Universal enveloping algebras of Leibniz algebras and (co)homology

Jean-Louis Loday, Teimuraz Pirashvili

Institutions: Centre national de la recherche scientifique

Published on: 01 Dec 1993 - Mathematische Annalen (Springer-Verlag)

Topics: Leibniz algebra, Universal enveloping algebra, Quantum group, Leibniz formula for determinants and Lie algebra

Math. Ann. 296, 139–158 (1993)

Mathematische Annalen

Universal enveloping algebras of Leibniz algebras and (co)homology

Jean-Louis Loday[1] **and Teimuraz Pirashvili**[2]

[1] Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7, rue René-Descartes, F-67084 Strasbourg Cedex, France
[2] A.M. Razmadze Mathematical Institute, Georgian Academy of Sciences, Rukhadze 1, 380093 Tbilisi, Georgia

Received September 17, 1992

Mathematics Subject Classification (1991): 17B35, 17B55, 17B68, 17D, 18G10, 18G35

0 Introduction

The homology of Lie algebras is closely related to the cyclic homology of associative algebras [LQ]. In [L] the first author constructed a "noncommutative" analog of Lie algebra homology which is, similarly, related to Hochschild homology [C, L]. For a Lie algebra $\mathfrak{g}$ this new theory is the homology of the complex

$$C_*(\mathfrak{g}) \qquad \ldots \to \mathfrak{g}^{\otimes n} \xrightarrow{d} \mathfrak{g}^{\otimes n-1} \to \ldots \to \mathfrak{g} \to k\,,$$

whose boundary map d is given by the formula

$$d(g_1 \otimes \ldots \otimes g_n) = \sum_{1 \leqq i < j \leqq n} (-1)^j (g_1 \otimes \ldots \otimes g_{i-1} \otimes [g_i, g_j] \otimes g_{i+1} \otimes \ldots \otimes \hat{g}_j \otimes \ldots \otimes g_n)\,.$$

Note that d is a lifting of the classical Chevalley-Eilenberg boundary map $\bar{d}: \Lambda^n \mathfrak{g} \to \Lambda^{n-1}\mathfrak{g}$. One striking point in the proof of $d^2 = 0$ is the following fact: the only property of the bracket, which is needed, is the so-called *Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]\,, \quad \text{for all } x, y, z \in \mathfrak{g}\,.$$

So, it is natural to introduce new objects: *the Leibniz algebras*, which are modules over a commutative ring k, equipped with a bilinear map $[-,-]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Leibniz identity. Since the Leibniz identity is equivalent to the classical Jacobi identity when the bracket is skew-symmetric, this notion is a sort of "noncommutative" analog of Lie algebras.

Hence for any Leibniz algebra there is defined a homology theory (and dually a cohomology theory) $HL_*(\mathfrak{g}) := H_*(C_*(\mathfrak{g}), d)$.

The principal aim of this paper is to answer affirmatively the following question. Is HL_* (resp. HL^*) a Tor-functor (resp. Ext-functor)? This leads naturally to the search for a universal enveloping algebra of a Leibniz algebra.

In Sect. 1 we give examples of Leibniz algebras and we show that the underlying module of a free Leibniz algebra is a tensor module. Then we define the notion of *representation* (and *co-representation*) of a Leibniz algebra. This enables us to define homology and cohomology with nontrivial coefficients.

In Sect. 2 we construct the universal enveloping algebra $UL(\mathfrak{g})$ of a Leibniz algebra $\mathfrak{g}$ [as a certain quotient of the tensor algebra $T(\mathfrak{g} \oplus \mathfrak{g})$] and prove that the category of $UL(\mathfrak{g})$-modules is equivalent to the category of $\mathfrak{g}$-representations. We show a Poincaré-Birkhoff-Witt theorem in this framework.

In Sect. 3 we prove the main theorem, that is the isomorphisms

$$HL_*(\mathfrak{g}, A) \cong \mathrm{Tor}_*^{UL(\mathfrak{g})}(U(\mathfrak{g}_{\mathrm{Lie}}), A),$$
$$HL^*(\mathfrak{g}, M) \cong \mathrm{Ext}^*_{UL(\mathfrak{g})}(U(\mathfrak{g}_{\mathrm{Lie}}), M).$$

Here $\mathfrak{g}_{\mathrm{Lie}}$ is the Lie algebra associated to $\mathfrak{g}$, $U(\mathfrak{g}_{\mathrm{Lie}})$ is the ordinary enveloping algebra of $\mathfrak{g}_{\mathrm{Lie}}$, A is a co-representation of $\mathfrak{g}$ and M a representation of $\mathfrak{g}$. The main tools that are used are Cartan's formulas and a Koszul type complex in the noncommutative framework. As a consequence we get the triviality of these theories for free Leibniz algebras.

In the last section we relate central extensions of $sl_n(A)$ with the Hochschild homology group $HH_1(A)$ of the associative algebra A (analog of a theorem of Bloch-Kassel-Loday). It is interesting to note that the Virasoro algebra is a universal extension of $\mathrm{Der}(\mathbb{C}[z, z^{-1}])$ both in the Lie framework and in the Leibniz framework.

In the whole paper k is a commutative ring with unit.

1 Representations of Leibniz algebras and (co)homology groups

(1.1) **Definition of Leibniz algebras.** A Leibniz algebra $\mathfrak{g}$ over k is a k-module equipped with a bilinear map, called *bracket*,

$$[-,-]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

satisfying the *Leibniz identity*:

(1.1.1) $$[x,[y,z]] = [[x,y],z] - [[x,z],y] \quad \text{for all } x,y,z \in \mathfrak{g}.$$

This is in fact a *right* Leibniz algebra. The dual notion of *left* Leibniz algebra is made out of the dual relation $[[x,y],z] = [x,[y,z]] - [y,[x,z]]$, for all $x,y,z \in \mathfrak{g}$. In this paper we are considering only right Leibniz algebras. A morphism of Leibniz algebras $\mathfrak{g} \to \mathfrak{g}'$ is a k-linear map which respects the bracket.

A Leibniz algebra is a *Lie algebra* if the condition

(1.1.2) $$[x,x] = 0 \quad \text{for all } x \in \mathfrak{g},$$

is fullfilled. Note that this condition implies the skew-symmetry property: $[x,y] + [y,x] = 0$. Then the Leibniz identity is equivalent to the Jacobi identity.

For any Leibniz algebra $\mathfrak{g}$ there is associated a Lie algebra $\mathfrak{g}_{\mathrm{Lie}}$, obtained by quotienting by the relation (1.1.2). The quotient map $\mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}}$ is universal for the maps from $\mathfrak{g}$ to any Lie algebra which respect the bracket. The image of $x \in \mathfrak{g}$ in $\mathfrak{g}_{\mathrm{Lie}}$ is denoted $\bar{x}$.

(1.2) *Examples.* (a) Obviously any Lie algebra is a Leibniz algebra.

(b) Let A be an associative k-algebra equipped with a k-module map $D: A \to A$ satisfying the condition

(1.2.1) $$D(a(Db)) = DaDb = D((Da)b) \quad \text{for any } a,b \in A.$$

Define a bilinear map on A by

$$[x,y]:=x(Dy)-(Dy)x\,.$$

Then, it is immediate to verify that this bracket satisfies the Leibniz relation. So A becomes a Leibniz algebra, that we denote by A_L. In general it is not a Lie algebra (unless $D=\mathrm{id}$). Here are examples of operators D which satisfy condition (1.2.1):

(b1) D is an algebra map, and is an idempotent ($D^2=D$).

(b2) A is a superalgebra (i.e. A is $\mathbb{Z}/2$-graded), so that any x can be uniquely written $x=x_+ + x_-$. Then take $D(x)=x_+$.

(b3) D is a square-zero derivation, that is $D(ab)=(Da)b+a(Db)$ and $D^2a=0$.

(c) Let A be an associative algebra and $b: A^{\otimes 3}\to A^{\otimes 2}$ the Hochschild boundary. Then $A\otimes A/\operatorname{Im} b$, equipped with the bracket $[a\otimes b, c\otimes d]=(ab-ba)\otimes(cd-dc)$ is a Leibniz bracket (cf. 4.4.).

(d) Let V be a k-module. The free Leibniz algebra $\mathscr{L}(V)$ over V is the universal Leibniz algebra for maps from V to Leibniz algebras. It can be constructed as a quotient of the free non-associative k-algebra over V like in [CE, p. 285]. Here is a more explicit description.

(1.3) **Lemma.** *The tensor module* $\bar{T}(V)=V\oplus V^{\otimes 2}\oplus\ldots\oplus V^{\otimes n}\oplus\ldots$ *equipped with the bracket defined inductively by*

(1.3.1) $$[x,v]=x\otimes v\,,\quad \textit{for}\quad x\in\bar{T}(V),\ v\in V\,,$$

(1.3.2) $$[x,y\otimes v]=[x,y]\otimes v-[x\otimes v,y]\,,\quad \textit{for}\quad x,y\in\bar{T}(V),\ v\in V\,,$$

is the free Leibniz algebra over V.

Proof. Let us first prove that we have defined a Leibniz algebra. Since $\bar{T}(V)$ is graded we can work by induction. The hypothesis implies that the Leibniz relation is true for any $z\in V\oplus V^{\otimes 2}\oplus\ldots\oplus V^{\otimes n-1}$. Let $z=t\otimes v\in V^{\otimes n}$, with $t\in V^{\otimes n-1}$ and $v\in V$. By applying (1.3.2) and the induction hypothesis one gets, on one hand,

$$\begin{aligned}[x,[y,z]]&=[x,[y,t\otimes v]]=[x,[y,t]\otimes v]-[x,[y\otimes v,t]]\\&=[x,[y,t]]\otimes v-[x\otimes v,[y,t]]-[[x,y\otimes v],t]+[[x,t],y\otimes v]\\&=[x,[y,t]]\otimes v-[x\otimes v,[y,t]]-[[x,y\otimes v],t]\\&\quad+[[x,t],y]\otimes v-[[x,t]\otimes v,y]\,.\end{aligned}$$

On the other hand, one gets,

$$\begin{aligned}[[x,y],z]&=[[x,y],t\otimes v]=[[x,y],t]\otimes v-[[x,y]\otimes v,t]\\&=[[x,y],t]\otimes v-[[x,y\otimes v],t]-[[x\otimes v,y],t]\end{aligned}$$

and

$$[x,z],y]=[[x,t\otimes v],y]=[[x,t]\otimes v,y]-[[x\otimes v,t],y]\,.$$

Now adding these three elements one gets

$$[x,[y,z]]-[[x,y],z]+[[x,z],y]=0$$

by the induction hypothesis, some cancellation and (1.3.2).

Let us now prove that the inclusion map $V\hookrightarrow\bar{T}(V)$ is universal among the k-linear maps $\phi: V\to\mathfrak{g}$ where $\mathfrak{g}$ is a Leibniz algebra. Define $f:\bar{T}(V)\to\mathfrak{g}$ inductively by

$$f(v)=\phi(v)\quad\text{and}\quad f(v_1\otimes\ldots\otimes v_n)=[f(v_1\otimes\ldots\otimes v_{n-1}),f(v_n)]\,,$$

where the latter is the bracket in $\mathfrak{g}$. Note that this definition is forced by relation (1.3.1). Since $\mathfrak{g}$ is a Leibniz algebra, f satisfies relation (1.3.2). This proves that $\bar{T}(V)$ is universal and therefore $\mathscr{L}(V) = \bar{T}(V)$. □

(1.4) *Remarks.* If V is one-dimensional, generated by x, then $\bar{T}(V) = kx \oplus kx^2 \oplus \ldots \oplus kx^n \oplus \ldots$ and the Leibniz structure is given by

$$[x^i, x^j] = \begin{cases} x^{i+1} & \text{if } j = 1\,, \\ 0 & \text{if } j \geqq 2\,. \end{cases}$$

For any V the Lie algebra associated to $\mathscr{L}(V)$ is the free Lie algebra $L(V)$, which can be identified with the primitive part of the tensor Hopf algebra $T(V) = k \oplus \bar{T}(V)$. Let us denote by $[-,-]_L$ the Leibniz bracket on $\bar{T}(V)$ and by $[-,-]$ the Lie bracket on $\bar{T}(V)$, i.e. $[a,b] = ab - ba$. Then $[\ldots[v_1, v_2]_L, v_3]_L \ldots, v_n]_L = v_1 \otimes v_2 \otimes \ldots \otimes v_n$ and the map $\gamma: \mathscr{L}(V) \to L(V)$ is given by $\gamma(v_1 \otimes \ldots \otimes v_n) = [\ldots[v_1, v_2], v_3] \ldots, v_n]$.

(1.5) *Representations and co-representations.* An *abelian extension of Leibniz algebras*

$$0 \to M \to \mathfrak{h} \to \mathfrak{g} \to 0$$

is an exact sequence of Leibniz algebras, which is split as a sequence of k-modules and which verifies $[M, M] = 0$.

Then M is equipped with two actions (left and right) of $\mathfrak{g}$,

$$[-,-] = \mathfrak{g} \times M \to M \quad \text{and} \quad [-,-]: M \times \mathfrak{g} \to M$$

which satisfy the following three axioms,

$$(MLL) \qquad [m,[x,y]] = [[m,x],y] - [[m,y],x]$$

$$(LML) \qquad [x,[m,y]] = [[x,m],y] - [[x,y],m]$$

$$(LLM) \qquad [x,[y,m]] = [[x,y],m] - [[x,m],y]$$

for any $m \in M$ and $x, y \in \mathfrak{g}$.

Note that the last two relations imply the following:

$$(ZD) \qquad [x,[m,y]] + [x,[y,m]] = 0\,.$$

By definition a *representation* of the Leibniz algebra $\mathfrak{g}$ is a k-module M equipped with two actions of $\mathfrak{g}$ satisfying these three axioms.

Dually, a *co-representation* of the Leibniz algebra $\mathfrak{g}$ is a k-module N equipped with two actions of $\mathfrak{g}$ satisfying the following three axioms

$$(MLL)' \qquad [[x,y],m] = [x,[y,m]] - [y,[x,m]]$$

$$(LML)' \qquad [y,[m,x]] = [[y,m],x] - [m,[x,y]]$$

$$(LLM)' \qquad [[m,x],y] = [m,[x,y]] - [[y,m],x]\,.$$

The last two relations imply

$$(ZD)' \qquad [y,[m,x]] + [[m,x],y] = 0\,.$$

A representation is called *symmetric* when

$$[m,x] + [x,m] = 0 \quad \text{for all } m \in M, x \in \mathfrak{g}\,.$$

Under this hypothesis any one of the six axioms implies the other five.

In particular a symmetric representation is also a symmetric co-representation and is equivalent to a module over $\mathfrak{g}_{\mathrm{Lie}}$ (that is a Lie representation).

A symmetric representation is uniquely determined by a right action and axiom (MLL).

The actions (left and right) of a Leibniz algebra on itself determine a representation.

A representation (resp. co-representation) is called *anti-symmetric* when

$$[x, m] = 0, \quad (\text{resp. } [m, x] = 0),\ x \in \mathfrak{g},\ m \in M .$$

A representation or co-representation is called *trivial* when

$$[x, m] = 0 = [m, x],\ x \in \mathfrak{g},\ m \in M .$$

A *morphism* $f: M \to M'$ of $\mathfrak{g}$-representations is a k-linear map which is compatible with the left and right actions of $\mathfrak{g}$ (and similarly for co-representations).

(1.6) *Action of a Leibniz algebra on another Leibniz algebra and crossed modules.* An exact sequence of Leibniz algebras

$$0 \to \mathfrak{g}' \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{g}'' \to 0$$

is said to be *split* when there exists a Leibniz morphism $s: \mathfrak{g}'' \to \mathfrak{g}$ such that $p \circ s = \mathrm{id}_{\mathfrak{g}''}$.

By using s and the Leibniz product $[-,-]_{\mathfrak{g}}$ of $\mathfrak{g}$ one gets two actions of $\mathfrak{g}''$ on $\mathfrak{g}'$:

$$[-,-]: \mathfrak{g}'' \times \mathfrak{g}' \to \mathfrak{g}', \qquad [x'', x'] := [s(x''), i(x')]_{\mathfrak{g}},$$
$$[-,-]: \mathfrak{g}' \times \mathfrak{g}'' \to \mathfrak{g}', \qquad [x', x''] := [i(x'), s(x'')]_{\mathfrak{g}},$$

These actions satisfy 6 relations, which are obtained from the Leibniz relation by taking one variable in $\mathfrak{g}$ and two in $\mathfrak{g}''$ (3 relations), and one variable in $\mathfrak{g}''$ and two in $\mathfrak{g}'$ (3 relations).

Let us define an *action of the Leibniz algebra* $\mathfrak{g}''$ on the Leibniz algebra $\mathfrak{g}'$ as two actions of $\mathfrak{g}''$ on $\mathfrak{g}'$ (denoted as above) satisfying these 6 relations.

It is clear that such a data enables us to reconstruct the semi-direct product $\mathfrak{g} = \mathfrak{g}' \ltimes \mathfrak{g}''$ (i.e. a split extension).

A *crossed module* is a homomorphism of Leibniz algebra $\mu: \mathfrak{g} \to \mathfrak{h}$ together with an action of $\mathfrak{h}$ on $\mathfrak{g}$ such that

(a) $\mu[h, g] = [h, \mu g]$, $\mu[g, h] = [\mu g, h]$,

(b) $[g, \mu g'] = [g, g'] = [\mu g, g']$, for $g, g' \in \mathfrak{g}$, $h \in \mathfrak{h}$.

(1.7) *Extensions of Leibniz algebras.* Let $\mathfrak{g}$ be a Leibniz algebras and M be a representation of $\mathfrak{g}$. An *abelian extension* of $\mathfrak{g}$ by M is a short exact sequence of Leibniz algebras

$$(\mathfrak{h}) \qquad 0 \to M \to \mathfrak{h} \to \mathfrak{g} \to 0$$

such that the sequence is split over k, the Leibniz bracket on M is trivial and the action of $\mathfrak{g}$ on M induced by the extension is the prescribed one. Two such extensions $(\mathfrak{h})$ and $(\mathfrak{h}')$ are isomorphic when there exists a Leibniz algebra map from $\mathfrak{h}$ to $\mathfrak{h}'$ which is compatible with the identity on M and on $\mathfrak{g}$. One denotes by $\mathrm{Ext}(\mathfrak{g}, M)$ the set of isomorphism classes of extensions of $\mathfrak{g}$ by M.

Let $f: \mathfrak{g}^{\otimes 2} \to M$ be a k-linear map. We define a bracket on $\mathfrak{h} = M \oplus \mathfrak{g}$ by

$$[(m_1, x_1), (m_2, x_2)] = ([m_1, x_2] + [x_1, m_2] + f(x_1, x_2), [x_1, x_2]) .$$

Then $\mathfrak{h}$ is a Leibniz algebra iff

(1.7.1)
$$[x, f(y,z)] + [f(x,z), y] - [f(x,y), z] - f([x,y], z) + f([x,z], y) + f(x, [y,z]) = 0$$

for all $x, y, z \in \mathfrak{g}$. If this condition holds, then we obtain an extension

$$0 \to M \xrightarrow{i} \mathfrak{h} \xrightarrow{p} \mathfrak{g} \to 0$$

of Leibniz algebras, where $i(m) = (m, 0)$, $p(m, x) = x$. Moreover this extension is split in the category of Leibniz algebras iff there exists a k-linear map $g:\mathfrak{g} \to M$ such that

(1.7.2)
$$f(x,y) = [x, g(y)] + [g(x), y] - g([x,y]), x, y \in \mathfrak{g}.$$

An easy consequence of these facts is the following natural bijection:

(1.7.3)
$$\mathrm{Ext}(\mathfrak{g}, M) \cong Z^2(\mathfrak{g}, M)/B^2(\mathfrak{g}, M).$$

Here $Z^2(\mathfrak{g}, M)$ is the set of all k-linear maps $f:\mathfrak{g}^{\otimes 2} \to M$ satisfying (1.7.1) and $B^2(\mathfrak{g}, M)$ is the set of such f which satisfy (1.7.2) for some k-linear map $g:\mathfrak{g} \to M$.

(1.8) *Cohomology of Leibniz algebras.* Let $\mathfrak{g}$ be a Leibniz algebra and M be a representation of $\mathfrak{g}$. Denote

$$C^n(\mathfrak{g}, M) := \mathrm{Hom}_k(\mathfrak{g}^{\otimes n}, M), \qquad n \geqq 0.$$

Let

$$d^n: C^n(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M)$$

be a k-homomorphism defined by

$$\begin{aligned}(d^n f)(x_1, \dots, x_{n+1}) &:= [x_1, f(x_2, \dots, x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}), x_i] \\ &+ \sum_{i \leqq i < j \leqq n} (-1)^{j+1} f(x_1, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \hat{x}_j, \dots, x_n).\end{aligned}$$

In the notation of Sect. 3 below we have

$$C^*(\mathfrak{g}, M) = \mathrm{Hom}_{UL(\mathfrak{g})}(W_*(\mathfrak{g}), M) \quad \text{and} \quad d^n = \mathrm{Hom}_{UL(\mathfrak{g})}(d_n, M)$$

and from Lemma 3.1 below it follows that

$$d^{n+1} d^n = 0, \quad \text{for} \quad n \geqq 0.$$

Therefore $(C^*(\mathfrak{g}, M), d)$ is a cochain complex, whose cohomology is called the *cohomology of the Leibniz algebra* $\mathfrak{g}$ with coefficients in the representation M:

$$HL^*(\mathfrak{g}, M) := H^*(C^*(\mathfrak{g}, M), d).$$

For $n = 0$, $HL^0(\mathfrak{g}, M)$ is the submodule of *left invariants* of M, i.e.

$$HL^0(\mathfrak{g}, M) = \{m \in M \mid [x, m] = 0 \text{ for any } x \in \mathfrak{g}\}.$$

For $n = 1$ a 1-cocycle is a k-module homomorphism

$$\delta: \mathfrak{g} \to M$$

satisfying the identity

$$\delta([x,y]) = [\delta(x),y] + [x,\delta(y)] .$$

Such a map is called a *derivation* from $\mathfrak{g}$ to M and the k-module of derivations is denoted $\mathrm{Der}(\mathfrak{g}, M)$. It is a coboundary if it has the form $ad_m(x) = [x,m]$ for some $m \in M$; ad_m is called an *inner derivation*. Therefore

$$HL^1(\mathfrak{g},M) = \mathrm{Der}(\mathfrak{g},M)/\{\text{inner derivations}\} .$$

When M is antisymmetric we have

$$HL^1(\mathfrak{g},M) = \mathrm{Der}(\mathfrak{g},M) = \{f:\mathfrak{g} \to M \mid f([x,y]) = [x,f(y)]\} .$$

It is clear that $1_\mathfrak{g} \in HL^1(\mathfrak{g},\mathfrak{g}^a)$, where $\mathfrak{g}^a$ is the antisymmetric representation, whose underlying k-module is $\mathfrak{g}$ and $\mathfrak{g}^a \times \mathfrak{g} \to \mathfrak{g}^a$ is the ordinary bracket on $\mathfrak{g}$. Therefore if $\mathfrak{g} \neq 0$, then $HL^1(\mathfrak{g},\mathfrak{g}^a) \neq 0$.

When M is symmetric, then $HL^1(\mathfrak{g},M) = HL^1(\mathfrak{g}_{\mathrm{Lie}},M) = H^1(\mathfrak{g}_{\mathrm{Lie}},M)$.

It is easy to check that the sets of 2-cocycles and 2-boundaries coincide with $Z^2(\mathfrak{g},M)$ and $B^2(\mathfrak{g},M)$ respectively. Therefore by (1.7.3) the group $HL^2(\mathfrak{g},M)$ classifies the equivalence classes of extensions of the Leibniz algebra $\mathfrak{g}$ by M.

(1.9) **Proposition.** *For any Leibniz algebra $\mathfrak{g}$ and any representation M, there is a natural bijection*

$$\mathrm{Ext}(\mathfrak{g},M) \cong HL^2(\mathfrak{g},M) . \qquad \square$$

Like in [C] we can easily show that crossed modules of Leibniz algebras are classified by HL^3.

(1.10) *Characteristic element of a Leibniz algebra.* Let $\mathfrak{g}$ be a Leibniz algebra. We denote by $\mathfrak{g}^{\mathrm{ann}}$ the kernel of the natural projection $\mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}}$. Therefore we have an exact sequence of Leibniz algebras,

(1.10.1) $$0 \to \mathfrak{g}^{\mathrm{ann}} \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}} \to 0 .$$

By definition of $\mathfrak{g}_{\mathrm{Lie}}$ the Leibniz algebra $\mathfrak{g}^{\mathrm{ann}}$ coincides with the right ideal of $\mathfrak{g}$ generated by the elements of the form $[x,x]$, $x \in \mathfrak{g}$. It follows from the Leibniz identity that

$$[x,[y,y]] = 0 , \quad \text{for} \quad x,y \in \mathfrak{g} .$$

Therefore (1.10.1) is an abelian extension of $\mathfrak{g}_{\mathrm{Lie}}$ by $\mathfrak{g}^{\mathrm{ann}}$. Moreover the induced structure of representation of $\mathfrak{g}_{\mathrm{Lie}}$ on $\mathfrak{g}^{\mathrm{ann}}$ is anti-symmetric. By (1.9) the extension (1.10.1) determines an element in $HL^2(\mathfrak{g}_{\mathrm{Lie}},\mathfrak{g}^{\mathrm{ann}})$. We call this element the *characteristic element* of the Leibniz algebra $\mathfrak{g}$ and denote it by $ch(\mathfrak{g}) \in HL^2(\mathfrak{g}_{\mathrm{Lie}},\mathfrak{g}^{\mathrm{ann}})$.

For example, when $\mathfrak{g}$ is a free Leibniz algebra with one generator, then $HL^2(\mathfrak{g}_{\mathrm{Lie}},\mathfrak{g}^{\mathrm{ann}}) \approx k$ and $ch(\mathfrak{g})$ is a generator.

Let M be a representation of $\mathfrak{g}$. Let us denote by M_{sym} the quotient of M by the relations $[x,m] + [m,x] = 0$ for $x \in \mathfrak{g}$, $m \in M$. This is a symmetric representation. The kernel of the projection map $M \twoheadrightarrow M_{\mathrm{sym}}$ is antisymmetric and is denoted by M_{anti}. Therefore we obtain a short exact sequence

$$0 \to M_{\mathrm{anti}} \to M \to M_{\mathrm{sym}} \to 0$$

and so a canonical element in $\mathrm{Ext}^1(M_{\mathrm{sym}},M_{\mathrm{anti}})$, where the Ext-group is taken in the category of representations of $\mathfrak{g}$. Note that the categories of antisymmetric representations and symmetric representations are both equivalent to the category of Lie representations of $\mathfrak{g}_{\text{Lie}}$.

(1.11) *Homology of Leibniz algebras.* Let $\mathfrak{g}$ be a Leibniz algebra and A be a co-representation of $\mathfrak{g}$. Denote $C_n(\mathfrak{g}, A) := A \otimes \mathfrak{g}^{\otimes n}$, $n \geqq 0$. We define a k-linear map

$$d_n = d_n^C : C_n(\mathfrak{g}, A) \to C_{n-1}(\mathfrak{g}, A)$$

by

$$\begin{aligned} d_n(m, x_1, \ldots, x_n) \\ &= ([m, x_1], x_2, \ldots, x_n) + \sum_{i=2}^{n} (-1)^i ([x_i, m], x_1, \ldots, \hat{x}_i, \ldots, x_n) \\ &+ \sum_{1 \leqq i < j \leqq n} (-1)^{j+1} (m, x_1, \ldots, x_{i-1}, [x_i, x_j], \ldots, \hat{x}_j, \ldots, x_n) . \end{aligned}$$

In the notation of Sect. 3 below we have

$$C_*(\mathfrak{g}, A) = W_*(\mathfrak{g}) \otimes_{UL(\mathfrak{g})} A \quad \text{and} \quad d_n^C = d_n^W \otimes 1_A ,$$

and from Lemma 3.1 below it follows that

$$d_n d_{n+1} = 0 , \qquad n \geqq 0 .$$

Therefore $(C_*(\mathfrak{g}, A), d)$ is a well-defined chain complex, whose homology is called the *homology of the Leibniz algebra* $\mathfrak{g}$ *with coefficients in the co-representation* A:

$$HL_*(\mathfrak{g}, A) := H_*(C_*(\mathfrak{g}, A), d) .$$

When A is symmetric, then $HL_*(\mathfrak{g}, A)$ coincides with the homology theory defined in [L] and [C]. A similar remark applies for cohomology.

(1.12) *Relation with the Chevalley-Eilenberg (co)homology of Lie algebras.* Let $\mathfrak{g}$ be a Leibniz algebra and M be a symmetric representation. Then M has a natural structure of $\mathfrak{g}_{\text{Lie}}$-module and the natural projection

$$C_n(\mathfrak{g}, M) = M \otimes \mathfrak{g}^{\otimes n} \to M \otimes \Lambda^n \mathfrak{g} , \qquad n \geqq 0 ,$$

is compatible with boundary maps. Therefore it induces a homomorphism

$$HL_*(\mathfrak{g}, M) \to H_*(\mathfrak{g}_{\text{Lie}}, M)$$

to the classical Chevalley-Eilenberg homology of the Lie algebra $\mathfrak{g}_{\text{Lie}}$, which is an isomorphism in dimensions 0 and 1 and a surjection in dimension 2. One has a similar homomorphism for cohomology

$$H^*(\mathfrak{g}_{\text{Lie}}, M) \to HL^*(\mathfrak{g}, M) .$$

2 Universal enveloping algebra of a Leibniz algebra

(2.1) Let $\mathfrak{g}^l$ and $\mathfrak{g}^r$ be two copies of the Leibniz algebra $\mathfrak{g}$ which is supposed to be free as a k-module. We denote by l_x and r_x the elements of $\mathfrak{g}^l$ and $\mathfrak{g}^r$ corresponding to $x \in \mathfrak{g}$. Consider the tensor k-algebra $T(\mathfrak{g}^l \oplus \mathfrak{g}^r)$, which is associative and unital.

Let I be the two-sided ideal corresponding to the relations

$$\begin{cases} \text{(i)} & r_{[x,y]} = r_x r_y - r_y r_x \,, \\ \text{(ii)} & l_{[x,y]} = l_x r_y - r_y l_x \,, \\ \text{(iii)} & (r_y + l_y) l_x = 0 \,, \quad \text{for any } x, y \in \mathfrak{g} \,. \end{cases}$$

(2.2) **Definition.** The *universal enveloping algebra* of the Leibniz algebra $\mathfrak{g}$ is the associative and unital algebra

$$UL(\mathfrak{g}) := T(\mathfrak{g}^l \oplus \mathfrak{g}^r)/I \,.$$

(2.3) **Theorem.** *The category of representations (resp. co-representations) of the Leibniz algebra $\mathfrak{g}$ is equivalent to the category of right (resp. left) modules over $UL(\mathfrak{g})$.*

Proof. Let M be a representation of $\mathfrak{g}$. Define a right action of $UL(\mathfrak{g})$ on the k-module M as follows. First $\mathfrak{g}^l$ and $\mathfrak{g}^r$ act on M by

$$m \cdot l_x = [x, m] \,, \qquad m \cdot r_x = [m, x] \,.$$

These actions are extended to an action of $T(\mathfrak{g}^l \oplus \mathfrak{g}^r)$ by composition and linearity. Axiom (MLL) (resp. (LML)) of representations implies that the elements of type (i) (resp. (ii)) act trivially. In presence of (LML), axiom (LLM) is equivalent to (ZD). This relation implies that elements of type (iii) act trivially. So M is equipped with a structure of right $UL(\mathfrak{g})$-module.

In the other direction it is immediate that, starting with a right $UL(\mathfrak{g})$-module, the restrictions of the actions to $\mathfrak{g}^l$ and $\mathfrak{g}^r$ give two actions of $\mathfrak{g}$ which make M into a representation.

The proof in the co-representation case is analogous. □

(2.4) **Proposition.** *The map $\eta : U(\mathfrak{g}_{\text{Lie}}) \oplus U(\mathfrak{g}_{\text{Lie}}) \otimes \mathfrak{g} \xrightarrow{\sim} UL(\mathfrak{g})$, $\bar{x} \mapsto r_x$, $1 \otimes y \to l_y$, is a $U(\mathfrak{g}_{\text{Lie}})$-module isomorphism. Under this isomorphism the product structure on the former module is induced by the product structure of $U(\mathfrak{g}_{\text{Lie}})$ and the formulas*

$$(1 \otimes x)\bar{y} = \bar{y} \otimes x + 1 \otimes [x, y] \,, \tag{2.4.1}$$

$$(1 \otimes y)(1 \otimes x) = -\bar{y} \otimes x \,, \quad \text{for} \quad x, y \in \mathfrak{g} \,. \tag{2.4.2}$$

Proof. Recall that the image of $x \in \mathfrak{g}$ in $\mathfrak{g}_{\text{Lie}}$ is denoted by $\bar{x}$. By (2.1.i) it is clear that $r_{[x,x]} = 0$, and so $\mathfrak{g}^r$ generates in $UL(\mathfrak{g})$ an algebra isomorphic to $U(\mathfrak{g}_{\text{Lie}})$. Hence the map η is well-defined.

Define a map $\theta : UL(\mathfrak{g}) \to U(\mathfrak{g}_{\text{Lie}}) \oplus U(\mathfrak{g}_{\text{Lie}}) \otimes \mathfrak{g}$ as follows: $\theta(r_x) = \bar{x}$ and $\theta(l_y) = 1 \otimes y$. Then θ is extended over $T(\mathfrak{g}^l \oplus \mathfrak{g}^r)$ by product, using formulas (2.4.1) and (2.4.2). Obviously formula (2.1.i) is fulfilled. Formula (2.1.ii) is a consequence of (2.4.1). Formula (2.1.iii) is a consequence of (2.4.2). □

(2.5) **Proposition.** *There are algebra homomorphisms*

$$d_0, d_1 : UL(\mathfrak{g}) \to U(\mathfrak{g}_{\text{Lie}}) \quad \textit{and} \quad s_0 : U(\mathfrak{g}_{\text{Lie}}) \to UL(\mathfrak{g}) \,,$$

which satisfy

$$d_0 s_0 = d_1 s_0 = \text{id} \,, \quad \textit{and} \quad (\operatorname{Ker} d_1)(\operatorname{Ker} d_0) = 0 \,.$$

Proof. Define $d_0, d_1 : UL(\mathfrak{g}) \to U(\mathfrak{g}_{\text{Lie}})$ by

$$\begin{cases} d_0(l_x) = 0 \\ d_0(r_x) = \bar{x} \end{cases} \qquad \begin{cases} d_1(l_x) = -\bar{x} \\ d_1(r_x) = \bar{x} \end{cases}$$

and $s_0 : U(\mathfrak{g}_{\text{Lie}}) \to UL(\mathfrak{g})$ by $s_0(\bar{x}) = r_x$.

It is clear that d_0, d_1 and s_0 are well-defined algebra homomorphisms (since $r_{[x,x]} = 0$).

The ideal $\operatorname{Ker} d_0$ (resp. $\operatorname{Ker} d_1$) is generated by the l_x's (resp. $(r_x + l_x)$'s), so the formula $(\operatorname{Ker} d_1)(\operatorname{Ker} d_0)$ follows from the relation $(r_x + l_x)l_y = 0$. □

(2.6) *Induced representation from Lie-modules.* Let M be a Lie-representation of the Lie algebra $\mathfrak{g}_{\mathrm{Lie}}$, that is a right $U(\mathfrak{g}_{\mathrm{Lie}})$-module. There are two ways to look at it as a module over $UL(\mathfrak{g})$: under d_0 or under d_1. The first one gives an anti-symmetric representation of $\mathfrak{g}$, and the second one gives a symmetric representation of $\mathfrak{g}$.

(2.7) *Examples.* (a) Suppose that $\mathfrak{g}$ is an *abelian Leibniz* (hence Lie) algebra, that is $[x, y] = 0$, for $x, y \in \mathfrak{g}$. Then $U(\mathfrak{g}_{\mathrm{Lie}}) = U(\mathfrak{g}) = S(\mathfrak{g})$ (symmetric algebra) and $UL(\mathfrak{g}) \cong S(\mathfrak{g}) \oplus S(\mathfrak{g}) \otimes \mathfrak{g}$, where the product is induced by the product of $S(\mathfrak{g})$ and

$$\begin{cases} (1 \otimes x)y = y \otimes x \in S(\mathfrak{g}) \otimes \mathfrak{g}\,, \\ (1 \otimes x)(1 \otimes y) = -y \otimes x \in S(\mathfrak{g}) \otimes \mathfrak{g}\,. \end{cases}$$

(b) Let V be a k-module and let $\mathscr{L}(V)$ be the free Leibniz algebra over V (cf. 1.3). It is well-known that $U(\mathscr{L}(V)_{\mathrm{Lie}}) = U(L(V)) \cong T(V)$. Since $\mathscr{L}(V) \cong \bar{T}(V) = T(V)/k$ as a k-module, one has an isomorphism of k-modules:

$$UL(\mathscr{L}(V)) \cong T(V) \oplus T(V) \otimes \bar{T}(V) \cong T(V) \otimes T(V)\,.$$

But the algebra structure is *not* the product of the two algebra structures. Denoting by r_v (resp. l_v) the generators of the first (resp. second) copy of $T(V)$, the product is induced by the classical product structure on the first copy of $T(V)$ and by

$$\begin{cases} l_v r_w = r_w l_v + l_{[v,w]} \\ l_v l_w = -r_w l_v\,. \end{cases}$$

For instance, if V is 1-dimensional, then $UL(\mathscr{L}(V))$ is isomorphic to the algebra $k\{x, y\}/(xy = 0)$, where $\{-, -\}$ means non-commutative polynomials.

(2.8) *A Poincaré-Birkhoff-Witt type isomorphism.* Let $\tau : V \twoheadrightarrow W$ be an epimorphism of k-modules. Define the associative algebra $SL(\tau)$ as the quotient of $S(W) \otimes T(V)$ by the 2-sided ideal generated by $1 \otimes xy + \tau(x) \otimes y$, for all $x, y \in V$.

Note that $UL(\mathfrak{g})$ is a filtered algebra, the filtration being induced by the filtration of $T(\mathfrak{g}^l \oplus \mathfrak{g}^r)$, that is $F_n UL(\mathfrak{g}) = \{\text{image of } k \oplus E \oplus \ldots \oplus E^{\otimes n} \text{ in } UL(\mathfrak{g})\}$, where $E = \mathfrak{g}^l \oplus \mathfrak{g}^r$.

The associated graded algebra is denoted $\operatorname{gr} UL(\mathfrak{g}) := \bigoplus_{n \geq 0} \operatorname{gr}_n UL(\mathfrak{g})$.

(2.9) **Theorem** (PBW). *For any Leibniz k-algebra $\mathfrak{g}$ such that $\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{Lie}}$ are free as k-modules, there is an isomorphism of graded associative k-algebras*

$$\operatorname{gr} UL(\mathfrak{g}) \cong SL(\mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}})\,.$$

Proof. Note that, as a k-module, $SL(\tau)$ is isomorphic to $S(W) \oplus S(W) \otimes V$. The classical PBW theorem gives an isomorphism $\operatorname{gr} U(\mathfrak{g}_{\mathrm{Lie}}) \cong S(\mathfrak{g}_{\mathrm{Lie}})$. By Proposition 2.4, the expected isomorphism is induced by the PBW isomorphism and the canonical isomorphism $\mathfrak{g}^l \cong \mathfrak{g}$. □

3 Cohomology and homology of Leibniz algebras as derived functors

In this section we prove that homology and cohomology of Leibniz algebras are suitable Tor and Ext groups respectively.

(3.0) Let $\mathfrak{g}$ be a Leibniz algebra and $UL(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. We define a chain complex $W_*(\mathfrak{g})$ in the category of right $UL(\mathfrak{g})$-modules as follows. Denote by $W_n(\mathfrak{g})$ the right $UL(\mathfrak{g})$-module $\mathfrak{g}^{\otimes n} \otimes UL(\mathfrak{g})$. Since $\mathfrak{g}^{\otimes n}$ is free over k, $W_n(\mathfrak{g})$ is free over $UL(\mathfrak{g})$. For short, we shall write $\langle x_1, \ldots, x_n\rangle r$ for $(x_1 \otimes \ldots \otimes x_n) \otimes r$, where $x_1, \ldots, x_n \in \mathfrak{g}$, $r \in UL(\mathfrak{g})$. Let

$$d_n : W_n(\mathfrak{g}) \to W_{n-1}(\mathfrak{g}), \qquad n \geqq 1,$$

be the homomorphism of right $UL(\mathfrak{g})$-modules given by

$$\begin{aligned} &d_n\langle x_1, \ldots, x_n\rangle \\ &\quad = \langle x_2, \ldots, x_n\rangle l_{x_1} + \sum_{i=2}^{n} (-1)^i \langle x_1, \ldots, \hat{x}_i, \ldots, x_n\rangle r_{x_i} \\ &\qquad + \sum_{1 \leqq i < j \leqq n} (-1)^{j+1} \langle x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_n\rangle . \end{aligned}$$

We shall prove that $(W_*(\mathfrak{g}), d)$ is a free resolution of $U(\mathfrak{g}_{\mathrm{Lie}})$ considered as a right $UL(\mathfrak{g})$-module under $d_1 : UL(\mathfrak{g}) \to U(\mathfrak{g}_{\mathrm{Lie}})$ (cf. 2.5). We first show that $(W_*(\mathfrak{g}), d)$ is a complex. The proof is along the same line as in the Lie case [CE, HS].

We define, for any $y \in \mathfrak{g}$, homomorphisms of right $UL(\mathfrak{g})$-modules

$$\begin{aligned} \theta(y) &: W_n(\mathfrak{g}) \to W_n(\mathfrak{g}), \qquad n \geqq 0, \\ i(y) &: W_n(\mathfrak{g}) \to W_{n+1}(\mathfrak{g}), \quad n \geqq 0, \end{aligned}$$

as follows:

$\theta(y)$ is left multiplication by l_y for $n = 0$, and

$$\begin{aligned} \theta(y)\langle x_1, \ldots, x_n\rangle &= -\langle x_1, \ldots, x_n\rangle r_y \\ &\quad + \sum_{i=1}^{n} \langle x_1, \ldots, [x_i, y], \ldots, x_n\rangle \quad \text{for} \quad n > 0, \\ i(y)\langle x_1, \ldots, x_n\rangle &= (-1)^n \langle x_1, \ldots, x_n, y\rangle . \end{aligned}$$

(3.1) **Proposition** (Cartan's formulas). *We have the following identities*

(i) $i(y)d_{n-1} + d_n i(y) = \theta(y)$,

(ii) $\theta(x)\theta(y) - \theta(y)\theta(x) = -\theta([x, y])$, *for* $n > 0$,

(iii) $\theta(x)i(y) - i(y)\theta(x) = i([y, x])$, *for* $n > 0$,

(iv) $\theta(y)d_n = d_n\theta(y)$, *for* $n > 0$,

(v) $d_n d_{n+1} = 0$.

Proof. i) The statement is easy when $n = 1$. Let us consider the case when $n > 0$. By definition one has

$$\begin{aligned} &i(y)\, d_{n-1}\langle x_1, \ldots, x_{n-1}\rangle \\ &\quad = (-1)^{n-2} \Bigg\{ \langle x_2, \ldots, x_{n-1}, y\rangle l_{x_1} + \sum_{i=2}^{n-1} (-1)^i \langle x_1, \ldots, \hat{x}_i, \ldots, x_{n-1}, y\rangle r_{x_i} \\ &\qquad + \sum_{1 \leqq i < j \leqq n-1} (-1)^{j+1} \langle x_1, \ldots, [x_i, x_j], \ldots, \hat{x}_j, \ldots, x_{n-1}, y\rangle \Bigg\}, \end{aligned}$$

and

$$\begin{aligned}
&d_n i(y)\langle x_1, \ldots, x_{n-1}\rangle \\
&\quad = (-1)^{n-1} d_n\langle x_1, \ldots, x_{n-1}, y\rangle \\
&\quad = (-1)^{n-1}\Bigg\{\langle x_2, \ldots, x_{n-1}, y\rangle l_{x_1} + \sum_{i=2}^{n-1}(-1)^i\langle x_1, \ldots, \hat{x}_i, \ldots, x_{n-1}, y\rangle r_{x_i} \\
&\qquad + (-1)^n\langle x_1, \ldots, x_{n-1}\rangle r_y \\
&\qquad + \sum_{1\leqq i<j\leqq n-1}(-1)^{j+1}\langle x_1, \ldots, [x_i, x_j], \ldots, \hat{x}_j, \ldots, x_{n-1}, y\rangle \\
&\qquad + \sum_{1\leqq i\leqq n}(-1)^{n+1}\langle x_1, \ldots, [x_i, y], \ldots, x_{n-1}\rangle\Bigg\}.
\end{aligned}$$

Therefore one gets

$$\begin{aligned}
&(i(y)d_{n-1} + d_n i(y))\langle x_1, \ldots, x_{n-1}\rangle \\
&\quad = -\langle x_1, \ldots, x_{n-1}\rangle r_y + \sum_{1\leqq i\leqq n}\langle x_1, \ldots, [x_i, y], \ldots, x_{n-1}\rangle \\
&\quad = \theta(y)\langle x_1, \ldots, x_{n-1}\rangle .
\end{aligned}$$

ii) We have

$$\begin{aligned}
\theta(x)\theta(y)\langle x_1, \ldots, x_n\rangle = \langle x_1, \ldots, x_n\rangle r_x r_y &- \sum_{i=1}^{n}\langle x_1, \ldots, [x_i, x], \ldots, x_n\rangle r_y \\
&- \sum_{i=1}^{n}\langle x_1, \ldots, [x_i, y], \ldots, x_n\rangle r_x \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^{n}\langle x_1, \ldots, [x_i, y], \ldots, [x_j, x], \ldots, x_n\rangle \\
&+ \sum_{i=1}^{n}\langle x_1, \ldots, [[x_i, y], x], \ldots, x_n\rangle .
\end{aligned}$$

Using the Leibniz identity and relation (i) of 2.1 we obtain

$$\begin{aligned}
&(\theta(x)\theta(y) - \theta(y)\theta(x))\langle x_1, \ldots, x_n\rangle \\
&\quad = \langle x_1, \ldots, x_n\rangle r_{[x,y]} + \sum_{i=1}^{n}\langle x_1, \ldots, [x_i, [y, x]], \ldots, x_n\rangle \\
&\quad = -\theta([x, y])\langle x_1, \ldots, x_n\rangle .
\end{aligned}$$

iii) By definition one has

$$\begin{aligned}&\theta(x)i(y)\langle x_1,\ldots,x_n\rangle\\&=(-1)^n\Big\{-\langle x_1,\ldots,x_n,y\rangle r_x+\sum_{i=1}^n\langle x_1,\ldots,[x_i,x],\ldots,x_n,y\rangle\\&\quad+\langle x_1,\ldots,x_n,[y,x]\rangle\Big\},\end{aligned}$$

and

$$\begin{aligned}&i(y)\theta(x)\langle x_1,\ldots,x_n\rangle\\&=(-1)^n\Big\{-\langle x_1,\ldots,x_n,y\rangle r_x+\sum_{i=1}^n\langle x_1,\ldots,[x_i,x],\ldots,x_n,y\rangle\Big\}.\end{aligned}$$

Therefore one obtains

$$\begin{aligned}(\theta(x)i(y)-i(y)\theta(x))\langle x_1,\ldots,x_n\rangle&=(-1)^n\langle x_1,\ldots,x_n,[y,x]\rangle\\&=i([y,x])\langle x_1,\ldots,x_n\rangle.\end{aligned}$$

(iv) We proceed by induction on n. For $n=1$ we have $\theta(y)d_1\langle x\rangle=l_yl_x$ and $d_0\theta(y)\langle x\rangle=-l_xr_y+l_{[x,y]}$. Therefore the statement in this case follows from the relations (ii) and (iii) of 2.1. For $n>1$ we have

$$\begin{aligned}&(\theta(y)d_n-d_n\theta(y))\langle x_1,\ldots,x_n\rangle\\&=(-1)^{n-1}\{\theta(y)d_ni(x_n)-d_n\theta(y)i(x_n)\}\langle x_1,\ldots,x_{n-1}\rangle.\end{aligned}$$

Thus it is sufficient to show that

$$\theta(y)d_ni(x)-d_n\theta(y)i(x)=0.$$

But

$$\begin{aligned}\theta(y)d_ni(x)-d_n\theta(y)i(x)&=\theta(y)\theta(x)-\theta(y)i(x)d_{n-1}\\&\quad-d_ni(x)\theta(y)-d_ni([x,y])\quad\text{(by (i) and (iii))},\\&=\theta(y)\theta(x)-\theta(y)i(x)d_{n-1}-\theta(x)\theta(y)+i(x)d_{n-1}\theta(y)\\&\quad-\theta([x,y])+i([x,y])d_{n-1}\quad\text{(by (i))},\\&=-\theta(y)i(x)d_{n-1}+i(x)\theta(y)d_{n-1}\\&\quad+i([x,y])d_{n-1}\quad\text{(by (ii) and inductive hypothesis)},\\&=0\quad\text{(by (iii))}.\end{aligned}$$

v) In low dimension we have

$$\begin{aligned}d_1d_2\langle x_1,x_2\rangle&=d_1(\langle x_2\rangle l_{x_1}+\langle x_1\rangle r_{x_2}-\langle[x_1,x_2]\rangle)\\&=l_{x_2}l_{x_1}+l_{x_1}r_{x_2}-l_{[x_1,x_2]}\quad\text{(by (iii) of 2.1)},\\&=-r_{x_2}l_{x_1}+l_{x_1}r_{x_2}-l_{[x_1,x_2]}=0\quad\text{(by (ii) of 2.1)}.\end{aligned}$$

To prove $d_nd_{n+1}=0$ we proceed by induction. We have, for $n\geqq 2$,

$$d_nd_{n+1}\langle x_1,\ldots,x_{n+1}\rangle=(-1)^nd_nd_{n+1}i(x_{n+1})\langle x_1,\ldots,x_n\rangle,$$

but by (i) we obtain

$$\begin{aligned} d_n d_{n+1} i(x) &= d_n \theta(x) - d_n i(x) d_n \\ &= d_n \theta(x) - \theta(x) d_n + i(x) d_{n-1} d_n = 0 \end{aligned}$$

by (iv) and the inductive hypothesis. □

(3.2) *Non-commutative Koszul complex.* For the proof that W_* is acyclic in positive dimensions we need one more complex, which corresponds to the Koszul complex in Lie theory.

Let $\tau : V \to W$ be an epimorphism of free k-modules, and $SL(\tau)$ be the algebra defined in (2.8). Let

$$U_n(\tau) = V^{\otimes n} \otimes SL(\tau)$$

and $d_n : U_n(\tau) \to U_{n-1}(\tau)$ be the homomorphism of right $SL(\tau)$-modules given by

$$\begin{aligned} d_n \langle v_1, \ldots, v_n \rangle &= \langle v_2, \ldots, v_n \rangle (1 \otimes v_1) \\ &\quad + \sum_{i=2}^{n} (-1)^i \langle v_1, \ldots, \hat{v}_i, \ldots, v_n \rangle (\tau v_i \otimes 1). \end{aligned}$$

It is not hard to show that $d_{n+1} d_n = 0$. Thus $(U_*(\tau), d)$ is a chain complex.

(3.3) **Lemma.** *$H_0(U_*(\tau), d) \cong S(W)$ and $H_i(U_*(\tau), d) = 0$ for $i > 0$.*

Proof. The first isomorphism follows from the isomorphism of k-modules $SL(\tau) \cong S(W) \oplus S(W) \otimes V$. This is also an isomorphism of rings if we define a product on the right-hand side by

$$(f + g \otimes v)(f' + g' \otimes v') = ff' + fg' \otimes v' + f'g \otimes v - gg'\tau(v) \otimes v',$$

where $f, g, f', g' \in S(W), v, v' \in V$. Therefore

$$U_n(\tau) \cong (V^{\otimes n} \otimes S(W)) \oplus (V^{\otimes n} \otimes S(W) \otimes V).$$

It follows from the definition of d that

$$d(V^{\otimes n} \otimes S(W) \otimes V) \subset V^{\otimes (n-1)} \otimes S(W) \otimes V$$

and the projection of $d|_{V^{\otimes n} \otimes S(W)}$ onto $V^{\otimes (n-1)} \otimes S(W) \otimes V$ coincides with the standard isomorphism

$$V^{\otimes n} \otimes S(W) \to V^{\otimes (n-1)} \otimes S(W) \otimes V.$$

This means that the kernel of the augmentation map, given by the first part of Lemma 3.3,

$$(U_*(\tau), d) \to S(W)$$

is the cone of the map

$$\alpha_* : (U'_*, d') \to (U''_*, d''),$$

where the chain complexes (U'_*, d'), (U''_*, d'') and the chain map α_* are defined as follows:

$$U'_n = V^{\otimes(n+1)} \otimes S(W), \qquad U''_n = V^{\otimes n} \otimes S(W) \otimes V,$$

$$d'_n(v_1, \ldots, v_{n+1}, f) = \sum_{i=2}^{n+1} (-1)^i (v_1, \ldots, \hat{v}_i, \ldots, v_{n+1}, \tau v_i f)$$

$$d''_n(v_1, \ldots, v_n, f, v) = \sum_{i=1}^{n} (-1)^i (v_1, \ldots, \hat{v}_i, \ldots, v_n, \tau v_i f, v)$$

$$\alpha_n(v_1, \ldots, v_{n+1}, f) = (v_2, \ldots, v_{n+1}, f, v_1).$$

Thus $H_i(U_*(\tau), d) = 0$ for $i > 0$, because α_* is an isomorphism. □

Now we prove the main result of this paper.

(3.4) **Theorem.** *Let $\mathfrak{g}$ be a Leibniz algebra, such that $\mathfrak{g}$ and $\mathfrak{g}_{\text{Lie}}$ are free as k-modules, M be a representation of $\mathfrak{g}$, and A be a co-representation of $\mathfrak{g}$. Then*

$$HL^*(\mathfrak{g}, M) \cong \operatorname{Ext}^*_{UL(\mathfrak{g})}(U(\mathfrak{g}_{\text{Lie}}), M),$$
$$HL_*(\mathfrak{g}, A) \cong \operatorname{Tor}^{UL(\mathfrak{g})}_*(U(\mathfrak{g}_{\text{Lie}}), A),$$

where the right $UL(\mathfrak{g})$-module structure on $U(\mathfrak{g}_{\text{Lie}})$ is given by the map $d_0 : UL(\mathfrak{g}) \to U(\mathfrak{g}_{\text{Lie}})$ defined in 2.5.

Proof. It follows from the definitions that

$$C^*(\mathfrak{g}, M) = \operatorname{Hom}_{UL(\mathfrak{g})}(W_*(\mathfrak{g}), M)$$
$$C_*(\mathfrak{g}, A) = W_*(\mathfrak{g}) \otimes_{UL(\mathfrak{g})} A.$$

On the other hand $W_*(\mathfrak{g})$ is a componentwise free complex in the category of right $UL(\mathfrak{g})$-modules. Therefore we need only to check that

$$H_0(W_*(\mathfrak{g})) \cong U(\mathfrak{g}_{\text{Lie}}),$$
$$H_i(W_*(\mathfrak{g})) = 0, \qquad i > 0.$$

The first isomorphism follows from Proposition 2.4. In order to prove the second one we consider the submodule

$$\mathscr{F}_i W_n(\mathfrak{g}) = \mathfrak{g}^{\otimes n} \otimes F_{i-n} UL(\mathfrak{g}) \subset W_n(\mathfrak{g}).$$

By definition of the boundary map d, we have

$$d_n(\mathscr{F}_i W_n(\mathfrak{g})) \subset \mathscr{F}_i W_{n-1}(\mathfrak{g}).$$

Therefore we obtain the filtered chain complex

$$0 \subset \mathscr{F}_0(W_*(\mathfrak{g})) \subset \ldots \subset \mathscr{F}_n(W_*(\mathfrak{g})) \subset \ldots \subset W_*(\mathfrak{g}).$$

By (2.9) we have an isomorphism

$$\bigoplus_{i \geq 0} \mathscr{F}_i(W_*(\mathfrak{g}))/\mathscr{F}_{i-1}(W_*(\mathfrak{g})) \approx U_*(\mathfrak{g} \to \mathfrak{g}_{\text{Lie}})$$

and the statement follows from Lemma 3.3. □

(3.5) **Corollary.** *Let $\mathfrak{g}$ be a free Leibniz algebra. Then*

$$HL_i(\mathfrak{g}, -) = 0\,, \quad \text{for} \quad i \geqq 2\,,$$
$$HL^i(\mathfrak{g}, -) = 0\,, \quad \text{for} \quad i \geqq 2\,.$$

Proof. By Proposition 1.9, we have $HL^2(\mathfrak{g}, -) = 0$. Therefore the projective dimension of $U(\mathfrak{g}_{\mathrm{Lie}})$ in the category of right $UL(\mathfrak{g})$-modules is less than or equal to 1. □

4 Central extensions of Leibniz algebras

In this section we prove that

$$HL_2(sl_n(A), k) \cong HH_1(A)\,,$$

when A is an associative and unital algebra (free over k), $n \geqq 5$ and k a commutative ring. Here $HH_1(A)$ denotes the Hochschild homology groups of A with coefficients in A. In particular, when A is commutative, then $HH_1(A)$ is the module of Kähler differentials $\Omega^1_{A|k}$. Note that there is no characteristic hypothesis on k. If k is of characteristic zero, then this isomorphism follows from previous results [C, L]. This isomorphism is the noncommutative analog of the isomorphism

$$H_2(sl_n(A), k) \cong HC_1(A)$$

proved by Bloch [B] when A is commutative and by Kassel and Loday [KL] in general.

(4.1) A *central extension* of a Leibniz algebra $\mathfrak{g}$ is an exact sequence of Leibniz algebras

$$(\mathfrak{h}) \qquad 0 \to \mathfrak{a} \to \mathfrak{h} \xrightarrow{p} \mathfrak{g} \to 0$$

such that $[\mathfrak{a}, \mathfrak{h}] = [\mathfrak{h}, \mathfrak{a}] = 0$ and $(\mathfrak{h})$ is split as exact sequence of k-modules. A central extension $(\mathfrak{h})$ is called *universal* if, for every central extension $(\mathfrak{h}')$ of $\mathfrak{g}$ there exists one and only one homomorphism $f : \mathfrak{h} \to \mathfrak{h}'$ satisfying $p = p'f$. Classical arguments based on the universal coefficient theorem show that the following proposition is true.

(4.2) **Proposition.** i) *A central extension $(\mathfrak{h})$ of $\mathfrak{g}$ is universal if and only if $\mathfrak{h}$ is perfect (i.e. $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$) and every central extension of $\mathfrak{h}$ splits.*

ii) *A Leibniz algebra $\mathfrak{g}$ admits a universal central extension if and only if $\mathfrak{g}$ is perfect.*

iii) *The kernel of the universal central extension is canonically isomorphic to $HL_2(\mathfrak{g}, k)$.* □

(4.3) *Noncommutative Steinberg algebra.* Let A be an associative algebra with unit over k.

(4.3.1) **Definition.** For $n \geqq 3$ the *noncommutative Steinberg algebra* $stl_n(A)$ is the Leibniz algebra defined by generators $v_{ij}(a)$, $a \in \mathscr{A}$, $1 \leqq i \neq j \leqq n$, subject to the relations

$$v_{ij}(\lambda a + \mu b) = \lambda v_{ij}(a) + \mu v_{ij}(b)\,, \quad \text{for} \quad \lambda, \mu \in k, \quad \text{and} \quad a, b \in A\,,$$
$$\begin{aligned}[v_{ij}(a), v_{ml}(b)] &= 0 \quad \text{if} \quad i \neq l \quad \text{and} \quad j \neq m \\ &= -v_{mj}(ba) \quad \text{if} \quad i = l \quad \text{and} \quad j \neq m \\ &= v_{il}(ab) \quad \text{if} \quad i \neq l \quad \text{and} \quad j = m\,.\end{aligned}$$

Let $sl_n(A)$ be the Lie algebra of matrices with entries in A whose trace in $A/[A,A]$ is zero. Let

$$\varphi : stl_n(A) \to sl_n(A)$$

be the map defined by

$$\varphi(v_{ij}(x)) = E_{ij}(x),$$

where $E_{ij}(x)$ is the matrix with only non-zero element x in place (i,j).

(4.4) **Theorem.** *For $n \geqq 3$ the kernel of φ is central in $stl_n(A)$ and is isomorphic to $HH_1(A)$. Moreover if $n \geqq 5$ then*

$$0 \to HH_1(A) \to stl_n(A) \xrightarrow{\varphi} sl_n(A) \to 0$$

is the universal central extension of $sl_n(A)$ (in the category of Leibniz algebras).

Proof. The proof is essentially the same as in [KL], except for the definition of $h(a,b)$. In our case we denote

$$H_{ij}(a,b) := [v_{ij}(a), v_{ji}(b)], \qquad 1 \leqq i \neq j \leqq n, \quad \text{and} \quad a,b \in A,$$
$$h_{ij}(a,b) := H_{ij}(a,b) - H_{ij}(ba,1).$$

It follows from the Leibniz identity that

(4.4.1) $$H_{ij}(a,bc) = H_{im}(ab,c) + H_{mj}(ca,b), \qquad m \neq i,j.$$

By using 4.4.1, we obtain

(4.4.2) $$h_{ij}(a,b) = h_{im}(a,b)$$

(4.4.3) $$h_{ij}(a,b) = H_{im}(ab-ba,1) + h_{mj}(a,b),$$

It follows from (4.4.1)–(4.4.3) that

(4.4.4) $$h_{ij}(a,bc) = h_{ij}(ab,c) + h_{ij}(ca,b).$$

Hence $\eta(a \otimes b) = h_{ij}(a,b)$ yields a homomorphism

$$\eta : A \otimes A/\operatorname{Im} b \to stl_n(A)$$

for which the following diagram is commutative

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & HH_1(A) & \longrightarrow & A\otimes A/\operatorname{Im} b & \xrightarrow{b} & A & \longrightarrow & HH_0(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle E_{ij}} & & \| & & \\
0 & \longrightarrow & \operatorname{Ker}\varphi & \longrightarrow & stl_n(A) & \xrightarrow{\varphi} & gl_n(A) & \longrightarrow & HH_0(A) & \longrightarrow & 0,
\end{array}$$

where b is the Hochschild boundary map. Similar arguments as in [KL] show that the restriction of η to $HH_1(A)$ is an isomorphism onto $\operatorname{Ker}\varphi$. □

(4.5) **Corollary.** *Let k be a commutative ring and A an associative and unital k-algebra which is free as a k-module. For any $n \geqq 5$ there is an isomorphism*

$$HL_2(sl_n(A),k) \cong HH_1(A).$$

In particular, if A is commutative, then

$$HL_2(sl_n(A),k) \cong \Omega^1_{A|k}. \qquad \square$$

From the universality of this extension it is clear that $stl_n(A)$ inherates a structure of $E_n(A)$-module, which is lifted from the adjoint representation on $sl_n(A)$. So $stl_n(A)$ is an extension of $sl_n(A)$ in the category of $E_n(A)$-modules. Theorem 4.4 and results of [DI] imply that $stl_n(A)$ is isomorphic, as $E_n(A)$-module, to the "additive Steinberg group" $St(A, A)$ of Dennis.

(4.6) *Characteristic element of the noncommutative Steinberg algebra.* Let $\mathfrak{g} = stl_n(A)$. It follows from the definition that $\mathfrak{g}_{\mathrm{Lie}} = st_n(A)$, where $st_n(A)$ is defined in [KL]. We recall that, for $n \geqq 5$ this is the universal central extension of $sl_n(A)$ in the category of Lie algebras. The commutative diagram

$$\begin{array}{ccccccccc}
 & & HC_0(A) & & & & & & \\
 & & \downarrow B & & & & & & \\
0 & \longrightarrow & HH_1(A) & \longrightarrow & stl_n(A) & \longrightarrow & sl_n(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & HC_1(A) & \longrightarrow & st_n(A) & \longrightarrow & sl_n(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

shows that

$$\mathfrak{g}^{\mathrm{ann}} = \mathrm{Ker}(\mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}}) \cong \mathrm{Im}\, B\,,$$

where B is Connes operator [LQ]. Hence

$$0 \to \mathrm{Im}\, B \to stl_n \to st_n \to 0$$

is a central extension. Moreover for $n \geqq 5$ this is a universal central extension by (4.2) and (4.5). Therefore

$$HL_2 st_n = \mathrm{Im}\, B\,.$$

Thus

$$HL^2(\mathfrak{g}_{\mathrm{Lie}}, \mathfrak{g}^{\mathrm{ann}}) = \mathrm{Hom}(HL_2 st_n, \mathrm{Im}\, B) \cong \mathrm{End}(\mathrm{Im}\, B)$$

and $ch(\mathfrak{g})$ corresponds to $\mathbf{1}_{\mathrm{Im}\, B}$.

(4.7) *Virasoro algebra.* By (4.6) the universal central extension of $sl_n(A)$ in the category of Lie algebras and in the category of Leibniz algebras do not coincide in general. What happens for the *Virasoro algebra* [KR] which is the universal central extension of the Lie algebra $\mathrm{Der}(\mathbb{C}[z, z^{-1}])$ in the category of Lie algebras? The answer is given by the following.

(4.7.1) **Proposition.** *The Virasoro algebra is the universal central extension of the Lie algebra* $\mathrm{Der}(\mathbb{C}[z, z^{-1}])$ *in the category of Leibniz algebras.*

Proof. (Compare with [KR]). It is sufficient to show that

$$H^2(\mathfrak{g}, \mathbb{C}) \to HL^2(\mathfrak{g}, \mathbb{C})$$

is an isomorphism, where $\mathfrak{g} = \mathrm{Der}(\mathbb{C}[z, z^{-1}])$. Since it is already injective it is sufficient to prove surjectivity.

The elements

$$d_n = z^{-n+1} \frac{d}{dz}, \qquad n \in \mathbb{Z}$$

form a basis for $\mathfrak{g}$. It is well-known that

$$[d_n, d_m] = (n - m) d_{n+m} .$$

Let $(d_n, d_m) \mapsto f(n, m)$ be a Leibniz 2-cocycle of $\mathfrak{g}$. Then

(4.7.2) $$(n - m) f(n + m, k) = (n - k) f(n + k, m) + (m - k) f(n, m + k) .$$

If we put $n = m = x$, $k = y - x$, we obtain

(4.7.3) $$(2x - y)(f(y, x) + f(x, y)) = 0 .$$

Take $m + n = 0$, $k = 0$ in (4.7.2) we obtain

(4.7.4) $$f(0, 0) = 0 .$$

Let $g(n) = \frac{1}{n} f(0, n)$ if $n \neq 0$ and $g(0) = 0$. It follows from (4.7.4) that

$$(f - \delta g)(0, n) = f(0, n) - ng(n) = f(0, n) - f(0, n) = 0 , \qquad n \in \mathbb{Z} .$$

Therefore we can assume that

$$f(0, n) = 0 , \qquad n \in \mathbb{Z} .$$

If we put $n = k = 0$ in (4.7.2), we obtain

$$f(m, 0) = 0 , \qquad m \in \mathbb{Z} .$$

Take $k = 0$ in (4.7.2) we obtain

(4.7.5) $$(n + m) f(n, m) = 0 .$$

It follows from (4.7.3)–(4.7.5) that $f(n, m) + f(m, n) = 0$ for all n, m and so f is also a Lie cocycle and hence Proposition 4.7.1 is proved. □

Remark. In fact $H^2(\mathrm{Der}(\mathbb{C}[z, z^{-1}]), \mathbb{C}) \cong HL^2(\mathrm{Der}(\mathbb{C}[z, z^{-1}]), \mathbb{C}) \cong \mathbb{C}$ and a generator is given by the cocycle f such that

$$f(n, -n) = n(n^2 - 1) \quad \text{for all } n \in \mathbb{Z} .$$

Acknowledgement. The second author is grateful to the CNRS for hospitality at the Institut de Recherche Mathématique Avancée, where this work was done.

References

[B] Bloch, S.: The dilogarithm and extensions of Lie algebras. Algebraic K-theory, Evanston 1980. (Lect. Notes Math., vol. 854, pp. 1–23) Berlin Heidelberg New York: Springer 1981

[CE] Cartan, H., Eilenberg, S.: Homological algebra. Princeton: Princeton University Press 1956

[C] Cuvier, C.: Homologie de Leibniz et homologie de Hochschild. C.R. Acad. Sci. Paris, Sér. A–B **313**, 569–572 (1991)

[DI] Dennis, K., Igusa, K.: Hochschild homology and the second obstruction for pseudo-isotopy. Algebraic K-theory, Oberwolfach 1980. (Lect. Notes Math., vol. 966, pp. 7–58) Berlin Heidelberg New York: Springer 1982

[HS] Hilton, P.J., Stammbach, U.: A course in homological algebra. (Grad. Texts in Math., vol. 4) Berlin Heidelberg New York: Springer 1971

[KL] Kassel, C., Loday, J.-L.: Extensions centrales d'algèbres de Lie. Ann. Inst. Fourier **33**, 119–142 (1982)

[KR] Kac, V.G., Rain, A.K.: Highest weight representation of infinite dimensional Lie algebras. (Adv. ser. in Math. Physics vol. 2) Singapore: World Scientific 1987

[L] Loday, J.-L.: Cyclic homology. (Grundl. Math. Wiss., Bd. 301) Berlin Heidelberg New York: Springer 1992

[LQ] Loday, J.-L., Quillen, D.: Cyclic homology and the Lie algebra homology of matrices. Comment. Math. Helv. **59**, 565–591 (1984)