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Universal limit mapping in grazing bifurcations

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Grazing bifurcations are nonsmooth bifurcations that occur in impacting mechanical oscillators. A sign of the lack of smoothness is a square root expression that arises in mappings describing the dynamics. Certain types of grazing bifurcations involve chaotic bands, or period adding cascades with or without chaotic bands. There is also a characteristic scaling behavior present. Here this type of bifurcation is investigated, and the self-similarity under scaling is used to derive a renormalized limit mapping. A study of the dynamics of the limit mapping identifying all attractors for all parameter values is also presented. [S1063-651X(97)09401-4]

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I. INTRODUCTION

When studying models of impacting mechanical systems, where the approximation of instantaneous velocity change at the moment of impact is used, one typically encounters grazing bifurcations of periodic orbits. The grazing bifurcation is not found in smooth dynamical systems, and is an effect of the fact that an impact with low velocity is sensitive to small changes in the initial conditions. The sensitivity is inversely proportional to impact velocity. Thus, if a stable periodic orbit is shifted under a parameter change, so that it has a zero velocity (grazing) impact, stability will typically change.

Grazing bifurcations have been the subject of several investigations recently. Whiston investigated the geometry of the singularities, the grazing bifurcations of a saddle point, and the creation of chaotic sets for a linear driven impact oscillator [1]. Nordmark studied the bifurcations of a node in single degree of freedom driven impact oscillators [2]. The existence of an attractor under conditions on the eigenvalues and an orientation condition was shown. The possibility of existence of chaos for an interval of the bifurcation parameter was demonstrated, as was a scaling behavior. Budd and Dux studied the linear impact oscillator with driving near resonance, and obtained an impact return mapping that is close to being one dimensional, when close to the bifurcation point [3]. Lamba and Budd made a careful estimate of the Lyapunov exponent in the chaotic band [4]. Fredriksson and Nordmark gave a derivation of the local form of the Poincaré mapping for systems with more than one degree of freedom [5] .

There have also been a number of studies of explicit mappings with square root terms. Nusse *et al.* [6] studied a mapping in one dimension. An equivalent mapping was studied by Foale and Bishop [7]. A two-dimensional mapping was thoroughly examined by Chin *et al.* [8,9].

In the current investigation we concentrate on the case where the bifurcation leads to an attractor with either stable periodic motion of long period, or chaotic motion. What is observed after the bifurcations is well summarized in Chin *et al.* [8]. We assume a bifurcation parameter ν such that there is a grazing bifurcation as ν increases through 0. Call the largest eigenvalue of the periodic orbit λ.

(i) For $\nu>0$ there is an attractor of size $O(\sqrt{\nu})$.

(ii) If $2/3<\lambda$ there is chaotic motion for $0<\nu<\nu'$.

(iii) If $\lambda<2/3$ there are stable periodic windows. The windows accumulate on $\nu=0$ and the periods increase by 1 from window to window. All orbits have a single low velocity impact. There is a scaling in the limit $\nu\rightarrow 0$ such that each periodic window is mapped onto the next if ν is scaled by λ^2.

(iv) If $1/4<\lambda<2/3$ the periodic windows do not overlap, and there is a chaotic band between windows.

(v) If $\lambda<1/4$ the periodic windows have a bit of overlap, and one or two of the stable orbits are always present. The purpose of the present investigation is to start with the form of the Poincaré mapping for a general impacting system, and derive an impact return mapping very similar to that of Budd and Dux [3]. Then we show how a one-dimensional limit mapping can be obtained through renormalization as we let the bifurcation parameter go to zero. The mapping obtained is piecewise continuous with an infinite number of branches. Other investigators have generally studied a low order approximation to the Poincaré mapping of the system. Using the limit mapping, we prove the existence of stable periodic orbits, and chaotic motion.

II. POINCARÉ MAPPINGS NEAR A GRAZING PERIODIC ORBIT

We will introduce a convenient form of the Poincaré mapping near a grazing periodic orbit (Fredriksson and Nordmark [5]). We assume that a section coordinate $x\in\mathbb{R}^N$ and a bifurcation parameter $\nu\in\mathbb{R}^M$ are introduced in such a way that the grazing fixed point is at $x=0$ when $\nu=0$.

The mapping near $x=0$, $\nu=0$ can be written as

$$p=f\circ g, \tag{1}$$

where f describes the nonimpact dynamics, and g corrects for low velocity impacts. $f\circ g$ is the composition of f and g, $(f\circ g)\ (x)=f(g(x))$. The function f

$$f(x,\nu):\mathbb{R}^N\times\mathbb{R}^M\rightarrow\mathbb{R}^N$$

is smooth with $f(0,0)=0$. We write

$$A=D_x f(0,0). \tag{2}$$

Points near $x=0$ may or may not undergo a low velocity impact. We will use the values of a smooth function h

$$h(x,\nu):\mathbb{R}^N\times\mathbb{R}^M\to\mathbb{R}$$

with $h(0,0)=0$ to separate the cases. We introduce

$$C=D_x h(0,0)\neq 0. \tag{3}$$

When $h>0$ (nonimpacting points) there is no low velocity impact, so g should then be the identity mapping. For $h=0$ (grazing points) we have a grazing impact, and when $h<0$ (impacting points) there is a low velocity impact. g can be written as

$$g(x,\nu):\mathbb{R}^N\times\mathbb{R}^M\to\mathbb{R}^N,$$

$$g(x,\nu)=\begin{cases} x & \text{if } h(x,\nu)\geqslant 0\\ b(x,y,\nu)y+x & \text{if } h(x,\nu)\leqslant 0,\end{cases} \tag{4}$$

where

$$y=\sqrt{-h(x,\nu)}, \tag{5}$$

and b is a smooth function

$$b(x,y,\nu):\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^M\to\mathbb{R}^N$$

with

$$B=b(0,0,0)\neq 0. \tag{6}$$

From this it follows that g is a continuous mapping, but $D_x g$ is unbounded for small negative h.

When $\nu=0$, we see that $x=0$ is a fixed point of f, and we assume that for small ν there is a fixed point $\tilde{x}(\nu)$ of f with $\tilde{x}(0)=0$. If $h(\tilde{x})>0$, then $\tilde{x}$ is also a fixed point of the full Poincaré mapping p. The grazing bifurcation occurs as $\tilde{x}$ tries to enter the impacting region $h<0$. For $h(\tilde{x})<0$ we introduce $\kappa=-h(\tilde{x})$.

We now make the following assumptions. (i) A has a single eigenvalue λ of largest modulus, and $0<\lambda<1$. To this eigenvalue corresponds a right eigenvector ϕ and a left eigenvector ϕ^*, with $\phi^*\phi=1$. (ii) $\phi^*B\neq 0$ and $C\phi\neq 0$. (iii) $CA^nB>0$ for all $n\geqslant 1$. These conditions are sufficient to guarantee an attractor of size $O(\sqrt{\kappa})$ for all sufficiently small positive κ. The key feature of the dynamics for small positive κ is that iterates repeatedly return to the impact region, but the typical time between impacts is large, of the order $-\log(\kappa)$.

A. The return mapping and its limit form

We will study the dynamics using a return mapping from the impact region back to itself. The mapping is $F=(f\circ g)^n=f^n\circ g$, where $n(x)$ is chosen such that $h(f^k\circ g)\geqslant 0$ for $1\leqslant k\leqslant n-1$, but $h(f^n\circ g)<0$. Thus $n(x)$ indicates the number of iterations needed to return to the impacting region. F is continuous in regions where n is constant, but discontinuous where n makes a jump. To find a useful limit form of the return mapping F, we will introduce the coordinate

$$z(x)=[\kappa+h(x)]/\kappa. \tag{7}$$

$z<1$ for impacting points, $z>1$ for nonimpacting points, $z=1$ for grazing points, and $z(\tilde{x})=0$. Then we have for impacting points $g=b\sqrt{\kappa}\sqrt{1-z}+x$. Now if x is of the order κ and we keep $z<1$ fixed, the only important contribution will be $B\sqrt{\kappa}\sqrt{1-z}$, with other terms vanishing in relation to this as $\kappa\to 0$.

Turning to f^n, we have for small $x-\tilde{x}$ that $f^n(x)=\tilde{x}+A^n(x-\tilde{x})+O(x-\tilde{x})^2$. For large n, A^n will be dominated by the effect of the eigenvalue largest in modulus, so $A^n=\phi\lambda^n\phi^*+$ smaller terms.

Putting this together we find

$$z(f^n\circ g(x))=\frac{\lambda^{n(x)}}{\sqrt{\kappa}}(C\phi)(\phi^*B)\sqrt{1-z(x)}+\cdots. \tag{8}$$

As has been pointed out earlier, there is a scaling relation such that the dynamics of F is similar if κ is scaled by a factor λ^2. To take advantage of this fact we introduce two new parameters μ and m, where $\mu\in(\lambda^2,1]$ and m is an integer. Any positive κ can be uniquely written as

$$\kappa=\mu(\lambda^{m-1}C\phi\phi^*B\sqrt{1-\lambda})^2. \tag{9}$$

We will also write $k(x)=m-n(x)$.

Now we can keep $z<1$ and μ fixed and let $m\to\infty$. All x dependence except through z will vanish, and we find

$$k(x)\to K(z), \tag{10}$$

$$z(f^n\circ g(x))\to G(z)=\frac{\lambda\sqrt{1-z}}{\sqrt{1-\lambda}\sqrt{\mu}\lambda^{K(z)}}. \tag{11}$$

As the range of G is $[\lambda,1)$, for the dynamics we need only study G on this interval. The different intervals on which K is constant and thus G continuous can be given explicitly. Let

$$J_k=1-\mu\lambda^{2k}(1-\lambda)\quad\text{for}\quad k\in\mathbb{Z} \tag{12}$$

and let

$$I_0=[\lambda,J_0], \tag{13}$$

$$I_k=(J_{k-1},J_k]\quad\text{for}\quad k>0. \tag{14}$$

Then K will have the value k if $z\in I_k$.

In the limit mapping G, the parameter λ should be treated as fixed, whereas μ is a bifurcation parameter. The value of λ is the only trace left of the original choice of the functions f, b, and h, as long as they meet the requirements stated above. Thus the mapping G is *universal* for a large class of impacting systems and not dependent on most details of the system at hand.

The mapping G is of course significantly different from the mapping F for finite m in that F is a mapping in $\mathbb{R}^N$ whereas G is a mapping in $\mathbb{R}$. Also F has a finite set of continuous branches, but the number of branches of G is infinite. The one-dimensional character of both f and g increases with increasing n, so the difference is only important where n is not large, that is, for z close to 1. There, however,

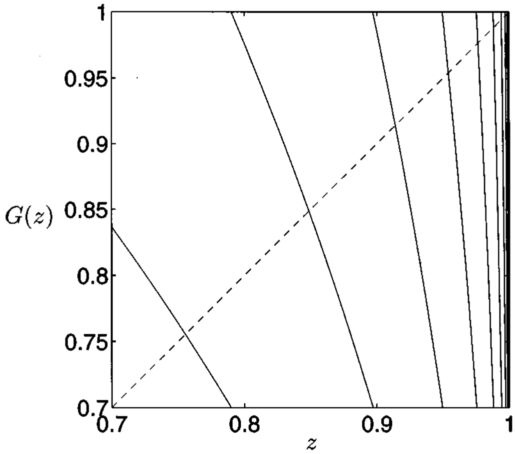

FIG. 1. The mapping G when $\lambda=0.7$ and $\mu=0.7$, together with the identity line.

G is very steep and the same holds for F, and we do not expect that this will have much influence on the dynamics.

To use G to make predictions about the dynamics of F for finite m, we should use

$$\kappa=\mu(\lambda^{m-1}C\phi\phi^*B\sqrt{1-\lambda})^2, \tag{15}$$

$$n(x)=m-K(z)+\cdots, \tag{16}$$

$$x=\tilde{x}+\phi\frac{\kappa}{C\phi}z+\cdots. \tag{17}$$

III. THE DYNAMICS OF G

To study the stability of the dynamics of G, the derivative is needed. We get

$$G'(z)=-\frac{\lambda}{2\sqrt{1-\lambda}\sqrt{\mu}\lambda^{K(z)}\sqrt{1-z}}. \tag{18}$$

G' is always negative, and if $z>J_1$, it is less than -1. Thus it it clear that in the interval $(J_1,1)$ there is an invariant set with chaotic motion for all parameter values. A more interesting question is that of attractors.

A. Chaos for all μ

The maximal value of G' is at $z=\lambda$, where

$$G'(\lambda)=-\frac{\lambda}{2(1-\lambda)\sqrt{\mu}}. \tag{19}$$

Now if $2/3<\lambda<1$ this is still less than -1 for all allowed μ. Thus there are no stable periodic orbits and the limit mapping is chaotic for all μ. The same should hold for F (and p) for all $0<\kappa<\kappa'$. Figure 1 shows an example where the expanding character of the mapping is seen.

B. Stable fixed points

We can easily see that there is a fixed point on each of the branches of G. If $\overline{z}$ is one of the fixed points, we can use Eqs. (11) and (18), together with $G(\overline{z})=\overline{z}$, to show that

$$G'(\overline{z})=-\frac{\overline{z}}{2(1-\overline{z})}. \tag{20}$$

The values of μ, λ, and $K(\overline{z})$ have canceled out of the calculation, so the slope is only dependent on $\overline{z}$, regardless of which branch the fixed point is on. The slope is between -1 and 0 if $0<\overline{z}<2/3$, and is less than -1 if $2/3<\overline{z}<1$.

Let us consider the fixed point on the leftmost branch $(K=0)$. When μ is close to 1, the fixed point will be close to λ, so if $\lambda<2/3$ it will be stable for all μ close to 1. If $1/4<\lambda<2/3$, the slope at the fixed point will decrease through -1 as μ decreases through

$$\mu_0=\frac{3\lambda^2}{4(1-\lambda)}. \tag{21}$$

This indicates the possibility of a period-doubling bifurcation at $\mu=\mu_0$ and $\overline{z}=2/3$. In fact, the period-doubling bifurcation is always *subcritical*, since the third derivative of $G\circ G$ at $z=2/3$ is 27/4 and thus positive.

This fixed point of G is of course a periodic point of period m for p, so this leads to a sequence of periodic windows where the period increases by 1 between consecutive windows. If we look at the two ends of the periodic window for the mapping p, the one of smaller κ is thus a subcritical period-doubling bifurcation and corresponds to $\mu=\mu_0$. The upper end corresponds to $\mu=1$ and is a grazing bifurcation. This grazing bifurcation is not of the type studied here, which leads to a continuously growing attracting set, as can be seen from the fact that the eigenvalue of largest modulus is $-\lambda/[2(1-\lambda)]$ and thus negative.

If $0<\lambda<1/4$ there will be no period-doubling bifurcation on branch 0 as μ decreases from 1 towards λ^2. This implies that for μ close to 1 the fixed point on the branch over I_1 will also be stable. When μ decreases from 1 it will lose stability in a subcritical period-doubling bifurcation at

$$\mu_1=\frac{3}{4(1-\lambda)}. \tag{22}$$

This means that there are two stable solutions for $\mu_1<\mu\leqslant 1$. For the mapping p they correspond to period m and $m-1$. There is always at least one stable periodic solution of p for κ small, and the bifurcations they undergo are as above.

In Fig. 2 we show the different regions where stable fixed points exist. Region N is outside of allowed parameter values. In region 1 there is a stable fixed point on branch 0. In region 2 there are two stable fixed points, one on each of the branches 0 and 1. The dash-dotted line marks the period-doubling bifurcation on branch 0 at $\mu=\mu_0$. The dashed line marks the period-doubling bifurcation on branch 1 at $\mu=\mu_1$. It can be remarked that the μ_1 curve is a continuation of the μ_0 curve, if we do not normalize the μ parameter to the interval $(\lambda^2,1]$.

C. When no fixed point is stable

Consider the case when $1/4<\lambda<2/3$ and $\lambda^2<\mu<\mu_0$. Then there exists a fixed point $\overline{z}$ with $2/3<\overline{z}<J_0$ with an eigenvalue less than -1. For $z>\overline{z}$, the derivative of G is

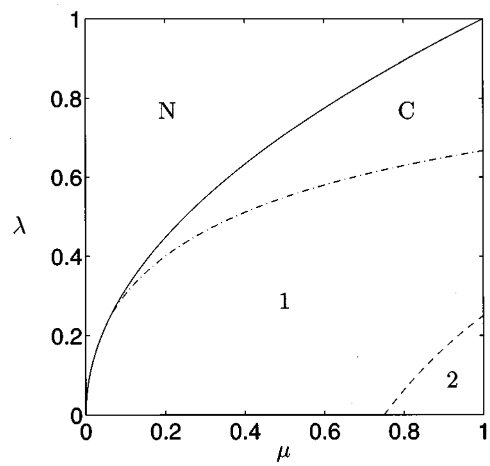

FIG. 2. The different parameter regions for the mapping G. Region N is outside of allowed parameter values. In region C motion is chaotic on the whole interval. In region 1 there is a single attractive fixed point. In region 2 there are two attractive fixed points.

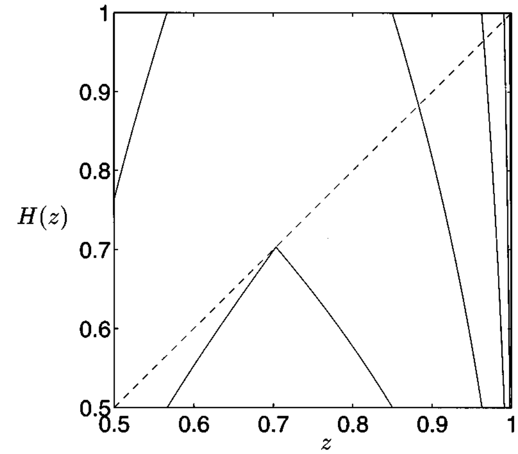

FIG. 3. The mapping H when $\lambda=0.5$ and $\mu=0.3$, together with the identity line.

less than -1 but this is not necessarily the case when $z\in[\lambda,\overline{z})$. To get around this, consider $G\circ G$ for $z\in[\lambda,\overline{z})$. The derivative of $G\circ G$ is larger than 1 near $z=\overline{z}$ and we can in fact show that the second derivative of $G\circ G$ is a positive number times the factor $2/3-G$. Since $G(z)>\overline{z}>2/3$, the second derivative is negative and the derivative of $G\circ G$ increases with z decreasing from $\overline{z}$. The fact that $G\circ G$ consists of more than one branch in the region $z\in[\lambda,\overline{z})$ does not cause any problems, since the derivative increases with decreasing z in each branch and when going from one branch to the next the derivative increases by a factor $1/\lambda$ (which can be seen from the fact that this is also the case for the mapping G). In short, if we define a mapping

$$H:[\lambda,1)\to[\lambda,1),$$

$$H(z)=\begin{cases} G(G(z)) & \text{if } z\in[\lambda,\overline{z}] \\ G(z) & \text{if } z\in[\overline{z},1), \end{cases} \tag{23}$$

the derivative will be less than -1 and decreasing as z increases from $\overline{z}$, and will be greater than 1 and increasing as z decreases from $\overline{z}$. Figure 3 illustrates this. Thus the mapping H will have no stable periodic points and the same goes for G. For the mapping p, if $1/4<\lambda<2/3$ and κ small, a stable period $m-1$ point and a stable period m point will be separated by a band of chaos.

Returning to Fig. 2, we have shown that for region C, the motion is chaotic.

D. Impossibility of other attractors

Assume $\lambda<2/3$. The subcritical period-doubling bifurcation creates an unstable orbit of period two. The amplitude rapidly grows as μ is changed until one of the two points in the period reaches λ where the orbit is destroyed in a grazing bifurcation. This happens at

$$\mu_g=\left[\frac{1}{2}\left(\sqrt{\frac{1+3\lambda}{1-\lambda}}-1\right)\right]^2. \tag{24}$$

For $0<\lambda<2/3$ we have $\lambda^2<\mu_g<1$, for $1/4<\lambda<2/3$ we have $\mu_0<\mu_g$, and for $0<\lambda<1/4$ we have $\mu_g<\mu_1$. We will now show that G can have no other attractors besides the fixed points on branches 0 and 1.

Suppose $1/4<\lambda<2/3$ with $\mu_g<\mu<1$, or $0<\lambda<1/4$ with $\mu_g<\mu<\mu_1$. There is a stable fixed point $\overline{z}_0$ in I_0 and an unstable fixed point $\overline{z}_1$ on I_1. Construct a mapping that is $G\circ G$ on $[\lambda,\overline{z}_1]$ and G on $(\overline{z}_1,1)$. For this mapping, all points in I_0 go to $\overline{z}_0$ monotonically, whereas for the other points, the mapping has a slope that is greater than 1 in modulus.

Suppose $1/4<\lambda<2/3$ with $\mu_0<\mu<\mu_g$, or $0<\lambda<1/4$ with $\lambda^2<\mu<\mu_g$. Now there is an unstable orbit of period two with members z_1 and z_2 on branch 0. Again studying $G\circ G$ on I_0, we find that points in the interval (z_1,z_2) go monotonically to $\overline{z}_0$, and that the mapping has slope greater than 1 in modulus outside of this interval.

Finally, for $0<\lambda<1/4$ with $\mu_1<\mu<1$, there are two stable fixed points $\overline{z}_0$ and $\overline{z}_1$, and an orbit of period two with members z_1 and z_2 on branch I_1. Points in I_0 go directly to $\overline{z}_0$ and points in (z_1,z_2) to $\overline{z}_1$. $G\circ G$ has slope greater than 1 in $(J_0,z_1]$ and G has slope less than -1 in $[z_2,1)$.

We have shown that each stable fixed point has an invariant interval associated with it, and it attracts this interval. We have also shown that outside of these intervals, either G or $G\circ G$ has slope larger than 1 in modulus. To show that there are no other attractors, we argue as follows. Consider an open interval outside of the intervals around the stable fixed points, and successive iterations of the interval. Since we have found that the mapping (or at least the double mapping) is expanding there, the interval increases in length over iterations. After a finite number of iterations at least one of two things must happen: we intersect one of the intervals around the stable fixed points, or the mapped interval contains one of the points J_i of discontinuity for G. In the latter case the interval $(z',1)$ for some $z'<1$ will be covered in the next iteration, and yet another iteration later some points are mapped into the intervals around the stable fixed points. To conclude: Any open set contains points that converge to one of the stable fixed points, and thus there are no other attractors. (As was pointed out earlier, there is chaotic motion for all values of λ and μ, but it is only attracting in region C.)

E. Summary

Referring to Fig. 2, we have shown the following: (i) In region C there is a chaotic attractor spanning the whole interval of allowed z values. (ii) In region 1 the only attractor is a stable fixed point on branch 0. (iii) In region 2 the only attractors are two stable fixed points, one on each of branches 0 and 1.

IV. REVIEW OF RESULTS AND DISCUSSION

For the limit mapping G, we have been able to identify all attractors and found explicit regions where there is a chaotic attractor, a single fixed point attractor, and two fixed point attractors. The findings agree with the results of earlier investigations of particular systems in 1, 2, and 3 dimensions. The introduction of the limit mapping G enables us to make an analysis that should be valid for all considered systems, in the limit of small bifurcation parameter.

The firm connection between results obtained for the limit mapping, and the dynamics for the full system, is not strictly shown in this paper. G is a pointwise limit, and as has been pointed out in a previous section, the fact that F for nonzero κ acts in N dimensions and only has a finite number of discontinuities prevents the convergence from being uniform in any normal sense of the word. Nevertheless, as differences are only pronounced near $z=1$, where both mappings are very steep, we do not expect that this will affect the validity of the given results.

This investigation has not been much concerned with the way particular aspects of the full system change the results when the bifurcation parameter cannot be regarded as small. For example, when $\lambda > 2/3$ it has often been observed that stable periodic windows appear for large enough values of the bifurcation parameter. Also, Budd and Dux [3] find a *supercritical* period-doubling bifurcation in some of the periodic windows. As we have shown here, the period-doubling bifurcation is always subcritical for small enough parameter values.

ACKNOWLEDGMENTS

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