \alpha, \beta, \theta, \nu)$.

Theorem 4.1. *Let the function* $f(z)$*, defined by* (1.7)*, be in the class* $\Lambda(c_m, \alpha, \beta, \theta, \nu)$*, then* $f(z)$ *is starlike of order* δ $(0 \leq \delta < 1)$ *in* $|z| < R_1$*, where* R_1 *is the largest* *value such that*

$$\sum_{m=2}^{n} \frac{c_m}{\phi(m)(1+2\alpha\nu)}\, R_1^{m-1} + \frac{1 \;-\; \sum_{m=2}^{n} c_m}{\phi(k)(1+2\alpha\nu)}\, R_1^{k-1} < \frac{1}{\alpha\nu\ (2\ \ -\beta(1+\theta))}.(k \geq n{+}1)$$

Proof. We must show that

$$\left|\frac{z\ f'(z))}{f(z)} - 1\right| \ <\ 1-\delta. \tag{4.1}$$

But

$$\begin{aligned}
&\left|\frac{zf'(z)}{f(z)} - 1\right| \\
&\leq \frac{\sum_{m=2}^{n} \frac{(m-1)\alpha\nu(2-\beta(1+\theta))}{\phi(m)(1+2\alpha\nu)}|z|^{m-1} + \sum_{k=n+1}^{\infty} (k-1)a_k|z|^{k-1}}{1 - \sum_{m=2}^{n} \frac{\alpha\nu(2-\beta(1+\theta))}{\phi(m)(1+2\alpha\nu)}|z|^{m-1} - \sum_{k=n+1}^{\infty} a_k|z|^{k-1}} \\
&< \frac{\sum_{m=2}^{n} \frac{(m-1)\alpha\nu(2-\beta(1+\theta))}{\phi(m)(1+2\alpha\nu)} R_1^{m-1} + \sum_{k=n+1}^{\infty} \frac{(k-1)\alpha\nu(2-\beta(1+\theta))}{\phi(k)(1+2\alpha\nu)} R_1^{k-1}}{1 - \sum_{m=2}^{n} \frac{\alpha\nu(2-\beta(1+\theta))}{\phi(m)(1+2\alpha\nu)} c_m R_1^{m-1} - \sum_{k=n+1}^{\infty} \frac{\alpha\nu(2-\beta(1+\theta))(1-\sum_{m=2}^{n} c_m)}{\phi(k)(1+2\alpha\nu)} R_1^{k-1}}.
\end{aligned}$$

Thus (4.1) holds true if the above term is less than $1-\delta$ or equivalently

$$\sum_{m=2}^{n} \frac{(m-\delta)\alpha\nu(2-\beta(1+\theta))}{\phi(m)(1+2\alpha\nu)} c_m R_1^{m-1} + \sum_{k=n+1}^{\infty} \frac{(k-\delta)\alpha\nu(2-\beta(1+\theta))}{\phi(k)(1+2\alpha\nu)} R_1^{k-1} < 1,$$

or

$$\sum_{m=2}^{n} \frac{c_m}{\phi(m)(1+2\alpha\nu)} R_1^{m-1} + \frac{1-\sum_{m=2}^{n} c_m}{\phi(m)(1+2\alpha\nu)} R_1^{k-1} < \frac{1}{\alpha\nu(2-\beta(1+\theta))},$$

and this completes the proof. □

Since $f(z)$ is convex if and only if zf' is starlike, so we conclude the following corollary.

Corollary . *Let $f(z)$ be in $\Lambda(c_m, \alpha, \beta, \theta, \nu)$, then $f(z)$ is convex of order δ $(0 \leq \delta < 1)$ in $|z| < R_2$, where R_2 is the largest value such that*

$$\sum_{m=2}^{n} \frac{m\ c_m}{\phi(m)(1+2\alpha\nu)}\ R_2^{m-1}\ +\ \frac{k(1-\sum_{m=2}^{n} c_m)}{\phi(m)(1+2\alpha\nu)}\ R_2^{k-1} < \frac{1}{\alpha\nu\ (2\ -\beta(1+\theta))}.$$

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