Universal Fluctuations in Correlated Systems

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The probability density function (PDF) of a global measure in a large class of highly correlated systems has been suggested to be of the same functional form. Here, we identify the analytical form of the PDF of one such measure, the order parameter in the low temperature phase of the 2D *XY* model. We demonstrate that this function describes the fluctuations of global quantities in other correlated equilibrium and nonequilibrium systems. These include a coupled rotor model, Ising and percolation models, models of forest fires, sandpiles, avalanches, and granular media in a self-organized critical state. We discuss the relationship with both Gaussian and extremal statistics.

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Self-similarity is an important feature of the natural world. It arises in strongly correlated many body systems when fluctuations over all scales from a microscopic length a to a diverging correlation length ξ lead to the appearance of "anomalous dimension" [1] and fractal properties. However, even in an ideal world, the divergence of ξ must ultimately be cut off by a macroscopic length L, allowing the definition of a range of scales between a and L, over which the anomalous behavior can occur. Such systems are found, for example, in critical phenomena, in self-organized criticality [2,3], or in turbulent flow problems. By analogy with fluid mechanics we shall call these finite-size critical systems "inertial systems" and the range of scales between a and L the "inertial range." One of the anomalous statistical properties of inertial systems is that, whatever their size, they can never be divided into mesoscopic regions that are statistically independent. As a result they do not satisfy the basic criterion of the central limit theorem and one should not necessarily expect global, or spatially averaged quantities to have Gaussian fluctuations about the mean value. In Ref. [4] (BHP) it was demonstrated that two of these systems, a model of finite size critical behavior and a steady state in a closed turbulent flow experiment, share the same non-Gaussian probability density function (PDF) for fluctuations of global quantities. Consequently it was proposed that these two systems—so utterly dissimilar in regard to their microscopic details—share the same statistics simply because they are critical. If this is the case, one should then be able to describe turbulence as a finite-size critical phenomenon, with an effective "universality class." As, however, turbulence and the magnetic model are very unlikely to share the same universality class, it was implied that the differences that separate critical phenomena into universality classes represent at most a minor perturbation on the functional form of the PDF. In this paper, to test this proposition, we determine the functional form of the BHP fluctuation spectrum and show that it indeed applies to a large class of inertial systems [5].

The magnetic model studied by BHP, the spin-wave limit to the two dimensional *XY* model, is defined by the harmonic Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \left[1 - \frac{1}{2} (\theta_i - \theta_j)^2 \right], \tag{1}$$

where J is the near neighbor exchange constant for angular variables θ_i that occupy a square lattice with periodic boundary conditions. The magnetization is defined as $m = 1/N \sum_i \cos(\theta_i - \overline{\theta})$, where $\overline{\theta}$ is the instantaneous mean orientation. This model is critical at all temperatures and for an infinite system has algebraic correlations on all length scales. In the finite system the lattice constant a and the system sizes $L = a\sqrt{N}$ define a natural inertial range. The model can be diagonalized in Fourier space, which makes it very convenient for analytical work. The PDF of the magnetization $P(m)$ can be expressed as the Fourier transform of a sum over its moments. In Ref. [6] it was shown that the moments vary as $\mu_n \propto g_n (g_2/2)^{-n/2} \sigma^n$, where σ^2 is the variance and the $g_k (k = 2, 3, 4, \dots)$ are sums related to the lattice Green function in Fourier space $G(\mathbf{q})$: $g_k = \sum_{\mathbf{q}} G(\mathbf{q})^k / N^k$. The fact that $\mu_n \propto \mu_1^n$ means that a change of N or T is equivalent to a linear transformation of the variate m; hence, the PDF can be expressed in a universal form. As shown in [6] the moment series can be resummed to give the following expression, exact to leading order in N:

$$P(y) = \int_{-\infty}^{+\infty} \frac{dx}{2\pi\sigma} \exp\left[iyx + \sum_{k=2}^{\infty} \frac{g_k}{2k} \left(ix \sqrt{\frac{2}{g_2}} \right)^k \right]. \tag{2}$$

Here $y = (m - \langle m \rangle)/\sigma$ and $\langle m \rangle$ is the mean of the distribution. Including only g_2 in (2) would give a Gaussian

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PDF with variance σ^2. However, the terms for $k > 2$ cannot be neglected and $\Pi(y) = \sigma P(y)$ is a non-Gaussian, universal function, independent of both the size of the system and the temperature [7]. Without loss of generality one can make the quadratic approximation $m = 1 - \sum_i(\theta_i - \overline{\theta})^2/2N$, which allows us to transform Eq. (2) to a form suitable for numerical integration [8]:

$$\Pi(y) = \int_{-\infty}^{+\infty} \sqrt{\frac{g_2}{2}} \frac{dx}{2\pi} \exp[i\Phi(x)],$$

$$i\Phi(x) = ixy\sqrt{\frac{g_2}{2}} - i\,\frac{x}{2}\,\mathrm{Tr}G/N \qquad (3)$$

$$- \frac{1}{2}\,\mathrm{Tr}\log(\mathbf{1} - ixG/N).$$

[The trace Tr of any function of G is defined as the sum for $\mathbf{q} \neq \mathbf{0}$ of the same function of $G(\mathbf{q})$.] In order to make an accurate test of this expression we have performed a high resolution molecular dynamics simulation of $P(m)$. Figure 1 compares the integrated Equation (3) with data for a system of 1024 classical rotors integrated over 10^8 molecular dynamics time steps in the low temperature phase. The agreement is globally excellent, particularly in the wings of the distribution and along the exponential tail for fluctuations below the mean.

The asymptotic values of $\Pi(y)$ are related to the saddle points of the integrand in (3). We find

$$\Pi(y) \propto |y| \exp\left(\frac{\pi}{2}\, by\right); \quad \text{for } y \ll 0, \qquad (4)$$

$$\Pi(y) \propto \exp\left(-\frac{\pi}{2}\, e^{b(y-s)}\right); \quad \text{for } y \gg 0, \qquad (5)$$

where $b = 8\pi\sqrt{g_2/2} \approx 1.105$ and $s = 0.745$. These forms give the correct asymptotic gradients of the molecular dynamics data on logarithmic and double logarithmic scales. The asymptotic forms are an accurate approximation to Eq. (2) for large $|y|$; however, deviations from

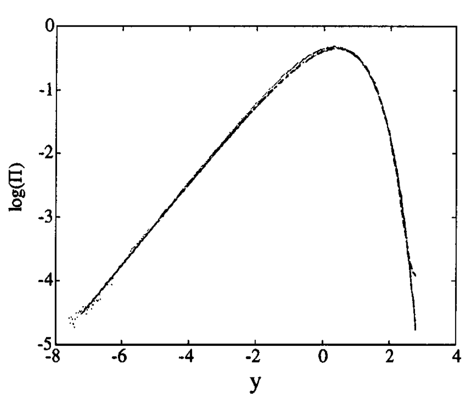

FIG. 1. The PDF $\Pi(y)$ as found from a numerical Fourier transform of Eq. (3) (long-dashed line), from Eq. (6) (solid line), and by molecular dynamics simulation of a system of $N = 1024$ classical rotors (dotted line).

the asymptotes are important over most of the physical range of y, which is typically limited to $\log|y| \sim O(1)$. Equations (4) and (5) serve as a guide to finding a good approximation to the functional form of $\Pi(y)$ in this range. To do this, we observe that the factor of y in (4) can be regarded as a constant in this regime, which along with (4) and (5) immediately suggests the form

$$\Pi(y) = K(e^{x-e^x})^a; \qquad x = b(y - s), \qquad a = \pi/2. \qquad (6)$$

This function must obey the three conditions of unit area, zero mean, and unit variance, which fixes b, s, and K to values slightly different from those found analytically: $b = 0.938$, $s = 0.374$, $K = 2.14$. An alternative approach is to choose the parameters in the generalized function $Ke^{a[b(y-s)-e^{b(y-s)}]}$ such that the first four Fourier coefficients match Eq. (2). In this case we find $a = 1.58$, $K = 2.16$, $b = 0.934$, $s = 0.373$, in satisfying agreement with the previous estimates. The ratios of the higher order Fourier coefficients differ from unity only very slowly, showing that Eq. (6) is an accurate approximation to $\Pi(y)$. This is directly confirmed by plotting Eq. (6) versus the molecular dynamics and exact results in Fig. 1, where the fit is seen to be of extremely high precision.

We now test the idea that the BHP fluctuation spectrum of the form of Eq. (6) is exhibited by many types of inertial system. Figure 2 shows the numerically simulated PDF of global quantities in several equilibrium and nonequilibrium models. The equilibrium models include the 2D Ising model at a temperature $T^*(N)$ just below the critical temperature and a 2D site percolation model on a square lattice for a site occupation probability $P^*(N)$ just above the percolation transition. The numerical results refer to the fluctuations of the absolute value of the magnetization and the fluctuations in the size of the spanning cluster, respectively. The nonequilibrium models are of the type that when driven slowly enter a scale-free or critical steady state defined as self-organized criticality (SOC) [2,3]. Here the global quantity is essentially a dissipation rate that fluctuates about a well defined mean value in the steady state. Details of the individual SOC models are as follows. (i) *The autoigniting forest fire model* [9] consists of "trees" planted at random on the vacant sites of a square lattice with probability p. In each time step the age T_i of a tree on site i is incremented by one unit. When $T_i = T_{\max}$ the tree ignites and T_i is reset to zero. Trees can also catch fire by being nearest neighbor to a site on fire. The energy, or wood stored in a tree, is proportional to T, and the figure shows the PDF of the total energy dissipated in fires at each time step. (ii) In the *Bak-Tang-Wiesenfeld (BTW) sandpile model* [10] a dynamical variable E_i is defined on lattice site i. The model is driven by adding units of the E field to randomly selected sites. When $E_i > E_{\max}$ the site variable is decreased by $E_{\max}$ and the E variable of the z neighbor sites is increased by $E_{\max}/z$. One or more of the neighbor sites may then aquire an E value larger than $E_{\max}$ and an avalanche is induced. The PDF shown refers to the fluctuations in the instantaneous number of relaxing sites. (iii) In the *Sneppen depinning model* [11] an interface moves through a static random field of pinning forces. The site along the interface that experiences the smallest pinning force is moved one unit ahead. If the local slope s_i exceeds 1, then the neighboring sites are moved one unit ahead until all $s_i \leq 1$. We call such a sequence of updates a microavalanche and calculate the PDF of the sum of areas covered by the progressing interface during an integral time scale T dependent on system size. (iv) The model for *granular media* is a "Tetris-like" 2D lattice gas ensemble of anisotropic particles settling under gravity in a finite box [12]. Because of the geometrical frustration, the total mass varies from one realization of the filling process to another. The PDF for fluctuations in bulk density of the particles is shown.

Referring to Fig. 2, the data sets for all models fall close to the BHP form, Eq. (6). In the equilibrium models (lower curves) the self-similarity is expected at the system-size dependent critical temperature $T^*(N)$ or percolation probability $P^*(N)$ only. The PDF for the 2D Ising model, for example, is temperature dependent, but makes a close approach to the BHP form around $T^*(N)$. We believe that the remaining deviations for fluctuations above the mean are due to the limited inertial range for the system sizes studied. For the nonequilibrium systems (upper curves) the data sets also show some deviation. This may simply be due to poor statistics, as deviations on either side of the mean are related by the constraints of normalization. In this respect, we note that extremely good statistics were required to get a satisfactory fit to Eq. (6) for the 2D XY model, while for more limited data sets systematic deviations occurred.

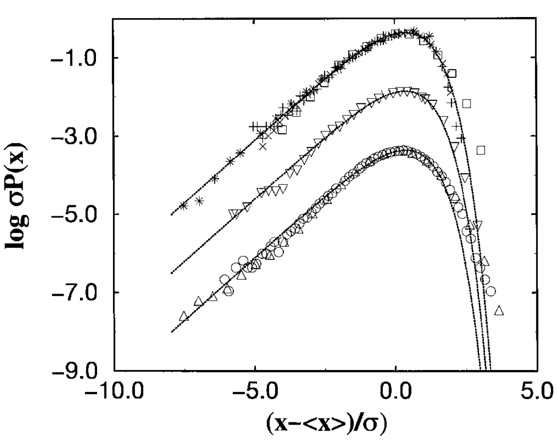

FIG. 2. Fluctuation spectra in equilibrium systems (lower curves): Ising ($\circ$), percolation ($\triangle$). The central curve (∇) corresponds to the fluctuation spectrum of the correlated extremal process—see text. The upper curves are the PDFs for the autoigniting forest fire model ($\times$), the Sneppen depinning model ($+$), the granular media model ($\square$), and the BTW sandpile model ($*$). The lines are Eq. (6). For clarity, the sets of curves are shifted downwards by 1.5 in log units.

Nevertheless it is clear that, to leading order, Eq. (6) correctly gives the behavior of the global fluctuation spectrum in all these systems, independently of the details of the each example. We propose that this is a consequence of the systems sharing the properties of finite size, strong correlations, and self-similarity.

To clarify this proposition, we return to our calculation on the 2D XY model. One can see explicitly that the BHP spectrum occurs through appearance of anomalous dimension and contributions on all length scales of the inertial range. The magnetization can be written, within the quadratic approximation, $m = 1 - \Sigma_{\mathbf{q}} m_{\mathbf{q}}$, where the $m_{\mathbf{q}}$s are the amplitudes from the individual spin-wave modes. These are statistically independent positive variates with PDF

$$P(m_q) = \sqrt{\frac{\beta q^2 a^2 N}{4\pi}}\, m_q^{-1/2} \exp(-\beta N q^2 a^2 m_q)\,, \quad (7)$$

whose mean and standard deviation therefore scale with q^{-2}. The "softest" modes have wave vector $q = 2\pi/L$ and hence, by themselves, make contributions of $O(1)$ to m, while the modes on the zone boundary with $q = \pi/a$ have only microscopic amplitude. The moments of $P(m)$ are determined by the mean magnetization. This is proportional to the integral over all contributions: $\sim \int_{2\pi/L}^{\pi/a} q^{-2} n(q)\, dq$, where $n(q) \sim q^{d-1}$ is the density of states. In one dimension the integral depends only on the lower limit $2\pi/L$ and only the soft modes count, while in three dimensions only the upper limit π/a is important and the multitude of modes near the zone boundary dominate the sum. In two dimensions, however, both limits of the integral are required and a detailed calculation gives $\langle m \rangle = 1 - \eta/2 \log(CL/a)$, with $C = 1.87$ [8] and critical exponent $\eta = T/2\pi J$. The relevance of fluctuations over all length scales of the zone therefore leads to the "anomalous" term $\log(L/a)$ and it ensures that the system cannot be cut into statistically independent parts.

The spin-wave approximation to the XY model is exactly equivalent to the Edwards-Wilkinson model of a growing interface in steady state [13], with the square of the interface width w equal to the sum over the amplitudes m_q: $w^2 = \sum_q m_q$. The fluctuations in the width of the interface have been studied by Foltin *et al.* [14] for the 1D case and by Rácz and Plischke [15] for the 2D interface. The BHP spectrum is found for the *critical* two dimensional case only and our calculation can be considered the completion of the study in Ref. [15].

The functional form of Eq. (6) suggests a relationship to Gumbel's first asymptote [16] for extreme value statistics, which have recently been discussed in relation to turbulence in one dimension [17]. The form (6) but with a taking *integer* values, where $a = 1, 2, 3, \ldots$ would correspond to the PDF for the first, second, third, ... largest of the N random numbers. However, the exponent $a = \pi/2$

suggests, as we have argued, that the fluctuations in m are not dominated by single independent variables. Rather, the analytic derivation of Eq. (6) shows that if extreme value statistics are involved they must be related to the statistics of some emergent coherent collective excitation of the system. This is borne out in the simulations of the Ising model and of all the SOC models studied. In the Ising model, it is found that both the full magnetization and the contribution to the magnetization from the largest connected cluster of parallel spins give the same PDF, within numerical error. For the Sneppen model the PDF of the *sum* over avalanches and that of the *largest* avalanche during the integral time T are both found to be of the BHP form, even though these quantities are not related by a simple scale. If the avalanches appearing during time T were uncorrelated one would expect that the PDF for the largest avalanche would be Gumbel's asymptote with $a = 1$. The modification towards our form indicates therefore that there are correlations between events during the period T. To test this idea, we have studied the PDF of the extreme values taken from sets of linearly correlated variables. The process consists of generating a vector $\vec{\chi} = (\chi_1, \dots, \chi_N)$, where χ_i are all independent and exponentially distributed. The maximum signal is obtained as $\xi_{\max} = \max\{\xi_1, \dots, \xi_N\}$, where the vector $\vec{\xi} = \mathbf{M}\vec{\chi}$ and $\mathbf{M}$ is an $N \times N$ matrix with random but fixed elements. The resulting spectrum, shown in Fig. 2, is found to be very close to the BHP spectrum.

In conclusion, our results infer that the non-Gaussian PDF of a global quantity in a critical system is a consequence of finite-size, strong correlations and self-similarity and is independent of universality class to leading order. Clearly many more studies of this point are required. Nonlinearity does not appear to be an essential feature, over and above the necessity, in a closed system, to couple the elementary degrees of freedom. Indeed, if nonlinearity were essential, it would seem impossible that the linear spin-wave theory could capture the fluctuations in the turbulence experiment [4]. Rácz and Plischke [15] have studied a series of linear and nonlinear models for growing interfaces. All show anisotropic PDFs for the interface width, with long tails in qualitative agreement with our data. It would very interesting to examine these models in detail to see how the strength of the nonlinearity affects the form of the fluctuations.

Finally, it seems that a relationship exists between the BHP curve and extremal statistics. Although we have shown that the BHP behavior is not simply due to extreme values of the statistically independent degrees of freedom of the 2D *XY* model, extreme values do appear to dominate the real space coherent structures that are excited in the critical (Ising model) or self-organized critical (Sneppen model) state.

Our findings thus establish a completely new and general consequence of self-similarity and they open the door to numerous studies that could lead to a unified global description of aspects of equilibrium and nonequilibrium behavior.

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