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UNIVERSAL DEFORMATION FORMULAE, SYMPLECTIC LIE GROUPS AND SYMMETRIC SPACES

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We define a class of symplectic Lie groups associated with solvable symmetric spaces. We give a universal strict deformation formula for every proper action of such a group on a smooth manifold. We define a functional space where performing an asymptotic expansion of the nonformal deformed product in powers of the deformation parameter yields an associative formal star product on the symplectic Lie group at hand. The cochains of the star product are explicitly given (without recursion) in the two-dimensional case of the affine group $ax + b$. The latter differs from the Giaquinto–Zhang construction, as shown by analyzing the invariance groups. In a Hopf algebra context, the above formal star product is shown to be a smash product and a compatible coproduct is constructed.

1. Introduction

The concept of universal deformation for abelian Lie group actions was introduced by Rieffel [1993] in the operator algebraic context. Later, the notion of a universal deformation formula (UDF) within a Hopf algebraic context was defined by Giaquinto and Zhang [1998] at the formal level. Here we study a similar notion within the framework of (nonabelian) solvable Lie group actions. Our construction of UDFs relies on both formal and nonformal aspects. The Lie groups considered here are symplectic semidirect products of abelian Lie groups, natural generalizations of the two-dimensional affine group $ax + b$. On each such group $\mathbb{S}$, we define a function space $\mathbb{A} \subset C^\infty(\mathbb{S})$ invariant under the left regular representation and endowed with a one-parameter family of products $\{\star_\theta\}_{\theta \in \mathbb{R}}$ in such a way that each pair $(\mathbb{A}, \star_\theta)$ is an associative (topological) algebra which the group $\mathbb{S}$ naturally acts on by automorphisms. As observed in [Bieliavsky 2002; Bieliavsky et al. 2003], such a data provides a UDF in the following way. Let M denote

a smooth manifold which the group $\mathbb{S}$ acts on by diffeomorphisms. Denote by $\alpha : \mathbb{S} \times C^\infty(M) \to C^\infty(M) : (s, u) \mapsto \alpha_s(u)$ the induced action at the level of functions and set $\alpha^x(u)(s) := \alpha_s(u)(x)$ for $u \in C^\infty(M)$, $x \in M$ and $s \in \mathbb{S}$. Then, the function space $\mathbb{B} := \{u \in C^\infty(M) \text{ such that } \alpha^x(u) \in \mathbb{A} \text{ for all } x \in M\}$ is naturally endowed with a one-parameter family of associative products $\{\star_\theta^M\}$ defined by the formula $u \star_\theta^M v(x) := (\alpha^x(u) \star_\theta \alpha^x(v))(e)$ (e denotes the identity element in $\mathbb{S}$). The problem is of course to show that the space $\mathbb{B}$ is sufficiently rich, in the sense that it contains at least the smooth compactly supported functions on M. This is the case for the class of groups $\mathbb{S}$ considered here provided the action of $\mathbb{S}$ on M is proper. Next, comes the question of defining an appropriate functional framework allowing to pass from our nonformal setting to the formal framework of star products [Bayen et al. 1978a; 1978b]. The above-mentioned nonformal universal deformation formulae are of the oscillatory — or WKB — type. This means that the product $a \star_\theta b$ is defined by an integral expression $a \star_\theta b = \int_{\mathbb{S} \times \mathbb{S}} K_\theta \, a \otimes b$ where the kernel K_θ has the oscillatory form $K_\theta = \theta^{-m} \mathbf{a}_\theta \, e^{(i/\theta)S}$, where $\mathbf{a}_\theta$ and S belong to $C^\infty(\mathbb{S} \times \mathbb{S} \times \mathbb{S}, \mathbb{R})$. In particular, for a and b smooth compactly supported, one may perform a stationary phase expansion of $a \star_\theta b$ in powers of θ yielding a formal product on $C^\infty(\mathbb{S})[\![\theta]\!]$. The property of associativity of this formal product depends on functional properties of the nonformal algebras $(\mathbb{A}, \star_\theta)$.

In Section 2, we prove that every exact symplectic semidirect product $\mathbb{S}$ of two abelian groups always acts strictly transitively on an elementary solvable symplectic symmetric space in the sense of [Bieliavsky 2002]. Conversely, we show that when complex every such space gives rise to an exact symplectic semidirect product of two abelian groups. This allows to identify in a $\mathbb{S}$-equivariant manner the manifolds underlying the group $\mathbb{S}$ and the corresponding symmetric space.

In Section 3, we observe that Section 2 together with the construction in [Bieliavsky 2002] of nonformal quantizations on elementary solvable symplectic symmetric spaces yield on every such semidirect product $\mathbb{S}$ a left-invariant nonformal deformation quantization. As an immediate consequence, the latter gives rise to strict deformation quantizations for the proper actions of $\mathbb{S}$ on smooth manifolds.

In Section 4, restricting to the case $\dim \mathbb{S} = 2$, we associate to our nonformal deformation an associative left-invariant formal star product on $C^\infty(\mathbb{S})[\![\theta]\!]$ for which we give the cochains totally explicitly (without any recursion). This is essentially done by defining a θ-independent functional space (denoted hereafter $\mathscr{E}$) closed under our one-parameter family of nonformal deformations $\{\star_\theta\}$ and by studying the smoothness (in a suitable sense) of our algebras $\{(\mathscr{E}, \star_\theta)\}_\theta$. For nonzero values of θ, the space $\mathscr{E}$ plays an analogous role as the Schwartz space does in the case of Weyl's quantization. As it clearly appears, the restriction to the dimension two is inessential.

In Section 5, we show that the above mentioned formal star product can be seen as a smash product of $C^\infty(\mathbb{R})[\![\theta]\!]$ and the Hopf algebra of polynomials on $\mathbb{R}$. From the latter, we deduce a compatible formal coproduct and a Hopf structure on $C^\infty(\mathbb{S})[\![\theta]\!]$.

2. Symplectic Lie algebras associated to a class of symmetric spaces

Definition 2.1. Following the terminology of [Lichnerowicz and Medina 1988], a *symplectic Lie algebra* is a pair $(\mathfrak{s}, \omega)$ where $\mathfrak{s}$ is a Lie algebra and $\omega \in \bigwedge^2(\mathfrak{s}^\star)$ is a nondegenerate Chevalley two-cocycle with respect to the trivial representation of $\mathfrak{s}$.

In this section, we associate symplectic Lie algebras to a class of (infinitesimal) symplectic symmetric spaces.

Definition 2.2 [Bieliavsky et al. 1995; Bieliavsky 1995]. Let $(\mathfrak{g}, \sigma)$ be an *involutive algebra*, meaning that $\mathfrak{g}$ is a finite dimensional real Lie algebra and σ is an involutive automorphism of $\mathfrak{g}$. Let Ω be a skewsymmetric bilinear form on $\mathfrak{g}$. Then the triple $(\mathfrak{g}, \sigma, \Omega)$ is called a *symplectic triple* if the following properties are satisfied.

(i) Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ (resp. $\mathfrak{p}$) is the $+1$ (resp. -1) eigenspace of σ. Then $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ and the representation of $\mathfrak{k}$ on $\mathfrak{p}$, given by the adjoint action, is faithful.

(ii) Ω is a Chevalley 2-cocycle for the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$ such that for any X in $\mathfrak{k}$, $i(X)\Omega = 0$. Moreover, the restriction of Ω to $\mathfrak{p} \times \mathfrak{p}$ is nondegenerate.

Two such triples $(\mathfrak{g}_i, \sigma_i, \Omega_i)$ $(i = 1, 2)$ are *isomorphic* if there exists a Lie algebra isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi \circ \sigma_1 = \sigma_2 \circ \psi$ and $\psi^*\Omega_2 = \Omega_1$.

Such a triple is called *indecomposable* if it cannot be expressed as a direct sum of symplectic triples.

Definition 2.3. A symplectic triple $t = (\mathfrak{g}, \sigma, \Omega)$ is called *holonomy isotropic*, or HI, if $[\mathfrak{k}, \mathfrak{p}]$ is an isotropic subspace of $(\mathfrak{p}, \Omega)$.

Proposition 2.1 [Bieliavsky 1998]. *A symplectic triple $t = (\mathfrak{g}, \sigma, \Omega)$ is holonomy isotropic if and only if $[\mathfrak{g}, \mathfrak{g}]$ is abelian.*

Definition 2.4. Let $t = (\mathfrak{g}, \sigma, \Omega)$ be HI and consider the extension sequence

$$(1) \qquad\qquad 0 \to [\mathfrak{g}, \mathfrak{g}] \to \mathfrak{g} \to \mathfrak{a} := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to 0.$$

The HI triple t is called *split* if this extension is split.

Lemma 2.2. *Let $t = (\mathfrak{g}, \sigma, \Omega)$ be HI split. Set $\mathfrak{b} = [\mathfrak{g}, \mathfrak{g}]$ and denote by $\rho : \mathfrak{a} \to$ End($\mathfrak{b}$) the splitting homomorphism. Then, realizing $\mathfrak{g}$ as the semidirect product $\mathfrak{g} = \mathfrak{b} \times_\rho \mathfrak{a}$, one can assume that $\mathfrak{a}$ is stable under σ.*

Proof. For $a \in \mathfrak{a} \subset \mathfrak{g}$, write $a = a_{\mathfrak{k}} + a_{\mathfrak{p}}$ according to the decomposition with respect to σ. Then for all $a, a' \in \mathfrak{a}$, one has $0 = [a, a'] = [a_{\mathfrak{p}}, a'_{\mathfrak{p}}] + b \quad b \in [\mathfrak{k}, \mathfrak{p}]$ since $\mathfrak{k}$ is abelian. This yields $[a_{\mathfrak{p}}, a'_{\mathfrak{p}}] = 0$.

Therefore, for $\mathrm{pr}_{\mathfrak{p}} : \mathfrak{g} \to \mathfrak{p}$ the projection parallel to $\mathfrak{k}$, the $\mathfrak{p}$-component $\mathrm{pr}_{\mathfrak{p}}(\mathfrak{a})$ is an abelian subalgebra of $\mathfrak{g}$ supplementary to $\mathfrak{b}$. A dimension count then yields the lemma. □

Lemma 2.3. *Assume that $t = (\mathfrak{g}, \sigma, \Omega)$ is HI split, indecomposable and nonflat. Set $0 \to \mathfrak{b} \to \mathfrak{g} \to \mathfrak{a} \to 0$ as in Lemma 2.2. Then $\mathfrak{a}$ and $\mathfrak{l} = [\mathfrak{k}, \mathfrak{p}]$ are in duality. In particular, there exists a $\mathfrak{k}$-invariant symplectic structure on $\mathfrak{p}$ for which $\mathfrak{a}$ is Lagrangian.*

Proof. Set $V := \mathfrak{l}^{\perp} \cap \mathfrak{a}$ and choose a subspace W of $\mathfrak{a}$ in duality with $\mathfrak{l}$. Counting dimensions yields $\mathfrak{a} = W \oplus V$. Moreover, in the decomposition $\mathfrak{p} = \mathfrak{l} \oplus W \oplus V$, the matrix of Ω is of the form

$$[\Omega] = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & B \\ 0 & -B' & A \end{pmatrix}.$$

Since $\det[\Omega] \neq 0$, one gets $\det \begin{pmatrix} -I & B \\ 0 & A \end{pmatrix} \neq 0$; hence $\det A \neq 0$ and V is symplectic. Now, $\Omega([\mathfrak{k}, V], \mathfrak{p}) = \Omega(V, \mathfrak{l}) = 0$, hence $[\mathfrak{k}, V] = 0$. Also $[V, \mathfrak{l}] = [V, [\mathfrak{k}, \mathfrak{p}]] = 0$ by Jacobi. Thus V is central, and therefore trivial by indecomposability. □

We now assume that $(\mathfrak{g}^1, \sigma^1)$ is the involutive Lie algebra underlying a split HI symplectic triple which is indecomposable and nonflat. We fix Ω^1 such that the HI symplectic triple $t^1 = (\mathfrak{g}^1, \sigma^1, \Omega^1)$ with $0 \to \mathfrak{b}^1 \to \mathfrak{g}^1 \to \mathfrak{a}^1 \to 0$ has $\mathfrak{a}^1$ and $\mathfrak{l}^1 = [\mathfrak{k}^1, \mathfrak{p}^1]$ dual Lagrangian subspaces. We then consider the associated exact triple [Bieliavsky 1998], which we denote by $t = (\mathfrak{g}, \sigma, \Omega)$ (if t^1 is already exact we set $t = t^1$). Since $\mathfrak{a}^1$ is isotropic, the triple t is elementary solvable with

$$0 \to \mathfrak{b} := [\mathfrak{g}, \mathfrak{g}] \to \mathfrak{g} \to \mathfrak{a} := \mathfrak{a}^1 \to 0.$$

We now follow a procedure as in [Bieliavsky 2002]. The map $\rho : \mathfrak{a} \to \mathrm{End}(\mathfrak{b})$ is injective (because Ω is nondegenerate), so we may identify $\mathfrak{a}$ with its image : $\mathfrak{a} = \rho(\mathfrak{a})$. Let $\Sigma : \mathrm{End}(\mathfrak{b}) \to \mathrm{End}(\mathfrak{b})$ be the automorphism induced by the conjugation with respect to the involution $\sigma|_{\mathfrak{b}} \in GL(\mathfrak{b})$, i.e. $\Sigma = Ad(\sigma|_{\mathfrak{b}})$. The automorphism Σ is involutive and preserves the canonical Levi decomposition $\mathrm{End}(\mathfrak{b}) = \mathscr{Z} \oplus sl(\mathfrak{b})$, where $\mathscr{Z}$ denotes the center of $\mathrm{End}(\mathfrak{b})$. Writing the element $a = \rho(a) \in \mathfrak{a}$ as $a = a_Z + a_0$ with respect to this decomposition, one has $\Sigma(a) = a_Z + \Sigma(a_0) = -a = -a_Z - a_0$, because the endomorphisms a and $\sigma|_{\mathfrak{b}}$ anticommute. Hence $\Sigma(a_0) = -2a_Z - a_0$ and therefore $a_Z = 0$. So, $\mathfrak{a}$ actually lies in the semisimple part $sl(\mathfrak{b})$. For any $x \in sl(\mathfrak{b})$, we denote by $x = x^S + x^N, \quad x^S, x^N \in sl(\mathfrak{b})$, its abstract Jordan–Chevalley decomposition. Observe that, for $sl(\mathfrak{b}) = sl_+ \oplus sl_-$, the

decomposition in (± 1)-Σ- eigenspaces, one has $\mathfrak{a} \subset sl_-$. Also, $\mathfrak{a}_N := \{a^N\}_{a \in \mathfrak{a}}$ is an abelian subalgebra in sl_- commuting with $\mathfrak{a}$. Set $\mathfrak{a}_S := \{a^S\}_{a \in \mathfrak{a}}$.

Consider the complexification $\mathfrak{b}^c := \mathfrak{b} \otimes \mathbb{C}$ and $\mathbb{C}$-linearly extend the endomorphisms $\{\rho(a)\}_{a \in \mathfrak{a}}$ and σ. Also consider the complex Lie algebra $sl(\mathfrak{b}^c) = sl(\mathfrak{b}) \otimes \mathbb{C}$ and $\mathbb{C}$-linearly extend to $sl(\mathfrak{b}^c)$ the involution Σ.

Let

$$\text{(2)} \qquad\qquad \mathfrak{b}^c =: \bigoplus_{\alpha \in \Phi} \mathfrak{b}_\alpha$$

be the weight space decomposition with respect to the action of $\mathfrak{a}_S$. Note that for all α, one has $\mathfrak{a}_N.\mathfrak{b}_\alpha \subset \mathfrak{b}_\alpha$. Moreover, for all $X_\alpha \in \mathfrak{b}_\alpha$ and $a^S \in \mathfrak{a}_S$, one has

$$\sigma(a^S.X_\alpha) = \alpha(a^S)\sigma(X_\alpha) = \sigma a^S \sigma^{-1}\sigma X_\alpha = \Sigma(a^S).\sigma(X_\alpha) = -a^S.\sigma(X_\alpha).$$

Therefore, $-\alpha \in \Phi$ and $\sigma \mathfrak{b}_\alpha = \mathfrak{b}_{-\alpha}$. Note in particular that $\sigma \mathfrak{b}_0 = \mathfrak{b}_0$.

Lemma 2.4. *If the triple t^1 is assumed indecomposable and nonflat, then*

$$\mathfrak{b}_0 = 0.$$

Proof. Assume $0 \in \Phi$. For all $\alpha \in \Phi$, the subspace

$$V_\alpha := \mathfrak{b}_\alpha \oplus \mathfrak{b}_{-\alpha}$$

of $\mathfrak{b}^c$ is stable under σ. In particular, the complexified involutive Lie algebra $(\mathfrak{g}^c, \sigma)$, where $\mathfrak{g}^c := \mathfrak{g} \otimes \mathbb{C}$, can be expressed as $\mathfrak{g}^c = \mathfrak{a}^c \times_\rho \mathfrak{b}^c$ with

$$\mathfrak{b}^c = \bigoplus_{\alpha \in \Phi^+} \mathfrak{b}_\alpha \oplus \mathfrak{b}_0,$$

where the *positive system* of weights Φ^+ is chosen so that

$$\Phi = \{0\} \cup \Phi^+ \cup (-\Phi^+)$$

(disjoint union). One therefore has the decomposition

$$V_\alpha = \mathfrak{k}_\alpha \oplus \mathfrak{l}_\alpha$$

into $(\pm)$-eigenspaces for σ. Moreover, since $\mathfrak{g}^c = [\mathfrak{g}^c, \mathfrak{g}^c]$ and $[\mathfrak{a}, \mathfrak{b}_\alpha] \subset \mathfrak{b}_\alpha$, one has

$$\mathfrak{l}_\alpha = [\mathfrak{a}, \mathfrak{k}_\alpha] \quad \text{and} \quad \mathfrak{k}_\alpha = [\mathfrak{a}, \mathfrak{l}_\alpha],$$

for all $\alpha \in \Phi^+ \cup \{0\}$. This implies $\mathfrak{l}_0 = [\mathfrak{a}_N, \mathfrak{k}_0]$ and $\mathfrak{k}_0 = [\mathfrak{a}_N, \mathfrak{l}_0]$. Hence $\mathfrak{l}_0 = [\mathfrak{a}_N, [\mathfrak{a}_N, \mathfrak{l}_0]]$ and an induction yields $\mathfrak{l}_0 = 0$. $\qquad\square$

Corollary 2.5. *A nilpotent HI split symplectic symmetric space is flat.*

Proposition 2.6. *Let $t = (\mathfrak{g} = \mathfrak{b} \times_\rho \mathfrak{a}, \sigma, \Omega = \delta\xi)$ be the exact triple associated with a nonflat indecomposable split symplectic triple t^1. Let Φ be the set of weights associated with the (complex) action of $\mathfrak{a}_S$ on $\mathfrak{b}^c$. Fix a positive system Φ^+ and set*

$$\mathfrak{b}^+ := \bigoplus_{\alpha \in \Phi^+} \mathfrak{b}_\alpha.$$

Then the pair $(\mathfrak{s}^c := \mathfrak{a}^c \times_\rho \mathfrak{b}^+, \Omega|_{\mathfrak{s}^c})$ is a (complex) symplectic Lie algebra.

Proof. By the proof of [Lemma 2.4](#), the restricted projection $\mathfrak{b}^+ \xrightarrow{p} \mathfrak{l}^c$ mapping X to $\frac{1}{2}(X - \sigma(X))$ is a linear isomorphism. Moreover, for all $X \in \mathfrak{b}^+$, $a \in \mathfrak{a}^c$, one has $\Omega(X, a) = \xi[p(c), a]$. The proposition follows from the nondegeneracy of the pairing $\mathfrak{a}^c \times \mathfrak{l}^c \to \mathbb{C}$. $\qquad\qquad\square$

Definition 2.5. Let t be an HI split symplectic triple. Decompose t into a direct sum of indecomposables and a flat factor. [Proposition 2.6](#) then canonically associates to t a (complex) symplectic Lie algebra $\mathfrak{s}^c(t)$, the *complex symplectic Lie algebra associated with t.*

Definition 2.6. A symplectic Lie algebra $(\mathfrak{s}, \omega)$ is called *elementary solvable* if

(i) it is a split extension of abelian Lie algebras $\mathfrak{a}$ and $\mathfrak{d}$:

(3) $$0 \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{s} \longrightarrow \mathfrak{a} \longrightarrow 0;$$

(ii) The cocycle ω is exact.

Conversely to [Proposition 2.6](#), one has

Proposition 2.7. *Every elementary solvable symplectic Lie algebra is associated with a split HI symplectic symmetric space.*

Proof. Denote by $\rho : \mathfrak{a} \to \mathrm{End}(\mathfrak{d})$ the splitting homomorphism and by $\overline{\rho} : \mathfrak{a} \to \mathrm{End}(\mathfrak{d})$ the opposite representation: $\overline{\rho}(a)(X) := -\rho(a)(X), \quad X \in \mathfrak{d}$. Set

$$\mathfrak{b} := \mathfrak{d} \oplus \mathfrak{d}$$

and let $\mathfrak{a}$ act on $\mathfrak{b}$ via $\rho \oplus \overline{\rho}$. Define the involution $\sigma_\mathfrak{b}$ of $\mathfrak{b}$ by

$$\sigma_\mathfrak{b}(X, Y) = (Y, X), \qquad X, Y \in \mathfrak{d}.$$

Set

$$\mathfrak{g} := \mathfrak{b} \times_{\rho \oplus \overline{\rho}} \mathfrak{a}$$

and define the involution σ of $\mathfrak{g}$ as

$$\sigma := \sigma_\mathfrak{b} \oplus (-\mathrm{id}_\mathfrak{a}).$$

One then observes that $(\mathfrak{g}, \sigma)$ is an involutive Lie algebra. We have $\mathfrak{k} = \{(X, X)\}_{X \in \mathfrak{d}}$ while $\mathfrak{p} = \{(X, -X)\}_{X \in \mathfrak{d}}$.

Let $\eta \in \mathfrak{d}^\star$ be such that $\delta\eta = \omega$ and define $\xi \in \mathfrak{k}^\star$ by

$$\xi(X, X) := \eta(X), \qquad X \in \mathfrak{d}.$$

Extending ξ to $\mathfrak{g}$ by 0 on $\mathfrak{p}$, one defines a symplectic coboundary on $\mathfrak{g}$:

$$\Omega := \delta\xi.$$

The triple $(\mathfrak{g}, \sigma, \Omega)$ then defines the desired elementary solvable symplectic symmetric space. $\qquad\square$

3. Strict deformation quantization for proper actions

We first recall that to an involutive Lie algebra, one associates a simply connected symmetric space M in the usual way (see [Kobayashi and Nomizu 1969], for example). When associated with a symplectic triple, the space M turns out to be naturally endowed with an invariant symplectic structure. The space M is then called a symplectic symmetric space. One then has

Theorem 3.1 [Bieliavsky 2002]. *Let M be the simply connected symplectic symmetric space associated with an HI split symplectic triple. Assume M is strictly geodesically convex with respect to its canonical affine connexion. There is a one-parameter family $\{K_\theta \in C^\infty(M \times M \times M, \mathbb{C})\}_{\theta \in \mathbb{R}}$ of smooth invariant three-point kernels, and a corresponding family $\{\mathcal{H}_\theta \in C^\infty(M)\}_\theta$ of invariant function spaces, such that*

(i) *$\mathcal{D}(M) := C_c^\infty(M) \subset \mathcal{H}_\theta$ for all θ;*

(ii) *the formula*

$$(4) \qquad u \star_\theta v(x); = \int_{M \times M} K(x, y, z)\, u(y)\, v(z)\, \mathrm{d}y\, \mathrm{d}z \quad (u, v \in \mathcal{D}(M))$$

extends to $\mathcal{H}_\theta$ as an associative $\mathbb{C}$-algebra product law denoted hereafter $\star_\theta$, and the automorphism group of M acts on the algebra $(\mathcal{H}_\theta, \star_\theta)$ by algebra automorphisms;

(iii) *for $u, v \in \mathcal{D}(M)$ and $x \in M$, one has the asymptotic expansion*

$$(5) \qquad u \star_\theta v(x) \sim uv(x) + \frac{\theta}{2i}\{u, v\}(x) + o(\theta^2),$$

where $\{,\}$ denotes the invariant symplectic Poisson bracket on $C^\infty(M)$.

We now pass to the announced nonformal UDFs. In this section, $(\mathfrak{s}, \Omega)$ denotes an *elementary solvable symplectic Lie algebra* associated with a strictly geodesically convex HI split symplectic symmetric space M. We denote by $\mathbb{S}$ the corresponding simply connected symplectic Lie group, whose identity element we write as e.

Now consider a proper smooth action of $\mathbb{S}$ on a manifold X:

$$\mathbb{S} \times X \to X : (g, x) \mapsto \tau_g(x).$$

And denote by

$$\mathbb{S} \times C^\infty(X) \to C^\infty(X) : (g, a) \mapsto \alpha_g(a)$$

the corresponding left action on functions on X. At last, for all $a \in C^\infty(X)$ and $x \in X$, set

$$\alpha^x(a) : \mathbb{S} \to \mathbb{C} : g \mapsto \alpha_g(a)(x) := \tau_{g^{-1}}^\star(a).$$

Theorem 3.2. *Define the space*

$$\mathbb{B}_\theta ; = \{a \in C^\infty(X) \text{ such that } \alpha^x(a) \in \mathcal{H}_\theta \text{ for all } x \in X\}.$$

Then

(i) $\mathcal{D}(X) := C_c^\infty(X) \subset \mathbb{B}_\theta$ *for all* θ, *and*

(ii) *the formula*

$$a \star_\theta^X b (x) := \left(\alpha^x(a) \star_\theta \alpha^x(b) \right)(e)$$

defines an associative $\mathbb{C}$-algebra product law on the function space $\mathbb{B}_\theta$.

Proof. Item (i) follows from the properness of the action and item (i) of the preceding theorem. Item (ii) follows from [Bieliavsky 2002, p. 282, item (ii)]; see also [Bieliavsky et al. 2003]. □

For more general actions, we shall remain at the formal level. We shall prove in the next section that the asymptotic expansion (5) actually defines an invariant formal star product on M. For simplicity, we shall treat only the case dim $\mathbb{S} = 2$ but as it will appear this is not essential.

4. Formal universal deformation formulae

We present the $ax + b$ group as $\mathbb{S} = \mathbb{R}^2 = \{(a, \ell)\}$ with multiplication law

$$(a, \ell).(a', \ell') := (a + a', \ e^{-2a'}\ell + \ell').$$

The Lie algebra $\mathfrak{s}$ of $\mathbb{S}$ is then realized as $\mathfrak{s} = \text{span}\{H, E\}$ with

$$(a, 0) =: \exp(aH) \quad \text{and} \quad (0, \ell) =: \exp(\ell E).$$

For $X \in \mathfrak{g}$, we denote by $\widetilde{X}$ the corresponding left-invariant vector field on $\mathbb{S}$.

We denote by $\mathscr{S}$ the Schwartz space on $\mathbb{S} = \{(a, \ell)\}$. For $u \in \mathscr{S}$, we denote by $\mathscr{F}(u)$ the partial Fourier transform

$$\mathscr{F}(u)(a, \xi) =: \hat{u}(a, \xi) =: \int e^{-i\xi\ell} u(a, \ell)d\ell.$$

We set $\tilde{\mathbb{S}} := \{(a, \xi)\}$ and we denote by $\tilde{\mathcal{S}}$ the space of Schwartz functions in these variables. Thus, one has the isomorphism $\mathcal{F} : \mathcal{S} \to \tilde{\mathcal{S}}$. For $\theta \in \mathbb{R}$, we define the following diffeomorphism $\phi_\theta : \tilde{\mathbb{S}} \to \tilde{\mathbb{S}}$:

$$\phi_\theta(a, \xi) := \begin{cases} (a, \frac{1}{2\theta} \sinh(2\theta\xi)) & \text{for } \theta \neq 0, \\ (a, \xi) & \text{for } \theta = 0. \end{cases}$$

Setting $\tau_\theta := \mathcal{F}^{-1} \circ (\phi_\theta^{-1})^\star \circ \mathcal{F}$, we formally define the following distribution space on $\mathbb{S}$:

$$\mathcal{E}_\theta := \tau_\theta(\mathcal{S}).$$

Lemma 4.1 [Bieliavsky 2002]. *For all $\theta \in \mathbb{R}$,*

$$\mathcal{S} \subset \mathcal{E}_\theta \subset \mathcal{S}' \cap C^\infty(\mathbb{S});$$

where $\mathcal{S}'$ denotes the space of tempered distributions in variables (a, ℓ).

Lemma 4.2.

$$(\phi_\theta^{-1})^\star(\tilde{\mathcal{S}}) = \begin{cases} (\phi_{1/2}^{-1})^\star(\tilde{\mathcal{S}}) & \text{for } \theta \neq 0, \\ \tilde{\mathcal{S}} & \text{for } \theta = 0. \end{cases}$$

Proof. First observe that $f \in (\phi_\theta^{-1})^\star \tilde{\mathcal{S}}$ if and only if for all $M, N \in \mathbb{N}$ one has $\int \xi^M \partial_\xi^N (\phi_\theta^\star f) \, d\xi < \infty$. Let us restrict ourself to the case $N = 1$, the general case being entirely similar. One has $\partial_\xi(\phi_\theta^\star f)(\xi) = \cosh(2\theta\xi)\phi_\theta^\star(\partial_\eta f) = \phi_\theta^\star(\sqrt{1 + (2\theta\eta)^2}\partial_\eta f)$. Therefore, the condition becomes

$$\int \text{arcsinh}(2\theta\eta)\partial_\eta f(\eta) \, d\eta < \infty.$$

Since the functions $\text{arcsinh}(2\theta\eta)\partial_\eta f(\eta)$ and $\text{arcsinh}\,\eta\,\partial_\eta f(\eta)$ have, up to a θ-dependant multiple, the same asymptotic behavior for large $|\eta|$, the finiteness of the above integral is equivalent to the finiteness of $\int \text{arcsinh}\,\eta\,\partial_\eta f(\eta)\,d\eta$. $\qquad\square$

In particular, for all $\theta \in \mathbb{R}$, one has

$$(\phi_\theta^{-1})^\star(\tilde{\mathcal{S}}) \subset (\phi_{1/2}^{-1})^\star(\tilde{\mathcal{S}}).$$

The above lemma leads us to define the following spaces:

$$\hat{\mathcal{E}} := (\phi_{1/2}^{-1})^\star(\tilde{\mathcal{S}}) \quad \text{and} \quad \mathcal{E} := \mathcal{F}^{-1}(\hat{\mathcal{E}}).$$

We then observe the following obvious fact.

Lemma 4.3. *Denote by $\mathcal{B}_\infty(\tilde{\mathbb{S}})$ the space of smooth functions $w \in C^\infty(\tilde{\mathbb{S}})$ such that for all multi-index α, $D^\alpha w \in C_0(\tilde{\mathbb{S}})$. Then,*

$$\hat{\mathcal{E}} \subset \mathcal{B}_\infty(\tilde{\mathbb{S}}) \quad \text{and} \quad \partial_\xi^2(\hat{\mathcal{E}}) \subset L^1(\tilde{\mathbb{S}}).$$

Lemma 4.4. *Let $u \in \mathscr{S}$. Then,*

(i) *there exists a dense open subset $\Omega \subset G$ where, for all $\theta \in \mathbb{R}$, the tempered distribution $\tau_\theta(u)$ coincides with a smooth function, $\tau_\theta^{\mathrm{loc}}(u) \in C^\infty(\Omega)$, in the sense that for all $\psi \in \mathscr{D}(\Omega)$, $\langle \tau_\theta(u), \psi \rangle = \langle \tau_\theta^{\mathrm{loc}}(u), \psi \rangle$.*

(ii) *The function $\mathbb{R} \to C^\infty(\Omega) : \theta \mapsto \tau_\theta^{\mathrm{loc}}(u)$ is smooth.*

Proof. Abel's criterion implies that for $\ell \neq 0$, the improper integral $\tau_\theta^{\mathrm{loc}}(u)(a, \ell) := \frac{1}{2\pi} \int e^{i\xi\ell} (\phi_\theta^{-1})^\star \hat{u}(\xi) \, \mathrm{d}\xi$ converges, since, by Lemma 4.3, $(\phi_\theta^{-1})^\star \hat{u} \in C_0(\tilde{\mathbb{S}})$. The function $\tau_\theta^{\mathrm{loc}}(u)$ is smooth in $(\theta, x) \in \mathbb{R} \times \Omega$ where $\Omega := \{(a, \ell) \mid \ell \neq 0\}$. Indeed, observing that $e^{i\xi\ell} = \frac{1}{i\ell} \partial_\xi e^{i\xi\ell}$, one obtains, using Lemma 4.3:

$$
\begin{aligned}
\tau_\theta^{\mathrm{loc}}(u)(a, \ell) &= \frac{i}{2\pi\ell} \lim_{r \to \infty} \int_{-r}^{r} e^{i\xi\ell} \partial_\xi (\phi_\theta^{-1})^\star \hat{u}(\xi) \, \mathrm{d}\xi \\
&= \frac{i}{2\pi\ell} \int e^{i\xi\ell} \partial_\xi (\phi_\theta^{-1})^\star \hat{u}(\xi) \, \mathrm{d}\xi.
\end{aligned}
$$

Hence, $\tau_\theta^{\mathrm{loc}}(u)(a, \ell) = \frac{-1}{2\pi\ell^2} \int e^{i\xi\ell} \partial_\xi^2 (\phi_\theta^{-1})^\star \hat{u}(\xi) \, \mathrm{d}\xi$, which is an existing integral. Now, by Lemma 4.2, one observes that setting $\hat{u}_\theta := \phi_{1/2}^\star (\phi_\theta^{-1})^\star \hat{u}$ defines a smooth family of Schwartz functions $\{\hat{u}_\theta\} \subset \tilde{\mathscr{S}}$ such that $(\phi_\theta^{-1})^\star \hat{u} = (\phi_{1/2}^{-1})^\star \hat{u}_\theta$. Therefore, one has

$$
\tau_\theta^{\mathrm{loc}}(u)(a, \ell) = \frac{-1}{2\pi\ell^2} \int e^{i\xi\ell} \partial_\xi^2 (\phi_{1/2}^{-1})^\star \hat{u}_\theta(\xi) \, \mathrm{d}\xi,
$$

whose integrand is bounded by the integrable function $\left| \partial_\xi^2 (\phi_{1/2}^{-1})^\star \hat{u}_{\theta_0} \right|$ for $\theta \in [-\epsilon, \epsilon]$. One proceeds similarly for derivatives. It follows that $\tau_\theta^{\mathrm{loc}}(u)$ defines an element of $C^\infty(\mathbb{R}, C^\infty(\Omega))$ [Trèves 1967, Theorem 40.1, p. 415]. Of course, $\langle \tau_\theta(u), \psi \rangle = \langle \tau_\theta^{\mathrm{loc}}(u), \psi \rangle$ for all $\psi \in \mathscr{D}(\Omega)$. □

Proposition 4.5 [Bieliavsky 2002]. (i) *The map $T_\theta := \mathscr{F}^{-1} \circ \phi_\theta^\star \circ \mathscr{F}$ establishes a linear isomorphism*

$$
T_\theta : \mathscr{E}_\theta \to \mathscr{S}.
$$

(ii) *Denote by $\star_\theta^0$ Weyl's product on $\mathscr{S}$. The formula*

$$
a \star_\theta b := \tau_\theta(T_\theta a \star_\theta^0 T_\theta b); \qquad a, b \in \mathscr{E}_\theta
$$

defines an associative product $\star_\theta$ on $\mathscr{E}_\theta$.

(iii) *At the level of smooth compactly supported $u, v \in C_c^\infty(\mathbb{S})$ and for all $x_0 \in \Omega$, one has the oscillatory integral formula*

$$
u \star_\theta v(x_0) =
$$
$$
\frac{1}{4\pi^2\theta^2} \int_{G \times G} \cosh 2(a_1 - a_2) \exp \frac{i \underset{0,1,2}{\oplus} \ell_2 \sinh 2(a_0 - a_1)}{2\theta} \, u(x_1) v(x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2,
$$

where $x_j = (a_j, \ell_j)$, $j = 0, 1, 2$.

(iv) *The three-point kernel*

$$K_\theta(x_0, x_1, x_2) := \frac{1}{4\pi^2\theta^2} \cosh(2(a_1 - a_2)) \, e^{\frac{i}{2\theta} \overset{\circlearrowright}{\underset{0,1,2}{\bigoplus}} \sinh(2(a_0 - a_1))\ell_2}$$

is left-invariant with respect to the diagonal action of $\mathbb{S}$ on $G \times G \times G$.

Remark 4.1. The formula in (iii) holds at every point $x_0 \in G$. Indeed, by left-invariance of the kernel, one has for every choice of $y \in \Omega$: $u \star_\theta v(x_0) = (L^\star_{x_0 y^{-1}} u \star_\theta L^\star_{x_0 y^{-1}} v)(y)$.

We set, for integral $n \geq 1$,

$$\beta(2n+1) := \frac{1}{2} \frac{(-1)^n}{4^n} \frac{1}{2n+1} \frac{(2n)!}{(n!)^2} = \frac{(-1)^n}{4^n} \frac{1}{2n+1} \binom{2n-1}{n-1},$$

and further $\beta(1) := \frac{1}{2}$ and $\beta(2n) := 0$ for $n \geq 0$.

Definition 4.1. Let k and N be positive integers such that $k \leq N$. We set

$$B_N^k := \sum_{\substack{k_1, \ldots, k_N \\ k = k_1 + \cdots + k_N \\ N = 1k_1 + 2k_2 + \cdots + Nk_N}} \frac{N!}{k_1! k_2! \cdots k_N!} \beta(1)^{k_1} \beta(2)^{k_2} \cdots \beta(N)^{k_N},$$

with the convention that $\beta(2n)^0 := 1$ and $B_0^0 := 1$.

Lemma 4.6. *Let $u \in C^\infty(\mathbb{S})$ and $N \in \mathbb{N}_0$. Then*

$$\frac{d^N}{d\theta^N}\big|_{\xi=0} \left(R^\star_{(\frac{1}{2} \operatorname{arcsinh} \xi, 0)} u \right) = \sum_{k=1}^{N} B_N^k \, \widetilde{H}^k . u,$$

where R denotes right multiplication on $\mathbb{S}$.

Proof. We shall use Faà di Bruno's formula [1857], which computes the higher order derivatives of composed functions:

$$(6) \quad \frac{d^n}{d\xi^n}[R(b(\xi))]$$
$$= \sum \frac{n!}{k_1! k_2! \cdots k_n!} \left(\frac{b^{(1)}(\xi)}{1!} \right)^{k_1} \left(\frac{b^{(2)}(\xi)}{2!} \right)^{k_2} \cdots \left(\frac{b^{(n)}(\xi)}{n!} \right)^{k_n} R^{(k)}(b(\xi)),$$

with $k := k_1 + \cdots + k_n$ and where $(k_1, \ldots, k_n)$ runs over all partitions of n (solutions of the equation $n = 1k_1 + 2k_2 + \cdots + nk_n$). We now observe the Taylor expansion

$$\tfrac{1}{2} \operatorname{arcsinh} \xi = \tfrac{1}{2}\xi + \sum_{n \geq 1} \frac{(-1)^n}{4^n} \frac{1}{2n+1} \binom{2n-1}{n-1} \xi^{2n+1} = \sum_{n \geq 0} \frac{(-1)^n}{2 \cdot 4^n} \frac{1}{2n+1} \frac{(2n)!}{(n!)^2} \xi^{2n+1}.$$

Plugging $b(\xi) := \frac{1}{2} \operatorname{arcsinh} \xi$ and $R(a) := R^\star_{(a,0)} u(x)$ into (6) at $\xi = 0$ then yields the lemma. $\qed$

Lemma 4.7. *Let γ and μ be maps from $\mathbb{R} \times \mathcal{G}$ to $C^\infty(\mathbb{S})$ such that*

(i) *for every $u \in \mathcal{G}$ the functions $\gamma(\,.\,, u)$ and $\mu(\,.\,, u)$ belong to $C^\infty(\mathbb{R}, C^\infty(\mathbb{S}))$;*

(ii) *for all θ the partial functions $\gamma(\theta, \,.\,)$ and $\mu(\theta, \,.\,)$ are $\mathbb{C}$-linear.*

Assume the function

$$\theta \mapsto \phi(\theta, u) := \mu(\theta, \gamma(\theta, u))$$

to be an element of $C^\infty(\mathbb{R}, C^\infty(\mathbb{S}))$ as well. Denoting by

$$\gamma(\theta, u) \sim \sum_m \theta^m \Gamma_m(u), \quad \mu(\theta, u) \sim \sum_m \theta^m M_m(u), \quad \phi(\theta, u) \sim \sum_m \theta^m \Phi_m(u)$$

the corresponding Taylor expansions, one has

$$\Phi_m(u) = \sum_{m=n+k} M_k(\Gamma_n(u)).$$

Proof. Let $R_N^\mu(u)$ be the rest at order N in the Taylor expansion of μ [Bourbaki 1967]. Then,

$$[\theta \mapsto R_N^\mu(\gamma(\theta, u))] \in o(\theta^N).$$

Indeed, $R_N^\gamma(u) = \theta^N \rho_N(u)$ where $\rho^N(u)$ is continuous in a neighborhood of $\theta = 0$. Therefore $\theta^{-(N+1)} R_N^\mu(\gamma(\theta, u)) = \sum_k^{N+1} \theta^{k-N-1} R_N^\mu(\Gamma_k(u)) + \theta R_N^\mu(\rho_{N+1}(u))$. The first $N + 2$ terms of this sum tend to 0 with θ. In the last term, one observes that $R_N^\mu(\rho_{N+1}(u))$ is continuous and hence bounded for $\theta \in [-\epsilon, \epsilon]$, which proves the assertion and with it the lemma. $\qquad\square$

Of course an analogous statement holds for bilinear maps.

Lemma 4.8. *For every smooth compactly supported $u, v \in C_c^\infty(\mathbb{S})$, the functions of θ given by $T_\theta(u)$, $u \star_\theta^0 v$ and $u \star_\theta v$ belong to $C^\infty(\mathbb{R}, C^\infty(\mathbb{S}))$.*

Proof. For every $x_0 \in G$,

$$T_\theta(u)(x_0) = \frac{1}{2\pi} \int e^{i\ell_0\xi} \phi_\theta^\star \hat{u}(a_0, \xi) \, d\xi.$$

The integrand $e^{i\ell_0\xi} \phi_\theta^\star \hat{u}(a_0, \xi)$ is smooth in the variables (a_0, ℓ_0, θ) as well as every of its partial derivatives. Moreover, each of these functions is bounded by an integrable function since $\phi_\theta^\star \hat{u}$ is Schwartz. Therefore the function $(x_0, \theta) \mapsto T_\theta(u)(x_0)$ is smooth.

The argument is similar for the other functions, simpler in fact since the integrand is in both cases compactly supported. $\qquad\square$

Theorem 4.9. *For every smooth compactly supported $u, v \in C_c^\infty(\mathbb{S})$ and for every $x \in G$, the function $[\mathbb{R} \to \mathbb{C} : \theta \mapsto u \star_\theta v(x)]$ is smooth. Its Taylor series at 0 defines*

an associative formal star product $\star_v$ on $C^\infty(\mathbb{S})[\![v]\!]$ whose explicit expression is

$$(7) \quad u \star_v v := uv + \sum_{n=1}^{\infty} \frac{v^n}{n!} (-2i)^n \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k}$$

$$\times \left(\left(\sum_{m=0}^{k} B_k^m \tilde{H}^m \tilde{E}^{n-k} . v \right) \left(\sum_{m=0}^{n-k} B_{n-k}^m \tilde{H}^m \tilde{E}^k . u \right) \right.$$

$$- 2 \left(\sum_{r=0}^{k} \binom{k}{r} (k-r)\beta(k-r) \sum_{p=0}^{r} B_r^p \tilde{H}^p \tilde{E}^{n-k} . v \right)$$

$$\times \left. \left(\sum_{s=0}^{n-k} \binom{n-k}{s} (n-k-s)\beta(n-k-s) \sum_{q=0}^{s} B_s^q \tilde{H}^q \tilde{E}^k . u \right) \right).$$

Proof. Using left invariance one gets

$$u \star_\theta v(x_0) = \int K_\theta(0, x_1, x_2) R^\star_{x_1} u|_{x_0} R^\star_{x_2} v|_{x_0} \, dx_1 \, dx_2.$$

Making the change of variables $\xi_i := \sinh(2a_i)$, $\eta_i := \frac{1}{2\theta}\ell_i$ yields

$$u \star_\theta v(x_0) = \frac{1}{4\pi^2} \int (1 - \gamma(\xi_1)\gamma(\xi_2)) \, e^{-i(\xi_1\eta_2 - \xi_2\eta_1)}$$

$$\times r^\star_{(\xi_1, 2\theta\eta_1)} u|_{x_0} \, r^\star_{(\xi_2, 2\theta\eta_2)} v|_{x_0} \, d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2,$$

where $\gamma(\xi) := \xi/\sqrt{1+\xi^2}$ and $r^\star_{(\xi, 2\theta\eta)} u|_x := u\left(a + \frac{1}{2}\operatorname{arcsinh}\xi, \, e^{-\operatorname{arcsinh}\xi}\ell + 2\theta\eta\right)$.
Now observe the asymptotic expansion ($\theta_0 \neq 0$)

$$u \star_\theta v(x_0) = \sum_{n=0}^{N} \frac{(\theta - \theta_0)^n}{n!} \frac{1}{4\pi^2} \int (1 - \gamma(\xi_1)\gamma(\xi_2)) \, e^{-i(\xi_1\eta_2 - \xi_2\eta_1)}$$

$$\times \frac{d^n}{d\theta^n} \left[r^\star_{(\xi_1, 2\theta\eta_1)} u|_{x_0} \, r^\star_{(\xi_2, 2\theta\eta_2)} v|_{x_0} \right]|_{\theta=\theta_0} \, d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2$$

$$+ o((\theta - \theta_0)^N).$$

One has

$$\frac{d^n}{d\theta^n} \left[r^\star_{(\xi_1, 2\theta\eta_1)} u|_{x_0} \, r^\star_{(\xi_2, 2\theta\eta_2)} v|_{x_0} \right]|_{\theta=\theta_0}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{d\theta^k} \left[r^\star_{(\xi_1, 2\theta\eta_1)} u|_{x_0} \right]|_{\theta=\theta_0} \frac{d^{n-k}}{d\theta^{n-k}} \left[r^\star_{(\xi_2, 2\theta\eta_2)} v|_{x_0} \right]|_{\theta=\theta_0}$$

$$= 2^n \sum_{k=0}^{n} \binom{n}{k} \eta_1^k \eta_2^{n-k} r^\star_{(\xi_1, 2\theta_0\eta_1)} \left(\tilde{E}^k . u \right)|_{x_0} \, r^\star_{(\xi_2, 2\theta_0\eta_2)} \left(\tilde{E}^{n-k} . v \right)|_{x_0},$$

where $\exp(aH) := (a, 0)$ and $\exp(\ell E) := (0, \ell)$. Therefore, an integration by parts yields (for $n \geq 1$)

$$\int \big(1 - \gamma(\xi_1)\gamma(\xi_2)\big) e^{-i(\xi_1\eta_2 - \xi_2\eta_1)} \frac{d^n}{d\theta^n} \big[r^\star_{(\xi_1, 2\theta\eta_1)} u|_{x_0} r^\star_{(\xi_2, 2\theta\eta_2)} v|_{x_0}\big]|_{\theta = \theta_0} \, d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2$$

$$= (-2i)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \bigg(\int e^{-i(\xi_1\eta_2 - \xi_2\eta_1)}$$

$$\times \Big(\partial^k_{\xi_2} r^\star_{(\xi_2, 2\theta_0\eta_2)} \big(\widetilde{E}^{n-k}.v\big) |_{x_0} \partial^{n-k}_{\xi_1} r^\star_{(\xi_1, 2\theta_0\eta_1)} \big(\widetilde{E}^k.u\big) |_{x_0}$$

$$- \partial^k_{\xi_2} \big[\gamma(\xi_2) \, r^\star_{(\xi_2, 2\theta_0\eta_2)} \big(\widetilde{E}^{n-k}.v\big) |_{x_0} \big] \, \partial^{n-k}_{\xi_1} \big[\gamma(\xi_1) \, r^\star_{(\xi_1, 2\theta_0\eta_1)} \big(\widetilde{E}^k.u\big) |_{x_0} \big]\Big)\bigg).$$

By observing that $\lim_{\eta \to 0} \partial_\xi r^\star_{(\xi, \eta)} u|_x = \partial_\xi r^\star_{(\xi, 0)} u|_x$, one gets

$$\lim_{\theta \to 0} \int \gamma(\xi_1)\gamma(\xi_2) \, e^{-i(\xi_1\eta_2 - \xi_2\eta_1)} \frac{d^n}{d\theta^n} \big[r^\star_{(\xi_1, 2\theta\eta_1)} u|_{x_0} r^\star_{(\xi_2, 2\theta\eta_2)} v|_{x_0}\big] \, d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2$$

$$= 4\pi^2 (-2i)^n \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \partial^k_\xi \big[\gamma(\xi) \, r^\star_{(\xi, 0)} \big(\widetilde{E}^{n-k}.v\big) |_{x_0} \big] |_{\xi=0}$$

$$\times \partial^{n-k}_\xi \big[\gamma(\xi) \, r^\star_{(\xi, 0)} \big(\widetilde{E}^k.u\big) |_{x_0} \big] |_{\xi=0},$$

which, by using Lemma 4.6 and the expansion

$$\gamma(\xi) = \xi + \sum_{m \geq 1} \frac{(-1)^m}{4^m} \frac{m+1}{m} \binom{2m}{m+1} \xi^{2m+1} = \sum_{m \geq 0} \frac{(-1)^m}{4^m} \frac{(2m)!}{(m!)^2} \xi^{2m+1},$$

is seen to be

$$4\pi^2 (-2i)^n \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \bigg(\Big(\sum_{r=0}^{k} \binom{k}{r} \gamma^{(k-r)}(0) \sum_{p=0}^{r} B^p_r \big(\widetilde{H}^p \widetilde{E}^{n-k}.v\big) \big|_{x_0} \Big)$$

$$\times \Big(\sum_{s=0}^{n-k} \binom{n-k}{s} \gamma^{(n-k-s)}(0) \sum_{q=0}^{s} B^q_s \big(\widetilde{H}^q \widetilde{E}^k.u\big) \big|_{x_0} \Big)\bigg).$$

This leads to the announced asymptotic expansion where ν is the formal parameter corresponding to θ.

Regarding associativity, one first observes that for all $x_0 \in \Omega$, one has the coincidence: $u \star_\theta v(x_0) = \tau^{\text{loc}}_\theta (T_\theta u \star^0_\theta T_\theta v)(x_0)$. Note that, independently, the left hand side is smooth in θ by virtue of Lemma 4.8, while the right hand side is smooth as well by virtue of Lemmas 4.4, 4.7 and 4.8. Lemma 4.7 therefore identifies the above-computed Taylor coefficients with the cochains of the star product obtained by intertwining Moyal's star product with a formal equivalence.   □

Remark 4.2. Our construction differs radically from the (formal) universal deformation formula of Giaquinto and Zhang [1998]: It was shown in [Bieliavsky et al. 2005b] that the (maximal) invariance diffeomorphism group of Giaquinto–Zhang star product on $ax + b$ is isomorphic to $\mathrm{Sp}(1, \mathbb{R}) \times \mathbb{R}^2$, while in our case the corresponding maximal invariance group is the automorphism group of the underlying symplectic symmetric space — in the two-dimensional case, the solvable group $\mathrm{SO}(1, 1) \times \mathbb{R}^2$.

5. Hopf structure

The formal star product $\star_\nu$ given by formula (7) can be described as an example of the following construction [Bieliavsky et al. 2005a].

Definition 5.1 [Bonneau and Sternheimer 2005]. Let B be a cocommutative bialgebra and C a B-bimodule algebra. The *L-R-smash product* $C \natural B$ is the algebra constructed on the vector space $C \otimes B$ where the multiplication is defined by

$$(8) \qquad (f \otimes a) \star (g \otimes b) = \sum_{(a)(b)} (f \leftharpoonup b_{(1)})(a_{(1)} \rightharpoonup g) \otimes a_{(2)} b_{(2)}$$

for $f, g \in C$ and $a, b \in B$.

On the first hand, the above construction provides deformation quantizations of $\mathscr{C}^\infty(G) \otimes \mathrm{Pol}(\mathscr{G}^*) \subset \mathscr{C}^\infty(T^*G)$, where $\mathscr{G} = \mathrm{lie}(G)$, as explained in [Bieliavsky et al. 2005a]. For $G = \mathbb{R}^n$ it reproduces the Moyal star product. On the other hand we have seen that, for $u, v \in \mathscr{C}^\infty(\mathbb{R}^2)$, $u \star_\nu v = T_\nu^{-1}(T_\nu u \star_\nu^0 T_\nu v)$ where $\star_\nu^0$ denotes the Moyal star product on $\mathscr{C}^\infty(\mathbb{R}^2)$. Restricting to $\mathscr{C}^\infty(\mathbb{R}) \otimes \mathrm{Pol}(\mathbb{R}) \subset \mathscr{C}^\infty(\mathbb{R}^2)$, the formal equivalence T_ν is viewed as $\mathrm{id} \otimes S_\nu$.

Proposition 5.1. *The formal star product $\star_\nu$ given by* (7) *coincides with the product underlying the L-R- smash product $\mathscr{C}^\infty(\mathbb{R})[\![\nu]\!] \natural \mathrm{Pol}(\mathbb{R})$ with $x \rightharpoonup f = f \leftharpoonup x = \frac{1}{2} S_\nu^{-1}(\frac{d}{dx}(S_\nu f))$ and we consider* $\mathrm{Pol}(\mathbb{R})$ *endowed with its the usual Hopf structure.*

Proof. See [Bieliavsky et al. 2005a]: the main fact is that $\mathscr{C}^\infty(\mathbb{R})[\![\nu]\!] \natural \mathrm{Pol}(\mathbb{R})$ carries the Moyal product for $x \rightharpoonup f = f \leftharpoonup x = \frac{1}{2} \frac{d}{dx}(f)$. $\qquad\square$

F. Panaite and F. Van Oystaeyen [2005] have shown that if B be a cocommutative Hopf algebra and C a B-module algebra, the map $\Phi : C \natural B \longrightarrow C \sharp B$ with $\Phi(f \natural a) = \sum_{(a)} f \leftharpoonup a_{(1)} \sharp a_{(2)}$ is an isomorphism of algebras, the multiplication on the smash product $C \sharp B$ being given by $(f \otimes a) \star (g \otimes b) = \sum_{(a)} f\ (a_{(1)} \bullet g) \otimes a_{(2)} b$ with $a \bullet g = a_{(1)} \rightharpoonup g \leftharpoonup J(a_{(2)})$, J the antipode of B. One therefore has

Proposition 5.2. *The star product $\star_\nu$ can also be seen as a classical smash product in the sense of Sweedler* [1968].

At last, it is shown in [Bieliavsky et al. 2005a] that, under conditions, if C is a Hopf algebra then so is $C \natural B$. Seing $\mathscr{C}^\infty(\mathbb{R})[\![\nu]\!]$ as $\mathscr{C}^\infty(\mathbb{R}) \otimes \mathbb{R}[\![\nu]\!]$ and considering

the natural Hopf structure on the tensor product of the Hopf algebras $\mathscr{C}^{\infty}(\mathbb{R})$ and $\mathbb{R}[\![\nu]\!]$ we show in [Bieliavsky et al. 2005a] that the needed conditions are fullfilled. Hence:

Theorem 5.3. *There exists a coproduct Δ_ν, a counit ϵ_ν and an antipode J_ν, compatible with $\star_\nu$, such that $\big(\mathscr{C}^{\infty}(\mathbb{R})[\![\nu]\!] \natural \operatorname{Pol}(\mathbb{R}), \; \star_\nu, \; \Delta_\nu, \; \epsilon_\nu, \; J_\nu\big)$ is a Hopf algebra.*

Replacing $\mathbb{R}$ by $\mathbb{R}^n$ these structural results still hold on the asymptotic expansion of the product described in formula (4).

Acknowledgment

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