UNIVARIATE MULTIQUADRIC APPROXIMATION: REPRODUCTION OF LINEAR POLYNOMIALS

by

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Abstract

It is known that multiquadric radial basis function approximations can reproduce low order polynomials when the centres form an infinite regular lattice. We make a start on the interesting question of extending this result in a way that allows the centres to be in less restrictive positions. Specifically, univariate multiquadric approximations are studied when the only conditions on the centres are that they are not bounded above or below. We find that all linear polynomials can be reproduced on $\mathbb{R}$, which is a simple conclusion if the multiquadrics degenerate to piecewise linear functions. Our method of analysis depends on a Peano kernel formulation of linear combinations of second divided differences, a crucial point being that it is necessary to employ differences in order that certain infinite sums are absolutely convergent. It seems that standard methods cannot be used to identify the linear space that is spanned by the multiquadric functions, partly because it is shown that this space provides uniform convergence to any continuous function on any finite interval of the real line.

1. Introduction

A radial basis function approximation has the form

$$s(x) = \sum_j \lambda_j \, \phi(\|x - x_j\|_2), \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $\{x_j\}$ is a fixed set of points in $\mathbb{R}^d$, where each λ_j is a real coefficient, and where ϕ is a fixed function from $\mathbb{R}^+$ to $\mathbb{R}$. Thus s is a linear combination of translates of the spherically symmetric function $\{\phi(\|x\|_2) : x \in \mathbb{R}^d\}$. In general ϕ does not have finite support, the simplest choice being the identity $\{\phi(r) = r : r \geq 0\}$. Hardy (1971) recommends an extension to this choice that avoids first derivative discontinuities, namely the "multiquadric" radial function

$$\phi(r) = \sqrt{r^2 + c^2}, \quad r \geq 0, \tag{1.2}$$

where c is a constant. It is very suitable in several practical calculations, and we are going to study some of its properties.

Much of the recent theory of radial basis function approximation addresses the case when the set of "centres" $\{x_j\}$ is the integer lattice $\mathbf{Z}^d \subset \mathbb{R}^d$. The usefulness of this case in practice is limited by the fact that it is more convenient to employ tensor product spline methods because of the finite support of B-splines, but some of the theoretical properties of multiquadric approximations on the infinite regular grid are stunning. In particular, the linear space of approximating functions, that is a consequence of the freedom in the coefficients $\{\lambda_j\}$, includes all polynomials of degree d. This result was discovered by Jackson (1987) when $c=0$ and d is odd, and it has been extended to all real c and all positive integers d by Buhmann (1988). Their analysis depends strongly on properties of Fourier transforms that are derived from the regular lattice. It seems possible, however, that the ability to reproduce all polynomials of degree d does not require the centres to form a regular grid. This conjecture is highly important to the development of general algorithms for multivariable approximation, because orders of accuracy are closely related to polynomial reproduction, and because radial basis functions are far more versatile than spline functions for fitting data at irregular points. Here we have in mind that the form (1.1) does not require $\mathbb{R}^d$ to be divided into regions, so there are no continuity conditions to be satisfied on interfaces between regions, which can cause severe difficulties when piecewise polynomials are employed.

The answer to the polynomial reproduction conjecture is known when $c=0$ and $d=1$, because in this case expression (1.1) is a piecewise linear function. If the centres are the infinite set $\{x_j : j \in \mathbf{Z}\}$ arranged in strictly ascending order, then, for every j, the set of approximating functions includes the "hat" function

$$\psi_j(x) = \frac{|x-x_{j-1}|}{2(x_j-x_{j-1})} - \frac{(x_{j+1}-x_{j-1})\,|x-x_j|}{2(x_j-x_{j-1})(x_{j+1}-x_j)} + \frac{|x-x_{j+1}|}{2(x_{j+1}-x_j)}, \quad x\in\mathbb{R}, \quad (1.3)$$

which satisfies the "cardinality conditions" $\{\psi_j(x_k)=\delta_{jk} : k\in\mathbf{Z}\}$. Hence the formula

$$s(x) = \sum_{j\in\mathbf{Z}} f(x_j)\,\psi_j(x), \quad x\in\mathbb{R}, \quad (1.4)$$

yields $s=f$ when f is any linear polynomial, provided that the infinite sequences $\{x_j : j=1,2,3,\ldots\}$ and $\{x_{-j} : j=1,2,3,\ldots\}$ both diverge. The generalization of the hat function to multiquadrics is the expression

$$\psi_j(x) = \frac{\phi(|x-x_{j-1}|)}{2(x_j-x_{j-1})} - \frac{(x_{j+1}-x_{j-1})\,\phi(|x-x_j|)}{2(x_j-x_{j-1})(x_{j+1}-x_j)} + \frac{\phi(|x-x_{j+1}|)}{2(x_{j+1}-x_j)}, \quad x\in\mathbb{R}, \quad (1.5)$$

where c is usually nonzero in the definition (1.2). We are going to prove that, for general c, formula (1.4) gives $s=f$ when f is any linear polynomial, without any further restrictions on the positions of the centres $\{x_j : j\in\mathbf{Z}\}$. In other words, Buhmann's polynomial reproduction result for multiquadrics when $d=1$ does not require the spacing between centres to be uniform. Unfortunately, most of our work is confined to the univariate case.

Some care is needed to ensure that each approximating function $\{s(x) : x\in\mathbb{R}\}$ is well-defined. Therefore, whenever we employ expression (1.1) or (1.4), we require the relevant sum to be absolutely convergent for all x. This condition defines an admissible set of approximating functions. The set that stems from the form (1.1) when $d=1$ is studied in Section 2. We find that it does not contain any nonzero polynomials, even when $c=0$. In Section 3, however, the B-spline representation of second order divided differences shows that the form (1.4) yields an admissible set that includes all linear polynomials, but no approximations to quadratic polynomials have uniformly bounded errors. Instead one could seek a reasonable approximation to a quadratic on a finite interval $[a,b]$ of $\mathbb{R}$. A generalization of this question is addressed in Section 4, where it is shown that, if $c\neq 0$, then any continuous function can be approximated uniformly on $[a,b]$, in stark contrast to the cases when either $c=0$ or $[a,b]$ is replaced by $\mathbb{R}$. These results are discussed briefly in Section 5 with an application to the multivariable case.

2. The set of approximations $\mathcal{S}_0$

Throughout the remainder of the paper we let $d=1$, and we let the centres be fixed points that satisfy the conditions that have been stated already, namely that $\{x_j : j \in \mathbf{Z}\}$ is a set of real numbers in strictly ascending order that is unbounded both above and below. Further, ϕ is the multiquadric radial function (1.2), where c is a constant that is allowed to be zero. In this section the coefficients $\{\lambda_j : j \in \mathbf{Z}\}$ of the expression

$$s(x) = \sum_{j=-\infty}^{\infty} \lambda_j \, \phi(|x - x_j|), \quad x \in \mathbb{R}, \tag{2.1}$$

have to satisfy the condition that the sum is absolutely convergent for every x. A convenient form of this restriction is given in the following lemma. We define $\mathcal{S}_0$ to be the set of functions s that are admitted by the condition on $\{\lambda_j\}$, and we are going to investigate whether $\mathcal{S}_0$ includes any nonzero polynomials.

Lemma 1. *The function (2.1) is in $\mathcal{S}_0$ if and only if the sum $\sum_j |\lambda_j x_j|$ is finite.*

Proof: If $s \in \mathcal{S}_0$, then the sum (2.1) is absolutely convergent when $x = 0$, which implies the inequality

$$\sum_{j=-\infty}^{\infty} |\lambda_j \, x_j| \le \sum_{j=-\infty}^{\infty} |\lambda_j \, \phi(|x_j|)| < \infty \tag{2.2}$$

as required. Conversely, if $\sum_j |\lambda_j x_j|$ is the finite number Λ_1, say, then, for any $x \in \mathbb{R}$, we have the relation

$$\begin{aligned} \sum_{j=-\infty}^{\infty} |\lambda_j \, \phi(|x - x_j|)| &\le \sum_{j \in I(x)} |\lambda_j \, \phi(|x - x_j|)| + \sqrt{2} \sum_{j \notin I(x)} |\lambda_j \, (x - x_j)| \\ &\le \sum_{j \in I(x)} |\lambda_j \, \phi(|x - x_j|)| + 2\sqrt{2}\, \Lambda_1, \end{aligned} \tag{2.3}$$

where $I(x)$ is the set

$$I(x) = \{j : |x - x_j| < |c|\} \cup \{j : |x_j| < |x|\}. \tag{2.4}$$

Expression (2.3) is bounded because, for each x, the number of elements in $I(x)$ is finite. Therefore the lemma is true.

It follows from this lemma that $\mathcal{S}_0$ is a linear space. Further, by giving special attention to the finite set $\{j : |x_j| < 1\}$, one can deduce that the sum $\sum_j |\lambda_j| = \Lambda_0$, say, is bounded. The definitions of Λ_0 and Λ_1 imply the inequality

$$|s(x)| \le \sum_{j=-\infty}^{\infty} |\lambda_j| \, (|x - x_j| + |c|) \le (|x| + |c|) \, \Lambda_0 + \Lambda_1, \quad x \in \mathbb{R}. \tag{2.5}$$

Therefore, if s is a polynomial, its degree is at most one.

Now, if s is a nonconstant linear polynomial, then $|s(x)-s(-x)|$ diverges as $x\to\infty$, but $|\phi(|x-x_j|)-\phi(|-x-x_j|)|$ remains finite. This remark provides a relation that excludes linear polynomials from $\mathcal{S}_0$, namely the condition

$$\begin{aligned}
|s(x)-s(-x)| &= \Big|\sum_{j=-\infty}^{\infty}\lambda_j\,[\phi(|x-x_j|)-\phi(|x+x_j|)\,]\,\Big| \\
&\le \sum_{j=-\infty}^{\infty}|\lambda_j|\,\big|\sqrt{(x-x_j)^2+c^2}-\sqrt{(x+x_j)^2+c^2}\big| \\
&= \sum_{j=-\infty}^{\infty}{}' \frac{|\lambda_j|\,|4\,x\,x_j|}{\sqrt{(x-x_j)^2+c^2}+\sqrt{(x+x_j)^2+c^2}} \\
&\le \sum_{j=-\infty}^{\infty}{}' \frac{|\lambda_j|\,|4\,x\,x_j|}{|2\,x|}\ \le\ 2\,\Lambda_1,
\end{aligned}\tag{2.6}$$

where the prime indicates that we drop any zero terms from the sum. It remains to consider whether $s\in\mathcal{S}_0$ can be a nonzero constant function.

In this case s would be bounded. Therefore the elementary inequality

$$\begin{aligned}
|s(x)| &= \Big|\sum_{j=-\infty}^{\infty}\lambda_j\sqrt{(x-x_j)^2+c^2}\,\Big| \\
&= \Big|\sum_{j=-\infty}^{\infty}\lambda_j\,|x-x_j|+\sum_{j=-\infty}^{\infty}\lambda_j\,[\sqrt{(x-x_j)^2+c^2}-|x-x_j|\,]\,\Big| \\
&\ge \Big|\sum_{j=-\infty}^{\infty}\lambda_j\,|x|\,\Big|-\sum_{j=-\infty}^{\infty}|\lambda_j\,x_j|-\sum_{j=-\infty}^{\infty}|\lambda_j\,c| \\
&= |x|\,\Big|\sum_{j=-\infty}^{\infty}\lambda_j\,\Big|-\Lambda_1-|c|\,\Lambda_0
\end{aligned}\tag{2.7}$$

gives $\sum_j\lambda_j=0$, which allows us to write s in the form

$$\begin{aligned}
s(x) &= \sum_{j=-\infty}^{\infty}\lambda_j\,[\sqrt{(x-x_j)^2+c^2}-\sqrt{x^2+c^2}\,] \\
&= \sum_{j=-\infty}^{\infty}{}'\lambda_j\,\frac{-2\,x\,x_j+x_j^2}{\sqrt{(x-x_j)^2+c^2}+\sqrt{x^2+c^2}},\qquad x\in\mathbb{R}.
\end{aligned}\tag{2.8}$$

Letting $x\to-\infty$ and $x\to+\infty$, it follows that $s(x)\to\sum_j\lambda_jx_j$ and $s(x)\to-\sum_j\lambda_jx_j$ respectively, which implies the limit

$$\lim_{x\to\infty}[\,s(x)+s(-x)\,]=0.\tag{2.9}$$

Therefore s is not a nonzero constant function.

These results suggest that multiquadric radial functions are unsuitable for the approximation of nonzero constant and linear polynomials. On the other hand, we know that these polynomials can be reproduced when $c=0$ and for nonzero c when $\{x_j\}$ is the set $\mathbf{Z}$ (Buhmann, 1988). The explanation of this apparent contradiction is that the linear space $\mathcal{S}_0$ is too small to include the most useful multiquadric approximations. Therefore a larger space is found in the next section that repairs this deficiency.

3. The set of approximations $\mathcal{S}_2$

We consider functions of the form

$$s(x) = \sum_{j=-\infty}^{\infty} \mu_j \psi_j(x), \quad x \in \mathbb{R}, \tag{3.1}$$

where each ψ_j is defined by equation (1.5). Therefore s can be regarded as a multiquadric approximation whose centres are still the points $\{x_j : j \in \mathbf{Z}\}$. In order that s is well-defined, we require the coefficients $\{\mu_j : j \in \mathbf{Z}\}$ to have the property that the sum (3.1) is absolutely convergent for every x. In this case we say that s is in the set $\mathcal{S}_2$.

We recall that, when $c=0$, ψ_j is the hat function (1.3). Then $s \in \mathcal{S}_2$ imposes no restriction on the coefficients $\{\mu_j\}$ because, for each x, there are at most two nonzero terms in the sum (3.1). In particular, $\mathcal{S}_2$ includes the piecewise linear interpolant (1.4) that satisfies $s=f$ when f is any linear polynomial, which answers the question that is raised in the final paragraph of Section 2. The remainder of this section, however, addresses the more interesting case when c is nonzero, so the multiquadric radial function $\{\phi(r)=(r^2+c^2)^{1/2} : r \in \mathbb{R}\}$ has continuous second derivatives. It is important to note that we can write $\phi(r)$ instead of $\phi(|r|)$ when r is negative.

Our method of analysis requires the function (3.1) to be expressed in terms of ϕ''. We use the B-spline representation of second divided differences

$$\frac{g(x_{j-1})}{(x_j-x_{j-1})} - \frac{(x_{j+1}-x_{j-1})\,g(x_j)}{(x_j-x_{j-1})(x_{j+1}-x_j)} + \frac{g(x_{j+1})}{(x_{j+1}-x_j)} = \int_{-\infty}^{\infty} B_j(\theta)\, g''(\theta)\, d\theta, \tag{3.2}$$

where g is any twice differentiable function and where B_j is the hat function

$$B_j(\theta) = \frac{|\theta-x_{j-1}|}{2(x_j-x_{j-1})} - \frac{(x_{j+1}-x_{j-1})\,|\theta-x_j|}{2(x_j-x_{j-1})(x_{j+1}-x_j)} + \frac{|\theta-x_{j+1}|}{2(x_{j+1}-x_j)}, \quad \theta \in \mathbb{R}, \tag{3.3}$$

satisfying the normalization condition $B_j(x_j)=1$ as in equation (1.3). It is straightforward to establish the identity (3.2) by integration by parts (see

Powell, 1981, for instance), the actual range of integration on the right hand side being $[x_{j-1}, x_{j+1}]$ because of the finite support of B_j. We make the choice $\{g(t) = \phi(x-t) : t \in \mathbb{R}\}$ for each $x \in \mathbb{R}$, in order to deduce from equation (3.2) that the definition (1.5) can be written in the form

$$\psi_j(x) = \tfrac{1}{2} \int_{-\infty}^{\infty} B_j(\theta)\, \phi''(x-\theta)\, d\theta, \quad x \in \mathbb{R}. \tag{3.4}$$

Since analytic differentiation of the multiquadric radial function provides the values

$$\left.\begin{aligned} \phi'(r) &= r/\sqrt{r^2+c^2} \\ \phi''(r) &= c^2/(r^2+c^2)^{3/2} \end{aligned}\right\}, \quad r \in \mathbb{R}, \tag{3.5}$$

we have the expression

$$\psi_j(x) = \tfrac{1}{2} c^2 \int_{-\infty}^{\infty} \frac{B_j(\theta)}{[\,(x-\theta)^2 + c^2\,]^{3/2}}\, d\theta. \tag{3.6}$$

Thus ψ_j is positive and the need for the assumption $c \neq 0$ is clear.

Now, when $s \in \mathcal{S}_2$, the sum (3.1) is absolutely convergent. Therefore, because the integrand of expression (3.6) is nonnegative, we can express s in the form

$$s(x) = \tfrac{1}{2} c^2 \int_{-\infty}^{\infty} \frac{K(\theta)}{[\,(x-\theta)^2 + c^2\,]^{3/2}}\, d\theta, \quad x \in \mathbb{R}, \tag{3.7}$$

where K is the function

$$K(\theta) = \sum_{j=-\infty}^{\infty} \mu_j B_j(\theta), \quad \theta \in \mathbb{R}. \tag{3.8}$$

Remembering the definition (3.3) of B_j, we see that K is just the piecewise linear interpolant to the coefficients $\{\mu_j : j \in \mathbf{Z}\}$ at the centres $\{x_j : j \in \mathbf{Z}\}$. This observation is only a small step away from our main conclusion that the set $\mathcal{S}_2$ contains all linear polynomials, once we have proved a result that is analogous to Lemma 1.

Lemma 2. *The function (3.1) is in the set $\mathcal{S}_2$ if and only if the function*

$$\hat{K}(\theta) = \sum_{j=-\infty}^{\infty} |\mu_j|\, B_j(\theta)\, (\theta^2+c^2)^{-3/2}, \quad \theta \in \mathbb{R}, \tag{3.9}$$

is absolutely integrable.

Proof: If $s \in \mathcal{S}_2$, then the sum (3.1) is absolutely convergent when $x=0$. It follows from equation (3.6) that we have the bound

$$\tfrac{1}{2} c^2 \int_{-\infty}^{\infty} \hat{K}(\theta)\, d\theta = \sum_{j=-\infty}^{\infty} |\mu_j|\, \psi_j(0) < \infty, \tag{3.10}$$

which shows the absolute integrability of $\hat{K}$. Conversely, if $\int |\hat{K}(\theta)|\, d\theta$ is the finite number Λ_3, say, then for any $x\in\mathbb{R}$ we find the inequality

$$\begin{aligned}\sum_{j=-\infty}^{\infty} |\mu_j\, \psi_j(x)| &= \tfrac{1}{2}\, c^2 \int_{-\infty}^{\infty} \frac{\sum_j |\mu_j|\, B_j(\theta)}{[\,(x-\theta)^2 + c^2\,]^{3/2}}\, d\theta \\ &\leq \tfrac{1}{2}\, c^2 \int_{J(x)} \frac{\sum_j |\mu_j|\, B_j(\theta)}{[\,(x-\theta)^2 + c^2\,]^{3/2}}\, d\theta + \sqrt{2}\, c^2 \Lambda_3, \qquad (3.11)\end{aligned}$$

where $J(x)$ is the set of values of θ that satisfy the condition

$$(x-\theta)^2 + c^2 \leq \tfrac{1}{2}\, (\theta^2 + c^2). \qquad (3.12)$$

Since $J(x)$ is bounded for each x, it follows that the right hand side of expression (3.11) is finite for each x, which gives the required result.

The characterization of the functions in $\mathcal{S}_2$ that is provided by the lemma is more convenient than the original definition of $\mathcal{S}_2$. Moreover, the lemma implies that $\mathcal{S}_2$ is a linear space. It is now straightforward to prove the principal results of this section, which are stated in two theorems.

Theorem 3. *If the coefficients of expression (3.1) have the values $\{\mu_j = 1 : j \in \mathbb{Z}\}$, then s is in the linear space $\mathcal{S}_2$, and it is the constant function $\{s(x) = 1 : x \in \mathbb{R}\}$. In other words the functions $\{\psi_j : j \in \mathbb{Z}\}$, which are positive because of equation (3.6), form a partition of unity.*

Proof: The values $\{\mu_j = 1 : j \in \mathbb{Z}\}$ and the definition (3.9) imply that $\hat{K}$ is the absolutely integrable function $\{(\theta^2 + c^2)^{-3/2} : \theta \in \mathbb{R}\}$, so s is in $\mathcal{S}_2$. Further, equations (3.7) and (3.8) and some elementary algebra yield the identity

$$\begin{aligned} s(x) &= \tfrac{1}{2}\, c^2 \int_{-\infty}^{\infty} [\,(x-\theta)^2 + c^2\,]^{-3/2}\, d\theta \\ &= \tfrac{1}{2}\, c^2 \int_{-\infty}^{\infty} (\theta^2 + c^2)^{-3/2}\, d\theta \\ &= \tfrac{1}{2} \left[\theta / \sqrt{\theta^2 + c^2} \right]_{-\infty}^{\infty} \\ &= 1, \quad x \in \mathbb{R}. \qquad (3.13)\end{aligned}$$

Therefore the theorem is true.

Theorem 4. *If f is any linear polynomial, then formula (1.4) yields a function in $\mathcal{S}_2$, and we have $s = f$.*

Proof: Let f be the polynomial $\{a\, x + b : x \in \mathbb{R}\}$, so formula (1.4) gives the coefficients $\{\mu_j = a\, x_j + b : j \in \mathbb{Z}\}$. It follows that the function (3.9) is absolutely integrable, which implies $s \in \mathcal{S}_2$. In this case expression (3.8) has the value $\{K(\theta)=a\,\theta+b : \theta\in\mathbb{R}\}$. Therefore equation (3.7) leads to the relation

$$\begin{aligned} s(x) &= \tfrac{1}{2}\,c^2 \int_{-\infty}^{\infty} (a\,\theta+b)\,[\,(x-\theta)^2+c^2\,]^{-3/2}\,d\theta \\ &= \tfrac{1}{2}\,c^2 \int_{-\infty}^{\infty} [\,(a\,x+b)-a\,\theta\,]\,(\theta^2+c^2)^{-3/2}\,d\theta \\ &= a\,x+b, \quad x\in\mathbb{R}, \end{aligned} \tag{3.14}$$

where the last line is a consequence of the identity (3.13) and the fact that the integral of an odd function is zero. The proof is complete.

Letting i be any integer, the choice of coefficients $\{\mu_j=|x_j-x_i| : j\in\mathbf{Z}\}$ in equation (3.1) is also remarkable. Lemma 2 states that they give an element of $\mathcal{S}_2$, and a straightforward calculation shows that it is the function $\{s(x)=[\,(x-x_i)^2+c^2\,]^{1/2}=\phi(|x-x_i|) : x\in\mathbb{R}\}$. Further, if we have the values

$$\mu_j = \sum_{i=-\infty}^{\infty} \lambda_i\,|x_j-x_i|, \quad j\in\mathbf{Z}, \tag{3.15}$$

where the sums $\sum_i|\lambda_i|$ and $\sum_i|\lambda_i x_i|$ are absolutely convergent, then s remains in $\mathcal{S}_2$ and it is expression (2.1). Therefore the set $\mathcal{S}_0$ that is studied in Section 2 is a linear subspace of $\mathcal{S}_2$.

The final result of this section is that the elements of $\mathcal{S}_2$ are unsuitable for the approximation of quadratic polynomials. Indeed, the statement of the following lemma shows that the average value of an element of $\mathcal{S}_2$ on the interval $[0,M]$ becomes much less than the average value of a positive quadratic polynomial on the same interval as $M\to\infty$.

Lemma 5. *If s is any element of $\mathcal{S}_2$, then it satisfies the condition*

$$\lim_{M\to\infty} M^{-3}\int_0^M s(x)\,dx = 0. \tag{3.16}$$

Proof: Equation (3.7) provides the identity

$$\int_0^M s(x)\,dx = \tfrac{1}{2}\,c^2 \int_{x=0}^{M}\int_{\theta=-\infty}^{\infty} \frac{K(\theta)}{[\,(x-\theta)^2+c^2\,]^{3/2}}\,d\theta\,dx, \tag{3.17}$$

and Lemma 2 gives the condition

$$\int_{-\infty}^{\infty} \frac{|K(\theta)|}{(\theta^2+c^2)^{3/2}}\,d\theta = \Delta \tag{3.18}$$

for some finite number Δ. When θ satisfies $|\theta| \geq 2M$ in equation (3.17), we deduce from $0 \leq x \leq M$ that the denominator $[(x-\theta)^2+c^2]^{3/2}$ is bounded below by $[\frac{1}{4}\theta^2+\frac{1}{4}c^2]^{3/2}$. Therefore expressions (3.17) and (3.18) imply the inequality

$$|\int_0^M s(x)\,dx| \leq \tfrac{1}{2}c^2\,[\,|\int_{x=0}^M \int_{\theta=-2M}^{2M} \frac{K(\theta)}{[(x-\theta)^2+c^2]^{3/2}}\,d\theta\,dx\,| + 8\,M\,\Delta\,]. \tag{3.19}$$

By changing the order of integration and making use of the elementary relations

$$0 \leq \tfrac{1}{2}c^2 \int_0^M \frac{1}{[(x-\theta)^2+c^2]^{3/2}}\,dx \leq 1, \tag{3.20}$$

we deduce the bound

$$|\int_0^M s(x)\,dx\,| \leq \int_{-2M}^{2M} |K(\theta)|\,d\theta + 4\,M\,\Delta\,c^2. \tag{3.21}$$

For any positive constant ϵ that satisfies $\epsilon < \Delta$, we define T by the equation

$$\int_{-T}^{T} \frac{|K(\theta)|}{(\theta^2+c^2)^{3/2}}\,d\theta = \Delta - \epsilon. \tag{3.22}$$

If $2M \leq T$, we are going to employ the inequality

$$\begin{aligned}\int_{-2M}^{2M} |K(\theta)|\,d\theta &\leq (T^2+c^2)^{3/2} \int_{-T}^{T} \frac{|K(\theta)|}{(\theta^2+c^2)^{3/2}}\,d\theta \\ &= (T^2+c^2)^{3/2}\,(\Delta-\epsilon),\end{aligned} \tag{3.23}$$

while, if $2M > T$, we use the relation

$$\begin{aligned}\int_{-2M}^{2M} |K(\theta)|\,d\theta &\leq (T^2+c^2)^{3/2}\,(\Delta-\epsilon) + \int_{-2M}^{-T} |K(\theta)|\,d\theta + \int_{T}^{2M} |K(\theta)|\,d\theta \\ &\leq (T^2+c^2)^{3/2}\,(\Delta-\epsilon) + (4\,M^2+c^2)^{3/2} \int_{|\theta|\geq T} \frac{|K(\theta)|}{(\theta^2+c^2)^{3/2}}\,d\theta \\ &= (T^2+c^2)^{3/2}\,(\Delta-\epsilon) + (4\,M^2+c^2)^{3/2}\,\epsilon.\end{aligned} \tag{3.24}$$

It follows from condition (3.21) that we have the bound

$$\frac{|\int_0^M s(x)\,dx|}{M^3} \leq \frac{(T^2+c^2)^{3/2}\,(\Delta-\epsilon)}{M^3} + \frac{(4\,M^2+c^2)^{3/2}\,\epsilon}{M^3} + \frac{4\,\Delta\,c^2}{M^2}. \tag{3.25}$$

The right hand side tends to 8ϵ as $M \to \infty$. Hence the lemma is a consequence of the fact that the positive constant ϵ can be arbitrarily small.

4. Uniform convergence on compact intervals

Lemma 5 shows that, if $s \in \mathcal{S}_2$ and if f is a quadratic polynomial, then the error

$$\|f-s\|_\infty = \sup\{|f(x)-s(x)| : x \in \mathbb{R}\} \tag{4.1}$$

is unbounded. In practice, however, we may require an approximation to $f(x)$ only for values of x that satisfy $a \le x \le b$ for finite a and b. It would be usual in this case to choose centres $\{x_i\}$ that lie in or near the interval $[a,b]$, but we continue to assume in this section that the centres have the properties that are summarised at the beginning of Section 2. We let f be any continuous function, and we seek the least maximum value of $|f-s|$ on $[a,b]$, where s has the form

$$s(x) = \sum_{j=-\infty}^{\infty} \lambda_j \, \phi(|x-x_j|), \quad x \in \mathbb{R}, \tag{4.2}$$

so we are reverting to the set $\mathcal{S}_0$. Actually, absolute convergence of the sum is not a problem, because we restrict attention to cases where the number of nonzero parameters $\{\lambda_j\}$ is finite.

When $c=0$, equation (4.2) allows s to be any continuous piecewise linear function whose knots are the points of the set $\{x_j : j \in \mathbf{Z}\} \cap [a,b]$. Therefore the least maximum value of the error when f is a quadratic polynomial, for example, is at least $h^2|f''|/16$, where h is the greatest distance between adjacent knots. In the more interesting case when c is nonzero, however, all the centres $\{x_j : j \in \mathbf{Z}\}$ are able to influence the form of s on $[a,b]$, so the scope of the approximating functions $\{s(x) : a \le x \le b\}$ is much greater than before. Indeed, the main result of this section is that continuous functions can be approximated to arbitrarily high accuracy on compact intervals.

This assertion is deduced from the Taylor series expansion of the function $\{\phi(|x-x_j|) : a \le x \le b\}$ when x_j is large and positive. We write the expansion in the form

$$\begin{aligned} \phi(|x-x_j|) &= x_j \left[1 - \frac{2x}{x_j} + \frac{x^2+c^2}{x_j^2} \right]^{1/2} \\ &= x_j \left[1 + \frac{\alpha_0(x)}{x_j} + \frac{\alpha_1(x)}{x_j^2} + \frac{\alpha_2(x)}{x_j^3} + \cdots \right], \quad a \le x \le b, \end{aligned} \tag{4.3}$$

each function $\{\{\alpha_\ell(x) : a \le x \le b\} : \ell = 0,1,2,\ldots\}$ being independent of x_j. Further, it follows from the coefficient of $x_j^{-\ell-1}$ in the identity

$$1 - \frac{2x}{x_j} + \frac{x^2+c^2}{x_j^2} = \left[1 + \frac{\alpha_0(x)}{x_j} + \frac{\alpha_1(x)}{x_j^2} + \frac{\alpha_2(x)}{x_j^3} + \cdots \right]^2 \tag{4.4}$$

that α_ℓ is defined by the formula

$$\alpha_\ell(x) = \tfrac{1}{2}\,[\,\beta_\ell(x) - \sum_{i=0}^{\ell-1} \alpha_i(x)\,\alpha_{\ell-i-1}(x)\,], \quad a \le x \le b, \tag{4.5}$$

where β_ℓ is the function

$$\beta_\ell(x) = \left\{ \begin{array}{cc} -2x, & \ell=0 \\ x^2+c^2, & \ell=1 \\ 0, & \ell\ge 2 \end{array} \right\}, \quad a \le x \le b. \tag{4.6}$$

Thus we find the expressions

$$\left. \begin{array}{ll} \alpha_0(x) = -x, & \alpha_1(x) = \tfrac{1}{2}\,c^2 \\ \alpha_2(x) = \tfrac{1}{2}\,c^2 x, & \alpha_3(x) = \tfrac{1}{2}\,c^2 x^2 - \tfrac{1}{8}\,c^4 \end{array} \right\}, \tag{4.7}$$

and we deduce by induction that α_ℓ is a polynomial of degree exactly $\ell-1$ for all $\ell \ge 2$, still assuming that c is nonzero.

We let ℓ be any positive integer, we let $x_{j(1)}$ be one of the centres $\{x_j : j \in \mathbf{Z}\}$ that is very large and positive, and we pick $\ell+1$ more centres $\{x_{j(k)} : k = 2,3,\ldots,\ell+2\}$ say, that satisfy the conditions $\{x_{j(k)} \ge 2x_{j(k-1)} : k=2,3,\ldots,\ell+2\}$, which is allowed by our assumptions for any choice of $x_{j(1)}$. The purpose of this construction is that the square system of equations

$$\sum_{k=1}^{\ell+2} w_k\, x_{j(k)}^{-t} = \delta_{t\ell}, \quad t=-1,0,\ldots,\ell, \tag{4.8}$$

in the coefficients $\{w_k : k=1,2,\ldots,\ell+2\}$ not only has a unique solution but also the products $\{w_k\, x_{j(k)}^{-\ell} : k=1,2,\ldots,\ell+2\}$ are uniformly bounded, which is a consequence of the explicit formula

$$w_k = x_{j(k)}^{\ell} \prod_{i=1,\, i\ne k}^{\ell+2} [\,1 - x_{j(k)}/x_{j(i)}\,]^{-1}, \quad k=1,2,\ldots,\ell+2. \tag{4.9}$$

Therefore equations (4.3) and (4.8) provide the relation

$$\sum_{k=1}^{\ell+2} w_k\, \phi(|x-x_{j(k)}|) = \alpha_\ell(x) + \mathcal{O}(x_{j(1)}^{-1}), \quad a \le x \le b. \tag{4.10}$$

Since $x_{j(1)}$ can be arbitrarily large and since ℓ can be any positive integer, it follows that all the functions $\{\alpha_\ell : \ell=1,2,3,\ldots\}$ can be approximated on $[a,b]$ to arbitrarily high accuracy from the linear space $\mathcal{S}_0$. Further, we recall from the construction of expression (4.7) that α_ℓ is a polynomial of degree exactly $\ell-1$. Therefore the uniform approximation of continuous functions on $[a,b]$ from $\mathcal{S}_0$ is a consequence of the well-known Weierstrass theorem. We state this conclusion formally.

Theorem 6. *Let $[a,b]$ be any finite interval of $\mathbb{R}$ and let $\mathcal{S}_0$ be the linear space that is defined in Section 2, the multiquadric parameter c being nonzero. Then, for any continuous function $\{f(x) : a \le x \le b\}$, there exists an approximation $s \in \mathcal{S}_0$ that satisfies the condition $\{|f(x)-s(x)| \le \epsilon : a \le x \le b\}$, where ϵ is any positive constant.*

5. Conclusions

When this work was begun, the main aim was to identify conditions on the centres $\{x_j : j \in \mathbf{Z}\}$ that would allow the reproduction of linear polynomials. It is therefore pleasing that our only requirement is that the centres are not bounded above or below. A crude explanation is contained in the remark that, if one seeks the multipliers of each of the functions $\{\phi(|x-x_i|) : i \in \mathbf{Z}\}$ in formula (1.4) when f is a linear polynomial, one finds that every multiplier is zero. Specifically, this remark shows that all centres at finite points are irrelevant, but it is debatable whether a function of the form (2.1) with zero parameters $\{\lambda_j : j \in \mathbf{Z}\}$ can be a nonzero linear polynomial. The introduction of the set $\mathcal{S}_2$ in Section 3 provides a satisfactory answer to this difficulty, and it suggests the use of B-splines that gives the main results.

The construction of spaces by taking linear intermediate combinations of the original radial functions is inelegant, however, and instead it would be usual to pick a metric space of real valued continuous functions, and to let the set of approximations be the closure of the span of the elements $\{\,\{\phi(|x-x_j|) : x \in \mathbb{R}\} : j \in \mathbf{Z}\}$. Two difficulties here, however, are the presence of unbounded functions and the fact that, due to Theorem 6, our approximating functions are dense in some of the usual choices of metric spaces. Therefore it is hard to decide what is meant by the term "reproduction of linear polynomials". Our approach seems to provide suitable results, but it may be useful to generalize $\mathcal{S}_2$ to a set of functions of the form

$$s(x) = \sum_{j=-\infty}^{\infty} \nu_j \, \chi_j(x), \quad x \in \mathbb{R}, \tag{5.1}$$

where each χ_j is in $\mathcal{S}_2$ and where the sum (5.1) is absolutely convergent for every x. In particular, when the set of centres is the integer grid $\{x_j = j : j \in \mathbf{Z}\}$, one can let the basis functions be the cardinal functions of interpolation that satisfy $\{\chi_j(\ell) = \delta_{j\ell} : j, \ell \in \mathbf{Z}\}$. Because $|\chi_j(x)|$ decays like $|x|^{-5}$ as $|x|$ becomes large [1], it is admissible to set $\{\nu_j = f(x_j) : j \in \mathbf{Z}\}$ in formula (5.1) when f is any quadratic polynomial. In this case it can be shown that $\|f-s\|_\infty$ is bounded. Therefore, in view of Lemma 5, the linear space $\mathcal{S}_2$

does not include all the well-defined multiquadric approximations that are of fundamental importance.

Theorem 6 explains some unfortunate numerical experiments that were described to me by Philip Smith of IMSL. The problem was to adjust both the centres and the coefficients of a multiquadric approximation in a least squares fitting calculation, the total number of centres being given. In fact the data could be fitted rather well by a low order polynomial. Therefore the centres diverged steadily away from the data points in order to provide the dominant terms of the Taylor series expansions that occur in Section 4, except that there were two variables, each x and each centre being a point in $\mathbb{R}^2$.

Uniqueness has not been mentioned so far, but it raises some interesting questions. In particular, if the function (3.1) is identically zero, each sum being absolutely convergent as usual, does it follow that every μ_j is zero? Nira Dyn (private communication) has proved this conjecture when there are certain bounds on maximum and minimum distances between adjacent centres. I have acquired a preference for the original conditions on the centres, however, and have shown that, if $s \in \mathcal{S}_0$ is zero, then the coefficients $\{\lambda_j : j \in \mathbf{Z}\}$ must vanish in expression (2.1). Further, Nira Dyn (private communication) has extended the work of Section 3 to the radial function

$$\phi(r) = (r^2+c^2)^{k+1/2}, \quad r \in \mathbb{R}, \tag{5.2}$$

where k is any nonnegative constant integer. She has discovered some excellent polynomial reproduction properties under the original conditions on the centres.

It was hoped that studying the univariate case would provide some ideas for investigating polynomial reproduction properties when there are two or more variables. It seems, however, that our main contribution to this important problem is that we have sustained the conjecture that there is no need for the centres to form a regular grid. Further, the fact that Theorem 4 is valid for all c provides a way of constructing some constant and linear polynomials in higher dimensions. Specifically, if a straight line in $\mathbb{R}^d$ contains a set of centres $\{x_j\}$ that is not bounded above or below, it is suitable to change the definition (1.5) to the expression

$$\psi_j(x) = \frac{\phi(\|x-x_{j-1}\|_2)}{2\|x_j-x_{j-1}\|_2} - \frac{\|x_{j+1}-x_{j-1}\|_2\phi(\|x-x_j\|_2)}{2\|x_j-x_{j-1}\|_2\|x_{j+1}-x_j\|_2} + \frac{(\phi\|x-x_{j+1}\|_2)}{2\|x_{j+1}-x_j\|_2}, \quad x \in \mathbb{R}^d. \tag{5.3}$$

Then, if f is constant or linear, formula (1.4) gives the polynomial $\{s(x) = f(Px) : x \in \mathbb{R}^d\}$, where Px is the orthogonal projection of x onto the line of centres. Thus d independent lines of centres can yield all linear polynomials in d variables, but further research has indicated that these conditions on the centres are stronger than necessary.

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